arXiv:1911.10429v1 [physics.plasm-ph] 23 Nov 2019

Nonlinear force-free configurations in cylindrical geometry

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(Received xx; revised xx; accepted xx)

We find a new family of solutions for force-free magnetic structures in cylindrical geometry. These solutions have radial power-law dependance and are periodic but non-harmonic in azimuthal direction; they generalize the vacuum z-independent potential fields to current-carrying configurations.

1. Introduction

Force-free magnetic configurations, satisfying condition $\mathbf{B} = \kappa \mathbf{J}$, where $\mathbf{B}$ is magnetic field and $\mathbf{J}$ is current density, are examples of magnetic structures that may represent the final stages of magnetic relaxation, or can be used as building block of plasma models (Lundquist 1951; Woltier 1958; Taylor 1974; Priest & Forbes 2000).

Particular *linear* examples of force-free equilibria, with spatially constant κ, were considered by Chandrasekhar & Kendall (1957). The most often-used configurations are Lundquist fields in cylindrical geometry (Lundquist 1951) and spheromaks in spherical geometry (Bellan 2000).

Using the self-similar assumption Lynden-Bell & Boily (1994) found non-linear self-similar solutions in spherical geometry. Their model of axially symmetric twisted configurations has been widely used in astrophysical and space applications (e.g. Thompson *et al.* 2002; Shibata & Magara 2011). In the spirit of Lynden-Bell & Boily (1994), in this paper we construct similar non-linear magnetic configurations in cylindrical geometry.

2. Self-similar configuration in cylindrical geometry

Shafranov (1966) and Grad (1967) formulated what is known as the Grad-Shafranov equation, separating complicated magnetic configuration in the set of nested/foliated flux surfaces, given by the condition that flux function P is constant on the surface, and the encompassed current flow. Let us look for force-free equilibria that are independent of coordinate z. The two Euler potentials α and β (or, equivalently, the related Clebsh variables) are

$$\begin{aligned} \alpha &= z \\ \beta &= P(r, \phi) \end{aligned} \tag{2.1}$$

while the magnetic field can be written as

$$\mathbf{B} = \nabla P \times \nabla z + g(P) \nabla z \tag{2.2}$$

where g is some function.

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Next we introduce a self-similar anzats

$$\begin{aligned} &P(r,\phi) = r^{-l} f(\phi) \\ &g(P) = \mathcal{C}|P|^p \\ &\mathbf{B} = \nabla P \times \nabla z + \mathcal{C}|P|^p \nabla z = \{f', lf, \mathcal{C}|f|^p\} r^{-(l+1)} \end{aligned} \tag{2.3}$$

Absolute value of $|P|$ in the non-linear term ensures that magnetic field is real ($f(\phi)$ can become negative). Below any appearance of f to non-integer power is to be understood to involve $\sqrt{f^2} = |f|$.

By dimensionality

$$p = 1 + 1/l \tag{2.4}$$

Equation for f becomes

$$lf'' + l^3 f + \mathcal{C}^2 (1+l) f^{(2+l)/l} = 0 \tag{2.5}$$

(note that the component B_z enters here as B_z^2. This justifies the use of $|P|$.)

For vacuum fields $\mathcal{C} = 0$ the above relations reproduce

$$\begin{aligned} &P \propto r^{-m} \sin(m\phi) \\ &B_r \propto r^{-(1+m)} f' \\ &B_\phi \propto m r^{-(1+m)} f \end{aligned} \tag{2.6}$$

with integer m.

The first integral is

$$f'^{,2} + l^2 f^2 + \mathcal{C}^2 |f|^{2(1+l)/l} = H_0, \tag{2.7}$$

By redefining $f \to \sqrt{H_0} f$ and $\mathcal{C} \to \mathcal{C} H_0^{-1/(2l)}$ the parameter H_0 can be set to unity,

$$f'^{,2} + l^2 f^2 + \mathcal{C}^2 |f|^{2(1+l)/l} = 1. \tag{2.8}$$

Equation (2.8) is the main equation describing non-linear force-free structures in cylindrical geometry. It depends on one parameter - the current strength $\mathcal{C}$. For a given $\mathcal{C}$ the value of l is then determined as an eigenvalue problem by requiring periodicity in ϕ, as we describe next.

We can solve for f in quadratures:

$$\phi = \int \left(\sqrt{1 - l^2 f - \mathcal{C}^2 |f|^{2(1+l)/l}} \right)^{-1} df \tag{2.9}$$

(so that the integration constant in Eq. (2.8) is just a phase ϕ where $f = 0$).

Periodicity in ϕ requires

$$\int_0^{f_{max}} \left(\sqrt{1 - l^2 f - \mathcal{C}^2 |f|^{2(1+l)/l}} \right)^{-1} df = \frac{\pi}{2m} \tag{2.10}$$

where $m = 1, 2..$ is an integer azimuthal number (see a comment after Eq. (3.5) why odd solutions, $\propto 2m+1$ in the denominator, are discarded), The value of f_{max} satisfies

$$1 - m^2 f_{max} - \mathcal{C}^2 f_{max}^{2(1+l)/l} = 0 \tag{2.11}$$

For given $\mathcal{C}$ the relations (2.10)- (2.11) constitute an eigenvalue problem on l. (For vacuum no current case $\mathcal{C} = 0$ this reduces to $l = m$, an integer - checkmark.) In practice, we follow the following procedure: for each $m = 1, 2..$ we assume some l and find $\mathcal{C}$ using relations (2.10)- (2.11). Thus, for each m there is a continuous relation $\mathcal{C}(l)$. (Physically, of course, it is the current $\mathcal{C}$ the determines the radial index l.)

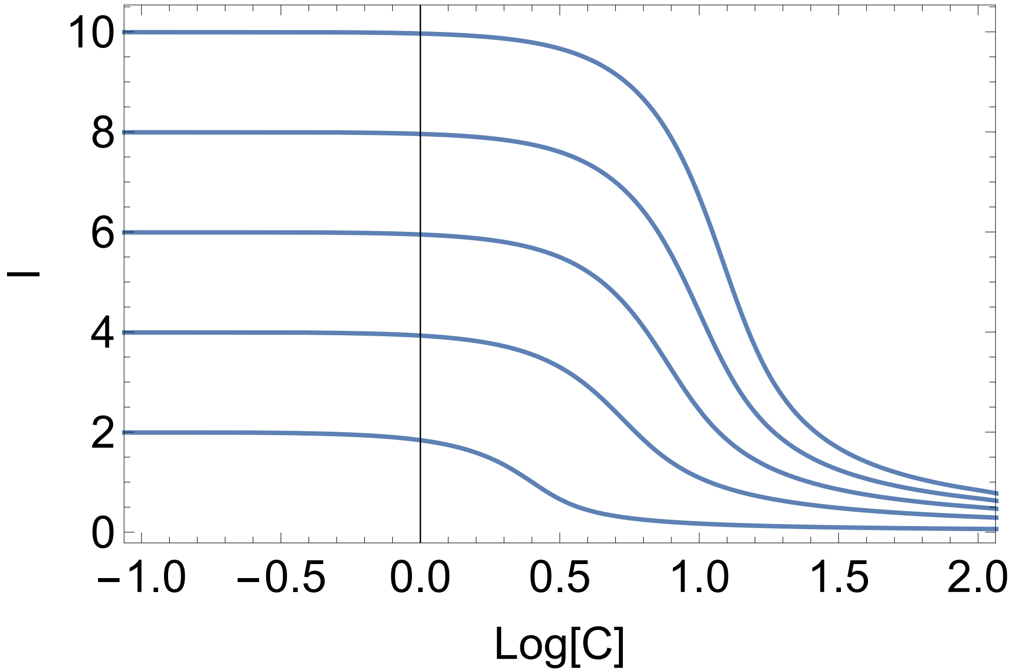

Figure 1: Dependence of the radial index l on the current parameter $\mathcal{C}$ for various harmonics $m = 2, 4, 6, 8, 10$.

Results are plotted in Figs 2-3. In Fig. 2 we plot a particular solution for $l = 1$ and $m = 2$. The flux functions forms a "petal" patter in azimuthal angle with number of "petals" equal $2m$. There is a corresponding axial, unidirectional magnetic field B_z.

In Fig. 1 we plot the curves $l(\mathcal{C})$ for various $m = 2, 4, 6, 8, 10$. Each curve starts at a point $\{\mathcal{C} = 0, l = 2m\}$. For non-zero current $\mathcal{C} > 0$ the radial dependence becomes more shallow, $l < 2m$.

In Fig. 3 we plot values of $\mathcal{C}$ as a function of azimuthal number m for different values of $l = 0.25...2$. Dashed lines are for convenience only, they connect points corresponding to the same radial parameter l.

3. Analysis of the solutions

In a formulation of force-free fields in the form

$$\operatorname{curl} \mathbf{B} = \kappa \mathbf{B} \tag{3.1}$$

the value of κ is

$$\kappa = \mathcal{C} \frac{1 + l}{lr} f^{1/l} \tag{3.2}$$

It is constant on flux surfaces P, Eq. 2.3.

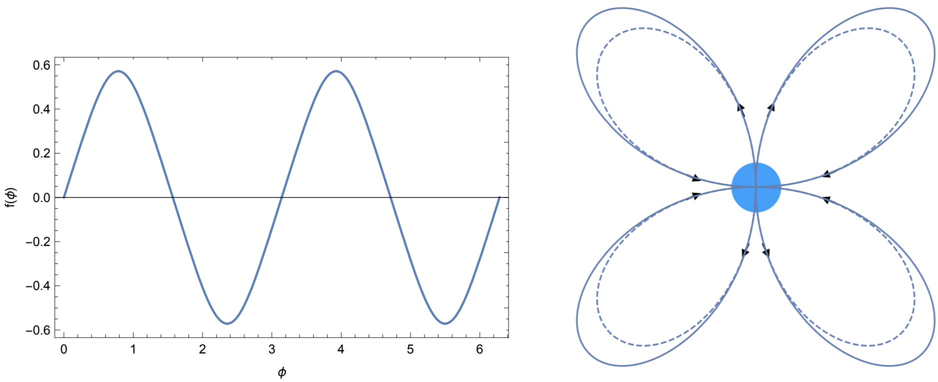

Figure 2: Left panel: example of solution $f(\phi)$ for $l = 1$, $m = 2$. In this case $\mathcal{C} = 2.517$. Right panel: structure of poloidal field. (Due to the assumed self-similar radial structure the solutions do not extend to $r = 0$). Dashed line is the corresponding vacuum case, Eq. 2.6.

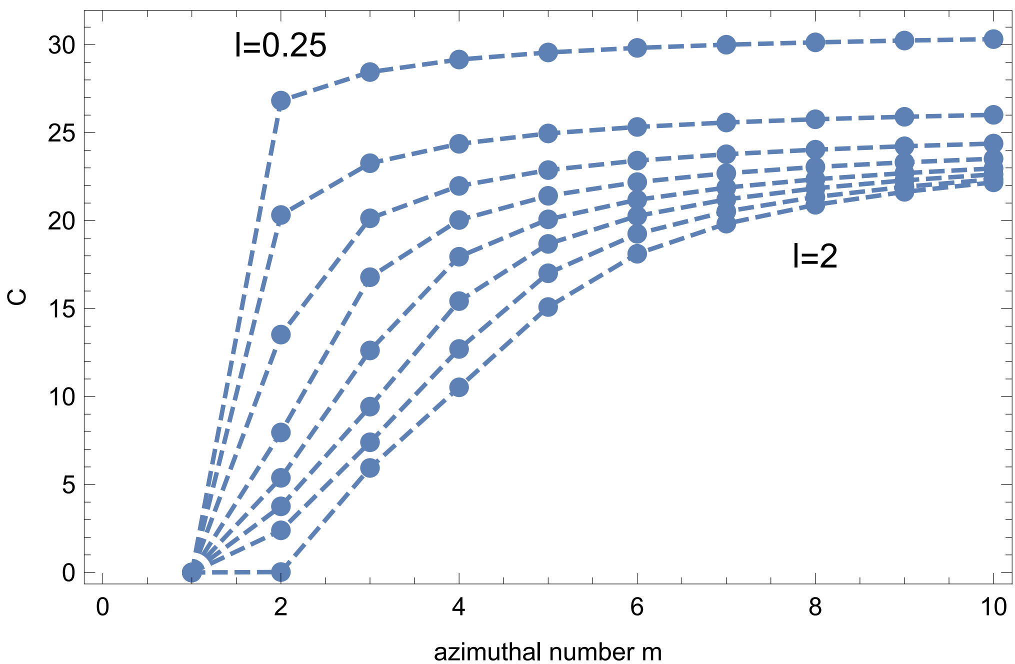

Figure 3: Values of $\mathcal{C}$ as a function of azimuthal number m for different values of $l = 0.25...2$ (in steps of 0.25). Top curves correspond to smaller l.

The current density (we incorporate factors of $4\pi/c$ into definition of magnetic field)

$$
\begin{aligned}
j_r &= \mathcal{C}\frac{1+ll}{l} r^{-(l+2)} \partial_\phi \left(|f|\right)^{(1+l)/l} \\
j_\phi &= \mathcal{C}(l+1) r^{-(l+2)} \left(|f|\right)^{(1+l)/l} \\
j_\phi &= r^{-(l+2)} \left(f'' + l^2\right) = -\mathcal{C}^2 \frac{1+ll}{l} r^{-(l+2)} \left(|f|\right)^{(2+l)/l}
\end{aligned} \tag{3.3}
$$

The total axial current is

$$I_z = \int_{r_0}^{\infty} r dr \int_0^{2\pi} d\phi j_z = -\frac{r_0^{-l}}{l} \int_0^{2\pi} d\phi \left(f'' + l^2 f\right). \tag{3.4}$$

where r_0 is the inner boundary. The total axial current vanishes if two conditions are satisfied

$$\int_0^{2\pi} f d\phi = 0$$
$$f'(2\pi) = f'(0) \tag{3.5}$$

All the solutions considered here satisfy these conditions: the second one requires even azimuthal numbers, $2m$. Generally, there is a larger family of self-similar force-free equilibria with non-zero total axial current.

There is non-zero toroidal current

$$j_\phi = \mathcal{C}(l+1) r^{-2-l} |f|^{(1+l)/l}$$
$$\int_0^{2\pi} d\phi j_\phi \neq 0 \tag{3.6}$$

The radial current density integrated over ϕ satisfies

$$j_r \propto \int_0^{2\pi} d\phi \partial_\phi \left(|f|\right)^{(1+l)/l} = \left(|f|\right)^{(1+l)/l}\Big|_0^{2\pi} = 0 \tag{3.7}$$

4. Discussion

In this paper we *make analytical progress with the highly nonlinear problem[s] of magnetohydrodynamics* (Lynden-Bell & Boily 1994). We find a class of non-linear self-similar force-free equilibria in cylindrical geometry. The solutions we find all connect to the vacuum case, in which case the flux function is $P_{vac} \propto r^{-m} \sin(m\phi)$. Structures with vanishing total axial current require even values of m (hence $m \rightarrow 2m$). For non-zero distributed current with the current parameter $\mathcal{C}$ the radial dependence changes to r^{-l}, with $l < 2m$, while remaining periodic in ϕ at $2m$. Solutions for a given m resemble vacuum solutions $\propto \sin(2m\phi)$, but they are not exactly harmonic in the nonlinear case.

For very large currents the solutions asymptote to $l \approx 0$, but never reach this limit. The case $l = 0$ corresponds to $B_r \propto 1/r$. Mathematically, this is the analogue of split monopole case in the spherical geometry - split monopole case can be achieved in spherical geometry (with corresponding anti-monopole in the opposite hemisphere), but is not possible in the cylindrical geometry.

Acknowledgments

This work had been supported by NASA grant 80NSSC17K0757 and NSF grants 10001562 and 10001521.

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