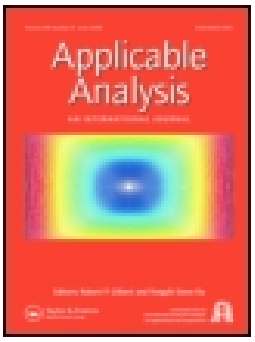

Applicable Analysis
An International Journal

ISSN: 0003-6811 (Print) 1563-504X (Online) Journal homepage: https://www.tandfonline.com/loi/gapa20

On the resilience of a fractional compartment model

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APPLICABLE ANALYSIS
https://doi.org/10.1080/00036811.2020.1712370

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Min Wang

Department of Mathematics, Kennesaw State University, Marietta, USA

ABSTRACT
In this paper, we consider the resilience problem for a type of fractional compartment models by studying an equivalent boundary value problem. A series of criteria on the existence, uniqueness, and multiplicity of solutions or positive solutions are obtained. The sensitivity analysis for a model parameter is then conducted. Examples are given to demonstrate the applications of the results as well.

ARTICLE HISTORY
Received 20 June 2019
Accepted 1 January 2020

COMMUNICATED BY
Scott Hansen

KEYWORDS
Green's function; positive solutions; fractional sensitivity equation

2010 MATHEMATICS SUBJECT CLASSIFICATIONS
34B08; 34B18

1. Introduction

In this paper, we consider a fractional compartment model

$$u' + aD^{\alpha}_{0+}u = f(u,t), \quad 0 < t < 1, \tag{1}$$

together with the boundary condition (BC)

$$u(0) = bu(1), \tag{2}$$

where $a > 0$, $b > 0$, $0 < \alpha < 1$, $f(x,t) \in C(\mathbb{R} \times [0,1]), \mathbb{R})$, and $D^{\alpha}_{0+}u$ denotes the left Riemann–Liouville fractional derivative of u defined by

$$\left(D^{\alpha}_{0+}u\right)(t) = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-s)^{-\alpha}u(s)\,\mathrm{d}s,$$

provided the right-hand side exists, where Γ is the Gamma function.

Fractional differential equations (FDEs) have been a focus of research for decades and have been applied in many areas. Compared with classical integer order differential equations (DEs), FDEs own superior merits of describing the long-term memory in time or long-range spatial interactions. In addition, the involvement of fractional order derivative(s) offers more degrees of freedom than integer order DEs even with exactly the same assumptions and conditions. These appealing features have attracted scholars to develop FDE models for problems arise in various fields of sciences and engineering, see for example [1–11].

To improve the interpretability of FDE models, Angstmann, Henry, and McGann [1,2] proposed a framework to develop fractional compartment models by analyzing the underlying physical process.

CONTACT Min Wang min.wang@kennesaw.edu

This framework was further applied in [3]. Following a similar idea, Graef, Ho, Kong, and Wang [5] investigated the bike share system. The inventory of a bike station, u_i, was described by

$$u_i' = q_i(t) - \omega_i(t)u_i - \Theta_i(t)a_i^{-\alpha_i}D_{0+}^{1-\alpha_i}\left(\frac{u_i}{\Theta_i}\right), \quad t > 0, \tag{3}$$

where q_i represents the arrival flux at a station, $\omega_i(t)u_i$ represents a Markov removal process that is independent of the history, and $\Theta_i(t)a_i^{-\alpha_i}D_{0+}^{1-\alpha_i}(u_i/\Theta_i)$ represents a non-Markov removal process that relates to the bike waiting time at a station with

$$\Theta_i(t) = \Theta_i(t,0) = \exp\left(-\int_0^t \omega(s)\,\mathrm{d}s\right).$$

The reader is referred to [5] for the details of the terms.

Remark 1.1: When the fractional term is ignored, Equation (3) becomes a first order ordinary DE model and $q_i(t) - \omega_i(t)u_i$ in (3) represents the net change determined by the current status, i.e. the arrival flux and the Markov removal process. The inclusion of the fractional term $\Theta_i(t)a_i^{-\alpha_i}D_{0+}^{1-\alpha_i}(u_i/\Theta_i)$ in (3) integrates the additional historical information, i.e. the waiting time, into the inventory model. Moreover, it is clear that $D_{0+}^{1-\alpha_i}(u_i/\Theta_i) \to u_i/\Theta_i$ as $\alpha_i \to 1$ and $D_{0+}^{1-\alpha_i}(u_i/\Theta_i) \to (u_i/\Theta_i)'$ as $\alpha_i \to 0$. Therefore, Equation (3) offers more degrees of freedom than the traditional integer order DE models by adjusting the parameter α_i. The inclusion of extra information and degrees of freedom ensures the flexibility, superiority, and practicality of FDE models.

One interesting question regarding model (3) is its resilience, i.e. under what circumstances the inventory will restore to certain level at time $\mathfrak{t}_1 > 0$ without extra interference. This question is equivalent to study the existence of solutions of the boundary value problem (BVP) consisting of Equation (3) and the periodic BC

$$u_i(0) = u_i(\mathfrak{t}_1), \quad i = 1, \ldots, N. \tag{4}$$

It is notable that a special case of BVP (3), (4) when $\omega \equiv 0$ and $N = 1$ has been considered by Lam and Wang [12]. For the general case, by a change of variable (see [5, Theorem 3.1]) and rescaling $[0, \mathfrak{t}_1]$ to $[0, 1]$, BVP (3), (4) can be transformed to an equivalent BVP

$$\begin{aligned} &v_i' = \mathfrak{t}_1 Q_i(\mathfrak{t}_1 t) - a_i^{-\alpha_i}D_{0+}^{1-\alpha_i}v_i, \\ &v_i(0) = \exp\left(-\int_0^{\mathfrak{t}_1} \omega_i(s)\,\mathrm{d}s\right)v_i(1), \end{aligned} \qquad 0 < t < 1,\ i = 1, \ldots, N, \tag{5}$$

with

$$Q_i(t) = \exp\left(\int_0^t \omega_i(s)\,\mathrm{d}s\right)q_i(t).$$

Clearly BVP (1), (2) covers BVP (5) as a special case when $N = 1$. By Remark 1.1 as well as the relation between (3) and (5), the nonlinear term f in (1) may be interpreted as the net change due to the current compartment status after certain appropriate transform, and the fractional term $aD_{0+}^{\alpha}u$ in (1) may be interpreted as the impact of the historical information to the rate of change of the compartment.

Because of the relation between BVP (1), (2) and BVP (5), the study of BVP (1), (2) will benefit the bike share system inventory management. Since model (3) can also find applications in other areas by appropriate modifications on the assumptions, the findings on BVP (1), (2) will contribute to the resilience problems in many areas. Therefore, there is a practical need to investigate BVP (1), (2).

Fractional BVPs have been an active research area for decades. The reader is referred to [12–17] and references therein for some recent advances. One major approach to study the existence, uniqueness, and multiplicity of solutions of the BVPs is the fixed point theory. The associated Green's functions play a crucial role in constructing the functional operators needed in this approach. Due to the unique feature of fractional calculus, the derivation of Green's functions for fractional BVPs is not trivial. In addition, it is challenging to prove the Green's functions' positivity, which is essential to many theorems on the existence of positive solutions. In this paper, we will follow an idea developed in [12] to construct the Green's function associated with BVP (1), (2). The positivity of the Green's function will be proved by leveraging the properties of Mittag–Leffler function. Because of the wide representation of BVP (1), (2), our results will benefit both theoretical research and applications.

Due to the measurement and data limitations in real life, the errors are inevitable when identifying the model parameters from the data. It is necessary to understand the impact of the parameter variation to the model performance. Parameter sensitivity analysis (SA) has been extensively applied to evaluate such impacts, see for example [11,18] and references therein. It is notable that in most existing literature, the underlying models are initial value problems (IVPs). The resulting sensitivity equations (SEs) are IVPs as well. On the other hand, the SE associated with BVP (1), (2) will be a fractional BVP, which is relatively rare. So our work will be supplementary to the exisiting works.

In this paper, we will study the existence and uniqueness of solutions to BVP (1), (2), as well as the SA with respect to the parameter a. The paper is organized as follows: after this introduction, the results on the uniqueness and existence of solutions are stated in Section 2. The SA is performed in Section 3. Two examples are given in that section as well. All the proofs are given in Section 4. Section 5 contains the conclusions and final discussions.

2. Existence and uniqueness of solutions

We first study the associated linear BVP

$$u' + aD^{\alpha}_{0+}u = h(t), \quad 0 < t < 1, \tag{6}$$

with BC (2).

Theorem 2.1: *Assume $bE_\alpha(-a) \neq 1$. For any $h \in C[0,1]$, BVP (6), (2) has a unique solution given by*

$$u(t) = \int_0^1 G(t,s)h(s)\,\mathrm{d}s,$$

with $G : [0,1] \times [0,1] \to \mathbb{R}$ defined by

$$G(t,s) = \begin{cases} \dfrac{bE_\alpha(-a(1-s)^\alpha)E_\alpha(-at^\alpha)}{1-bE_\alpha(-a)} + E_\alpha(-a(t-s)^\alpha), & 0 \le s \le t \le 1, \\ \dfrac{bE_\alpha(-a(1-s)^\alpha)E_\alpha(-at^\alpha)}{1-bE_\alpha(-a)}, & 0 \le t < s \le 1, \end{cases} \tag{7}$$

where $E_\alpha(t)$ is the Mittag-Leffler function defined by

$$E_\alpha(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(n\alpha+1)}.$$

Remark 2.1: It is easy to see that G defined by (7) is the Green's function for the BVP consisting of

$$u' + aD^{\alpha}_{0+}u = 0, \quad 0 < t < 1,$$

and BC (2).

The following bounds of G are needed in our proofs. Define

$$\overline{G} = \frac{b+1-bE_\alpha(-a)}{1-bE_\alpha(-a)} \quad \text{and} \quad \underline{G} = \frac{b[E_\alpha(-a)]^2}{1-bE_\alpha(-a)}. \tag{8}$$

Theorem 2.2: *Let G, $\underline{G}$, and $\overline{G}$ be defined by* (7) *and* (8) *respectively. Assume $bE_\alpha(-a) < 1$. For any $(t,s) \in [0,1] \times [0,1]$, we have*

$$0 < \underline{G} < G(t,s) \le \overline{G}. \tag{9}$$

Remark 2.2: By (8), it is clear that $\overline{G}$ and $\underline{G}$ are continuous functions with respect to the parameter a.

Now we are ready to study the nonlinear BVP (1), (2). Without further mentioning, we assume $bE_\alpha(-a) < 1$ in the remaining of this paper.

The first result is on the uniqueness of solutions.

Theorem 2.3: *Let $\overline{G}$ be defined by* (8). *Assume $f(x,t)$ satisfies the Lipschitz condition in x with $\kappa \in [0, \overline{G}^{-1})$, i.e. there exists $\kappa \in [0, \overline{G}^{-1})$ such that for any $x_1, x_2 \in \mathbb{R}$ and $t \in [0,1]$,*

$$|f(x_1,t) - f(x_2,t)| \le \kappa |x_1 - x_2|. \tag{10}$$

Then BVP (1), (2) *has a unique solution.*

Motivated by the applications, we are particularly interested in the positive solutions of BVP (1), (2). Next, we consider the existence and multiplicity of positive solutions. Define

$$\gamma = \frac{\underline{G}}{\overline{G}} = \frac{b[E_\alpha(-a)]^2}{b+1-bE_\alpha(-a)}. \tag{11}$$

By (9), we have $0 < \gamma < 1$.

For $u \in C[0,1]$, we denote by $\|u\|$ the standard maximum norm of u. The next theorem is a result on the existence of positive solutions of BVP (1), (2).

Theorem 2.4: *If there exist $0 < r_* < r^*$ [respectively, $0 < r^* < r_*$], such that*

$$f(x,t) \le \overline{G}^{-1} r_* \quad \textit{for all } (x,t) \in [\gamma r_*, r_*] \times [0,1] \tag{12}$$

and

$$f(x,t) \ge \underline{G}^{-1} r^* \quad \textit{for all } (x,t) \in [\gamma r^*, r^*] \times [0,1]. \tag{13}$$

Then BVP (1), (2) *has at least one positive solution u with $r_* \le \|u\| \le r^*$ [respectively, $r^* \le \|u\| \le r_*$].*

By applying Theorem 2.4 multiple times, it is easy to see the following results hold.

Corollary 2.5: *Let* $\{r_i\}_{i=1}^{N} \subset \mathbb{R}$ *such that* $0 < r_1 < r_2 < r_3 < \cdots < r_N$. *Assume either*

(a) *f satisfies* (12) *with* $r = r_i$ *when i is odd, and satisfies* (13) *with* $r = r_i$ *when i is even; or*
(b) *f satisfies* (12) *with* $r = r_i$ *when i is even, and satisfies* (13) *with* $r = r_i$ *when i is odd.*

Then BVP (1), (2) *has at least N−1 positive solutions* u_i *with* $r_i < \|u_i\| < r_{i+1}$, $i = 1, 2, \ldots, N-1$.

Corollary 2.6: *Let* $\{r_i\}_{i=1}^{\infty} \subset \mathbb{R}$ *such that* $0 < r_1 < r_2 < r_3 < \cdots$. *Assume either*

(a) *f satisfies* (12) *with* $r_* = r_i$ *when i is odd, and satisfies* (13) *with* $r^* = r_i$ *when i is even; or*
(b) *f satisfies* (12) *with* $r_* = r_i$ *when i is even, and satisfies* (13) *with* $r^* = r_i$ *when i is odd.*

Then BVP (1), (2) *has an infinite number of positive solutions.*

Remark 2.3: By [1–3,5], the term $f(x, t)$ in model (1) represents the flux entering the compartment. Theorems 2.3 and 2.4, and Corollaries 2.5 and 2.6 provide the criteria on f for the existence, uniqueness, and multiplicity of solutions or positive solutions. Appropriate policies may be made to adjust the flux to meet the conditions in these results. Therefore, our results may be used as references for resilience policy making.

3. Sensitivity analysis of parameter *a*

In this section, we study the impact of the parameter a to the solution of BVP (1), (2). We denote the solution as $u(t; a)$ to emphasize that it relies on the parameter a and define

$$\|u(\cdot; a)\| = \max_{t\in[0,1]} |u(t; a)|,$$

$$\|u(\cdot; a_1) - u(\cdot; a_0)\| = \max_{t\in[0,1]} |u(t; a_1) - u(t; a_0)|.$$

Our first result is on the continuous dependence of solution of BVP (1), (2) with respect to the parameter a.

Definition 3.1: Let $u(t; a_0)$ be a solution of BVP (1), (2) with respect to a_0. The solution is said to be continuously dependent on a at a_0 if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $a > 0$ with $|a - a_0| < \delta$, $\|u(\cdot; a) - u(\cdot; a_0)\| < \varepsilon$.

By (7) and (8), it is clear that G, $\overline{G}$, and $\underline{G}$ depend on a. Hence we may treat $\overline{G} = \overline{G}(a)$ as a function of a. Therefore, the range of the Lipschitz constant κ in (10) also relies on a. In the sequel, by "(10) holds when $a = a_0$" we mean that there exists $\kappa \in [0, 1/\overline{G}(a_0))$ such that (10) holds.

Theorem 3.2: *Assume* (10) *holds when* $a = a_0$. *Then the solution of BVP* (1), (2) *is continuously dependent on a at* a_0.

Next, we derive the SE associated with BVP (1), (2).

Theorem 3.3: *Assume* (10) *holds at a. Let* $f(x, t)$ *be continuous in* (x, t) *and have continuous first partial derivative with respect to x, for all* $(x, t) \in \mathbb{R} \times [0, 1]$. *Then* $u(t; a)$ *is continuously differentiable with respect to a. Let* $u_a = \partial u(t; a)/\partial a$. *Then* u_a *satisfies the SE*

$$\begin{aligned} &u_a' + aD_{0+}^{\alpha} u_a = f_u(u, t)u_a - D_{0+}^{\alpha} u, \quad 0 < t < 1, \\ &u_a(0) = bu_a(1), \end{aligned} \tag{14}$$

where $f_u(u, t)$ *denotes the partial derivative of f with respect to the first variable u.*

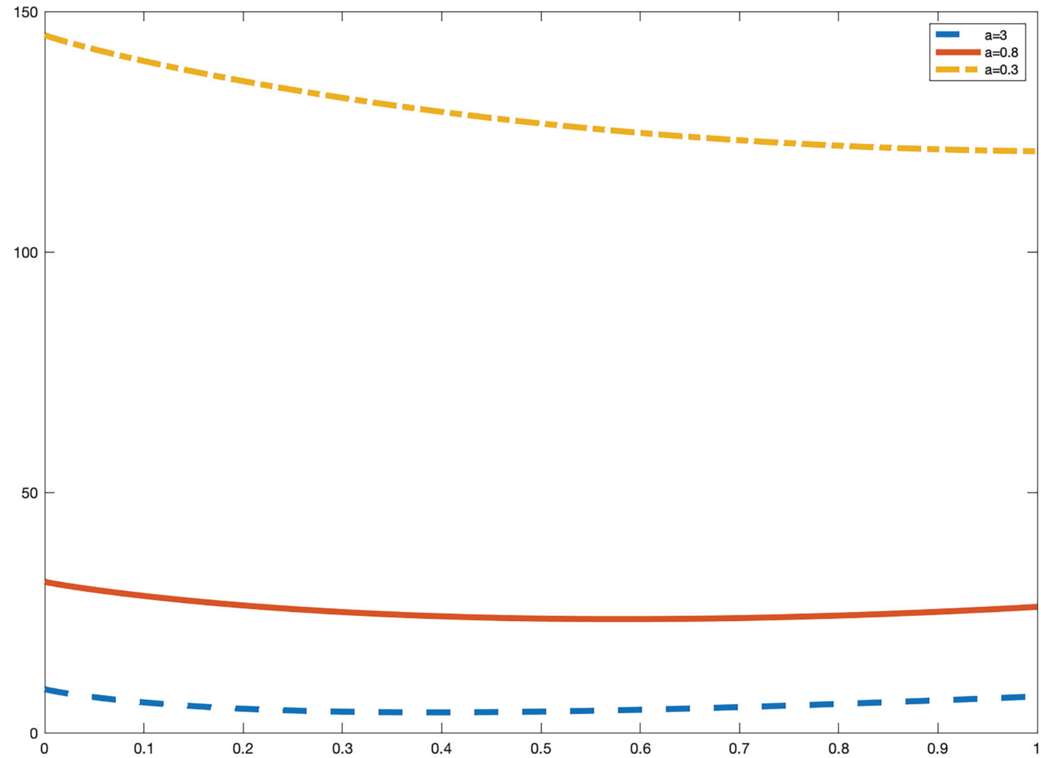

Figure 1. Solutions of BVP (15) with respect to various values of a.

Remark 3.1: (a) By (10), it is easy to see that $|f_u(u,t)| \in [0, 1/\overline{G}(a))$, $t \in (0,1)$.

(b) Theorem 3.3 provides an effective approach to estimate model performance variations due to the errors caused by parameter fitting.

We will conclude this section with two examples to demonstrate the applications of our results.

Example 3.4: Consider the linear BVP

$$\begin{aligned} &u' + aD^{\alpha}_{0+}u = 100\sin\left(\frac{t}{\pi}\right), \\ &u(0) = bu(1), \end{aligned} \tag{15}$$

where $\alpha = 0.9$, $b = 1.2$, $a = 0.3, 0.8, 3$. By Theorem 2.1, BVP (15) has a unique solution for each a. The plots of the solutions are given in Figure 1. The figure shows a decreasing trend with respect to a.

By Theorem 3.3, the associated SE is

$$\begin{aligned} &u_a' + aD^{\alpha}_{0+}u_a = -D^{\alpha}_{0+}u, \\ &u_a(0) = bu_a(1). \end{aligned} \tag{16}$$

The SE is solved for each a and the plots of u and u_a are given in Figures 2–4. These figures show that $u_a < 0$ on $[0,1]$. This is consistent to the decreasing trend observed in Figure 1. In addition, by studying the magnitudes of u_a in these figures, it is clear that u is more sensitive to a when a is close to

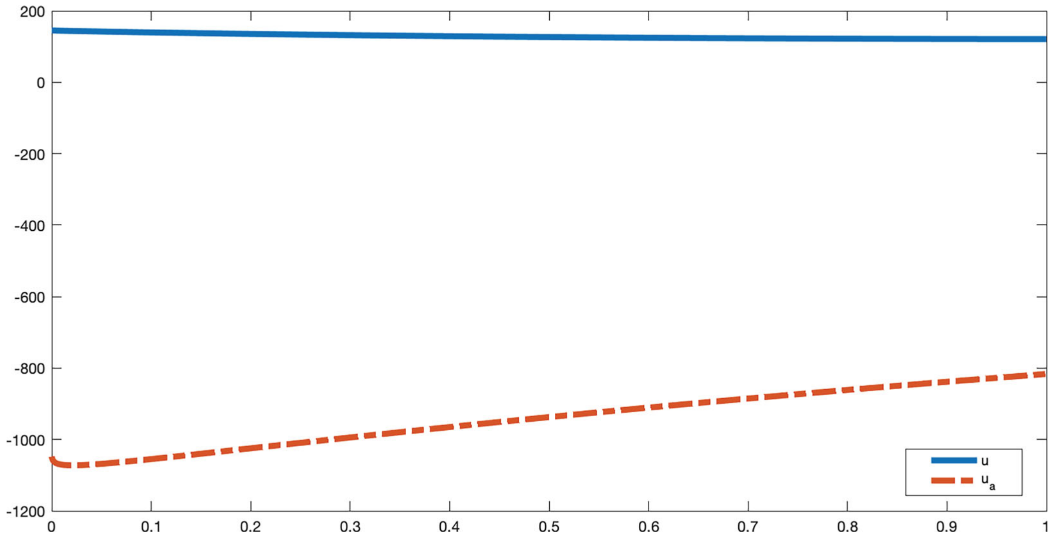

Figure 2. u and u_a when $a = 0.3$.

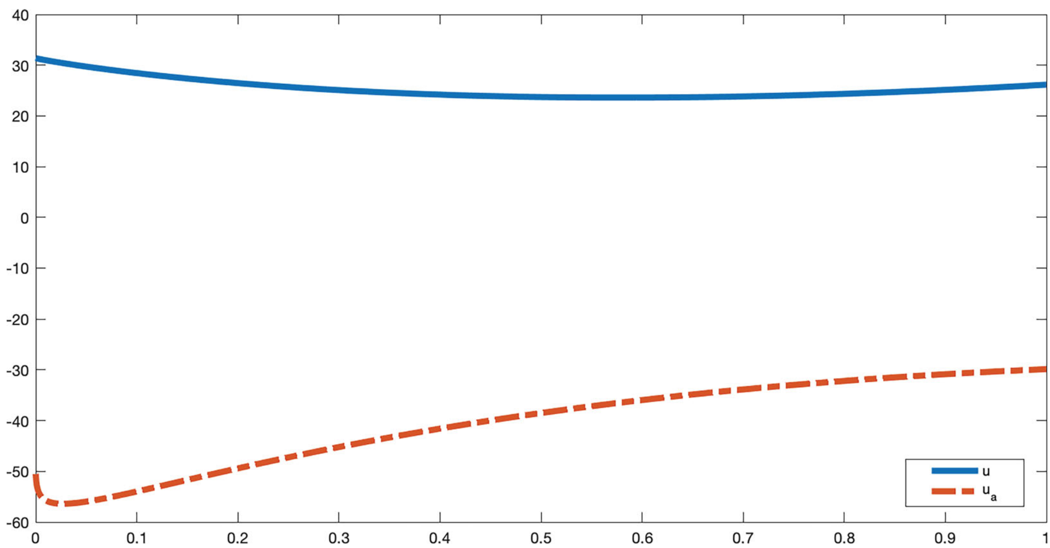

Figure 3. u and u_a when $a = 0.8$.

0, and for fixed a, u is more sensitive when t is close to 0. These observations help us better understand the impact of a to the solutions.

Example 3.5: Consider the nonlinear BVP

$$\begin{aligned} u' + aD^{\alpha}_{0+}u &= \frac{1}{2}\tanh(u) + e^{-t}, \\ u(0) &= bu(1), \end{aligned} \tag{17}$$

where $\alpha = 0.5$, $b = 0.8$, and $a = 4, 5, 6$. It is easy to verify that all the conditions of Theorem 2.3 are satisfied. Therefore, BVP (17) has a unique solution for every a. The solutions are plotted in Figure 5.

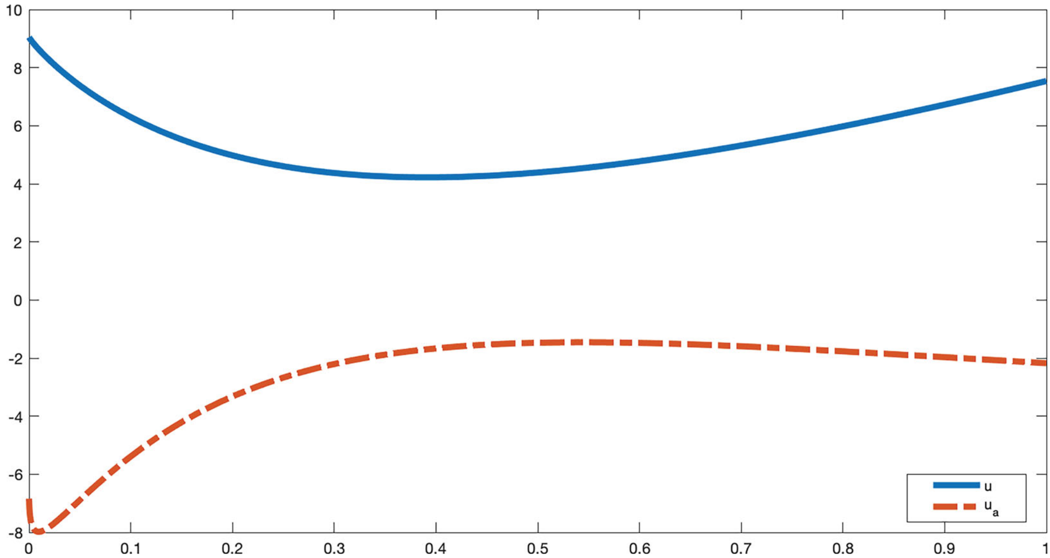

Figure 4. u and u_a when $a = 3$.

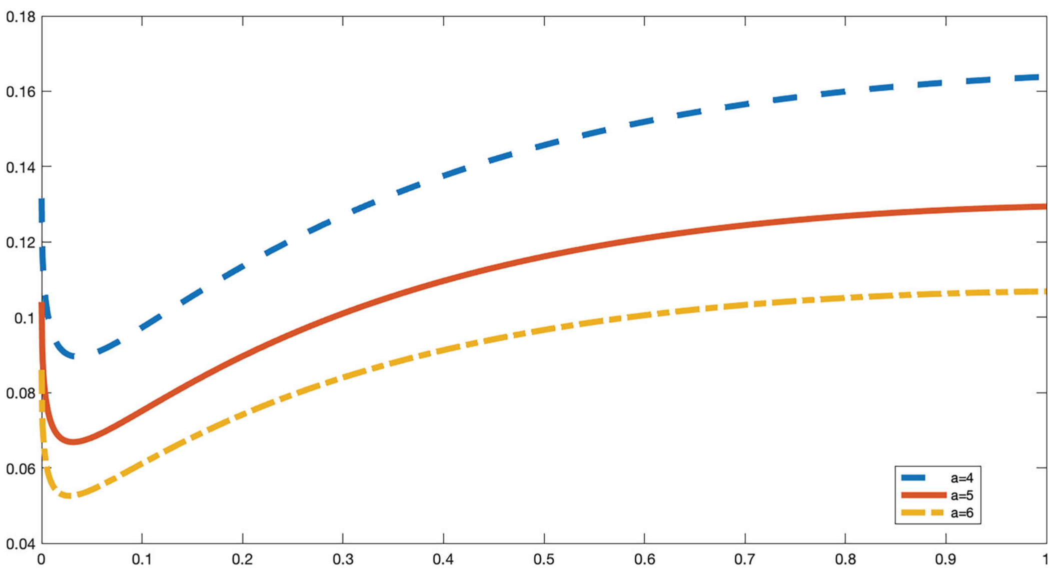

Figure 5. Solutions of BVP (17) with respect to various values of a.

By Theorem 3.3, the associated SE is

$$\begin{aligned} &u_a' + aD_{0+}^{\alpha}u_a = \frac{1}{2}\text{sech}^2(u)u_a - D_{0+}^{\alpha}u, \\ &u_a(0) = bu_a(1). \end{aligned} \tag{18}$$

By Theorem 2.3, SE (18) has a unique solution as well. The plots of u and u_a for each a are given in Figures 6–8 respectively. Similar conclusions to BVP (15) can be drawn from these figures.

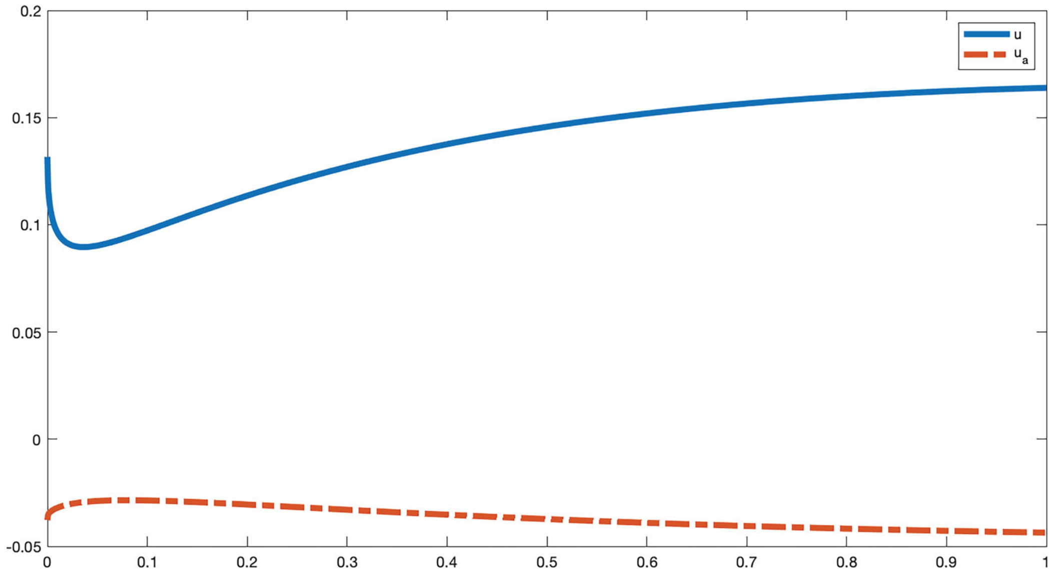

Figure 6. u and u_a when $a = 4$.

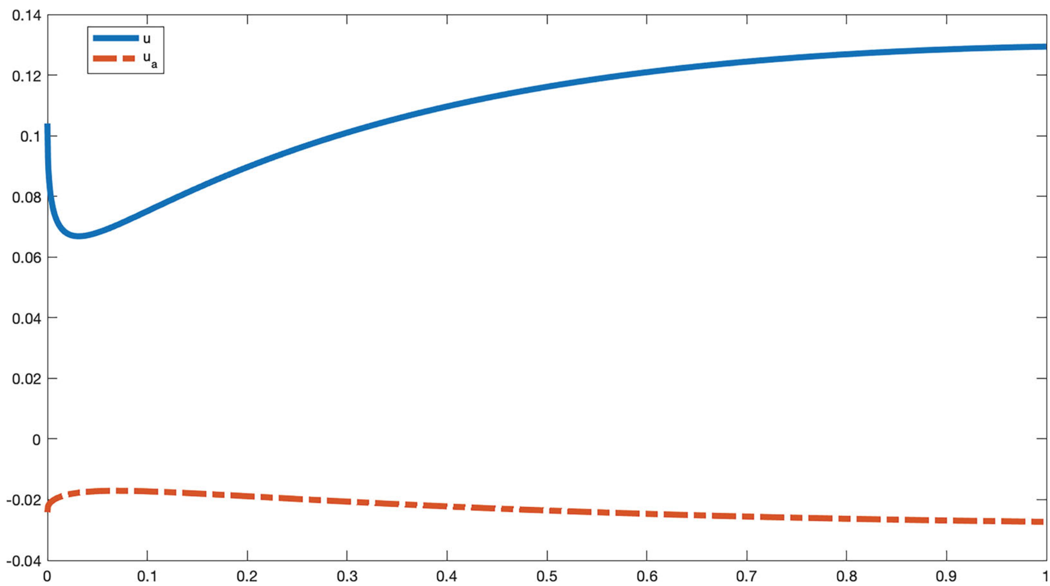

Figure 7. u and u_a when $a = 5$.

4. Proofs

We give all the proofs in this section.

Proof of Theorem 2.1: The proof is in the same way as the proof of [12, Theorem 2.1]. We omit the details. ■

Proof of Theorem 2.2: Similar to the proof of [12, Lemma 3.2], by taking $\partial G/\partial s$, we can show that for any $t \in [0, 1]$,

$$\frac{E_\alpha(-at^\alpha)}{1 - bE_\alpha(-a)} \le G(t,s) \le \frac{bE_\alpha(-a(1-t)^\alpha)E_\alpha(-at^\alpha)}{1 - bE_\alpha(-a)} + 1, \quad 0 \le s \le t,$$

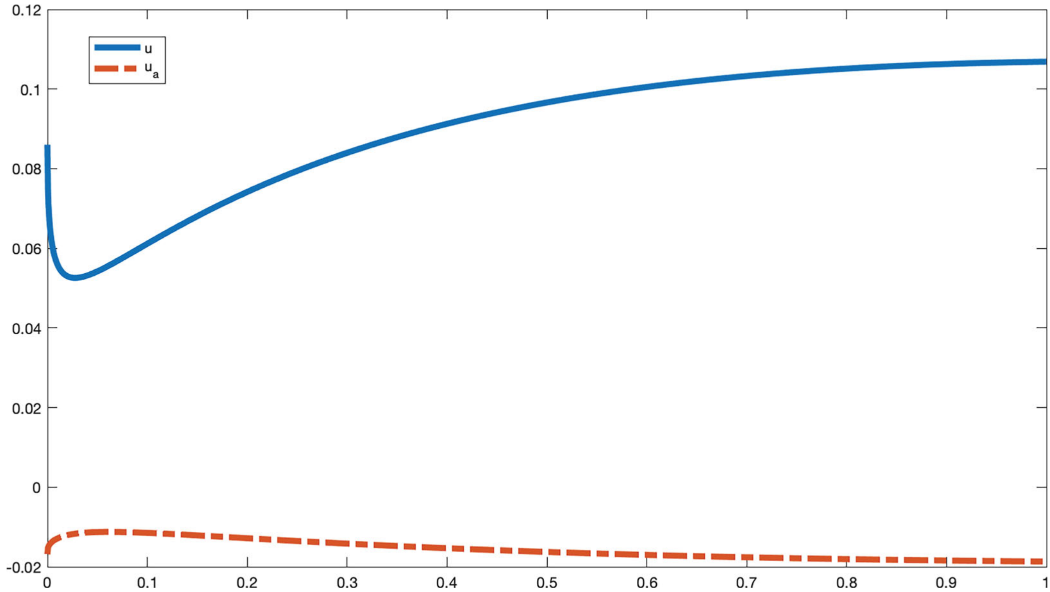

Figure 8. u and u_a when $a = 6$.

$$\frac{bE_\alpha(-a(1-t)^\alpha)E_\alpha(-at^\alpha)}{1-bE_\alpha(-a)} < G(t,s) \le \frac{bE_\alpha(-at^\alpha)}{1-bE_\alpha(-a)}, \quad t < s \le 1.$$

Note that

$$b[E_\alpha(-a)]^2 \le bE_\alpha(-a(1-t)^\alpha)E_\alpha(-at^\alpha) \le b.$$

Then (9) follows immediately. ■

We will use the fixed point theory to prove the existence and uniqueness of solutions of the nonlinear BVP (1), (2). The following operator plays an important role.

Let G be defined by (7). Define an operator $T: \ C[0,1] \to C[0,1]$ by

$$(Tu)(t) = \int_0^1 G(t,s)f(u(s),s)\,\mathrm{d}s, \quad t \in [0,1]. \tag{19}$$

By Theorem 2.1, it is easy to see that u is a solution of BVP (1), (2) if and only if u is a fixed point of T.

Define a cone K in $C[0,1]$ by

$$K = \left\{ u \in C[0,1] \mid u(t) \ge 0, \ t \in [0,1] \text{and} \min_{t\in[0,1]} u(t) \ge \gamma \|u\| \right\} \tag{20}$$

with γ defined by (11). Then by the same argument as [12, Lemma 3.3], we can show

Lemma 4.1: *$T(K) \subset K$ and T is completely continuous.*

Theorems 2.3 and 2.4 can be proved by the standard procedure using contraction mapping theorem and Krasnosel'skii's fixed point theorem, see for example [12,19]. We omit the details.

Proof of Theorem 3.2: By Theorem 2.3, BVP (1), (2) has a unique solution $u(t; a_0)$ when $a = a_0$ and $0 \le \kappa \overline{G}^{a_0} < 1$, where $\overline{G}^{a_0}$ is the upper bound defined by (8) with respect to a_0. By Remark 2.2, there exists $\delta_1 > 0$ such that when $|a_1 - a_0| < \delta_1$,

$$0 \le \kappa \overline{G}^{a_1} < 1, \tag{21}$$

where $\overline{G}^{a_1}$ is the upper bound of the Green's function G with respect to a_1. Without lose of generality, we assume $a_0 < a_1$. Then for any $t \in [0, 1]$,

$$\begin{aligned}
|u(t; a_1) - u(t; a_0)| &= \left| \int_0^1 G(t, s; a_1) f(u(s; a_1), s)\, \mathrm{d}s - \int_0^1 G(t, s; a_0) f(u(s; a_0), s)\, \mathrm{d}s \right| \\
&= \left| \int_0^1 G(t, s; a_1) f(u(s; a_1), s)\, \mathrm{d}s - \int_0^1 G(t, s; a_1) f(u(s; a_0), s)\, \mathrm{d}s \right. \\
&\quad \left. + \int_0^1 G(t, s; a_1) f(u(s; a_0), s)\, \mathrm{d}s - \int_0^1 G(t, s; a_0) f(u(s; a_0), s)\, \mathrm{d}s \right| \\
&\le \int_0^1 G(t, s; a_1) \left| f(u(s; a_1), s) - f(u(s; a_0), s) \right| \mathrm{d}s \\
&\quad + \int_0^1 |G(t, s; a_1) - G(t, s; a_0)|\, |f(u(s; a_0), s)|\, \mathrm{d}s \\
&\le \int_0^1 G(t, s; a_1) \kappa\, |u(s; a_1) - u(s; a_0)|\, \mathrm{d}s \\
&\quad + \int_0^1 |G(t, s; a_1) - G(t, s; a_0)|\, |f(u(s; a_0), s)|\, \mathrm{d}s \\
&\le \kappa \overline{G}^{a_1} \| u(\cdot; a_1) - u(\cdot; a_0) \| \\
&\quad + \int_0^1 |G(t, s; a_1) - G(t, s; a_0)|\, \overline{f}\, \mathrm{d}s,
\end{aligned} \tag{22}$$

where

$$\overline{f} = \max_{(x,t) \in [0, \|u(\cdot; a_0)\|] \times [0,1]} |f(x, t)|.$$

By (7), for any $(t, s) \in [0, 1] \times [0, 1]$, there exists $\xi \in (a_0, a_1)$ such that

$$|G(t, s; a_1) - G(t, s; a_0)| = \left| \left. \frac{\partial G}{\partial a} \right|_{\xi} (a_1 - a_0) \right| \le \overline{G}_a |a_1 - a_0| \tag{23}$$

with

$$\overline{G}_a := \max_{(t,s,a) \in [0,1] \times [0,1] \times [a_0, a_1]} \left| \frac{\partial G(t, s; a)}{\partial a} \right|.$$

Therefore, by (22), (21), and (23),

$$\| u(\cdot; a_1) - u(\cdot; a_0) \| \le \kappa \overline{G}^{a_1} \| u(\cdot; a_1) - u(\cdot; a_0) \| + \overline{G}_a\, |a_1 - a_0|\, \overline{f},$$

or

$$\| u(\cdot; a_1) - u(\cdot; a_0) \| \le \frac{\overline{G}_a \overline{f}}{1 - \kappa \overline{G}^{a_1}}\, |a_1 - a_0|\,. \tag{24}$$

Therefore, for any $\varepsilon > 0$, let

$$\delta = \min\left\{\delta_1, \frac{1 - \kappa \overline{G}^{a_1}}{2\overline{G}_a \overline{f}}\varepsilon\right\}.$$

By (24), we have

$$\|u(\cdot; a_1) - u(\cdot; a_0)\| < \varepsilon$$

when $|a_1 - a_0| < \delta$. Hence $u(t; a)$ is continuously dependent on a at a_0. ∎

The following lemma is needed to prove Theorem 3.3.

Lemma 4.2: *Let*

$$\tilde{G}(t,s) = \begin{cases} \dfrac{1}{1-b}, & 0 \le s \le t \le 1, \\ \dfrac{b}{1-b}, & 0 \le t \le s \le 1. \end{cases} \tag{25}$$

For any $h \in C[0,1]$, *BVP*

$$\begin{aligned} u' &= h(t), \\ u(0) &= bu(1), \end{aligned} \tag{26}$$

has a unique solution given by

$$u(t) = \int_0^1 \tilde{G}(t,s)h(s)\,\mathrm{d}s, \quad t \in [0,1]. \tag{27}$$

Proof: It is easy to see that BVP (26) is a special case of BVP (6), (2) when $a = 0$. Then the conclusion follows immediately from (7) with $a = 0$. ∎

Proof of Theorem 3.3: By Lemma 4.2, the solution of BVP (1), (2) must satisfy

$$u(t) = \int_0^T \tilde{G}(t,s)\left[f(u(s),s) - aD_{0+}^{\alpha}u(s)\right]\mathrm{d}s. \tag{28}$$

Since f and its partial derivative with respect to u are continuous in (u,t), we have

$$u_a(t) = \int_0^T \tilde{G}(t,s)\left[f_u(u(s),s)u_a - aD_{0+}^{\alpha}u_a(s) - D_{0+}^{1-\alpha_i}u(s)\right]\mathrm{d}s. \tag{29}$$

This is equivalent to SE (14). ∎

5. Conclusion and discussion

In this paper, we consider a type of fractional BVPs motivated by the resilience problems of fractional compartment models. A series of results on the existence, uniqueness, and multiplicity of solutions or positive solutions of the fractional BVP (1), (2) are obtained. Then sensitivity analysis for the model parameter a is conducted as well. The obtained results will contribute to the study of resilience of fractional compartment models and model calibration. Due to the extensive applications of the compartment models, our results may find broad impacts in many areas.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

M. Wang's research was supported by the National Science Foundation under Grant No. 1830489.

ORCID

Min Wang http://orcid.org/0000-0002-1019-2523

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