# Nonreciprocity in acoustic and elastic materials

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## ABSTRACT

The law of reciprocity in acoustics and elastodynamics codifies a relation of symmetry between action and reaction in fluids and solids. In its simplest form, it states that the frequency response functions between any two material points remain the same after swapping source and receiver, regardless of the presence of inhomogeneities and losses. As such, reciprocity has served as a powerful modeling and experimental tool for numerous applications that make use of acoustic and elastic wave propagation. A recent change in paradigm has prompted us to see reciprocity under a new light: as an obstruction to the realization of wave-bearing media in which the source and receiver are not interchangeable. Such materials may enable the creation of devices such as acoustic one-way mirrors, diodes and topological insulators. Here, we review how reciprocity breaks down in materials with structured space- and time-dependent constitutive properties and report on recent advances in the modeling and fabrication of these materials as well as on experiments demonstrating nonreciprocal acoustic and elastic wave propagation therein. The success of these efforts has significant promise to enable robust, unidirectional acoustic and elastic wave steering capabilities largely beyond what is currently possible in conventional materials, metamaterials or phononic crystals.

## Introduction

Looking at the rippled surface of a pond, we hardly expect water rings to shrink and converge to their center. Rather it would seem that the growth of water rings with time is an unshakable law of physics; it is not. Under the right circumstances, converging rings can appear, for example, by carefully placing a circular rim on the water surface or, more dramatically, by switching off gravity for a split second[1]. Wave motion of various shapes and in different substances usually abides by this principle of time-reversal symmetry that all possible motion, reversed in time, is equally possible. A subtler law of symmetry in wave motion is known as the law of reciprocity (Box 1). Reciprocity is the property that the vibrations imparted by a source at a position in space and received at a separate location remains the same when source and receiver are interchanged. For instance, two water striders floating on the surface of the pond are equally vulnerable to the ripples generated by one and the other: if the first shakes its legs with a given force sending ripples displacing the water under the second by a given amount; the second can reciprocate by displacing the same amount of water under the first while applying the same effort. What is remarkable about these two symmetries, time-reversal and reciprocity, is that they are typically oblivious to composition and geometry. In particular, they hold in the presence of any amount of elastic scattering and reflection by arbitrarily shaped heterogeneities and boundaries. Accordingly, they have been leveraged to enable a number of widely applicable techniques in experimental characterization and numerical modeling such as time-reversed acoustics and the boundary element method.[2–9] Perhaps more extraordinary is that, while the wave equation is no longer invariant under time reversal in the presence of dissipation, as is the case in viscous fluids, the fundamental property of reciprocity is unchanged.[10] [11, §1.9]

From a different perspective, reciprocity can be seen as a hindrance. For instance, under reciprocity, there is no way to tune transmission to two different levels in two opposite directions. As a result, the creation of acoustic and elastic wave devices that exploit unidirectional transmission, such as "acoustic diodes",[12] cannot exist in the presence of reciprocity. The ability to create materials and systems that enable nonreciprocal wave transport in acoustic and elastic media is therefore of significant interest in broad areas of engineering and science.[13,14] The purpose of the present work is to review designs and strategies that overcome this fundamental limitation of classical acoustic and phononic systems and enable purposeful and tunable nonreciprocal wave propagation.

Reciprocally exchanged vibrations do not only match in amplitude, but have identical time profiles. In particular, their spectra feature the same frequency and phase content. Accordingly, the shift in frequency observed in the Doppler effect is fundamentally nonreciprocal. Indeed, if in the original configuration a source communicates to a receiver through a barrier moving with the sound, then, in the flipped configuration, the barrier moves against the sound. This means that the Doppler shift changes signs when source and receiver are swapped, contradicting reciprocity. We should therefore expect reciprocity to fail in systems with mean flow[15] or external biasing such as magnetic fields applied to ferromagnetic materials[16] or Coriolis forcing,[17], all of which break time reversal symmetry. Time reversibility and reciprocity are physical analogs, but their opposites are not necessarily equivalent. Examples of time irreversible mechanisms include both external biasing and also energy dissipation effects such as viscosity. Both are associated with a loss of time-reversibility, but in physically significant different ways. In particular, reciprocity is maintained in linear acoustics with viscous damping (Box 2). More generally, for time-invariant materials, we anticipate the failure of reciprocity in the presence of mechanisms that generate harmonics, such as nonlinear and time-dependent constitutive material properties. This review is organized around the four following research areas that have been explored to create nonreciprocal materials in order to serve directional wave control purposes

- Kinetic media, i.e., media with moving parts or circulating flows;
- Activated media, i.e., media with time-dependent properties;
- Topologically protected one-way edge states, i.e., waves localized at the boundaries of some kinetic or activated media;
- Nonlinear media, i.e., media displaying nonlinear elastic behavior.

Before proceeding further, it is worthwhile to stress that spatial symmetry and reciprocity, while interdependent, are fundamentally different (Box 3). Thus, nonreciprocity will not occur in configurations where the source and receiver are interchangeable under spatial symmetry. Conversely, while spatial asymmetry does not imply nonreciprocity, it can enable it. Asymmetric wave propagation can be useful, be it reciprocal or not, and has been increasingly investigated in recent years in connection to an emergent class of media known as topological insulators. These are materials that stop waves from propagating through their bulk while favoring their transmission along edges and interfaces in specific directions. The interplay between asymmetry and nonreciprocity in topological insulators constitutes one of the thrusts of the present review. Another thrust is the physical consequences of spatial asymmetry combined with nonlinear material properties: the departure from linearity opens up the possibility of nonreciprocal dynamics, but the asymmetry is an essential ingredient. Finally, we present a summary and an outlook on future work.

**Box 1 | A brief history of reciprocity.**

Early hints of reciprocity can be found in the "Mécanique Analytique" of Lagrange[18] as reported by Lamb[19]. Since then, important particular cases were dealt with by Helmholtz[20] in an 1860 paper on sound in tubes and by Clebsch[21] in his work of 1862 on systems of rods. Clebsch noted that the coefficients of the dynamical matrix relating nodal displacements to external loads are symmetric, although he did not formulate that property as a statement of reciprocity. This was later done by Maxwell[22] while investigating the same topic in a paper published in 1864. One of Maxwell's theorems reads: "The extension in *BC*, due to unity of tension along *DE*, is always equal to the tension in *DE* due to unity of tension in *BC*." Betti extended the concept of reciprocity to the work done by static forces on an elastic body in 1872[23]. In acoustics and elastodynamics, a reciprocal theorem, namely that "The vibration excited at *A* will have at *B* the same relative amplitude and phase as if the places were exchanged", was stated and proven by Rayleigh[24] in 1873. He seems to be the first to refer to the "reciprocal character" of the property. Rayleigh appreciated the extent and generality of the reciprocal theorem and considered the presence of heterogeneities and of linear damping. Various equivalent mathematical formulations of reciprocity exist.[25,26] In harmonic elastodynamics, reciprocity in a medium holds if for all displacement fields $\vec{u}_1$ and $\vec{u}_2$ respectively radiated by systems of forces $\vec{F}_1$ and $\vec{F}_2$, the virtual work done by system $\vec{F}_1$ in the motion $\vec{u}_2$ is equal to the virtual work done by system $\vec{F}_2$ in the motion $\vec{u}_1$: $W(\vec{F}_1,\vec{u}_2) = W(\vec{F}_2,\vec{u}_1)$ (Fig. 1). The above theorems attributed to Maxwell, Betti and Rayleigh are then concerned with cases where the forces are concentrated to a single member or point. The reciprocity theorem applies equally to media with linear mechanisms for wave attenuation, including viscous loss [25, Ch. 7] [11, §1.9], as well as media with distributed[26] or moving loads[27].

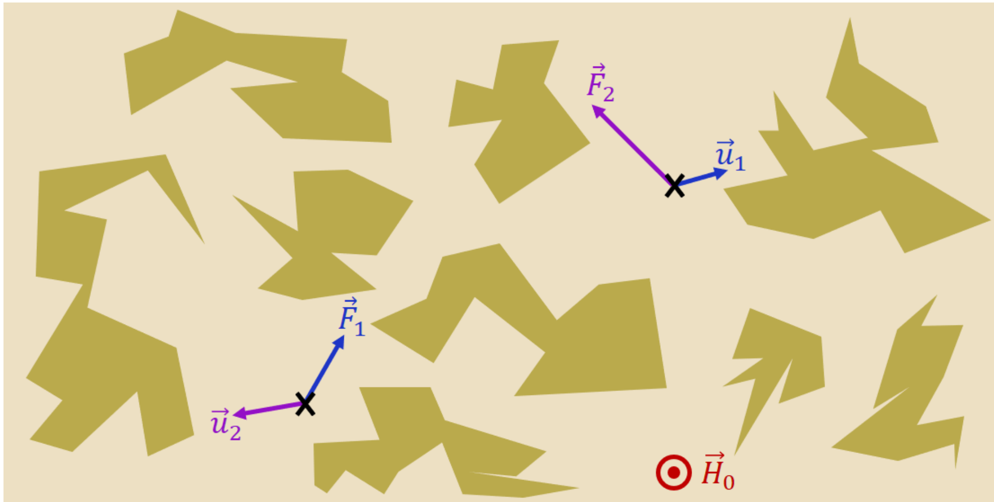



**Figure 1.** Box 1-Reciprocity theorems generally consider two distinct excitation problems in a given medium, an example of which is given in this figure. In the first (blue arrows), a source of waves, represented by the force $\vec{F}_1$, is placed at a given point at the bottom left of the figure, and creates a response (represented by the velocity vector $\vec{u}_1$) at another point at the top right of the figure. The second problem (purple arrows) considers a reciprocal situation, in which another force $\vec{F}_2$ is imparted at the second point, creating a velocity $\vec{u}_2$ at the first. The reciprocity theorems are mathematical statements linking the response of the medium in both situations, regardless of inhomogeneities or the presence of non-conservative processes such as thermo-viscous losses. For instance, in elastodynamics, reciprocity requires equality between virtual works constructed from these two situations, $W(\vec{F}_1, \vec{u}_2) = W(\vec{F}_2, \vec{u}_1)$. In linear, time-invariant acoustic and elastic media, the theorem can only be invalidated if time-reversal symmetry is broken at a microscopic scale. This can be achieved by introducing a momentum in the wave-bearing medium, represented by $\vec{H}_0$, e.g. through the use of a mean flow field that does not change direction on time reversal.[28] The only other ways to violate reciprocity are to break the linearity or time-invariance of the medium itself.

**Box 2 | The principle of microscopic reversibility and reciprocity.** At first glance, reciprocity may appear to be a consequence of time-reversal symmetry of the propagation medium. Indeed, when all terms in the wave equation are invariant under time reversal, solutions come in time-reversed pairs; one for $t$ and another for $-t$. This means that if a wave can propagate in one direction, the time-reversed wave, which propagates in the opposite direction, is also a viable physical solution. However, reciprocity is a more fundamental symmetry of the response function between two points in a medium. It compares the displacement/velocity at point $B$ due to a perturbative force applied at point $A$, to that obtained at point $A$ when the same force perturbation is imparted at point $B$. Specifically, a medium is reciprocal if the response functions $\alpha_{ik}$ (also known as generalized susceptibilities) governing the relation $x_i = \alpha_{ik} f_k$ between a quantity $x_i$ and a dual perturbation $f_k$, potentially applied at a different location, satisfy the symmetry property $\alpha_{ik} = \alpha_{ki}$. Onsager[29] and Casimir[30] proved that a linear time-invariant medium, slightly perturbed away from thermodynamic equilibrium, and whose governing equations at a microscopic scale obey time-reversal symmetry, is necessarily a reciprocal medium. The latter hypothesis is known as the principle of microscopic reversibility. To illustrate the principle, consider this example. Suppose that heat conduction in an anisotropic solid obeys Fourier's law $q_i = -k_{ij} \partial_j T$. It is well-known that heat conduction is macroscopically irreversible in general, that is: a temperature evolution reversed in time is not physically admissible as it contradicts the second law of thermodynamics. However, heat conduction is microscopically reversible since the equations governing the motion of particles on the microscopic scale obey time-reversal symmetry. Consequently, in the linear regime, heat conduction is reciprocal and $k_{ij} = k_{ji}$. Note also that Rayleigh's reciprocity theorem in its historical form admits simple direct proofs for the propagation of acoustic fields at the macroscopic scale, but can readily be obtained from the Onsager-Casimir theorem of reciprocity by letting $x_i$ be a component of the particle velocity and $f_k$ be a component of the volume force. One can break reciprocity by lifting one or several of the assumptions behind the Onsager-Casimir theorem such as linearity, time invariance and/or microscopic reversibility. For instance, microscopic reversibility is invalid in the presence of a magnetic field, Coriolis forces, or of a mean flow. In each of these examples, a microscopic momentum bias, which does not change on time reversal, is imparted at the microscale leading to a violation of reciprocity for macroscopically observable fields. In summary, reciprocity is separate from and independent of the general notion of time reversal symmetry. Reciprocity holds in the presence of material dissipation, including viscous losses, which is not time-reversible [25, Ch. 7][11], but it fails under conditions of mean flow, or magnetic forcing in magneto-sensitive materials.

**Box 3 | Asymmetric and nonreciprocal wave propagation.** It is useful to distinguish between asymmetric and nonreciprocal

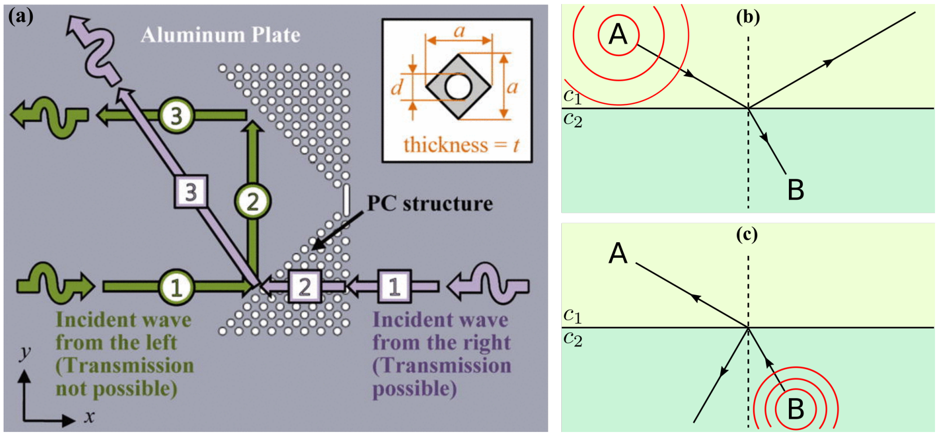


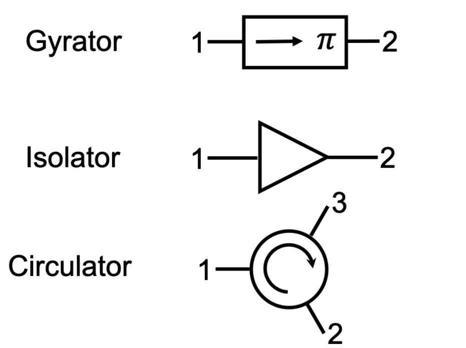

**Figure 2.** Box 3 - (a) Schematic representation of linear asymmetric wave propagation in a plate with systematically arranged holes (Oh et al.[31]). A wave incident from the left is reflected back (180°) after interacting with the two prismatic arrays of scatterers, while a wave incident from the right is transmitted with a 45° change in the direction of propagation. (b, c) Wave transmission between two media with different speeds of sound $c_1 > c_2$: while transmission can be highly asymmetric in a global sense, it remains reciprocal for point-to-point signaling.

wave propagation because they are not generally equivalent (Fig. 2). In systems with asymmetric wave propagation, the monitored output changes if the locations of the input and output are interchanged. In this sense, nonreciprocity can lead to asymmetric wave propagation, but the inverse is not necessarily true. For example, it is possible in a reciprocal system to excite an elastic wave at a source location and obtain a different wave type at the receiver due to a linear mode conversion mechanism.[32] This process, although intriguing and useful,[31–35] is reciprocal. Care should also be taken in the choice of the input and output parameters when checking for reciprocity. As a general rule, the input and output variables to be interchanged while verifying reciprocity should be dual in a sense, e.g., their product yields a power or an energy [36, Ch. 2]. Thus, reciprocity ensures the invariance of the ratio of the output velocity to the input force; it does not say anything about the ratio of the input and output velocities, or forces. The reciprocity relation takes a statistical form (ensemble average) when it relates the radiation impedance of an elastic structure to the diffuse sound field generated by it.[37] Acoustical reciprocity does not hold if the medium is in motion when the propagation direction has some component orthogonal to the direction of mean flow, such as one may experience in a windy environment.[38] Loss of reciprocity was noted explicitly by Morse and Ingard when discussing the refraction of sound between two fluid media with differing mean flow velocities parallel to their interface. [39, Ch. 11] For waves in a steady flow, reciprocity is restored[19] if the direction of the flow is reversed when the source and receiver are interchanged – this is also known as the flow reversal theorem.[28]. Another interesting scenario is when the boundary or supports of a linear acoustic medium moves. The boundary conditions (the constraint equations) in a system with a moving boundary/support are explicit functions of time. Reciprocity does not hold under these conditions [40, Ch. 11]. As a case in point, an asymmetric waveguide with a prescribed harmonic displacement input can exhibit asymmetric end-to-end wave transmission[41]. Some relevant misconceptions regarding nonreciprocity and asymmetric propagation were addressed recently by Maznev et al. in the context of acoustic diodes and isolators[12].

**Box 4 | Gyrators, Isolators and Circulators.** Many nonreciprocal mechanical systems are built on analogies with multi-port electronic devices. The reciprocity theorem applied to a multi-port acoustical system where the inputs are $x_i$ and outputs are $f_k$ implies that the scattering matrix $S_{ik}$, such that $x_i = S_{ik} f_k$, is symmetric: $S_{ik} = S_{ki}$. The most common linear devices that violate the reciprocity condition are gyrators, isolators, and circulators. These systems are therefore described by an asymmetric scattering matrix $S_{ik} \neq S_{ki}$. Gyrators and isolators have two ports, whereas circulators have three ports. Let us first consider a general two port scattering system, described by a two-by-two scattering matrix $S = [S_{11}, S_{12}; S_{21}, S_{22}]$, and ask the following question: Can we have nonreciprocity in a lossless two-port system? Since energy conservation implies unitarity for $S$, one can use the general paramerization of two-by-two unitary matrices, defining the angles $\theta$, $\alpha$, $\phi$ and $\Phi$ and write

$$S = \begin{pmatrix} e^{i\phi}\cos\theta & e^{i\alpha}\sin\theta \\ -e^{-i\alpha}\sin\theta e^{i\Phi} & e^{-i\phi}\cos\theta e^{i\Phi} \end{pmatrix}$$

This form of the scattering matrix implies that the off-diagonal coefficients can differ in phase, but never in amplitude. This means that lossless two-port scattering systems can only be nonreciprocal in the transmission phase. Gyrators are two-port nonreciprocal systems that transmit the same power between their two ports, but achieve the maximum nonreciprocal phase difference of $\pi$ between $S_{12}$ and $S_{21}$. However, when one wants to create a one-way propagation device for waves, one needs a nonreciprocal response in the transmission amplitude. Tellegen proved that any nonreciprocal response, in phase or amplitude, can be achieved with combinations of gyrators and other reciprocal elements[42]. In particular, to achieve nonreciprocity in amplitude in a two-port system, we must simply relax the assumption of unitarity. For instance, ideal isolators correspond to a sub-unitary scattering matrix of $S = [0,0;1,0]$, which can only be obtained if absorption losses are present. We also note that the super-unitary version of an isolator ($S = [0,0;1,1]$), requires active systems. Interestingly, the generalization of passive isolators to three ports systems, known as circulators, correspond instead to the unitary scattering matrix shown in Fig. 3. Circulators transmit waves in a uni-rotational fashion, from port 1 to 3, 3 to 2, and 2 to 1, but never in reverse. They are capable of isolation since one can use a circulator to build an isolator between, for instance, ports 1 and 2. In this case, the third port must be matched so that it acts as a perfect absorber, enabling passive two-port isolation at ports 1 and 2. These devices are often used in electronic systems to isolate sources from unwanted back-reflection that originates at loads, or separate the emission and reception channels in full-duplex communication systems and have been demonstrated in acoustical systems[15], which are relevant in many imaging or underwater communication systems.




$$S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

**Figure 3.** Box 4 - The most common nonreciprocal wave components are gyrators, isolators and circulators, commonly found in electromagnetic devices. The gyrator is a two port system that is nonreciprocal in its transmission phase, but not in amplitude. It transmits from port 1 to port 2 with a 180 degrees phase shift, and from port 2 to port 1 with no phase shift. The isolator, instead, is nonreciprocal in magnitude. It has a unitary transmission from port 1 to 2, and full absorption from port 2 to 1. The circulator is the generalization of the isolator to a three port system, transmitting waves in a rotational fashion, from port 1 to 3, 3 to 2, and 2 to 1.

## Nonreciprocity in kinetic media

As explained in detail in Box 2, all linear time-invariant wave systems are inherently reciprocal, unless there exists a quantity $H_0$, external to the system, that biases the motion by breaking time-reversal symmetry on a microscopic scale. This external bias $H_0$ must be time-odd, i.e., it flips sign under time-reversal. For instance, elastic waves in ferromagnetic crystals can be nonreciprocal in the presence of an external magnetic field, although magneto-elastic phenomena tend to be very weak in natural materials.[16] A simple example of a nonreciprocal linear time-invariant medium is a fluid in constant motion with velocity $v_0$, assumed to be smaller than the speed of sound $c_0$. The motion, which is imparted by an external system (like a pump) independently of the presence or absence of sound in the medium, imparts a mean momentum to the acoustic medium. That momentum can be interpreted as a time-odd bias. If one then imagines two points placed along the flow, it is clear that the speed of sound along the flow $c_0 + v_0$ will differ from that against the flow $c_0 - v_0$, and thus the transmitted signals along or against the flow will differ by a phase (assuming the absence of losses, cf. Box 4). This simple idea can be exploited to build acoustic gyrators[43] in air pipes under steady flow and isolators[44]. Figure 4a illustrates how waves excited along the flow pick up a phase of 90 degrees, whereas in the opposite direction, the phase is -90 degrees (Fig. 4b). The 180 phase difference between forward and backward waves (Fig. 4c) occurs for a specific waveguide length, for which the system acts as a gyrator. This example allows us to also point out another challenge in building nonreciprocal systems: since the strength of the bias $v_0$ is generally small compared to $c_0$, appreciable nonreciprocal transmission phases occur only after long distances, namely for systems much larger than the wavelength. Using grating elements to slow down sound it is possible to shrink the size of these nonreciprocal devices, and realize compact isolators and gyrators for guided and radiated waves.[44]

Interestingly, the effect of fluid motion on acoustic wave propagation can lead to similar phenomena as observed for electrons in a magnetic field. For instance, it has been shown that an acoustic beam passing through a vortex acquires a nonreciprocal phase shift analogous to the Aharonov-Bohm phase for an electron going through a region of space with non-zero magnetic vector potential.[45] An acoustic analog of the Zeeman effect (splitting of the electronic energy levels in an atom subject to a magnetic field) has also been demonstrated in a ring resonator subject to constant rotating air flow.[15] Such a design was then exploited to construct an acoustic circulator whose resonant nature allowed the use of slow fluid motion to achieve large nonreciprocal behavior. This device is shown in Fig. 4d-f. In Fig. 4d, the cylindrical acoustic cavity, coupled to three external waveguides, contains air at rest. Consequently, reciprocity holds and the scattering matrix is symmetric (waves incident at a given port split equally between the two other ports). By properly rotating the air filling the cylindrical cavity, it is possible to strongly break reciprocity and create an acoustic circulator (Fig. 4e). This was validated in a series of experiments using airborne audible sound and air motion implemented through standard fans.

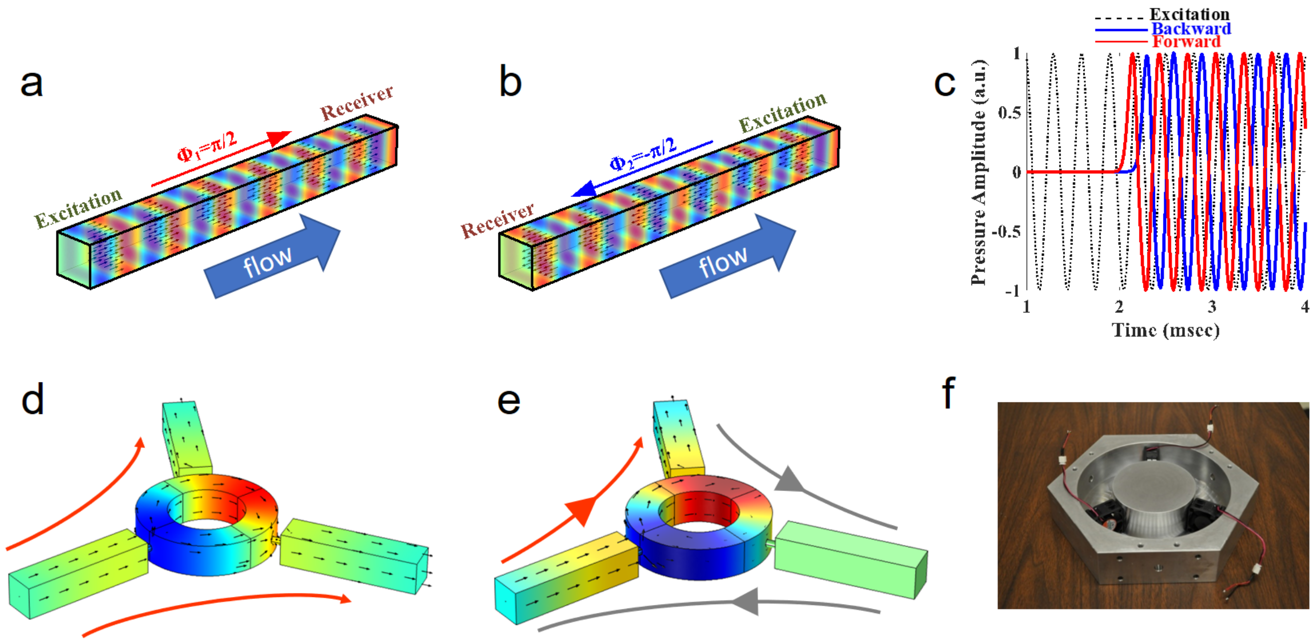



**Figure 4.** Acoustic reciprocity from fluid motion. Sound propagates in an acoustic pipe under constant air flow with different speeds depending on whether it travels (a) along or (b) against the flow. Therefore, the transmission coefficients for forward and backward acoustic propagation can differ in phase (c). In this example, the velocity of the flow and the length of the waveguide are adjusted for a 180 phase difference between forward and backward transmission, making an acoustic gyrator. (d) A cylindrical cavity connected to three waveguides acts as a reciprocal power splitter at its dipolar resonance frequency. (e) The system can be turned into a nonreciprocal circulator by rotating the air inside at a specific subsonic speed depending on the total quality factor of the dipolar resonance. (f) A photograph of the first prototype of an acoustic circulator (top plate not shown), where fluid motion is realized using fans.[15]

## Nonreciprocity in activated materials

### Background

Activated materials modulate their constitutive properties in space and time in response to an external stimulus. These types of materials violate time-invariance and are thus capable of exhibiting a wide array of nonreciprocal wave phenomena. Figure 5a depicts an activated two-phase laminate where, as time passes, the spatial phase profile is translated at constant speed. Alternatively, a phononic crystal is activated by varying the properties of a unit cell as a function of time (Figure 5b). Lastly, rather than modulating bulk properties, an activated structure can be obtained by modulating an interaction coefficient between the structure and, for instance, attached resonators or the ground (Figure 5c), or more generally by modulating any boundary condition.

From a modeling perspective, wave motion in activated materials is described by equations with time-dependent constitutive parameters. For instance, the wave equation $c^2(x,t)\nabla^2\phi(x,t) = \ddot{\phi}(x,t)$, where the modulated physical property $c$, the speed of sound, depends on both space and time variables. Emphasis here is on modulations in the form of progressive periodic

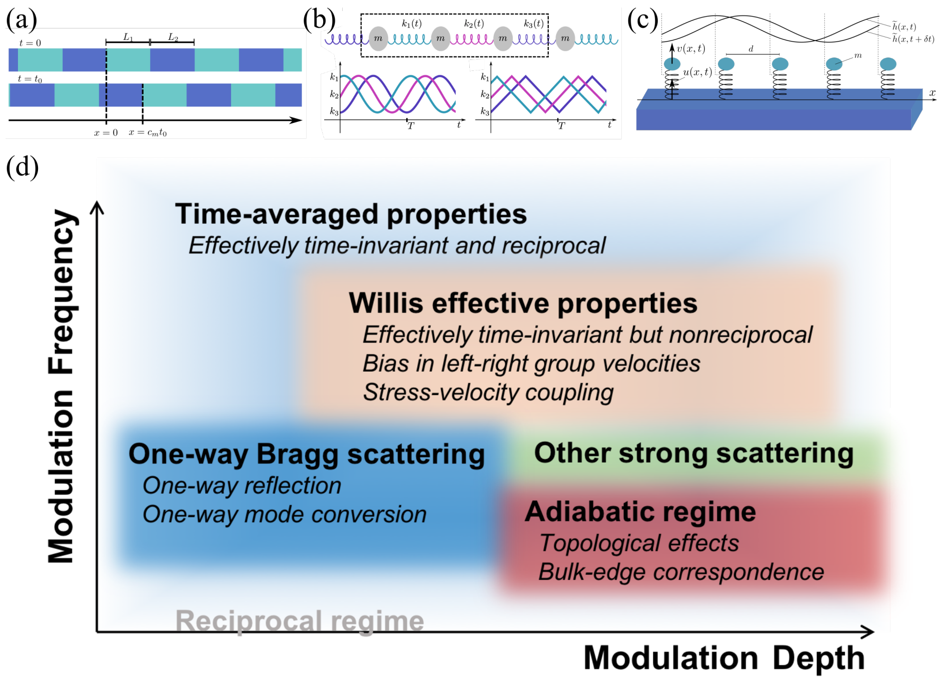



**Figure 5.** Activated media at a glance: (a) an activated two-phase laminate;[46] (b) an activated phononic crystal;[47] (c) an activated metabeam;[48] (d) a chart of reciprocal and nonreciprocal regimes.

waves referred to as pump waves, e.g., $c(x,t) = c_o + \tilde{c}(x - vt)$ where $\tilde{c}$ is periodic. Such progressive modulations create a space-time bias which enables nonreciprocal wave motion as a function of the modulation frequency $f_{\mathrm{M}}$ and depth, e.g., $|\tilde{c}|/c_o$ (Figure 5d). We first explore the Bragg scattering regime for small-amplitude medium-speed modulations; faster modulations can be described by an equivalent medium with effective properties whereas slower, but higher amplitude modulations lead to an adiabatic regime.

## One-way Bragg mirrors

Waves incident on a sonic or phononic crystal are scattered as they encounter geometric irregularities, boundaries and changes in material properties. Simply speaking, incident wave fronts are partially reflected by periodically spaced objects with acoustic impedance that differs from the background medium. What is remarkable is that even when individual objects reflect weakly, strong reflection from the ensemble, known as Bragg reflection, can still be observed under a condition of constructive interference dependent on wavelength and angle of incidence. Conversely, destructive interference means that the wave has not been effectively reflected and was rather transmitted unaltered. Either way, if the crystal is stationary, the incident, transmitted and reflected waves all have the same frequency. In contrast, a Doppler-like effect occurs when the crystal has properties that are modulated in space and time by the action of a pump wave. Thus, the frequency of the reflected wave is shifted up or down depending on the relative motion of the incident wave and the pump wave.[49] The Doppler shift in turn modifies the condition for constructive interference which now involves an extra parameter: the sense of propagation. A one-way Bragg mirror[50–52] can therefore be conceived by tuning the Doppler shift to favor reflection when the incident and pump waves propagate, say, in the same direction; and maintain transmission when they propagate in opposite directions. Anomalous Doppler effects may be observed[53] in a periodic acoustic medium when the sound source is moving. Thus, sound from a stationary source at a frequency inside the band gap of the periodic medium cannot be heard by an observer in the far field (even if the observer moves), while the sound from a moving source can be heard by a far field stationary observer.

The first experimental evidence[54] of nonreciprocal Bragg scattering in activated materials was observed in a structure of ring magnets sliding over a common axial rail and housed by grounded solenoids (Figure 6a,b). Each ring then repels or attracts its host solenoid by a force proportional to the current input of the solenoid. By modulating the current, the structure effectively behaves as a grounded spring-mass chain with a modulated grounding stiffness $k_g(x,t) = k_o + \delta k \sin(q_{\mathrm{M}}x - 2\pi f_{\mathrm{M}}t)$. Inspection of the frequency response function shows that reciprocity breaks down at select frequencies $f_{i,j}$ where, in this case, the modulation favors co-propagated signals over counter-propagated ones. These frequencies satisfy the Bragg condition $f_i - f_j = n f_{\mathrm{M}}$, where $n$ is an integer.

In a second experimental demonstration,[55] the solenoids were supported on a beam substrate using pairs of cantilevers. Now each solenoid oscillates co-axially with a permanent magnet directly bonded to the host beam. The structure is activated by electrically driving the magnetic coupling between the solenoids and the permanent magnets (Figure 6c,d). This configuration allowed direct measurement of the Doppler shift by comparing the transmission and reflection spectra. It is observed that a dip in transmission at a frequency $f_1$ is accompanied by peak in reflection at a frequency $f_2$ such that $f_1 - f_2 = f_M$. Thus, the structure operates as a one-way mirror at the pair $(f_1, f_2)$. If frequency $f_1$ passes when incident from the right, then it is reflected into $f_2$ when incident from the left; the same holds upon substituting $f_1$ for $f_2$ and left for right.

A closer look shows that a full, however small, range of frequencies centered on $f_1$ and $f_2$ behave similarly. These define the mirror bandgaps, i.e., frequency bands over which signals cannot penetrate into the mirror. Therefore, a one-way mirror is easily recognized through its band diagram since it features directional bandgaps that do not extend over the whole Brillouin zone and are restricted to regions of positive or negative incidence (Figure 6e). Maximum one-way reflection capabilities are recovered when the directional bandgaps do not overlay, that is when the Doppler shift is larger than bandwidth of the gap.

When the background supports multiple propagating modes, as in dispersive waveguides and metamaterials, the frequency shift from one-way reflection can drastically modify the incident mode. For instance, in a resonant metamaterial, an incident acoustic mode can be directionally reflected into an optical mode.[48,56] Furthermore, one-way mode conversion can be triggered in transmission;[57–60] although such nonreciprocal transitions are yet to be observed. Other mechanisms for activated nonreciprocity include one-dimensional piezoelectric structures with space-time modulated electrical boundary conditions[61,62], linear piezophononic media under electrical bias[63] and media with space-time modulated effective mass.[64] Space-time modulation of related material properties can lead to other nonreciprocal effects: for instance, a traveling-wave modulation of thermal conductivity and specific heat capacity causes the heat flux to have different properties when it propagates in the same or opposite direction to the modulation.[65]

### Strong scattering and nonreciprocal Willis coupling

Pump waves with larger amplitudes cause stronger scattering; propagated modes no longer contain a handful of components but rather a complete spectrum of frequencies and wavenumbers. In that case, either full Floquet-Bloch analysis or time-domain simulations can be used to construct the band diagrams.[68–70] Schematically, the spatial profile of the modulation dictates the band structure; each band is then sheared, or tilted, by an angle that is a function of the modulation frequency.[46,47] Therefore, large amplitude fast modulations lead to directional gaps significantly wider than in the case of weak scattering. The consequence is that one-way total reflection can be ensured by modulating smaller regions. Conversely however, one-way total transmission is no longer possible due to the large time-varying impedance mismatch between the modulated region and the background.

Beyond directional gaps, strong scattering breaks reciprocity and time-reversal symmetry within passing bands as well. Indeed, the modulation-induced tilt favors, via acceleration, waves propagated either along or against the pump wave while opposing their time reversed version through deceleration (Fig. 6e). Taken to extremes, supersonic modulations can 'freeze' propagation in a given direction, or even reverse it. The modulated medium then appears to lack modes propagated in a given direction while exhibiting an excess of modes propagated in the opposite direction[67,71] (Fig. 6e). This phenomenon is fundamentally different from one-way Bragg reflection, but can still be leveraged for the purposes of one-way mirroring especially at low frequencies where wide bandgaps are seldom available.

The bias between positive and negative group velocities admits an interesting interpretation in terms of constant effective material parameters valid when the wavelength of the pump wave is small in comparison to the typical wavelength of the propagated waves. Hooke's law then ceases to apply as it systematically yields reciprocal behaviors and is replaced by a constitutive law of the Willis type[72,73] where the Willis coupling coefficients in this case account for the nonreciprocal nature of the modulated microstructure. A constitutive law of the Willis form is characterized by an effective bulk modulus $\kappa^{\text{eff}}$, an effective mass density $\rho^{\text{eff}}$ and a third parameter $S$ that couples stress, $\sigma$, to velocity, $\vec{v}$, as well as momentum, $\vec{\mu}$ to strain, $\varepsilon$, namely $\sigma = \kappa^{\text{eff}} : \varepsilon + s \cdot \vec{v}$; $\vec{\mu} = -S : \varepsilon + \rho^{\text{eff}} \cdot \vec{v}$. While being linear and macroscopically time-invariant, this form of Willis coupling introduces a first-order time derivative to the motion equation thus breaking time reversal symmetry and, consequently, reciprocity.[46,67,74,75] In the terminology of Box 2, such Willis media are nonreciprocal because they break microscopic reversibility. Note that Willis coupling constants resulting from media with spatio-temporally modulated material properties are analogous to emergent nonreciprocal bianisotropic electromagnetic Tellegen media that result from modeling moving media in a static reference frame[76–78]. In that sense, this nonreciprocal Willis coupling has an influence similar to that of an externally applied magnetic field on the motion of charged particles. Note also that while reciprocal Willis coupling has been experimentally observed,[79] experimental observation of nonreciprocal Willis coupling in a dynamic medium is particularly challenging as it requires a fast, near sonic, strong modulation of both elastic and inertial properties.

As the modulation speed tends to infinity, the material properties become solely time dependent, independent of spatial position. A structure with material properties varying periodically in time can display bandgaps in the wave number domain ,

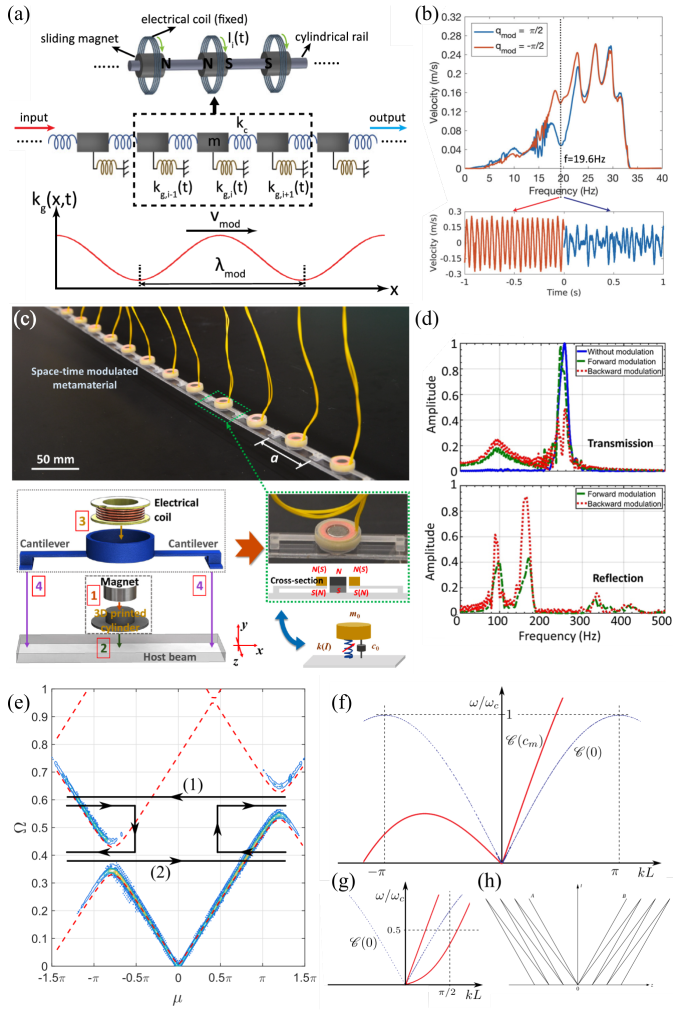


**Figure 6.** Experimental demonstrations of nonreciprocity in dynamic media: (a) a magnetically activated phononic lattice;[54] (b) its measured end-to-end frequency response function; (c) a magnetically activated dynamic beam;[55] (d) its measured transmission and reflection spectra; (e) a typical dispersion diagram of a weakly modulated medium featuring a pair of one-way bandgaps;[52] (f) example of bias in group velocities of a strongly modulated medium;[66] (g) reversal of group velocities under a high-frequency strong modulation;[66] (h) example of characteristic lines in a modulated medium with reversed group velocities.[67]

i.e. tilted by 90° in the frequency vs wavenumber band diagram. These vertical bandgaps produce wave motion with complex frequencies whose amplitude grows and/or decays everywhere in space exponentially in time. This amplification effect has been of interest for at least 60 years, first for electrical transmission lines with time-varying capacitance[80]. Recent interest has turned to time dependent acoustic[81] and elastic media[82], and space-time checkerboard patterns which lead to novel wave effects.[83] In practice, internal dissipation counteracts exponential parametric growth with the result that the amplitude of the time-dependent modulation must exceed a critical value if it is to produce parametric growth.[84]

## Slow modulations and topological properties

Phase accumulation during slow modulations can have order one effects by virtue of Berry phase and associated topological invariants. Consider the scattering caused by slow pump waves, for which the phase shift $\dot{\gamma}\mathrm{d}t$ induced by the modulation over a small period of time $\mathrm{d}t$ is insufficient to trigger any transitions between dispersion branches. Accordingly, no significant nonreciprocal effects are expected to emerge over short observation periods. Remarkably however, the phase shift $\gamma = \int_0^T \dot{\gamma}\mathrm{d}t$ accumulated over longer periods of the order of a modulation cycle $T = 1/f_{\mathrm{M}}$ need not vanish and can cause significant overall changes in frequency and in propagation velocity. The phase increment $\dot{\gamma}$ is known as Berry's connection, whereas $\gamma$ is known as Berry's phase which were originally introduced by Berry[86] in order to explain nonclassical, i.e., quantal, interference

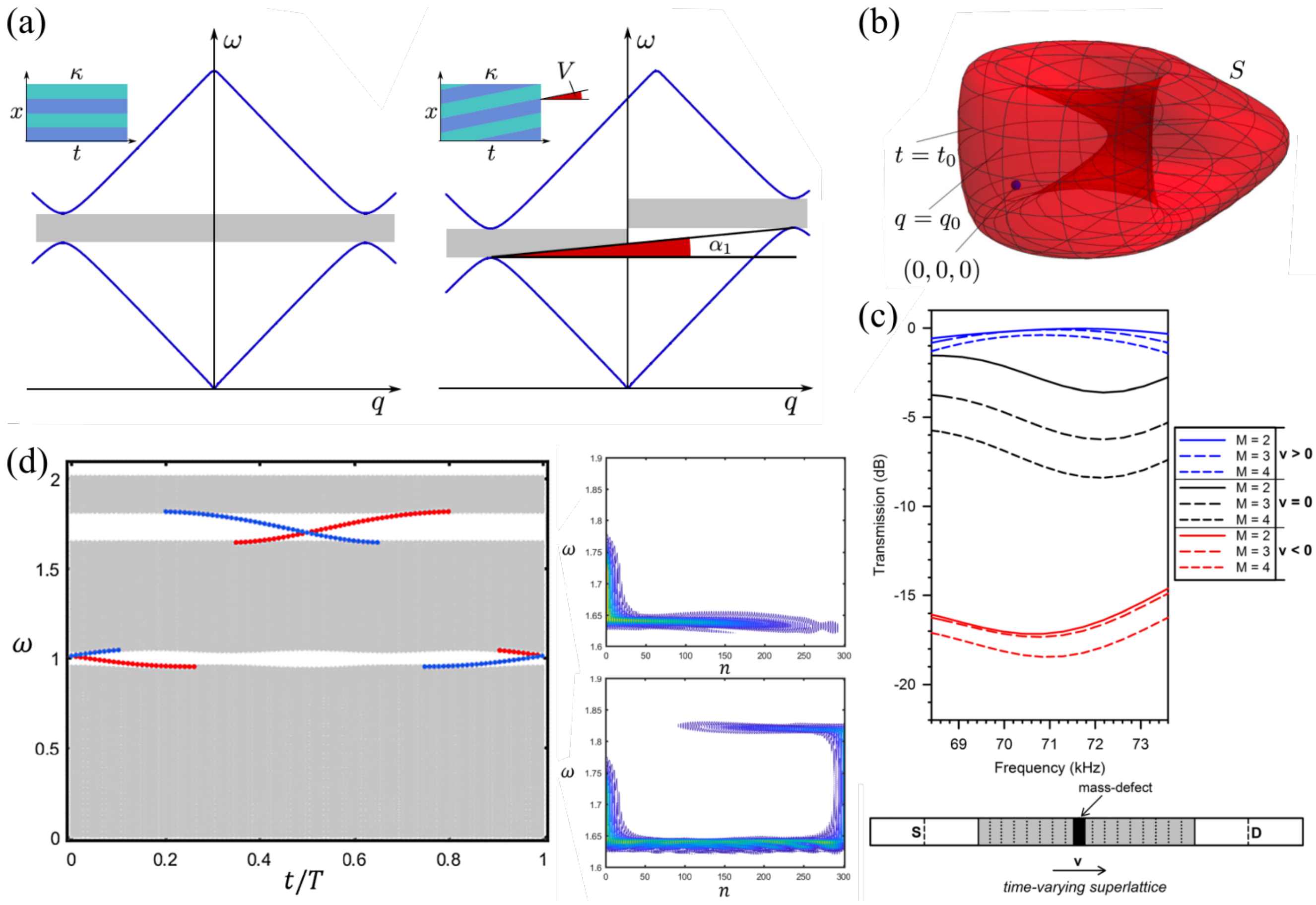



**Figure 7.** Topological effects in dynamic media: (a) definition of tilt angle;[47] (b) quantization of the tilt angle as a winding (Chern) number;[47] (c) robust transmission in dynamic media;[51] (d) topological one-way edge states predicted by bulk-edge correspondence.[85]

phenomena induced by slowly changing environments such as externally applied magnetic fields and vector potentials. In the present context, Berry's phase, understood as slow, but continual accumulation of Doppler shifts, takes opposite values for modes propagating in opposite directions. It therefore explains how and why the bands of a periodic medium whose material properties are modulated in space and time are sheared and tilted the way they do under the influence of a modulation, albeit a slow one (Figure 7a). As a matter of fact, each infinitesimal segment of a passing band shadowing an element d$q$ of the Brillouin zone gets rotated by an amount equal to the Berry's curvature $\partial_q \dot{\gamma}/2\pi$; the whole band will tilt by an amount proportional to the total Berry's curvature $\int_{q\in \mathrm{BZ}} \int_0^T \partial_q \dot{\gamma} \mathrm{d}t \mathrm{d}q$.[46,47] Total curvature is an example of a quantized topological invariant. Its value, an integer multiple of $2\pi$, which is known as the Chern number, is immune to small perturbations affecting the pump wave and the background medium (Figure 7b). Thus, a nonzero Chern number ensures the existence of a robust directional bandgap and constitutes one measure of nonreciprocity in a modulated medium.

Depending on applications, one-way mirroring can be regarded as a type of immunity against back-scattering by defects.[51] That is because in a directional gap, waves are constrained to propagate in a unique sense so that defects can only scatter forward, i.e., in the direction of the incoming wave, but not against it. A pump wave therefore usually guarantees consistently high levels of transmission in specific directions (Figure 7c). In some instances, however, the transmission is halted by a phenomenon of localization at boundaries. The localized vibration is then pumped up or down in frequency until it reaches a passing band at which time it can be sent back as a form of back-scattering. In the position-frequency space, such vibrations manifest as edge modes circulating either clockwise or anticlockwise along the boundaries of the domain[47,60,85] (Figure 7d). The difference between the number of clockwise and anticlockwise edge modes is also a topological invariant; in fact, by a principle of bulk-edge correspondence,[87] that number is exactly equal to the Chern number. Consequently, a nonzero Chern number, in addition to controlling band tilting, indicates an imbalance between counter-propagating edge modes in clear violation of time-reversal symmetry.

## Nonreciprocity and topologically protected edge states

### Background

Wave motion supported by the existence of topologically protected edge states (TPESs)[88–91] can be extremely robust to backscattering and capable of nonreciprocal behavior under certain circumstances. TPESs are modes appearing at the interface between two different insulators, i.e., band gap materials, and are typically immune to scattering by defects. At the origin of their peculiar qualities is the concept of "topology." In this case, topology does not refer to the shape or geometry of the underlying medium but, in the context of crystals, to the topological features of the dispersion of Bloch eigenmodes over the Brillouin zone. In contrast to other properties such as phase and group velocities or the gap width, a topological property is, by definition, insensitive to continuous perturbations except those strong enough to close the band gap.[90] In other words, the topology of an insulating crystal only changes when it is so perturbed that it stops being an insulator. Two insulators are then topologically inequivalent if they exhibit different topologies. Interestingly, at an interface between two inequivalent insulators, the topology changes, and therefore the system is forced to close its band gap locally, which means that it must support localized edge states: the so-called TPESs. TPESs display an inherent robustness to perturbations, including geometrical imperfections, structural disorder, and addition of impurities. This robustness is a result of the difference in the topology of the surrounding bulk insulators, independent of the interface details, which can only be destroyed by modifying the entirety of the bulk insulator, namely closing their band gaps, which requires unlikely global and large perturbations.

Of particular relevance to this review, are nonreciprocal TPESs which propagate along the interface in one direction only, never in reverse. It is therefore important here to distinguish two classes of two-dimensional topological insulators, depending on whether the corresponding edge modes are nonreciprocal or not: (*i*) the Chern insulators, in which a nontrivial topology is created by breaking time-reversal symmetry, and as a by-product, support unidirectional nonreciprocal TPESs; and (*ii*) the spin-Hall insulators, which rely on preserved time, and sometimes space, symmetries and for which the TPESs come in time-reversed pairs (called spins) propagating in opposite directions on the interface, hence preserving reciprocity. We stress here that only the first kind of edge modes break reciprocity, and will be the principal focus of this section of our review. However, the two time-reversed states of the second kind are decoupled; thus they may be considered as featuring some form of one-way propagation for a certain class of loading restricted by the underlying preserved symmetry. For this reason, we also briefly mention works on reciprocal TPESs. These exceptional properties of both reciprocal and nonreciprocal TPESs have largely fueled the interest of a very wide community of researchers over the past few years, from condensed matter systems,[92] to photonics,[89] acoustics,[91] elastodynamics or mechanics[88].

Starting from the work in condensed matter physics of Haldane,[92] who predicted the possibility of unidirectional electronic edge states, TPESs in quantum systems have been the subject of intense research due to their potential in a number of technological areas. The existence of nonreciprocal electronic propagation, which was demonstrated experimentally by Zhang and colleagues[93] for example, is based on breaking time reversal symmetry to produce one-way *chiral* edge modes between bulk bands that are characterized in terms of their Chern number as a topological invariant. More recently, Kane and Mele[94,95] discovered reciprocal topological modes in systems with intrinsic spin orbit (ISO) coupling that exhibit the quantum spin Hall effect (QSHE). These systems, whose unique behavior was demonstrated experimentally in[96], do not require breaking of time reversal symmetry, and are therefore of particular interest as guidelines to establish analogies in other physical domains. In fact, these quantum mechanical phenomena have been pursued in diverse areas of physics as in the case of the work of Haldane and Raghu[97], who demonstrated boundary modes in electromagnetic systems following Maxwell's relations. They have been subsequently investigated in acoustics[98–100], photonics[90,101], mechanics[102–106], as well as in coupled wave domains such as opto-mechanics[107]. In all of these different domains, properties such as lossless propagation, existence of waves confined to a boundary or to an interface, immunity to back-scattering and localization in the presence of defects and imperfections are all related to band topology. While the existence of TPESs does not necessarily imply nonreciprocal wave motion, reciprocity breaking is one way to induce topological order in acoustics and mechanics. In addition, geometrical symmetry breaking, as in the case of chiral symmetries, while not breaking reciprocity, can be used to support topologically robust one-way wave motion that does not back-scatter in presence of disorder, provided that this disorder does not couple opposite chiral states. In this section, we describe nonreciprocal topological insulators and we briefly review reciprocal concepts supporting edge bound propagation of sound and elastic waves in an attempt to provide a broader context.

### Nonreciprocity through quantum Hall effect analogs

According to the early studies in condensed matter physics, one could identify two broad ways to realize topologically protected wave propagation in acoustic and elastic media. The first one seeks to mimic the quantum Hall effect (QHE) by breaking reciprocity through active components. Changing of the parity of active devices or modulating the physical properties in time, for example, has been shown to alter the direction and nature of edge waves[51,66]. Examples include magnetic fields in biological systems[108], lattices of gyroscopes[109,110] (Fig. 8b) and acoustic circulators operating on the basis of a flow-induced bias[99,111–116]. In particular, Prodan and Prodan[108] demonstrated edge modes in biological structures where time reversal symmetry is broken by Lorentz forces on ions produced by weak magnetic fields. Also, Zhang et al.[117] developed a systematic way to analyze eigenvalue problems which break time reversal symmetry by gyroscopic forces. The experimental demonstration on a lattice of gyroscopes is presented in[109], while theoretical studies by Bertoldi and coworkers[118] have investigated the existence of chiral edge modes in hexagonal and square gyroscopic lattices (Fig. 8d). Another example is the work of Khanikaev and coworkers, where moving fluid breaks time-reversal symmetry at the inclusion level inducing topological order [111](Fig. 8c), lately implemented in experiments[114] consistent with the work of Kariyado and Hatsugai[17], where Coriolis forces are generated by spinning the entire lattice. Systems supporting TPESs that rely on the modulation of the strength of the interactions (stiffness) in time include the works of Fleury and coworkers[99], Yang et al.[115], Deymier and coworkers[51], Carusotto et al.[119] and Chen et al.[85], among others. In general, this body of works pursues methods for breaking time reversal symmetry through the introduction of an external bias, which modulates the strength of interactions or the inertial properties of the system and provides the ability to produce chiral TPES. As such, these states are associated with wave motion that is strictly nonreciprocal. For these reasons, the establishment of QHE analogues can be considered as examples of media referred to in this manuscript as “activated materials."

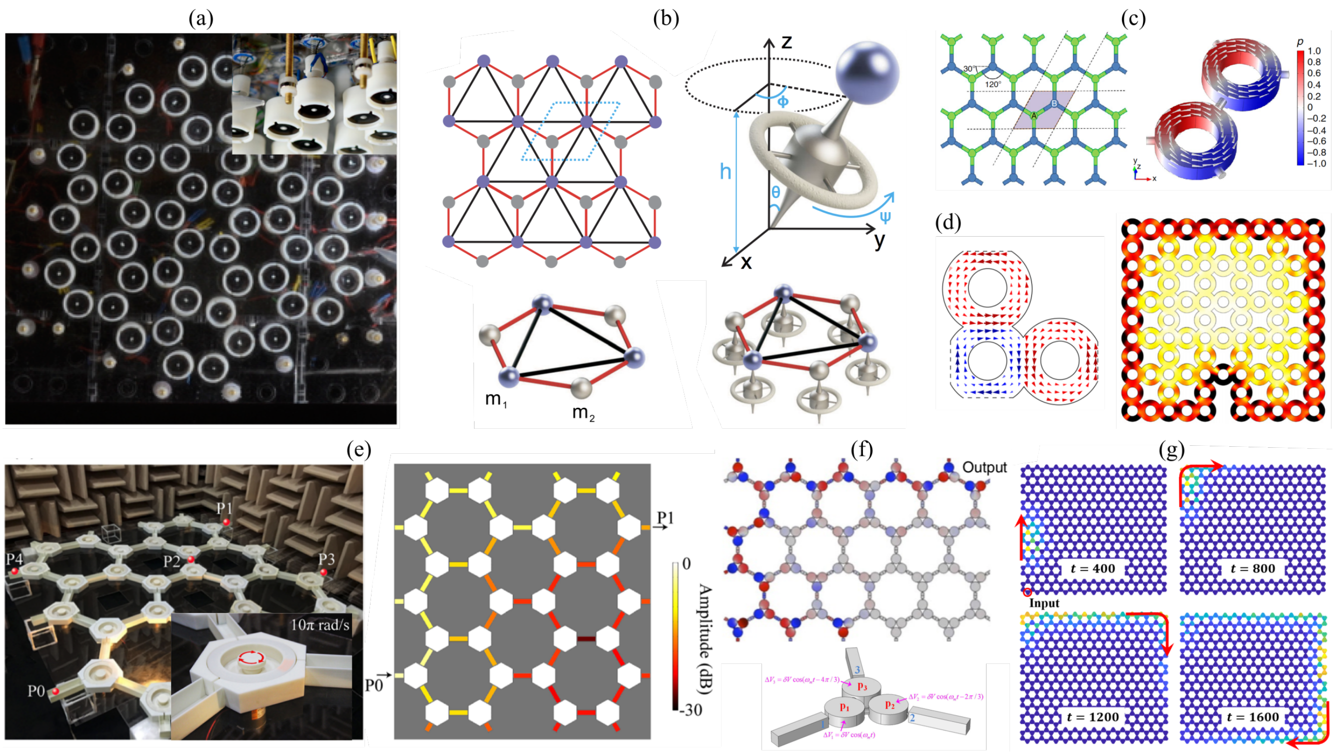



**Figure 8.** Examples of QHE analogues in mechanics and acoustics: (a) lattice of coupled gyroscopes from,[109] (b) schematic of hexagonal gyroscopic lattice proposed in,[118] (c) first proposal of lattices of acoustic circulators as Chern insulators.[111] (d) Chern sonic insulator in a fluid network with a self-organized flow of self-propelled particles.[112] (e) Experimental realisation of the concept in (c), based on acoustic circulators.[114] (f) Floquet Chern topological insulator based on a lattice of time-modulated circulators.[99,120] (g) Elastic analog of (f) realized with phase-delayed activated spring tripods.[85]

### Reciprocal concepts based on quantum spin (QSHE) and quantum valley Hall effects (QVHE) analogs

The need for an external bias, and for an associated external source of power has motivated the search for QSHE analogues that employ only passive components. This is particularly compelling for mechanical metamaterials, for which the need for an external bias to break reciprocity may be a significant challenge towards the practical application of one-way wave motion. For this reason, material platforms that are fully reciprocal and that feature both forward and backward propagating *helical* edge states have attracted significant interest. Due to geometrical asymmetries, two oppositely propagating modes support opposite chirality, and can propagate with robust properties protected by topology, as long as disorder, truncations or lattice defects do not couple the two helicities. This is generally guaranteed as long as the entire system, including the defects, respect a given symmetry, or an accidental spin degeneracy, in addition to time-reversal symmetry itself. Examples of QSHE analogs are illustrated in several studies by way of both numerical[102,103,121,122] and experimental[106,123,124] investigations, which involve coupled pendulums[123], plates with two scale holes[102] and resonators[103,106], as well as electrical circuits[124]. Among these, the work of Süsstrunk and Huber[123] provided one of the first experimental demonstration of these edge states in a square lattice network of double pendulums connected by a network of springs and levers. Also, Mousavi and coworkers[102] demonstrated numerically helical edge modes in perforated thin plates by coupling symmetric and antisymmetric Lamb wave modes. Foehr and co-workers exploited similar concepts for the realization of topological waveguides based on resonant spirals cut out from thin plates[125]. The results in[102] have also provided guidelines for the experimental demonstration of a QSHE analog in the form of a continuous elastic plate, which is presented in Miniaci et al.[126]. Another example is the work of Huang and coworkers, where a mechanical analog of the in-plane QSHE was realized in a hexagonal kagome lattice using the Brillouin zone folding technique[122]. Chiral symmetries can also be used to induce topological order in acoustic systems[116,127], also extendable to higher-order topological phases, such as robust corner states[128].

A parallel line of work, again resulting in reciprocal systems, employs valley degrees of freedom (which are natural time-reversed modes belonging to band extrema with opposite Bloch wave numbers) and the topological differences generated by symmetry inversions within a unit cell to achieve TPES in structures that emulate the quantum valley Hall effect (QVHE). Notably, the QVHE exploits valley states, which coexist in frequency but not in Bloch wavenumber, instead of spin states, which would be perfectly degenerate time-reversed states. The advantage is that each lattice site needs to have only one degree of freedom, which provides the opportunity to obtain configurations of reduced geometrical complexity. Valley degrees of freedom arise naturally in systems with time reversal symmetry and have been predicted theoretically in graphene[129,130], where wave-functions at opposite valleys feature opposite polarizations and thus emulate spin orbit interactions. This concept was extended by Ma and Shvets[131] to a photonic crystal exhibiting topologically protected valley edge states, while Dong and coworkers[132] illustrated valley modes in photonic crystals with an hexagonal lattice of inclusions. Recently, this approach has also been extended to acoustic waves propagating in a 1D phononic crystal, where the Zak phase is the topological invariant employed to assess the non-trivial nature of the bulk gaps[133]. A subsequent extension to 2D acoustic domains employs triangular stubs to break inversion symmetry by varying their orientation with the lattice[134–136]. Initial studies in mechanical metamaterials include the numerical work of Pal and al.[137], followed by the experimental demonstration of a QVHE-analog in the form of an hexagonal lattice with lumped masses inserted at locations that break the $C_{3v}$ symmetry inherent to the hexagonal geometry, while preserving the $C_3$ symmetry.[138] A similar approach was implemented in[139], while the idea of enlarging a unit cell to induce zone folding at the $\Gamma$ point in reciprocal space that leads to a double degeneracy was pursued in[98] as well as through an array of resonators[140]. Also, Chen et al.[141] investigated topological Stoneley waves propagated along the edges of kagome lattices by using asymptotic homogenization technique that transforms the discrete motion equation of the lattice into a continuum partial differential equation. This study was the first adaption of the QVHE to in-plane motion of mechanical lattices. As explained above, the edge modes of a QVH insulator are strictly reciprocal, but apparent nonreciprocity can be observed for defects that do not couple valleys together, in particular valley-preserving interface turns.[142]

## Nonreciprocity in Nonlinear Media

Material nonlinearity is perhaps most commonly associated with the absence of reciprocity in elastic and acoustic media where the effective material properties do not change with time. While reciprocity does not generally hold in nonlinear materials, these media are not always nonreciprocal. Spatial asymmetry is a necessary additional ingredient for breaking reciprocity, and even then reciprocity is not guaranteed for all parameter ranges. We focus the discussion on the familiar point-to-point form of reciprocity as it applies – or not – to nonlinear, non-gyroscopic systems with time-invariant properties and boundary conditions. Given that the vast majority of the literature in this topic concerns one-dimensional waveguides, we emphasize systems that can be adequately modeled using a combination of scalar wave fields and coupled oscillators.

The salient feature of nonlinear media is that their dynamic behavior depends on the amplitude of motion. Nonlinear, nonreciprocal devices rely on this dependence to operate, which itself is a function of various parameters and can lead to different nonlinear phenomena. The most common mechanisms responsible for nonreciprocal dynamics are generation of higher harmonics, nonlinear resonances and bifurcation (i.e., a sudden change in the topology of the phase space due to a small change in a system parameter)[143,144]. Operating at relatively lower amplitudes of motion, where the frequency of the nonlinear response remains mostly preserved, may also lead to nonreciprocity both at finite frequencies[145,146] and at zero frequency (static nonreciprocity)[147,148]. Whether static or dynamic, two ingredients are necessary for circumventing the constraint of reciprocity: (i) nonlinearity, (ii) asymmetry.

Before diving into a more systematic description of these approaches, it is worth stressing that nonlinearity-based nonreciprocity holds a few fundamental limitations compared to the externally biased systems considered in the previous sections. First, a passive nonreciprocal device can indeed support drastically different transmissions for oppositely propagating waves, but it cannot ensure isolation when the system is excited simultaneously from both sides[149]. In this sense, these devices cannot operate as conventional isolators to protect a source from back-reflections, since these reflections can trickle through the isolated port in the presence of an outgoing signal. This is a result of the fact that the superposition principle does not apply to nonlinear systems. In addition, there is a trade-off between the degree of nonreciprocity achievable in passive, nonlinear resonators and the magnitude of forward transmission[150,151]. In the following, we provide an overview of the different strategies available to realize nonreciprocity based on nonlinearities.

### Weakly nonlinear media

It is helpful to begin with examples of reciprocal nonlinear response. The most straightforward scenario is a geometrically symmetric system, for which exchanging the source and receiver has no effect on signal propagation. Any form of spatial symmetry guarantees reciprocal propagation of waves regardless of the type and degree of nonlinearity. An example would be a granular chain of identical, spherical beads with the input and output located at the two ends[152].

Mirror symmetry of a reciprocal nonlinear medium may be broken in various ways. Examples include gradually changing the properties of the medium[153,154], using asymmetric unit cells in periodic waveguides[155], incorporating nonlinear defects or interfaces[156–162], breaking the symmetry of the functional form of the nonlinear internal forces[146,148,163–165], or having the input and output points at asymmetric locations[166]. An asymptotic analysis may be adopted to demonstrate these results in a general form for a weakly nonlinear system subject to an impulsive load[166]. It can be shown in this context that even the boundary conditions can play a role in determining whether the response in a specific configuration is reciprocal.

A nonlinear medium can generate multiple harmonics of a wave traveling through it. The amplitudes of the extra harmonics, and whether they are super- or sub-harmonics, depend on the nature of the nonlinearity and the frequency of the incident wave. At low energies (weak nonlinearity), the dominant frequency is typically the second harmonic — it is the third harmonic if the nonlinear force is odd-symmetric (e.g., cubic restoring force). Reciprocity is broken if the transmission of (some of) the harmonics are altered upon interchanging the source and receiver locations. Liang et al.[158,167] used this principle to operate an acoustic diode. They attached a layer of a nonlinear medium to a linear periodic waveguide and fixed the frequency of the incident wave within the band gap of the linear medium such that its second harmonic would fall within a pass band. By choosing the amplitude of the incident wave properly, they created the following scenario: if the wave comes from the nonlinear side, its second harmonic is generated and passes through the linear medium to the other side; if the wave comes from the linear side, it is filtered out before reaching the nonlinear medium and negligible energy is transmitted to the other side.

Another nonlinear phenomenon that can be used for nonreciprocal transmission is the energy dependence of resonance frequencies. For instance, consider wave transmission through a linear layered medium that contains an asymmetric nonlinear portion[143]. It is relatively straightforward to set up the layers such that the wave amplitudes are different on the two sides of the nonlinear layer. For the near-resonance dynamics, the combination of asymmetry and nonlinearity leads to different transmissions through the nonlinear layer when the source and receiver are interchanged. The appeal of this scenario is that the frequency of the input is mostly preserved. The work of Cui et al.[145] provides an experimental demonstration of this phenomenon using granular chains.

Whether the realization of nonreciprocity is based on the generation of extra (sub- or super-) harmonics or the shifts of resonances, the underlying principle is the same. We can regard both cases as $1:n$ nonlinear resonances, where $n$ denotes the dominant frequency of the response dynamics. We have $n=2$ for second-harmonic generation, for example, and $n=1$ when we are relying on shifts occurring at the same frequency as the external driving. These principles work for both transient and steady-state dynamics. Equation(s) governing wave amplitude under a weakly nonlinear response may be obtained asymptotically[168]. The result of this analysis is a nonlinear Schrodinger equation (or a set of such equations) better known as the amplitude equation in the mechanics community.

### Bifurcation-based nonreciprocity

Bifurcations are considered to be the point of departure from a weakly nonlinear behavior characterized by a change in the topology of the phase space of the response[169,170]. Different types of bifurcations can occur depending on the nature of the response (e.g., static equilibrium or periodic motion).

One familiar bifurcation in mechanics is the bistable behavior of elastic members in compression; e.g., the buckling of a column. Nadkarni et al.[171,172] utilized bistability within the unit cell of a lattice material to demonstrate long-range unidirectional transmission of waves. They incorporated asymmetry in the potential energy of each unit cell, such that different energies are required for transitioning from one energy well to the other. Provided that the initial configuration of the lattice is chosen appropriately, it becomes possible to transfer energy through the entire lattice in the direction of decreasing potential energy. This unidirectional transmission occurs with minimal loss and dispersion (i.e., change in the wave form) because the underlying mechanism is a change in the static equilibrium of the unit cells.

A similar bistability can happen in a medium in periodic motion, for example in response to an external harmonic excitation. For instance, consider a one-dimensional, layered medium consisting of nonlinear unit cells, with an asymmetric potential energy function. The system can be assembled such that when the excitation source is placed in different locations (e.g., at either end of the medium), the information travels through different energy landscapes. Accordingly, as the driving amplitude increases, the response corresponding to one of the two source locations loses stability first[144]. As a result, the two outputs become markedly different. In this scenario, the response is understandably nonreciprocal even before reaching the bifurcation point. If the bifurcation results in a significant change in the response, particularly in terms of energy transmission, then the post-bifurcation nonreciprocity is more pronounced. To achieve this, one strategy is to use the supratransmission phenomenon[173–175].

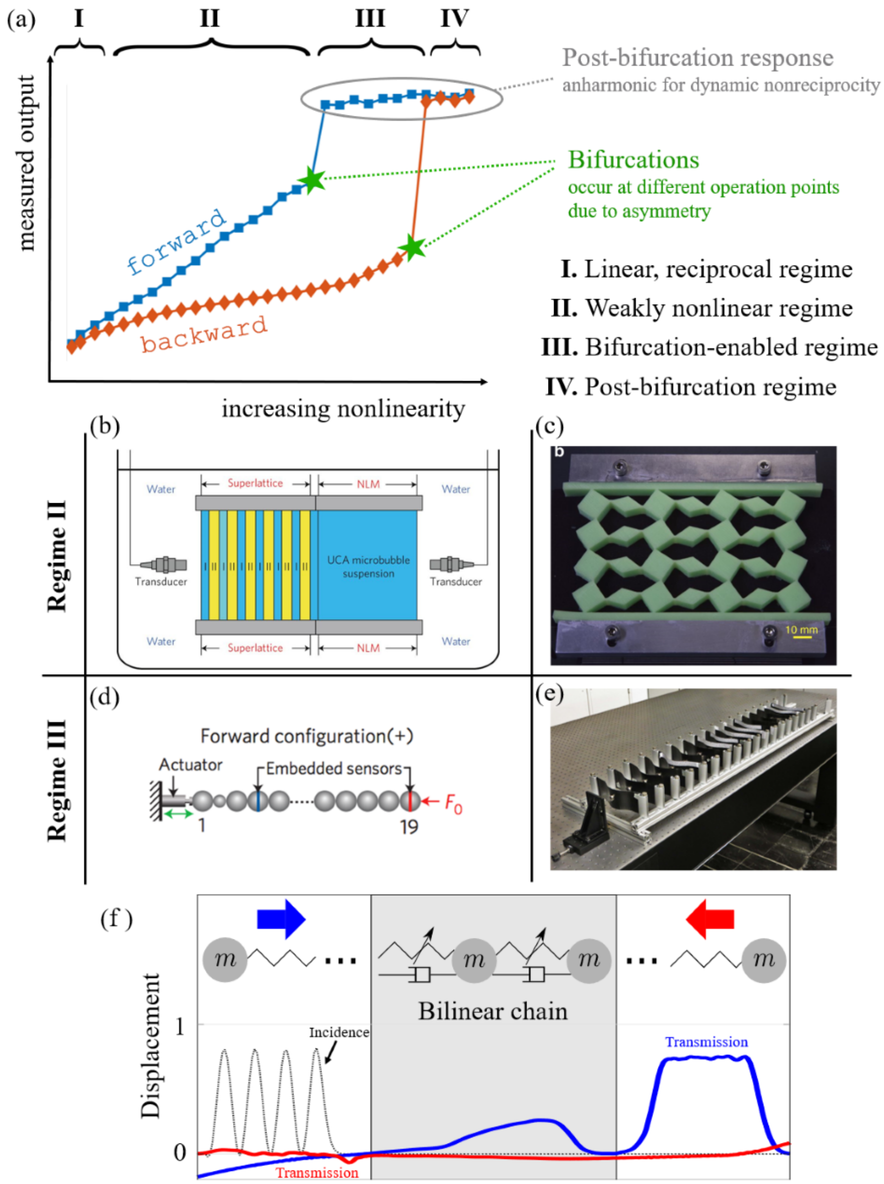



**Figure 9.** Overview of nonreciprocity in nonlinear media. (a) The degree of nonlinearity increases along the horizontal axis; for example, due to an increase in the input energy or forcing amplitude. The monitored output of the nonlinear medium is plotted on the vertical axis; for example, the amplitude of motion. Generically, four different response regimes exist for amplitude dependent nonlinear systems. Regimes II and III are useful for the design of nonreciprocal devices and materials, with examples provided in panels (b-e). Bilinear media depend only on the sign of the displacement or strain. An example is a chain of springs and masses with a finite region of bilinear springs (f). Panel (b) is adopted from Ref.[158], panel (c) from Ref.[147] and panel (d) from Ref.[176]. Panel (e) is a picture of the setup in Refs.[171,172], and panel (f) is based on Ref.[177]

.

Consider a nonlinear periodic medium that is subject to harmonic excitations, with a frequency within the linear band gap. No energy is transmitted through the medium at low driving amplitudes due to the dispersive property of the periodic medium. Beyond a critical driving amplitude, however, the low-energy harmonic motion loses stability and a nonlinear wave with a broadband frequency content is generated. Thus, the amount of transmitted energy increases by orders of magnitude, and its spectral contents lies within the linear pass band of the medium (again, owing to dispersion). If the nonlinear medium is not symmetric, then the onset of supratransmission may depend on the location of the source. Boechler et al.[176] used this principle to develop a rectifier for elastic waves by embedding a defective unit within a uniform granular chain to break the symmetry of the system. This enabled them to measure a very high contrast (4 orders of magnitude) between the transmitted energies when the source and receiver were interchanged.

The performance of bifurcation-based nonreciprocity depends on the driving amplitude, among other parameters. For example, there is a value of driving amplitude above which bifurcations occur for both the forward and backward configurations. Although the response would remain nonreciprocal, the nonreciprocity could be less marked. Additionally, the post-bifurcation response is often quasi-periodic or chaotic[144, 174, 175], making it cumbersome to compare the two outputs in a reciprocity test. Relying on bifurcations for achieving nonreciprocity is naturally more suitable when the device is operated near the bifurcation point. The post-bifurcation behavior of the response, as well as the instability mechanism, depends on various properties of the system, such as the type of nonlinearity, energy loss, structural imperfections, finite size effects and operating frequency. For supratransmission, these effects are discussed by Yousefzadeh and Phani[175, 178].

### Nonreciprocity in bilinear media

Bilinearity is a special case of elastic nonlinearity characterized by a stiffness that transitions between two linear states at a critical load. When this transition occurs at zero loading, the effective stiffness depends on whether the local one-dimensional deformation is in a state of compression or extension. In this case, unlike other nonlinear stiffness models, the bilinear relation is amplitude-independent in the sense that even for very small amplitudes of oscillations, the constitutive law is nonlinear: the nonlinearity enters only through the sign of the displacement. Bilinear, also known as bimodular, material response has been proposed as a model for studying contact and for elastic solids containing cracks. Bilinear media can exhibit unique families of complex dynamic behavior owing to the non-smooth nature of their constitutive material law[170]. Bilinearity may be regarded as the simplest departure from linearity that can produce nonreciprocal static and dynamic behavior with the requirement, as always, that asymmetry is a necessary although not sufficient element.

A single bilinear spring is spatially symmetric and is therefore a reciprocal system. Nonreciprocity can be obtained in two-degree-of-freedom spring-mass systems, with asymmetric bilinear spring arrangements.[148]. Bifurcation-based nonreciprocity may be realized in lattice materials with bilinear elasticity[146]. Amplitude-independent nonreciprocal systems may be designed by combining multiple bilinear springs in one dimensional spring-mass chains.[177] In this model, the indispensable ingredient of asymmetry is achieved using spatially varying bilinear spring properties, see Figure 9.

The schematic in Figure 9 summarizes the different scenarios discussed for nonreciprocity in nonlinear media. Whether nonreciprocity is based on weakly nonlinear behavior, bifurcation or sign-dependent bilinearity, we emphasize that simultaneous presence of nonlinearity and asymmetry are necessary, *but not sufficient*, conditions for the existence of nonreciprocal dynamics.

### Nonlinear dynamic media

We have been careful so far in our discussion to avoid any overlap between passive nonlinear media and dynamic linear media. Unfortunately, it is not possible to make a general claim on whether nonlinearity intensifies the nonreciprocal bias of a modulated medium or pacifies it. Like any other nonlinear problem, the answer depends on a variety of different parameters, such as the type and strength of nonlinearity, the spatial and temporal nature of the excitations, the frequency, wavenumber and amplitude of modulations, and the energy loss within the system. Conclusions can be made only on a specific, case-by-case analysis.

One scenario is that nonlinear forces monotonically intensify the nonreciprocal bias in a time-modulated medium. This is very similar to how nonlinearity affects nonreciprocity in a passive medium, with the main difference being that the linear regime would no longer be reciprocal in a modulated medium (Regime I in Figure 9). The possibility of the existence of this scenario was verified numerically[179] for the setup of Wang et al.[54] with the amplitude of modulations acting as the control parameter for nonlinearity (the horizontal axis in Figure 9).

## Conclusions and perspective

As a principle of nature, reciprocity is remarkably tenacious. It persists in the presence of heterogeneity on any length scale (spatial asymmetry) and internal damping (time irreversibility). Reciprocal wave motion is guaranteed if the medium is time invariant, passive, and linear with microscopic time reversibility, circumstances which are the case for most materials in day to day life. For these reasons, one can conclude that reciprocity is a robust fundamental principle that is hard to beat.

In this review, we have presented many specific approaches that have been proposed and employed to purposefully violate one or more of the requirements for reciprocal wave motion in acoustic and elastic materials. The most familiar situation for breaking reciprocity is the presence of mean flow. In this scenario, flow breaks the microscopic time reversibility, leading to nonreciprocal effects analogous to the gyrator when the flow is parallel to wave motion, or the circulator when they are not in the same direction (Figs. 3 and 4). The condition of time-invariance is broken in activated media where material properties are varied in both space and time. These materials can display strong nonreciprocal effects if the space-time modulation is large enough (Fig. 5). Various mechanisms are available to realize the level of active control necessary to modulate material properties, such as piezoelectric materials and magnetic elements (Fig. 6), which make activated materials a practical and promising candidate for applications in elastic wave control that seek to make use of nonreciprocal signal transmission. The dynamics of space-time modulated materials can be explained in terms of new descriptors of physical states, such as the Willis dynamic effective medium equations and by topological invariants in the frequency-wavenumber domain (Fig. 7). Aside from activated materials, there is a deeper connection between one-way wave propagation and the topological description of quasi-periodic systems. While this has analogies in electronic systems, the mechanical realizations are quite distinct (Fig. 8), and can exhibit robust one-way propagation effects. Finally, material nonlinearity combined with spatial asymmetry provides a distinct and significant route to nonreciprocity (Fig. 9). Nonlinearity remains an important tool for realizing nonreciprocity in elastic and acoustic materials. The optimal utilization of nonlinearity for operation of nonreciprocal devices remains an active area of development despite the intrinsic limitations of passive, nonlinear materials in comparison to linear, externally-biased materials. Even more remarkable outcomes are expected when nonlinearity is utilized in combination with another reciprocity-breaking mechanism. For example, recent studies have focused on the nonlinear dynamics of topologically protected edge states[180,181]. The combined effects of nonlinearity and spatiotemporal modulations have not yet received much attention in the literature. In the light of recent advances in modeling complex nonlinear and spatio-temporally varying elastic materials[70,182] and experimental realizations of dynamic elastic media[54,55,183,184], we expect the study of nonreciprocity to extend to modulated nonlinear media in the near future. Beyond the innate appeal of the underlying physical phenomena, the interest in the dynamics of these media is partly motivated by the fact that nonlinear forces are inevitably present in experiments. Another new perspective was recently presented by Bartolo and Carpentier[185] in connection to the elasticity of surfaces with non-orientable topology; e.g., a Möbius strip. Despite having a linear constitutive law, the deformation of a Möbius strip subject to shear stress is intrinsically nonlinear and the elastic response is not reciprocal. This counterintuitive property of non-orientable surfaces can potentially lead to yet another class of nonreciprocal elastic materials.

All of the techniques presented here and elsewhere to elude reciprocity have their own benefits and drawbacks. Active materials, which includes systems with mean flow as well as spatiotemporal modulation of effective properties, require constant sources of energy. Nonlinear materials, on the other hand, are passive systems requiring no external impetus. However, they use the input power of the source, which may come with other drawbacks and limitations on the isolation of simultaneous sources, isolation level, or bandwidth, as previously noted in photonics.[149,186] Nonlinear materials also rely on more complicated constitutive relations that are not easily designed or created using existing fabrication technologies. Topological wave systems are inherently narrowband in frequency and wavenumber space and confined to the boundaries of engineered material domains. Yet individual systems can take advantage of these hurdles in a constructive manner to effectuate nonreciprocal wave motion. The task before the material designer is to consolidate all possible effects in order to generate a nonreciprocal response for a given application. Due to the complexities listed above, many of the proposed nonreciprocal acoustics examples are not fully realizable, with further developments requiring new mechanisms. Some possible avenues include the use of transmission frequency windows via coupling of band gaps with nonlinear frequency shifting. A particularly interesting possibility[187] combines acoustic waves under the small-on-large effects of elastic pre-stress with nonlinearity in two distinctly different modes: dynamically to detune the signal frequency, and in a static manner to change the effective elastic stiffness. Other possible routes combine bilinearity as the frequency converter with a linear wave filter[165].

We note that the review has focused on the direct problem: how to design materials with embedded nonreciprocal mechanisms in order to add an additional degree of control to propagating waves. However, there has been little or no consideration of inverse problems posed by nonreciprocity and how they could lead to new techniques for assessing material properties. Just as one can "hear the shape of a drum"[188] if one knows the modal frequencies, so it may be possible to infer nonlinear material effects or modulation depth and frequency from measured violations of reciprocity. For instance, there have been reports (anecdotal) that the deviations from reciprocity for multiple point-to-point ultrasonic measurements in concrete undergoing fatigue loading correlate with the duration and intensity of loading. Proposed mechanisms include increased nonlinearity due to the growth of microcracks. Unambiguous identification of nonreciprocity requires careful prior calibration of the acousto-electric equipment, but if implemented properly, nonreciprocal data measurement could become a useful tool for ultrasonic nondestructive evaluation (NDE), a multi-billion dollar industry. This is but one practical example of the implications of the value in understanding and designing nonreciprocal acoustic and elastic materials.

This review has outlined many strategies, in theory and practice, for achieving nonreciprocity of acoustic and elastic wave motion in materials. However, this is a highly active and rich area of study and we anticipate significant advances to be made in the years to come. There is certainly a need for greater one-way transmission efficiency, frequency bandwidth and amplitude independence, and research with that focus will certainly be built upon the scientific principles and advances reviewed here.

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## Acknowledgements

ANN, GH, and MRH acknowledge support from NSF EFRI Award No. 1641078. MRH acknowledges support from ONR YIP award No. N00014-18-1-2335. CD acknowledges support from NSF EFRI Award No. 1741565. AA acknowledges support from NSF EFRI Award No. 1641069, the DARPA Nascent program, and AFSOR MURI Award No. FA9550-18-1-0379. RF acknowledges support from the Swiss National Science Foundation under SNSF grant No. 172487 and the SNSF Eccellenza award No. 181232.


## Author contributions

HN, BY, MR, and RF provided initial drafts of portions of the manuscript. All authors subsequently integrated, reviewed, and revised the full manuscript. MH and GH coordinated manuscript writing and organization.

## Competing interests

The authors declare no competing interests.