Existence of suitable weak solutions to the Navier-Stokes equations for intermittent data

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Abstract

Local in time weak solutions to the 3D Navier-Stokes are constructed for a class of initial data in L^2_{loc}. In contrast to other constructions (e.g. [23, 18, 22]), the initial data is not required to be uniformly locally square integrable and, in particular, can exhibit growth in a local L^2 sense. This class of initial data includes vector fields in the critical Morrey space and discretely self-similar vector fields in L^2_{loc}.

1 Introduction

Let u be the velocity field associated with a viscous incompressible fluid and p the associated pressure. The Navier-Stokes equations describe the evolution of u and p [6, 12, 27, 23, 24, 30, 31]. In particular, we have

$$\begin{aligned}&\partial_t u - \Delta u + u\cdot\nabla u + \nabla p = 0,\\ &\nabla\cdot u = 0,\end{aligned} \tag{1.1}$$

in the sense of distributions. The system (1.1) is set on $\mathbb{R}^3\times(0,T)$ where $T>0$ can be $+\infty$. Also, u evolves from a prescribed, divergence-free initial data $u_0\colon \mathbb{R}^3\to\mathbb{R}^3$.

In the classical paper [25], J. Leray constructed global-in-time weak solutions to (1.1) on $\mathbb{R}^4_+=\mathbb{R}^3\times(0,\infty)$ for any divergence-free vector field $u_0\in L^2(\mathbb{R}^3)$. Leray's solution u satisfies the following properties:

1. $u\in L^\infty(0,\infty;L^2(\mathbb{R}^3))\cap L^2(0,\infty;\dot H^1(\mathbb{R}^3))\cap C_{\mathrm{w}}(0,T;L^2)$,

2. u satisfies the weak form of (1.1),

$$\iint\big(-u\cdot\partial_t\zeta+\nabla u:\nabla\zeta+(u\cdot\nabla)u\cdot\zeta\big)=0,\qquad \zeta\in C_c^\infty(\mathbb{R}^4_+;\mathbb{R}^3)\text{ s.t. }\operatorname{div}\zeta=0,$$

3. $u(t) \to u_0$ in $L^2(\mathbb{R}^3)$ as $t \to 0^+$,

4. u satisfies the *global energy inequality*: For all $t > 0$,

$$\int_{\mathbb{R}^3} |u(x,t)|^2\,dx + 2\int_0^t \int_{\mathbb{R}^3} |\nabla u(x,t)|^2\,dx\,ds \le \int_{\mathbb{R}^3} |u_0(x)|^2\,dx.$$

In his book [23], Lemarié-Rieusset introduced a local analogue of Leray weak solutions evolving from locally integrable data $u_0 \in L^2_{\mathrm{uloc}}$. Here, L^q_{uloc}, for $1 \le q \le \infty$, is the space of functions on $\mathbb{R}^3$ with finite norm

$$\|u_0\|_{L^q_{\mathrm{uloc}}} := \sup_{x\in\mathbb{R}^3} \|u_0\|_{L^q(B(x,1))} < \infty.$$

We also denote

$$E^q = \mathrm{Cl}_{L^q_{\mathrm{uloc}}}(C_0^\infty(\mathbb{R}^3)),$$

the closure of $C_0^\infty(\mathbb{R}^3)$ in the L^q_{uloc}-norm.

The following definition is motivated by those found in [16, 18, 23]. Note that Q^* is a slightly larger set than Q so that $Q \subset Q^*$ and Q^{**} is a slightly larger set than Q^* so that $Q^* \subset Q^{**}$ (see Section 2).

Definition 1.1 (Local energy solutions)**.** *A vector field $u \in L^2_{\mathrm{loc}}(\mathbb{R}^3\times[0,T))$ is a local energy solution to (1.1) with divergence-free initial data $u_0 \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$ if the following conditions hold:*

1. *$u \in L^\infty(0,T;L^2_{\mathrm{loc}}) \cap L^2_{\mathrm{loc}}(\mathbb{R}^3\times[0,T])$,*

2. *for some $p \in L^{3/2}_{\mathrm{loc}}(\mathbb{R}^3\times(0,T))$, the pair (u,p) is a distributional solution to (1.1),*

3. *for all compact subsets K of $\mathbb{R}^3$ we have $u(t) \to u_0$ in $L^2(K)$ as $t \to 0^+$,*

4. *u is suitable in the sense of Caffarelli-Kohn-Nirenberg, i.e., for all cylinders Q compactly supported in $\mathbb{R}^3\times(0,T)$ and all non-negative $\phi \in C_0^\infty(Q)$, we have the local energy inequality*

$$2\iint |\nabla u|^2\phi\,dx\,dt \le \iint |u|^2(\partial_t\phi + \Delta\phi)\,dx\,dt + \iint (|u|^2+2p)(u\cdot\nabla\phi)\,dx\,dt, \quad (1.2)$$

5. *the function $t \mapsto \int u(x,t)\cdot w(x)\,dx$ is continuous on $[0,T)$ for any compactly supported $w \in L^2(\mathbb{R}^3)$,*

6. *for every cube $Q \subset \mathbb{R}^3$, there exists $p_Q(t) \in L^{3/2}(0,T)$ such that for $x \in Q^*$ and $0 < t < T$,*

$$p(x,t) - p_Q(t) = -\frac{1}{3}\delta_{ij}f(x) + \mathrm{p.v.}\int_{y\in Q^{**}} K_{ij}(x-y)(u_i(y,s)u_j(y,s))\,dy$$
$$+ \int_{y\notin Q^{**}} (K_{ij}(x-y) - K_{ij}(x_Q-y)(u_i(y,s)u_j(y,s))\,dy,$$

where x_Q is the center of Q and $K_{ij}(y) = \partial_i\partial_j(4\pi|y|)^{-1}$.

This differs from the usual definitions of local Leray solutions (e.g. in [23, 18, 28, 16]) and local energy solutions (e.g. in [17, 5]) because we are not assuming the data or the solutions are uniformly locally square integrable. In particular, we neither assume $u_0 \in L^2_{\mathrm{uloc}}$ nor that u satisfies

$$\operatorname*{ess\,sup}_{0\le t<R^2} \sup_{x_0\in\mathbb{R}^3} \int_{B_R(x_0)} |u(x,t)|^2\,dx + \sup_{x_0\in\mathbb{R}^3} \int_0^{R^2} \int_{B_R(x_0)} |\nabla u(x,t)|^2\,dx\,dt < \infty$$

for any $R > 0$. When referencing local Leray solutions, we mean local energy solutions that, additionally, satisfy these assumptions. These modifications reflect the main goal of this paper, which is to construct local energy solutions for initial data that is not uniformly locally square integrable.

Other interesting classes of solutions can be found in [21, 8, 31], e.g. very weak solutions or weak solutions in a class inspired by the BMO^{-1} space of Koch-Tataru [19]. However, the local energy class is particularly useful in that it is very general but retains enough structure to make progress on theoretical problems such as regularity and uniqueness. Several examples identified in [26] concerning this usefulness are the following:

- Local Leray solutions satisfy the local energy inequality of Caffarelli, Kohn, and Nirenberg. Consequently, partial regularity results based on that of Caffarelli, Kohn, and Nirenberg [7] are available for local Leray solutions. Additionally, the local energy inequality allows for the analysis of dynamics, e.g. the turbulence theory of Dascaliuc and Grujić [9].

- Local Leray solutions appear as the limit when re-scaling solutions near a possible singularity. Since the energy is a supercritical quantity, it blows up in the scaling limit, even though the limiting solution still solves the Navier-Stokes equations. The local energy, however, does not blow up, and the resulting solution belongs to the local Leray class.

- Local Leray solutions make sense in several critical infinite energy spaces such as the Lebesgue space L^3, the Lorentz space $L^{3,\infty}$, and the Morrey space $M^{2,1}$, all of which embed in L^2_{uloc} but not in L^2. Critical spaces are spaces for which the norm of u is scaling invariant. These spaces are borderline cases for many important questions like regularity and uniqueness. For example, $L^\infty(0,T;L^3)$ is a regularity class for Leray solutions [10], but this is unknown for $L^\infty(0,T;L^3_w)$, even though L^3_w is only marginally larger than L^3. Local Leray solutions played a key role in [15] and there is compelling evidence that they play an important role in establishing non-uniqueness of solutions in the Leray class [14, 13]. See also [5] which examines local energy solutions and their existence, regularity and uniqueness properties for data in Morrey spaces.

Local energy solutions are known to exist locally in time for initial data in $L^2_{\mathrm{uloc}}(\Omega)$ where Ω is $\mathbb{R}^3$ [23, 24, 18, 22] or $\mathbb{R}^3_+$ [26]. Global existence is known provided the initial data decays at spatial infinity in an appropriate sense, e.g. $u_0 \in E^2$ [23, 18] or u_0 has oscillation decay [22], or if the initial data is self-similar [24, 15, 4].

Our goal is to construct local-in-time suitable weak solutions for some L^2_{loc} initial data that is not uniformly locally square integrable. To accomplish this, we construct solutions for data in weighted, adapted local energy spaces. These are built off of a specific cover of $\mathbb{R}^3$ by cubes. Let $S_0 = \{x : |x_i| \le 2; i = 1,2,3\}$ and let $R_n = \{x : |x_i| < 2^n; i = 1,2,3\}$. Denote $S_n = \overline{R_{n+1} \setminus R_n}$ for $n \in \mathbb{N} \setminus \{1\}$ and $S_1 = \overline{R_2 \setminus S_0}$. Then $|S_n| = 56 \cdot 2^{3n}$ for $n \in \mathbb{N}$.

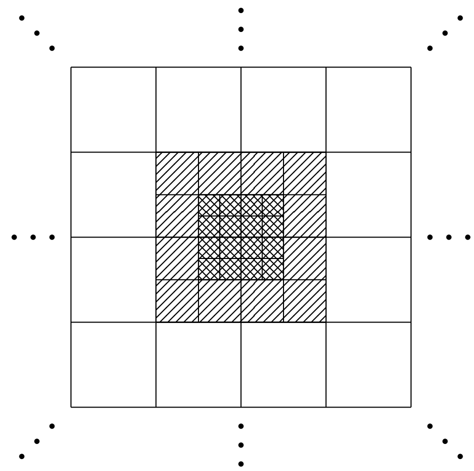

Figure 1: A two-dimensional illustration of S_0, S_1 and S_2. The cross hatched region is S_0, the hatched region is S_1 and the remaining region is S_2. The subsequent shells are just dyadic dilations of S_2.

Partition S_0 into 64 cubes of side-length 1 and S_n into 56 cubes of side-length 2^n. Then, S_0 is comprised of a $4 \times 4 \times 4$ grid of cubes while each shell S_n boxes in $\cup_{i=0}^{n-1} S_0$. Let $\mathcal{C}$ be the collection of these cubes. Note that the number of cubes in $\cup_{i=0}^{n-1} S_i$ grows linearly in n.

The main features of the collection $\mathcal{C}$ are the following:

(i) The side-length of a cube is proportional to its distance from the origin.

(ii) Adjacent cubes have comparable volume.

(iii) If $|Q'| < |Q|$ then the distance between the centers of Q and Q' is $\sim |Q|^{1/3}$.

(iv) The number of cubes Q' satisfying $|Q'| < |Q|$ is bounded above by $\sim |Q|^{1/3}$.

For convenience we also refer to the collection of cubes in $\mathcal{C}$ contained in S_n as S_n and accordingly write $Q \in S_n$.

Our initial data space is an analogue of L^2_{uloc} but adapted to the cover $\mathcal{C}$ and weighted.

Definition 1.2. *We say $f \in M^{p,q}_{\mathcal{C}}$ if*

$$\|f\|^p_{M^{p,q}_{\mathcal{C}}} := \sup_{Q \in \mathcal{C}} \frac{1}{|Q|^{q/3}} \int_Q |f(x)|^p \, dx < \infty.$$

Let $\mathring{M}^{p,q}_{\mathcal{C}}$ be the subset of $M^{p,q}_{\mathcal{C}}$ so that

$$\frac{1}{|Q|^{q/3}} \int_Q |u_0|^p \, dx \to 0 \text{ as } |Q| \to \infty, Q \in \mathcal{C}.$$

Comments on Definition 1.2

1. We clearly have the embedding $M^{2,q}_{\mathcal{C}} \subset M^{2,q'}_{\mathcal{C}}$ whenever $q < q'$. Additionally, $M^{2,q}_{\mathcal{C}} \subset \mathring{M}^{2,q'}_{\mathcal{C}}$ whenever $q < q'$.

2. Recall that f is in the critical (with respect to the scaling of the Navier-Stokes equations) Morrey space $M^{2,1}$ if

$$\sup_{x_0\in\mathbb{R}^3,r>0}\frac{1}{|B_r(x_0)|^{1/3}}\int_{B_r(x_0)}|f|^2\,dx<\infty.$$

   $M^{2,1}_{\mathcal{C}}$ is clearly a much weaker space than $M^{2,1}$ and does not assert any control (except square integrability) at small scales.

3. Note that $M^{2,q}_{\mathcal{C}}\subset L^2_{\mathrm{loc}}$ for all q, but $M^{2,q}_{\mathcal{C}}$ is not directly comparable to L^2_{uloc} when $q<3$. Indeed,

$$f(x)=\sum_{k\in\mathbb{N}}2^{qk/2}\chi_{B_1(2^ke_1)}(x)\in M^{2,q}_{\mathcal{C}}\setminus L^2_{\mathrm{uloc}},$$

   while $g(x)=1\in L^2_{\mathrm{uloc}}\setminus M^{2,q}_{\mathcal{C}}$. On the other hand, $L^2_{\mathrm{uloc}}\subset M^{2,3}_{\mathcal{C}}$.

4. The set L^p is dense in $\mathring{M}^{p,q}_{\mathcal{C}}$. To see this, let $u_0\in\mathring{M}^{p,q}_{\mathcal{C}}$ and $\epsilon>0$ be given. Then, there exists N so that

$$\|u_0\chi_{\mathbb{R}^3\setminus B_N(0)}\|^p_{M^{p,q}_{\mathcal{C}}}<\epsilon.$$

   Then, $u_0(1-\chi_{\mathbb{R}^3\setminus B_N(0)})\in L^p$ and

$$\|u_0(1-\chi_{\mathbb{R}^3\setminus B_N(0)})-u_0\|^p_{L^p(B_N(0))}<\epsilon.$$

Our main result concerns the local existence of solutions to the Navier-Stokes equations with data in $\mathring{M}^{2,2}_{\mathcal{C}}$.

Theorem 1.3. *Assume $u_0\in\mathring{M}^{2,2}_{\mathcal{C}}$ is divergence-free. Let $T=c^{-1}\min\{1,\|u_0\|^{-4}_{M^{2,2}_{\mathcal{C}}}\}$, for a sufficiently large constant $c>0$. Then, there exists $u\colon\mathbb{R}^3\times(0,T)\to\mathbb{R}^3$ and $p\colon\mathbb{R}^3\times(0,T)\to\mathbb{R}$ so that (u,p) is a local energy solution to the Navier-Stokes equations and*

$$\operatorname*{ess\,sup}_{0<t<T}\|u(t)\|^2_{M^{2,2}_{\mathcal{C}}}+\sup_{Q\in\mathcal{C}}\frac{1}{|Q|^{2/3}}\int_0^T\int_Q|\nabla u|^2\,dx\,dt\le C\|u_0(t)\|^2_{M^{2,2}_{\mathcal{C}}},$$

where $C>0$ is a constant.

Comments on Theorem 1.3:

1. Existence of solutions with data in $M^{2,q}_{\mathcal{C}}$ is related to the existence of self-similar and discretely self-similar solutions with data in L^2_{loc}. Recall that if there exists $\lambda>1$ so that $u_0(x)=\lambda u_0(\lambda x)$ for all x, then u_0 is said to be discretely self-similar, while u_0 is self-similar if this holds for all $\lambda>0$.

   In [24], Lemarié-Rieusset constructed self-similar solutions for self-similar initial data in L^2_{loc}. Later in [8], Chae and Wolf constructed discretely self-similar solutions for discretely self-similar data in L^2_{loc}. These solutions were not shown to satisfy the local energy inequality. In [4], the first author and Tsai constructed discretely self-similar solutions for the same data as [8] satisfying the local energy inequality by extending a construction in [3]. As mentioned in [4, Comment 4], discretely self-similar data in

L^2_{loc} is not necessarily in L^2_{uloc}, so there was no general existence theory including this class of data. Note that

$$\frac{1}{|Q|^{1/3}}\int_Q |u_0(x)|^2\,dx \le C\int_{B_{\sqrt{2}}(0)} |u_0(y)|^2\,dy,$$

after re-scaling the solution and changing variables. Thus any DSS data in L^2_{loc} belongs to $M^{2,1}_{\mathcal{C}}$ and, thus, also to $\mathring{M}^{2,2}_{\mathcal{C}}$. So $\mathring{M}^{2,2}_{\mathcal{C}}$ can be viewed as a natural class containing all discretely self-similar L^2_{loc} initial data for which a local existence theory is now available. Finding such a functional setting was a motivation for this paper.

2. Unlike other constructions of local energy solutions, Theorem 1.3 applies to some initial data that is not uniformly locally square integrable. The example $f(x)$ in the third comment following Definition 1.2 illustrates this as it is not uniformly locally square integrable but (after modifying it to make it divergence free) is a valid initial data for Theorem 1.3.

3. The decay assumption on u_0 means that u_0 is a limit of L^2 functions u_0^n in $M^{2,2}_{\mathcal{C}}$. This is convenient for an approximation argument since solutions for the initial data u_0^n are well understood.

   To eliminate the decay assumption, we would need a local existence theory for a regularized problem and a local pressure expansion for this system. The solutions to the regularized problem have no decay and are not necessarily bounded. Indeed, $e^{t\Delta}u_0$ are not necessarily bounded for $u_0 \in M^{2,2}_{\mathcal{C}}$. This complicates the analysis of the pressure. Recently Kwon and Tsai introduced a new approximation scheme which localizes the pressure but not the solution [22]. We expect this approach would allow the assumption that $u_0 \in \mathring{M}^{2,1}_{\mathcal{C}}$ to be weakened to $u_0 \in M^{2,1}_{\mathcal{C}}$. For the sake of simplicity we do not pursue this here, but note it in case such an improvement is useful in a future application.

4. Considering the scale of $M^{2,q}$ spaces, Theorem 1.3 gives existence of solutions for $u_0 \in M^{2,q}$ for $q < 2$. It is worth noting that solutions with a priori bounds in the $M^{2,q}$ class can also be constructed when $u_0 \in \mathring{M}^{2,q}$ and $q < 2$, but the time scale becomes $T = c^{-1}\min\{1, \|u_0\|^{-4}_{M^{2,q}}\}$. The proof is identical to that of Theorem 1.3. It seems difficult to extend the results to $q > 2$, which would be interesting because $L^2_{\mathrm{uloc}} \subset M^{2,3}$, and local existence is known in L^2_{uloc}. The reason that $q = 2$ is an endpoint case for our argument has to do with the treatment of the cubic term in the a priori estimates in Section 3.

5. The decay of the initial data does not lend itself to the usual extension argument to go from local to global existence in [23, 18, 22]. This is because, while the data is decaying, it is doing so at progressively larger scales. Since it is not becoming small at a single scale, the ensuing solution cannot be made small in the far-field at any fixed time. Thus the splitting argument of [23, 18, 22] cannot be used and a new approach is needed.

While this paper was under review and prior to the revision, Fernandez-Dalgo and Lemarié-Rieusset posted the paper [11] on the arxiv in which they construct global solutions in the framework of weighted spaces $L^2_{w_\gamma}$ where $w_\gamma = (1+|x|)^{-\gamma}$ with $0 < \gamma \le 2$ and

$$\|u_0\|^2_{L^2_{w_\gamma}} := \int_{\mathbb{R}^3} |u_0|^2 w_\gamma(x)\,dx.$$

The largest space in this scale occurs when $\gamma = 2$. This space is smaller than $\mathring{M}^{2,2}_{\mathcal{C}}$. Namely, one can check that

$$f(x) = \sum_{Q\in\mathcal{C}} \chi_Q(x)|Q|^{-1/6}(\ln|x_Q|)^{-1/2} \in \mathring{M}^{2,2}_{\mathcal{C}} \setminus L^2_{w_2}.$$

It is easy to adjust this example so that f is divergence-free. A global construction for data in a related but non-comparable space (to that in [11]) appears in [5].

Also note an important existence result due to Basson in two spatial dimensions obtained in [1], where the initial data is required to satisfy $\sup_{R\geq 1} R^{-2}\int_{|x|<R}|u_0(x)|^2\,dx < \infty$.

The paper is organized as follows. The main technical ingredients are the analysis of the pressure which is carried out in Section 2 and new a priori bounds in the $M^{2,q}_{\mathcal{C}}$ setting which are contained in Section 3. The compactness argument of [23, 18] puts these ingredients together in Section 4 where we prove Theorem 1.3.

2 Pressure formula and its bound

The pressure estimates in [23] and [18] can be adapted to the $M^{2,q}_{\mathcal{C}}$ framework. In this section we verify that the pressure formula converges when u is in a reasonable solution class for data in $M^{2,q}_{\mathcal{C}}$. First we introduce some notation. Denote

$$G_{ij}f = R_iR_jf = -\frac{1}{3}\delta_{ij}f(x) + \text{p.v.}\int K_{ij}(x-y)f(y)\,dy,$$

where R_i denotes the i-th Riesz transform and

$$K_{ij}(y) = \partial_i\partial_j\frac{1}{4\pi|y|} = \frac{-\delta_{ij}|y|^2 + 3y_iy_j}{4\pi|y|^5}.$$

For a given cube Q, let $x_Q \in \mathbb{R}^3$ be the center of Q. Fix $Q \in \mathcal{C}$. Let Q^* be the union of Q and all adjacent cubes in $\mathcal{C}$, and let Q^{**} be the same union but for Q^*.

For $x \in Q^*$, let

$$\begin{aligned} G^Q_{ij}f(x) = &-\frac{1}{3}\delta_{ij}f(x) + \text{p.v.}\int_{y\in Q^{**}} K_{ij}(x-y)f(y)\,dy \\ &+ \int_{y\notin Q^{**}} \big(K_{ij}(x-y) - K_{ij}(x_Q-y)\big)f(y)\,dy. \end{aligned} \tag{2.1}$$

Our pressure expansion is: For $Q \in \mathcal{C}$ and $t \in (0,T)$, there exists $p_Q(t) \in L^{3/2}(0,T)$ such that

$$p(x,t) - p_Q(t) = (G^Q_{ij}u_iu_j)(x,t), \qquad x \in Q^* \tag{2.2}$$

where p is the pressure corresponding to a local energy solution u.

Above, the pressure formula needed to be modified in comparison with the usual Riesz transform formula because f is not required to decay at spatial infinity. Note that this is the typical modification of the singular integrals for spaces with no decay at infinity [23, 24, 18, 16, 29]. When $u \in L^p$ has compact support, G^Q_{ij} agrees with G_{ij} up to a constant.

In this section, we bound $p(x,t) - p_Q(t)$ when u is in a solution class associated to the spaces $M^{2,q}_{\mathcal{C}}$. We assume for some $T > 0$ that

$$\alpha_T(u) := \sup_{0<t<T} \|u(t)\|^2_{M^{2,q}_{\mathcal{C}}} < \infty,$$

and

$$\beta_T(u) := \sup_{Q\in\mathcal{C}} \frac{1}{|Q|^{q/3}} \int_0^T \int_Q |\nabla u|^2 \,dx\,dt < \infty.$$

When T is clear we sometimes write α and β in place of α_T and β_T. These assumptions imply

$$\gamma(u) := \sup_{Q\in\mathcal{C}} \frac{1}{|Q|^{q/3}} \int_0^T \int_Q |u|^3 \,dx\,dt < \infty.$$

We now show that

$$\int_0^T \int_{Q^*} |p - p_Q(t)|^{3/2} \,dx\,dt$$

is bounded in terms of $\alpha(u)$ and $\gamma(u)$ and powers of $|Q|$.

Lemma 2.1. *Assume u is a local energy solution to the Navier-Stokes equations on $\mathbb{R}^3 \times [0,T]$ with pressure p satisfying* (2.2)*. Then, for $Q \in \mathcal{C}$,*

$$\frac{1}{|Q|^{1/3}} \int_0^T \int_{Q^*} |p - p_Q(t)|^{3/2} \,dx\,dt$$
$$\leq C \sup_{Q'\cap Q^{**}\neq\emptyset} \frac{1}{|Q'|^{1/3}} \int_0^T \int_{Q'} |u|^3 \,dx\,dt + CT|Q|^{q/2-1/3}\alpha_T^{3/2}.$$

Proof. For $x \in Q^*$, write $p(x,t) - p_Q(t)$ as $I_{\text{near}}(x,t) + I_{\text{far}}(x,t)$ where $I_{\text{near}}(x,t)$ is sum of the first two terms on the right hand side of (2.1) and $I_{\text{far}}(x,t)$ is the last term.

The required estimate for I_{near} follows from the Calderón-Zygmund inequality and noting that the argument for the boundedness is local. In particular,

$$\frac{1}{|Q|^{1/3}} \int_0^T \int_{Q^*} |I_{\text{near}}|^{3/2} \,dx\,dt \leq \frac{C}{|Q|^{1/3}} \int_0^T \int_{Q^{**}} |u|^3 \,dx\,dt.$$

Since $|Q| \sim |Q'|$ for all $Q' \subset Q^{**}$ and there are a fixed number of such Q' (independent of Q), we have

$$\frac{1}{|Q|^{1/3}} \int_0^T \int_{Q^{**}} |u|^3 \,dx\,dt \leq C \sup_{Q'\cap Q^{**}\neq\emptyset} \frac{|Q'|^{1/3}}{|Q|^{1/3}} \frac{1}{|Q'|^{1/3}} \int_0^T \int_{Q'} |u|^3 \,dx\,dt$$
$$\leq C \sup_{Q'\cap Q^{**}\neq\emptyset} \frac{1}{|Q'|^{1/3}} \int_0^T \int_{Q'} |u|^3 \,dx\,dt.$$

To estimate I_{far} note that when $y \notin Q^{**}$ and $x \in Q^*$, we have

$$|K_{ij}(x-y) - K_{ij}(x_Q - y)| \leq \frac{C|Q|^{1/3}}{|x-y|^4}.$$

This estimate (for balls instead of cubes) may be found in [18] and [16]. Thus, for $x \in Q^*$,

$$
\begin{aligned}
|I_{\text{far}}(x,t)| &\le C|Q|^{1/3} \int_{\mathbb{R}^3 \setminus Q^{**}} \frac{1}{|x-y|^4} |u|^2 \, dy \\
&\le C \sum_{Q' \in \mathcal{C}_1} |Q|^{1/3} \int_{Q'} \frac{1}{|x-y|^4} |u|^2 \, dy + C \sum_{Q' \in \mathcal{C}_2} |Q|^{1/3} \int_{Q'} \frac{1}{|x-y|^4} |u|^2 \, dy,
\end{aligned}
$$

where $\mathcal{C}_1$ is the collection of $Q' \in \mathcal{C}$ such that $Q' \cap Q^{**} = \emptyset$ and $|Q'| < |Q|$ while $\mathcal{C}_2$ is the collection of $Q' \in \mathcal{C}$ such that $Q' \cap Q^{**} = \emptyset$ and $|Q'| \ge |Q|$. Let $n \in \mathbb{N}$ be such that $Q \in S_n$. If $|Q'| < |Q|$, then $Q' \in S_j$ for some $j < n$. Also, the number of cubes Q' such that $|Q'| \le |Q|$ is bounded above by $|Q|^{1/3}$. Furthermore, $|Q|^{1/3} \sim |x_Q - x_{Q'}|$. Also, if $Q' \cap Q^{**} = \emptyset$, $y \in Q'$, and $x \in Q$, then $|x - y| \sim |x_Q - x'_Q|$. Hence,

$$
\begin{aligned}
\sum_{Q' \in \mathcal{C}_1} |Q|^{1/3} \int_{Q'} \frac{1}{|x-y|^4} |u|^2 \, dy &\le C \sum_{Q' \in \mathcal{C}_1} \frac{|Q|^{1/3}}{|Q|^{4/3}} \int_{Q'} |u|^2 \, dy \\
&\le C \sup_{|Q'|<|Q|} \frac{|Q|^{1/3}}{|Q|} \int_{Q'} |u|^2 \, dy = C \sup_{|Q'|<|Q|} \frac{|Q'|^{q/3}}{|Q|^{2/3}} \frac{1}{|Q'|^{q/3}} \int_{Q'} |u|^2 \, dy \\
&\le C|Q|^{q/3-2/3} \sup_{Q' \in \mathcal{C}} \frac{1}{|Q'|^{q/3}} \int_{Q'} |u|^2 \, dx.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\sum_{Q' \in \mathcal{C}_2} |Q|^{1/3} \int_{Q'} \frac{1}{|x-y|^4} |u|^2 \, dy = \sum_{m \ge n} \sum_{Q' \in S_m \cap \mathcal{C}_2} \frac{|Q|^{1/3}}{|x_Q - x_{Q'}|^4} \int_{Q'} |u|^2 \, dx \\
&\quad \le C|Q|^{1/3} \sum_{m \ge n} \frac{1}{2^{(4-q)m}} \sup_{Q' \in \mathcal{C}} \frac{1}{|Q'|^{q/3}} \int_{Q'} |u|^2 \, dx \le \frac{C|Q|^{1/3}}{2^{(4-q)n}} \sup_{Q' \in \mathcal{C}} \frac{1}{|Q'|^{q/3}} \int_{Q'} |u|^2 \, dx \\
&\quad \le C|Q|^{q/3-1} \sup_{Q' \in \mathcal{C}} \frac{1}{|Q'|^{q/3}} \int_{Q'} |u|^2 \, dx,
\end{aligned}
$$

and thus

$$
\begin{aligned}
\frac{1}{|Q|^{1/3}} \int_0^T \int_{Q^*} |I_{\text{far}}(x,t)|^{3/2} \, dx \, dt &\le C \frac{T(|Q|^{q/2-1} + |Q|^{q/2-3/2})}{|Q|^{1/3}} |Q| \alpha_T^{3/2} \\
&\le CT|Q|^{q/2-1/3} \alpha_T^{3/2}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{1}{|Q|^{1/3}} \int_0^T \int_{Q^*} |p(x,t) - p_Q(t)|^{3/2} \, dx \, dt \\
&\qquad \le C \sup_{Q' \cap Q^{**} \neq \emptyset} \frac{|Q'|^{1/3}}{|Q|^{1/3}} \frac{1}{|Q'|^{1/3}} \int_0^T \int_{Q'} |u|^3 \, dx \, dt + CT|Q|^{q/2-1/3} \alpha_T^{3/2},
\end{aligned}
$$

and the proof is concluded. □

Note that we have not shown that, for a given solution, the pressure satisfies the local expansion (2.2), but rather that if the pressure satisfies the local expansion, then it is bounded in the above sense.

3 A priori bounds

In order to approximate a solution in the local energy class by Leray solutions we need an estimate for them in the $M^{2,q}_{\mathcal{C}}$ spaces.

Let ϕ be a radial smooth cutoff function such that $\phi = 1$ in $[-1/2, 1/2]^3$, $\phi = 0$ off of $[-3/4, 3/4]^3$ with ϕ non-increasing in $|x|$. For $Q \in \mathcal{C}$, let ϕ_Q be the translation and dilation of ϕ so that ϕ_Q equals 1 on Q and vanishes off of Q^*. Then, $\|\partial^\lambda \phi_Q(x)\|_{L^\infty} \le C(\lambda)/|Q|^{|\lambda|/3}$ where C does not depend on Q and λ is any multi-index. Denote

$$\alpha_t = \operatorname*{ess\,sup}_{0<s<t} \sup_{Q\in\mathcal{C}} \frac{1}{|Q|^{q/3}} \int_Q |u(x,s)|^2 \phi_Q \, dx,$$

and

$$\beta_t = \sup_{Q\in\mathcal{C}} \frac{1}{|Q|^{q/3}} \int_0^t \int_Q |\nabla u|^2 \phi_Q \, dx\, ds.$$

The following statement provides a priori estimates for the existence of suitable weak solutions with data $u_0 \in M^{2,q}_{\mathcal{C}}$.

Theorem 3.1. *Assume $u_0 \in M^{2,q}_{\mathcal{C}}$, for $0 \le q \le 2$, is divergence-free, and let (u,p) be a local energy solution with initial data u_0 on $\mathbb{R}^3 \times (0,T)$ where*

$$T = \frac{1}{C} \min\left\{1, \|u_0\|^{-4}_{M^{2,q}_{\mathcal{C}}}\right\} \tag{3.1}$$

for a sufficiently large universal constant C. Assume additionally that

$$\alpha_T + \beta_T < \infty,$$

and α_t and β_t are continuous in t. Then

$$\sup_{0<t<T} \|u(t)\|^2_{M^{2,q}_{\mathcal{C}}} + \sup_{Q\in\mathcal{C}} \frac{1}{|Q|^{q/3}} \int_0^T \int_Q |\nabla u|^2 \, dx\, dt \le C\|u_0(t)\|^2_{M^{2,q}_{\mathcal{C}}}. \tag{3.2}$$

It is important that Theorem 3.1 also applies to suitable Leray weak solutions as shown at the end of this section.

Proof of Theorem 3.1. Assume u and p are as in the statement of Theorem 3.1. Fix $Q \in \mathcal{C}$. The local energy inequality and the item 5 of Definition 1.1 give

$$\begin{aligned}
&\frac{1}{2}\int |u(x,t)|^2 \phi_Q(x)\, dx + \int_0^t \int |\nabla u(x,s)|^2 \phi_Q(x)\, dx\, ds \\
&\quad \le \frac{1}{2}\int |u(x,0)|^2 \phi_Q(x)\, dx + \frac{1}{2}\int_0^t \int |u(x,s)|^2 \Delta\phi_Q(x)\, dx\, ds \\
&\qquad + \frac{1}{2}\int_0^t \int \big(|u(x,s)|^2 u(x,s)\cdot\nabla\phi_Q(x) + 2p(x,s)u(x,s)\cdot\nabla\phi_Q(x)\big)\, dx\, ds.
\end{aligned}$$

Clearly,

$$\begin{aligned}
\frac{1}{2}\int_0^t \int |u|^2 \Delta\phi_Q \, dx\, ds &\le \frac{Ct}{|Q|^{2/3}} \operatorname*{ess\,sup}_{0\le s\le t} \sup_{Q'\cap Q^*\neq\emptyset} \int_{Q'} |u|^2\, dx \\
&\le Ct \operatorname*{ess\,sup}_{0\le s\le t} \sup_{Q'\cap Q^*\neq\emptyset} \frac{|Q'|^{q/3}}{|Q|^{2/3}} \frac{1}{|Q'|^{q/3}} \int_{Q'} |u|^2\, dx \le C\, t\, \alpha_t,
\end{aligned}$$

where we used $q \le 2$ and the fact that the smallest cubes in $\mathcal{C}$ have volume bounded away from zero so that $|Q'|^{q/3}/|Q|^{2/3} \le C$ when $Q' \cap Q^* \neq \emptyset$.

For the cubic and pressure terms we have

$$\begin{aligned}
&\int_0^t \int \left(\frac{|u|^2}{2} u \cdot \nabla \phi_Q(x) + pu \cdot \nabla \phi_Q(x) \right) dx\, ds \\
&\quad \le \frac{1}{|Q|^{1/3}} \int_0^t \int_{Q^*} (|u|^3 + |p - p_Q|^{3/2})\, dx\, ds \\
&\quad \le C \sup_{Q' \cap Q^{**} \neq \emptyset} \frac{|Q'|^{1/3}}{|Q|^{1/3}} \frac{1}{|Q'|^{1/3}} \int_0^t \int_{Q'} |u|^3\, dx\, ds + Ct|Q|^{q/3}\alpha_t^{3/2} \\
&\quad \le C \sup_{Q' \cap Q^{**} \neq \emptyset} \frac{1}{|Q'|^{1/3}} \int_0^t \int_{Q'} |u|^3\, dx\, ds + Ct|Q|^{q/3}\alpha_t^{3/2},
\end{aligned}$$

where we used Lemma 2.1 and $q/2 - 1/3 \le q/3$.

Recall that for any cube Q' the Gagliardo-Nirenberg inequality implies

$$\int_{Q'} |u|^3\, dx \le C \left(\int_{Q'} |u|^2\, dx \right)^{3/4} \left(\int_{Q'} |\nabla u|^2\, dx \right)^{3/4} + \frac{C}{|Q'|^{1/2}} \left(\int_{Q'} |u|^2\, dx \right)^{3/2}.$$

Hence, for any $Q' \in \mathcal{C}$,

$$\begin{aligned}
&\frac{1}{|Q'|^{1/3}} \int_0^t \int_{Q'} |u|^3\, dx\, ds \\
&\quad \le Ct^{1/4}|Q'|^{q/2-1/3} \left(\frac{1}{|Q'|^{q/3}} \operatorname*{ess\,sup}_{0 \le s \le t} \int_{Q'} |u|^2\, dx \right)^{3/4} \left(\frac{1}{|Q'|^{q/3}} \int_0^t \int_{Q'} |\nabla u|^2\, dx\, ds \right)^{3/4} \\
&\qquad + Ct|Q'|^{q/2-5/6} \left(\frac{1}{|Q'|^{q/3}} \operatorname*{ess\,sup}_{0 \le s \le t} \int_{Q'} |u|^2\, dx \right)^{3/2} \\
&\quad \le Ct^{1/4}|Q'|^{q/2-1/3}(\alpha_t + \beta_t)^{3/2},
\end{aligned}$$

where we used $t \le C$ by (3.1) and the fact that the smallest cubes in $\mathcal{C}$ have volume bounded away from 0 so that $|Q'|^{-5/6} \le C|Q'|^{-1/3}$. At this point we have shown

$$\begin{aligned}
&\int_Q |u(x,t)|^2\, dx + \int_0^t \int_Q |\nabla u(x,s)|^2\, dx\, ds \\
&\quad \le C\, t\, \alpha_t + Ct|Q|^{q/3}\alpha_t^{3/2} + C \sup_{Q' \cap Q^{**} \neq \emptyset} \frac{|Q'|^{1/3}}{|Q|^{1/3}} \frac{1}{|Q'|^{1/3}} \int_0^t \int_{Q'} |u|^3\, dx\, ds \\
&\quad \le C\, t\, \alpha_t + Ct|Q|^{q/3}\alpha_t^{3/2} + Ct^{1/4} \sup_{Q' \cap Q^{**} \neq \emptyset} |Q'|^{q/2-1/3}(\alpha_t + \beta_t)^{3/2}.
\end{aligned}$$

So, again using the fact that the volumes of cubes in $\mathcal{C}$ are bounded away from zero and dividing through by $|Q|^{q/3}$ we obtain

$$\begin{aligned}
&\frac{1}{|Q|^{q/3}} \int_Q |u(x,t)|^2\, dx + \frac{1}{|Q|^{q/3}} \int_0^t \int_Q |\nabla u(x,s)|^2\, dx\, ds \\
&\quad \le C\, t\, \alpha_t + Ct\alpha_t^{3/2} + Ct^{1/4} \sup_{Q' \cap Q^{**} \neq \emptyset} \frac{|Q'|^{q/2-1/3}}{|Q|^{q/3}} (\alpha_t + \beta_t)^{3/2} \\
&\quad \le C\, t\, \alpha_t + Ct^{1/4}(\alpha_t + \beta_t)^{3/2},
\end{aligned}$$

provided $q \leq 2$ and noting that $t \leq C$ by (3.1). Therefore,

$$\alpha_t + \beta_t \leq Ct(\alpha_t + \beta_t) + Ct^{1/4}(\alpha_t + \beta_t)^{3/2}.$$

Since $\alpha_0 + \beta_0 = \alpha_0 = \|u_0\|^2_{M^{2,q}_{\mathcal{C}}}$ and $\alpha_t + \beta_t$ is continuous in t, it follows that

$$\alpha_t + \beta_t \leq 2\|u_0\|^2_{M^{2,1}_{\mathcal{C}}},$$

on some time interval $[0, T)$ where T is maximal, that is, $\alpha_T + \beta_T = 2\|u_0\|^2_{M^{2,q}_{\mathcal{C}}}$. Let

$$T_0 = \max\{(2C)^{-1}, (C4\alpha_0^{1/2})^{-4}\}.$$

If $T < T_0$, then $\alpha_T < \alpha_0$, which is impossible. Therefore, $T_0 \leq T$, that is

$$\operatorname*{ess\,sup}_{0<t<T_0} \sup_{Q \in \mathcal{C}} \frac{1}{|Q|^{q/3}} \int_Q |u|^2\,dx + \sup_{Q \in \mathcal{C}} \frac{1}{|Q|^{q/3}} \int_0^{T_0} \int_Q |\nabla u|^2\,dx\,dt \leq 2\|u_0\|^2_{M^{2,q}_{\mathcal{C}}},$$

and (3.2) is established. □

We also need for Theorem 3.1 to hold for suitable weak solutions in the Leray class. For a definition of a suitable weak solution in the Leray class, see [31, Definition 3.1].

Lemma 3.2. *Theorem 3.1 applies to suitable weak solutions in the Leray class.*

Proof. Assume (u, p) is a suitable weak solution in the sense of [31, Definition 3.1]. Then, u satisfies the items 1–5 in Definition 1.1. For item 6, which concerns the existence of pressure note that

$$\lim_{|x_0| \to \infty} \int_0^{R^2} \int_{B_R(x_0)} |u|^2\,dx\,dt = 0,$$

for any $R > 0$. Indeed, for a fixed R we have

$$\int_0^{R^2} \int_{B_R(x_0)} |u|^2\,dx\,dt \leq \int_0^{R^2} \int |u|^2\,dx < \infty.$$

Thus by the dominated convergence theorem

$$\lim_{|x_0| \to \infty} \int_0^{R^2} \int_{B_R(x_0)} |u|^2\,dx\,dt = \int_0^{R^2} \lim_{|x_0| \to \infty} \int_{B_R(x_0)} |u|^2\,dx\,dt = 0.$$

In [17] the above property is shown to imply the local pressure expansion for balls, which is equivalent to our pressure expansion for cubes. Thus item 6 is satisfied. Hence, any suitable weak solution in the sense of [31, Definition 3.1] is a local energy solution. Note also that

$$\operatorname*{ess\,sup}_{0<t<T} \|u(t)\|^2_{M^{2,q}_{\mathcal{C}}} + \sup_{Q \in \mathcal{C}} \frac{1}{|Q|^{q/3}} \int_0^T \int_Q |\nabla u|^2\,dx\,dt$$
$$\leq C \operatorname*{ess\,sup}_{0<t<T} \|u\|^2_{L^2} + C \int_0^T \|u\|^2_{H^1}\,dt < \infty.$$

It remains to show that $\alpha_t + \beta_t$ is continuous in t. Our proof is based on the argument in [20, Lemma 2]. Denote by $\mathcal{L}_Q \subseteq [0,T)$ the set of Lebesgue points of the function $t \mapsto \int |u(x,s)|^2 \phi_Q(s)\,ds$. We first show that

$$\operatorname*{ess\,sup}_{0<s<t} \int |u(x,s)|^2 \phi_Q(x)\,dx + \int_0^t \int |\nabla u(x,s)|^2 \phi_Q(x)\,dx\,ds \tag{3.3}$$

is continuous in t for every fixed Q. Since $\int_0^t \int |\nabla u(x,s)|^2 \phi_Q(x)\,dx\,ds$, is continuous in t, we only need to show the continuity of the first term in (3.3). Using a sequence of functions of the form $\phi_Q(x)\psi(t)$, we obtain from (1.2)

$$\begin{aligned}
&\int |u(x,t_2)|^2 \phi_Q(x)\,dx + \int_{t_2}^{t_2} \int |\nabla u(x,s)|^2 \phi_Q(x)\,dx\,ds,\\
&\quad\le \int |u(x,t_1)|^2 \phi_Q(x)\,dx\\
&\qquad + \iint |u|^2 (\partial_t \phi_Q + \Delta \phi_Q)\,dx\,dt\\
&\qquad + \iint (|u|^2 + 2p)(u\cdot\nabla\phi_Q)\,dx\,dt, \qquad t_1, t_2 \in \mathcal{L}_Q, \quad 0 \le t_1 \le t_2 < T.
\end{aligned}$$

Choosing $t_1 = t \in \mathcal{L}_Q$ and $t_2 = t+h$, we get

$$\limsup_{h\to 0+, t+h\in\mathcal{L}_Q} \int |u(x,t+h)|^2 \phi_Q(x)\,dx \le \int |u(x,t)|^2 \phi_Q(x)\,dx, \qquad t \in \mathcal{L}_Q, \tag{3.4}$$

while setting $t_2 = t \in \mathcal{L}_Q$ and $t_1 = t-h$, we get

$$\liminf_{h\to 0+, t-h\in\mathcal{L}_Q} \int |u(x,t-h)|^2 \phi_Q(x)\,dx \ge \int |u(x,t)|^2 \phi_Q(x)\,dx, \qquad t \in \mathcal{L}_Q. \tag{3.5}$$

Since $\operatorname{ess\,sup}_{0<s<t} \int |u(x,s)|^2 \phi_Q(x)\,dx$ is non-decreasing in t, (3.4) and (3.5) imply continuity of $\operatorname{ess\,sup}_{0<s<t} \int |u(x,s)|^2 \phi_Q(x)\,dx$ as a function of t for any fixed Q.

Next, we establish the continuity of α_t. Fix $t \in [0,T)$, and let $\epsilon > 0$. Note that $\operatorname{ess\,sup}_{0<s<T} \|u(s)\|_2^2 < \infty$. Thus, there exists m so that

$$\operatorname*{ess\,sup}_{0<s<T} \frac{1}{|Q|^{q/3}} \int_Q |u(x,s)|^2\,dx \le \frac{\epsilon}{2}, \qquad Q \in S_n, \quad n \ge m.$$

We have two possibilities: Either

$$\operatorname*{ess\,sup}_{0<s<t} \sup_{Q\in\mathcal{C}} \frac{1}{|Q|^{q/3}} \int |u(x,s)|^2 \phi_Q\,dx\,ds \le \frac{\epsilon}{2}, \tag{3.6}$$

or there exists $Q_0 \in S_n$ with $n < m$ so that

$$\operatorname*{ess\,sup}_{0<s<t} \sup_{Q\in\mathcal{C}} \frac{1}{|Q|^{q/3}} \int |u(x,s)|^2 \phi_Q\,dx\,ds = \operatorname*{ess\,sup}_{0<s<t} \frac{1}{|Q_0|^{q/3}} \int_{Q_0} |u(x,s)|^2 \phi_Q\,dx. \tag{3.7}$$

Assume the first case, (3.6), holds. Then for each $Q \in S_n$, where $n < m$, there exists $\delta_Q > 0$ so that

$$\operatorname*{ess\,sup}_{0<s<\tau} \frac{1}{|Q|^{q/3}} \int |u(x,s)|^2 \phi_Q(x)\,dx \le \epsilon$$

whenever $|t-\tau| < \delta_Q$. Then, using the first part of the proof,

$$\left|\operatorname*{ess\,sup}_{0<s<\tau} \frac{1}{|Q|^{q/3}} \int |u(x,s)|^2 \phi_Q(x)\, dx - \operatorname*{ess\,sup}_{0<s<t} \frac{1}{|Q|^{q/3}} \int |u(x,s)|^2 \phi_Q(x)\, dx\right| \leq \epsilon$$

for all $Q \in S_n$ where $n < m$ and $|t-\tau| < \delta_Q$. Letting $\delta = \min_{Q \in \bigcup_{n=0}^m S_n} \delta_Q$ gives

$$\left|\operatorname*{ess\,sup}_{0<s<t} \sup_{Q\in\mathcal{C}} \frac{1}{|Q|^{q/3}} \int |u(x,s)|^2 \phi_Q\, dx\, ds - \operatorname*{ess\,sup}_{0<s<\tau} \sup_{Q\in\mathcal{C}} \frac{1}{|Q|^{q/3}} \int_Q |u(x,s)|^2 \phi_Q\, dx\right| \leq \frac{3\epsilon}{2}$$

for $t - \tau < \delta$ by (3.6) and

$$\operatorname*{ess\,sup}_{0<s<\tau} \sup_{Q\in\mathcal{C}} \frac{1}{|Q|^{q/3}} \int_Q |u(x,s)|^2 \phi_Q\, dx \leq \frac{3\epsilon}{2}.$$

If the second case, (3.7), holds, then let S be the collection of $Q \in \bigcup_{n=0}^{m-1} S_n$ so that

$$\operatorname*{ess\,sup}_{0<s<t} \sup_{\tilde{Q}\in\mathcal{C}} \frac{1}{|\tilde{Q}|^{q/3}} \int |u(x,s)|^2 \phi_{\tilde{Q}}\, dx\, ds > \operatorname*{ess\,sup}_{0<s<t} \frac{1}{|Q|^{q/3}} \int_Q |u(x,s)|^2 \phi_Q\, dx.$$

Let $S' = \cup_{n=0}^{m-1} S_n \setminus S$. Let

$$\begin{aligned}\bar{\epsilon} = \min\Bigg\{ \frac{\epsilon}{2}, \min_{Q\in S}\Bigg(&\operatorname*{ess\,sup}_{0<s<t} \sup_{\tilde{Q}\in\mathcal{C}} \frac{1}{|\tilde{Q}|^{q/3}} \int |u(x,s)|^2 \phi_{\tilde{Q}}\, dx\, ds \\ &- \operatorname*{ess\,sup}_{0<s<t} \frac{1}{|Q|^{q/3}} \int_Q |u(x,s)|^2 \phi_Q\, dx\Bigg)\Bigg\}.\end{aligned}$$

For each $Q \in S'$, there exists $\delta_Q > 0$ so that

$$\begin{aligned}&\operatorname*{ess\,sup}_{0<s<t} \sup_{Q\in\mathcal{C}} \frac{1}{|Q|^{q/3}} \int |u(x,s)|^2 \phi_Q\, dx\, ds + \frac{\bar{\epsilon}}{2} \\ &\quad\geq \operatorname*{ess\,sup}_{0<s<\tau} \frac{1}{|Q|^{q/3}} \int_Q |u(x,s)|^2 \phi_Q\, dx \\ &\quad\geq \operatorname*{ess\,sup}_{0<s<t} \sup_{Q\in\mathcal{C}} \frac{1}{|Q|^{q/3}} \int |u(x,s)|^2 \phi_Q\, dx\, ds - \frac{\bar{\epsilon}}{2}\end{aligned}$$

whenever $|t-\tau| < \delta_Q$. On the other hand, for each $Q \in S$, there exists $\delta_Q > 0$ so that

$$\operatorname*{ess\,sup}_{0<s<\tau} \frac{1}{|Q|^{q/3}} \int_Q |u(x,s)|^2 \phi_Q\, dx \leq \operatorname*{ess\,sup}_{0<s<t} \sup_{Q\in\mathcal{C}} \frac{1}{|Q|^{q/3}} \int |u(x,s)|^2 \phi_Q\, dx\, ds - \frac{\bar{\epsilon}}{2}.$$

Hence,

$$\left|\operatorname*{ess\,sup}_{0<s<\tau} \frac{1}{|Q|^{q/3}} \int_Q |u(x,s)|^2 \phi_Q\, dx - \operatorname*{ess\,sup}_{0<s<t} \frac{1}{|Q|^{q/3}} \int_Q |u(x,s)|^2 \phi_Q\, dx\right| \leq \epsilon$$

provided $|t-\tau| < \min_{S\cup S'} \delta_Q$.

The proof of the continuity of β_t is similar, but simpler since there is no supremum over the time interval in the definition of β. □

4 Construction

4.1 Approximating the initial data

We begin with a lemma on approximation of functions in $\mathring{M}^{2,2}_{\mathcal{C}}$ with those in L^2.

Lemma 4.1. *Assume $f \in \mathring{M}^{2,2}_{\mathcal{C}}$ is divergence-free. For every $\epsilon > 0$ there exists a divergence-free $g \in L^2$ such that $\|f - g\|_{M^{2,2}_{\mathcal{C}}} \leq \epsilon$.*

To make sure the approximation is divergence-free, we utilize Bogovskii's map [2] (see also [12, 31]) which we recall next.

Lemma 4.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, where $2 \leq n < \infty$. There is a linear map Ψ that maps a scalar $f \in L^2(\Omega)$ with $\int_\Omega f = 0$ to a vector field $v = \Psi f \in W^{1,2}_0(\Omega; \mathbb{R}^n)$ and*

$$\operatorname{div} v = f, \quad \|v\|_{W^{1,2}_0(\Omega)} \leq c(\Omega)\|f\|_{L^2(\Omega)}.$$

The constant $c(\Omega)$ depends on the size of Ω and, when Ω are shells of the form $\{x : 2^n \leq |x| \leq 2^{n+1}\}$, it is of the form $C2^n$.

Proof of Lemma 4.1. Assume $f \in \mathring{M}^{2,2}_{\mathcal{C}}$ is divergence-free, and let $\epsilon > 0$. Assume n is large enough so that $\|f\chi_{\mathbb{R}^3 \setminus B_{2^n}(0)}\|_{M^{2,2}_{\mathcal{C}}} \leq \epsilon/C$ for some constant C to be identified later. Let $Z = Z_n \in C^\infty(\mathbb{R}^3)$ satisfy $Z(x) = 1$ if $|x| \leq 2^n$, $Z(x) = 0$ if $|x| \geq 2^{n+1}$, and assume it is radial, and non-increasing for $2^n \leq |x| \leq 2^{n+1}$. For every n, we may choose Z so that $\|\nabla Z\|_\infty \leq C2^{-n}$ where C is independent of n. Then,

$$\nabla \cdot (fZ) = f \cdot \nabla Z$$

because f is divergence-free. Note that

$$\int f \cdot \nabla Z \, dx = 0$$

because Z has compact support and f is divergence free. Denote by Φ the image of $-f \cdot \nabla Z$ under the Bogovskii map with $q = 2$ and domain $A_n = \{x : 2^n \leq |x| \leq 2^{n+1}\}$. Then, $\Phi \in W^{1,2}_0(A_n)$ and

$$\nabla \cdot (Zf + \Phi) = 0.$$

Furthermore,

$$\int_{A_n} |\Phi|^2 \, dx \leq C 2^{2n} \int_{A_n} |f|^2 |\nabla Z|^2 \, dx \leq C \int_{A_n} |f|^2 \, dx.$$

Let $g = Zf + \Phi$.

Note that there exist only finitely many cubes $Q' \in \mathcal{C}$ that intersect A_n and this number is bounded independently of n. Hence,

$$\|\Phi\|^2_{M^{2,2}_{\mathcal{C}}} \leq \sup_{Q' \cap A_n \neq 0} \frac{1}{|Q'|^{2/3}} \int_{Q'} |\Phi|^2 \, dx \leq C \frac{1}{2^{2n}} \int_{A_n} |\Phi|^2 \, dx,$$

where we have also used the fact that if Q' intersects A_n, then $|Q'| \sim 2^{3n}$. Thus,

$$\begin{aligned}\frac{1}{2^{2n}}\int_{A_n}|\Phi|^2\,dx &\le C\frac{1}{2^{2n}}\int_{A_n}|f|^2\,dx\\ &\le C\frac{1}{2^{2n}}\sum_{Q'\cap A_n\neq 0}\int_{Q'}|f\chi_{\mathbb{R}^3\setminus B_{2^n}(0)}|^2\,dx\\ &\le C\sum_{Q'\cap A_n\neq 0}\frac{1}{|Q'|^{2/3}}\int_{Q'}|f\chi_{\mathbb{R}^3\setminus B_{2^n}(0)}|^2\,dx\\ &\le C\|f\chi_{\mathbb{R}^3\setminus B_{2^n}(0)}\|^2_{M^{2,2}_{\mathcal{C}}}\le C\epsilon^2.\end{aligned}$$

Therefore,

$$\|f-g\|_{M^{2,2}_{\mathcal{C}}}\le\|f-Zf\|_{M^{2,2}_{\mathcal{C}}}+\|\Phi\|_{M^{2,2}_{\mathcal{C}}}\le C\epsilon,$$

and the proof is concluded. □

4.2 Proof of Theorem 1.3

Proof of Theorem 1.3. Let $u_0\in\mathring{M}^{2,2}_{\mathcal{C}}$ be divergence-free. By Lemma 4.1, for each $n\in\mathbb{N}$ there exists $u_0^n\in L^2$ so that $\|u_0-u_0^n\|_{M^{2,2}_{\mathcal{C}}}\le 1/2^n$. By the classical theory, there exists a global in time suitable weak solution u^n in the Leray class with the initial data u_0^n and an associated pressure p^n (see e.g. [31, Chapter 3]), which satisfies the pressure expansion (2.2).

Let $Q_n=\cup_{m\le n;Q'\in S_m}Q'$. We adopt the same convention when defining Q_n^* and Q_n^{**} as when $Q\in\mathcal{C}$; namely, Q_n^* is the union of Q_n and all cubes in $\mathcal{C}$ adjacent to Q_n and Q_n^{**} the union of Q_n^* and the cubes in $\mathcal{C}$ adjacent to Q_n^*.

Define $\bar{p}^n$ recursively as follows:

- If $x\in Q_0$ then $\bar{p}^n(x)=G_{ij}^{Q_1}(u_i^nu_j^n)(x)$, where we use 0 instead of x_Q in (2.1).
- If $x\in Q_{k+1}$ then $\bar{p}^n(x)=G_{ij}^{Q_{k+1}}(u_i^nu_j^n)(x)+c_{n,k}$, where

  $$c_{n,k}=G_{ij}^{Q_k}(u_i^nu_j^n)(x)-G_{ij}^{Q_{k+1}}(u_i^nu_j^n)(x),\qquad x\in Q_{k+1}$$

  is a constant (in x) chosen so that the definition is unambiguous. Note that

  $$c_{n,k}=\int_{Q^{**}_{k+1}\setminus Q^{**}_k}K_{ij}(-y)u_i^nu_j^n(y)\,dy.$$

It is easy to see that $\nabla p^n=\nabla\bar{p}^n$ and therefore $(u^n,\bar{p}^n)$ is also a global-in-time local energy solution to (1.1). We therefore redefine p^n to be $\bar{p}^n$. Defined in this way p^n still satisfies the local pressure expansion for cubes in $\mathcal{C}$.

By Lemma 3.2 we may apply Theorem 3.1 with $q=2$ to obtain a priori estimates for u^n on the time interval $[0,T)$ depending only on $\|u_0\|_{M^{2,2}_{\mathcal{C}}}$.

We now construct a solution to (1.1) with the initial data u_0 following the inductive procedure from [18]. Fix $k\in\mathbb{N}$ and denote by B_{2^k} the ball of radius 2^k centered at the origin. From Theorem 3.1 we have

$$\sup_{0<t<T}\int_{B_{2^k}}|u^n(x,t)|^2\,dx+\int_0^T\int_{B_{2^k}}|\nabla u^n(x,t)|^2\,dx\,dt\le C(k,u_0).$$

It follows that

$$\int_0^T \int_{B_{2^k}} |u^n|^{10/3}\,dx\,dt \le C(k,u_0)$$

and

$$\int_0^T \int_{B_{2^k}} |p^n|^{3/2}\,dx\,dt \le C(k,u_0),$$

where we used $c_{n,k} \le C(k)\|u^n\|^2_{L^2(Q^{**}_{k+1})} \le C(k,u_0)$. Note that the constants change from line to line but depend only on k and u_0. As in [18, p. 154], using (1.1), we additionally have that for any $w \in C_0^\infty(B_{2^k})$,

$$\int_0^T \int_{B_{2^k}} \partial_t u^n \cdot w\,dx\,dt \le C(k,u_0)\left(\int_0^T \int_{B_{2^k}} |\nabla w|^3\,dx\,dt\right)^{1/3},$$

implying

$$\|\partial_t u^n\|_{X_k} \le C(k,u_0),$$

where X_k is the dual space of $L^3(0,T;\mathring{W}^{1,3}(B_{2^k}))$.

The preceding four estimates and compactness arguments imply that there exists a sub-sequence $\{(u^{1,n},p^{1,n})\}$ and a couple (u_1,p_1) such that

$$\begin{aligned} u^{1,n} &\overset{*}{\rightharpoonup} u_1 \quad \text{in } L^\infty(0,T;L^2(B_2))\\ u^{1,n} &\rightharpoonup u_1 \quad \text{in } L^2(0,T;H^1(B_2))\\ u^{1,n} &\to u_1 \quad \text{in } L^3(0,T;L^3(B_2)) \end{aligned}$$

and

$$p^{1,n} \rightharpoonup p_1 \quad \text{in } L^{3/2}(0,T;L^{3/2}(B_2))$$

as $n \to \infty$.

We repeat this procedure for $k=2$. Let $\{u^{2,n}\}$ be a subsequence of $\{u^{1,n}\}$ that converges to a vector field u_2 on $B_2 \times (0,T)$ in the sense

$$\begin{aligned} u^{2,n} &\overset{*}{\rightharpoonup} u_2 \quad \text{in } L^\infty(0,T;L^2(B_4))\\ u^{2,n} &\rightharpoonup u_2 \quad \text{in } L^2(0,T;H^1(B_4))\\ u^{2,n} &\to u_2 \quad \text{in } L^3(0,T;L^3(B_4)), \end{aligned}$$

as $n \to \infty$. Additionally, there exists p_2 so that

$$p^{2,n} \rightharpoonup p_2 \quad \text{in } L^{3/2}(0,T;L^{3/2}(B_4)).$$

Iterating this argument we obtain a collection $\{u^{k,n}\}$ and a sequence u_k so that u_k is defined on $B_{2^k} \times (0,T)$ and

$$\begin{aligned} u^{k,n} &\overset{*}{\rightharpoonup} u_k \quad \text{in } L^\infty(0,T;L^2(B_{2^k}))\\ u^{k,n} &\rightharpoonup u_k \quad \text{in } L^2(0,T;H^1(B_{2^k}))\\ u^{k,n} &\to u_k \quad \text{in } L^3(0,T;L^3(B_{2^k})), \end{aligned}$$

as $n \to \infty$ for each k. Additionally, there exists p_k so that

$$p^{k,n} \rightharpoonup p_k \quad \text{in } L^{3/2}(0,T;L^{3/2}(B_{2^k})).$$

Let $u^{(k)} = u^{k,k}$ and $p^{(k)} = p^{k,k}$. Note that if $n > m$, then $u_n = u_m$ on B_{2^m} and $p_n = p_m$ on B_{2^m}. Hence, we unambiguously define u and p by letting them equal u_n and p_n respectively on B_{2^n}. Then,

$$\begin{aligned} u^{(k)} &\overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,T;L^2_{\text{loc}}) \\ u^{(k)} &\rightharpoonup u \quad \text{in } L^2(0,T;H^1_{\text{loc}}) \\ u^{(k)} &\to u \quad \text{in } L^3(0,T;L^3_{\text{loc}}), \end{aligned}$$

as $k \to \infty$. Also,

$$p^{(k)} \rightharpoonup p \quad \text{in } L^{3/2}(0,T;L^{3/2}_{\text{loc}}).$$

These convergence properties ensure that u and p satisfy (1.1) in the sense of distributions. Furthermore, the convergence properties of $u^{(k)}$ and $p^{(k)}$ imply that the local energy inequality, which is satisfied by all $u^{(k)}$ and $p^{(k)}$, is inherited by u and p. Also, $\|\partial_t u\|_{X_k} \le C(k, u_0)$. Fix $w \in L^2(B_{2^k})$. Then the function

$$t \mapsto \int_{B_{2^k}} u(x,t)\cdot w(x)\,dx$$

is continuous in time because $\|\partial_t u\|_{X_k} \le C(k,u_0)$ and

$$\operatorname*{ess\,sup}_{0<t<T} \int_{B_{2^{k+1}}} |u|^2\,dx + \int_0^T \int_{B_{2^{k+1}}} |\nabla u|^2\,dx\,dt < \infty.$$

That the limit u is in the correct local energy class follows from the local energy inequality. Indeed, let $\phi_Q = 1$ on Q and equal zero off of Q^* be the usual non-negative cut-off function. The local energy inequality, weak continuity in time, and the convergence properties of $u^{(k)}$ and $p^{(k)}$ give

$$\begin{aligned} &\int \frac{|u(x,t)|^2}{2}\phi_Q(x)\,dx + \int_0^t \int |\nabla u(x,s)|^2 \phi_Q(x)\,dx\,ds \\ &\quad \le \int \frac{|u(x,0)|^2}{2}\phi_Q(x)\,dx + \int_0^t \int \frac{|u|^2}{2}\Delta\phi_Q(x)\,dx\,ds \\ &\qquad + \int_0^t \int \left(\frac{|u|^2}{2} u\cdot\nabla\phi_Q(x) + pu\cdot\nabla\phi_Q(x)\right)\,dx\,ds \\ &\quad \le \int \frac{|u(x,0)|^2}{2}\phi_Q(x)\,dx + \lim_{k\to\infty}\int_0^t \int \frac{|u^{(k)}|^2}{2}|\Delta\phi_Q(x)|\,dx\,ds \\ &\qquad + \lim_{k\to\infty}\int_0^t \int \left(\frac{|u^{(k)}|^3}{2}|\nabla\phi_Q(x)| + |p^{(k)} - p_Q^{(k)}(t)|^{3/2}|\nabla\phi_Q(x)|\right)\,dx\,ds \end{aligned}$$

for all $t \ge 0$ where we used the fact that $\int_0^t \int p_Q^{(k)}(t) u^{(k)}\cdot\nabla\phi_Q\,dx\,dt = 0$.

Dividing by $|Q|^{2/3}$ and using the estimates in Section 3 gives

$$\operatorname*{ess\,sup}_{0<t<T_0} \frac{1}{|Q|^{2/3}} \sup_{Q\in\mathcal{C}} \int_Q |u|^2\,dx + \sup_{Q\in\mathcal{C}} \frac{1}{|Q|^{2/3}} \int_0^{T_0} \int_Q |\nabla u|^2\,dx\,dt \le C\|u_0\|^2_{M^{2,1}_{\mathcal{C}}}.$$

We remark that the estimates of Section 2 cannot be applied directly to u and p at this point because we do not have the local pressure expansion for p yet, and to establish it we need the above estimate for u.

Convergence to the initial data in L^2_{loc} follows from the weak continuity in time and the local energy inequality.

We finally establish the local pressure expansion. Define $\bar{p}$ recursively by

- If $x \in Q_0$ then $\bar{p}(x) = G^{Q_1}_{ij}(u_i u_j)(x)$
- If $x \in Q_{k+1}$ then $\bar{p}(x) = G^{Q_{k+1}}_{ij}(u_i u_j)(x) + c_k$, where

$$c_k = G^{Q_k}_{ij}(u_i u_j)(x) - G^{Q_{k+1}}_{ij}(u_i u_j)(x) \quad \text{on } Q_{k+1},$$

  is a constant (in x) . Note that

$$c_k = \int_{Q^{**}_{k+1} \setminus Q^{**}_k} K_{ij}(-y) u_i u_j \, dy.$$

Note that if $Q^{**} \subset Q_{k+1}$ and $k \in \mathbb{N}$ is minimal, then, for all $x \in Q^*$ and at a fixed t (which we suppress),

$$\begin{aligned}
&G^{Q}_{ij}(u_i u_j)(x) - \bar{p}(x,t) - c_k \\
&= \text{p.v.} \int_{y \in Q^{**}} K_{ij}(x-y)(u_i u_j)(y)\, dy - \text{p.v.} \int_{y \in Q^{**}_{k+1}} K_{ij}(x-y)(u_i u_j)(y)\, dy \\
&\qquad + \int_{y \notin Q^{**}} (K_{ij}(x-y) - K_{ij}(x_Q - y))(u_i u_j)(y)\, dy \\
&\qquad - \int_{y \notin Q^{**}_{k+1}} (K_{ij}(x-y) - K_{ij}(-y))(u_i u_j)(y)\, dy \\
&= - \int_{Q^{**}_{k+1} \setminus Q^{**}} K_{ij}(x_Q - y) u_i u_j(y)\, dy \\
&\qquad + \int_{y \notin Q^{**}_{k+1}} (K_{ij}(-y) - K_{ij}(x_Q - y)) u_i u_j(y)\, dy.
\end{aligned} \tag{4.1}$$

Since $x_Q \in Q_{k+1}$, the last term converges absolutely whenever $u \in M^{2,2}_{\mathcal{C}}$. Furthermore, the right hand side above does not depend on x and so is a constant depending on Q and t. We therefore have for $(x,t) \in Q^*$ that

$$\bar{p}(x,t) = G^{Q}_{ij}(u_i u_j)(x,t) + p_Q(t), \tag{4.2}$$

where $p_Q(t)$ is defined by collecting the constants appearing above. It is easy to check that the function $\bar{p}$ defined this way satisfies the analog of (4.2) for all cubes in $\mathbb{R}^3$, not just for the cubes in $\mathcal{C}$. It remains to show that $\bar{p} = p$, as this will show that p satisfies the equation (2.1) modulo a function of time, i.e. it satisfies the local pressure expansion.

We claim that

$$G^{Q}_{ij}(u^{(k)}_i u^{(k)}_j) \to G^{Q}_{ij}(u_i u_j), \tag{4.3}$$

in $L^{3/2}(0,T;L^{3/2}(Q))$ for any cube $Q\in\mathcal{C}$, i.e.,

$$\begin{aligned}
&-\frac{1}{3}\delta_{ij}(u_i^{(k)}u_j^{(k)})(x,s)+\text{p.v.}\int_{y\notin Q^{**}}K_{ij}(x-y)(u_i^{(k)}u_j^{(k)})(y,s)\,dy\\
&\qquad+\text{p.v.}\int_{y\in Q^{**}}(K_{ij}(x-y)-K_{ij}(x_Q-y))(u_i^{(k)}u_j^{(k)})(y,t)\,dy\\
&\to-\frac{1}{3}\delta_{ij}(u_iu_j)(x,s)+\text{p.v.}\int_{y\notin Q^{**}}K_{ij}(x-y)(u_iu_j)(y,s)\,dy\\
&\qquad+\text{p.v.}\int_{y\in Q^{**}}(K_{ij}(x-y)-K_{ij}(x_Q-y))(u_iu_j)(y,t)\,dy.
\end{aligned}\tag{4.4}$$

Since $u^{(k)}\to u$ strongly in $L^3(0,T;L^3_{\text{loc}})$, we have $\delta_{ij}u_i^{(k)}u_j^{(k)}\to\delta_{ij}u_iu_j$ in $L^{3/2}(0,T;L^{3/2}(Q^{**}))$. Also, using the Calderón-Zygmund theory, we have

$$\begin{aligned}
&\text{p.v.}\int_{y\in Q^{**}}K_{ij}(x-y)(u_iu_j)(y)\,dy\\
&\qquad-\frac{1}{3}\delta_{ij}u_iu_j+\text{p.v.}\int_{y\in Q^{**}}K_{ij}(x-y)(u_i^{(k)}u_j^{(k)})(y)\,dy\\
&=\text{p.v.}\int_{y\in Q^{**}}K_{ij}(x-y)((u_iu_j)(y)-((u_i^{(k)}u_j^{(k)})(y))\,dy\to 0,
\end{aligned}$$

in $L^{3/2}(0,T;L^{3/2}(Q^{**}))$.

Therefore, the first two terms on the left hand side of (4.4) converge to the corresponding two on the right. To get the convergence of the remaining term, for $R>0$ denote by $Q_R(0)$ the cube centered at zero with side length R. Note that

$$\begin{aligned}
&\left|\int_{y\notin Q^{**}}(K_{ij}(x-y)-K_{ij}(x_Q-y))((u_iu_j)(y)-(u_i^{(k)}u_j^{(k)})(y))\,dy\right|\\
&\quad\le\left|\int_{y\in Q_R(0)\cap(Q^{**})^c}(K_{ij}(x-y)-K_{ij}(x_Q-y))((u_iu_j)(y)-(u_i^{(k)}u_j^{(k)})(y))\,dy\right|\\
&\qquad+\left|\int_{y\in Q_R(0)^c\cap(Q^{**})^c}(K_{ij}(x-y)-K_{ij}(x_Q-y))((u_iu_j)(y)-(u_i^{(k)}u_j^{(k)})(y))\,dy\right|\\
&\quad=:I_1^R+I_2^R.
\end{aligned}$$

For the first term, we have

$$\begin{aligned}
\int_0^T\int_Q|I_1^R|^{3/2}\,dx\,dt&\le C|Q|\int_0^T\left(\int_{Q_R(0)}\big|(u_iu_j)(y,t)-(u_i^{(k)}u_j^{(k)})(y,t)\big|\,dy\right)^{3/2}dt\\
&\le C(Q,R)\int_0^T\int_{Q_R(0)}\big|(u_iu_j)(y,t)-(u_i^{(k)}u_j^{(k)})(y,t)\big|^{3/2}\,dy\,dt\to 0.
\end{aligned}$$

Let $\epsilon>0$ be given. Then

$$\begin{aligned}
|I_2^R(x,t)|&\le C\sum_{Q'\in\mathcal{C};|Q'|^{1/3}>R}|Q|^{1/3}\int_{Q'}\frac{1}{|x-y|^4}\big|(u_iu_j)(y,t)-(u_i^{(k)}u_j^{(k)})(y,t)\big|\,dy\\
&\le C\sum_{Q'\in\mathcal{C};|Q'|^{1/3}>R}\frac{|Q|^{1/3}}{|Q'|^{4/3}}\int_{Q'}\big|(u_iu_j)(y,t)-(u_i^{(k)}u_j^{(k)})(y,t)\big|\,dy.
\end{aligned}$$

Since all $u^{(k)}$ and u are uniformly bounded in $M^{2,2}_{\mathcal{C}}$ in terms of $\|u_0\|_{M^{2,2}_{\mathcal{C}}}$, we have

$$|I_2^R(x,t)| \leq C\|u_0\|^2_{M^{2,2}_{\mathcal{C}}} \sum_{Q'\in\mathcal{C};|Q'|>R} \frac{|Q|^{1/3}}{|Q'|^{2/3}}.$$

The sum above can be expanded over cubes in nested shells where the number of cubes in each shell is bounded. We therefore have

$$|I_2^R(x,t)| \leq \frac{C}{R^2}|Q|^{1/3}\|u_0\|^2_{M^{2,2}_{\mathcal{C}}} \sum_{n=\lfloor \log_2(R)\rfloor}^{\infty} \frac{1}{2^{2n}}.$$

Clearly, we may choose R so that

$$\int_0^T \int_Q |I_2^R(x,t)|^{3/2}\,dx\,dt \leq \epsilon.$$

Therefore, we have the convergence (4.3) in $L^{3/2}(0,T;L^{3/2}(Q))$ for all $Q\in\mathcal{C}$. Now note that $p^{(k)}$ also satisfies (4.1), i.e. for $x\in Q$ we can write

$$p^{(k)}(x,t) = G^Q_{ij}(u_i^{(k)}u_j^{(k)})(x,t) + p_Q^{(k)}(t),$$

where p_Q consists of the constants in (4.1) with u replaced by $u^{(k)}$. We have shown that $p^{(k)} \to p$ and $G^Q_{ij}(u_i^{(k)}u_j^{(k)}) \to G^Q_{ij}(u_iu_j)$, both in $L^{3/2}(0,T;L^{3/2}(Q))$. To show that p satisfies the local pressure expansion, it is enough to show that $p_Q^{(k)}(t)\to p_Q(t)$ in $L^{3/2}(0,T)$. Expanding $p_Q(t)-p_Q^{(k)}(t)$ leads to several integrals over bounded regions and an integral over unbounded regions. The integrals over bounded regions all converge to zero by convergence properties of $u^{(k)}$ to u. The remaining term is

$$\int_{(Q^{**}_{k+1})^c} (K_{ij}(-y) - K_{ij}(x_Q-y))(u_i^{(k)}u_j^{(k)} - u_iu_j)(y,t)\,dy,$$

and we need to explain why this vanishes as $k\to\infty$. We have

$$|K_{ij}(-y) - K_{ij}(x_Q-y))| \lesssim \frac{1}{|y|^4},$$

for $|y|$ large and so can treat this in an analogous way to I_2^R. This gives convergence of the sequence at every time t. Convergence in $L^{3/2}(0,T)$ follows by the dominated convergence theorem. This proves that

$$p^{(k)}(x,t) = G^Q_{ij}(u_i^{(k)}u_j^{(k)})(x,t) + p_Q^{(k)}(t) \to G^Q_{ij}(u_iu_j)(x,t) + p_Q(t)$$

in $L^{3/2}(0,T;L^{3/2}(Q))$, which implies $p = G^Q_{ij}(u_iu_j) + p_Q(t)$, i.e. p satisfies the local pressure expansion for u. □

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