On the global regularity for a 3D Boussinesq model without thermal diffusion

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Abstract

In a recent paper [Y], Ye proved the global persistence of regularity for a 3D Boussinesq model in $H^s(\mathbb{R}^3) \times H^s(\mathbb{R}^3)$ with $s > 5/2$. In this paper, we show that the global persistence and uniqueness still hold when $s > 3/2$.

1 Introduction

In [C1], Chae proposed a modified Navier-Stokes model and addressed the global regularity persistence with initial data $u_0 \in H^s(\mathbb{R}^3)$ and $s > 5/2$. The modified Navier-Stokes model reads

$$
\begin{aligned}
&u_t - \Delta u + \mathcal{R} \times (u \times \omega) = 0\\
&\nabla \cdot u = 0,
\end{aligned}
$$

where the operator $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3)$ is the vector of Riesz transforms defined by using the Fourier transform as

$$(\mathcal{R}_j f)\hat{}(\xi) = \frac{\xi_j}{i|\xi|}\hat{f}(\xi), \qquad j = 1, 2, 3.$$

Subsequently, Ye in [Y] proved the global regularity and persistence for the 3D Boussinesq model

$$u_t - \Delta u + \mathcal{R} \times (u \times \omega) + \mathcal{R} \times \mathcal{R} \times (\rho e_3) = 0 \tag{1.1}$$

$$\rho_t + u \cdot \nabla \rho = 0 \tag{1.2}$$

$$\nabla \cdot u = 0 \tag{1.3}$$

in $H^s(\mathbb{R}^3) \times H^s(\mathbb{R}^3)$ for $s > 5/2$. For the 2D Boussinesq equations, the global existence and persistence of regularity have been topics of high interest since the seminal work of Chae [C2] and of Hou and Li [HL], who proved the global existence of a unique solution. Namely, the global persistence holds for (u_0, ρ_0) in $H^s \times H^{s-1}$ for integers $s \geq 3$ [HL], while we have the global persistence in $H^s \times H^s$ for integers $s \geq 3$ by [C1]. The persistence in $H^s \times H^{s-1}$ for the intermediate values $1 < s < 3$ was then settled in [HKZ]. For other results on the global existence and persistence of solutions, see

[ACW, BrS, CG, CW, DP, HK, KWZ, LPZ, T]. The persistence of regularity in the Sobolev spaces $W^{s,q} \times W^{s-1,q}$ when $q \neq 2$ was studied in [KWZ], where it was proven that the persistence holds if $(s-1)q > 2$. Later, Kukavica and the author of this paper addressed the global regularity persistence for the fractional Boussinesq equations in [KW1] and long time behavior of the solutions in [KW2].

In this paper, we prove that the global regularity persistence and the uniqueness still hold with initial data $(u_0, \rho_0) \in H^s \times H^s$ and $s > 3/2$. The paper is organized as follows. In Section 2, we introduce basic notations and state the main theorem on the persistence. In Section 3, we prove Theorem 2.1, while the uniqueness of solutions is obtained in Section 4.

2 Notation and the main result on global persistence

We consider the 3D Boussinesq model (1.1)–(1.3) with the initial condition $u(x,0) = u_0$. The following is the main result of this paper.

Theorem 2.1. *Let $s > 3/2$, and assume that $\|u_0\|_{H^s} < \infty$ with $\nabla \cdot u_0 = 0$ and $\|\rho_0\|_{H^s} < \infty$. Then there exists a unique solution (u,ρ) to the equations (1.1)–(1.3) such that $u \in C\left([0,\infty); H^s(\mathbb{R}^3)\right) \cap L^2_{loc}\left([0,\infty); H^{s+1}(\mathbb{R}^3)\right)$ and $\rho \in C\left([0,\infty); H^s(\mathbb{R}^3)\right)$.*

The operator Λ^α is defined by

$$\Lambda^\alpha = (-\Delta)^{\alpha/2}, \qquad 1 < \alpha < 2,$$

or, using the Fourier transform $(\Lambda^\alpha f)\hat{}(\xi) = |\xi|^\alpha \hat{f}(\xi)$ for $\xi \in \mathbb{R}^3$. In the next lemma, we recall the product rule for fractional derivatives.

Lemma 2.2 (Product estimate). *Let $s > 0$. For all $f, g \in H^s(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, the inequality*

$$\|\Lambda^s(fg)\|_{L^q(\mathbb{R}^3)} \leq C\|f\|_{L^{q_1}(\mathbb{R}^3)}\|\Lambda^s g\|_{L^{\tilde{q}_1}(\mathbb{R}^3)} + C\|\Lambda^s f\|_{L^{q_2}(\mathbb{R}^3)}\|g\|_{L^{\tilde{q}_2}(\mathbb{R}^3)}$$

holds, where $q_1, \tilde{q}_1, \tilde{q}_2 \in [q,\infty]$ and $q_2 \in [q,\infty)$ satisfy $1/q = 1/q_1 + 1/\tilde{q}_1 = 1/q_2 + 1/\tilde{q}_2$ and $C = C(q_1, \tilde{q}_1, \tilde{q}_2, q_2, s)$. In particular,

$$\|\Lambda^s(fg)\|_{L^2(\mathbb{R}^3)} \leq C\|f\|_{L^\infty(\mathbb{R}^3)}\|\Lambda^s g\|_{L^2(\mathbb{R}^3)} + C\|\Lambda^s f\|_{L^2(\mathbb{R}^3)}\|g\|_{L^\infty(\mathbb{R}^3)}.$$

For the proof, cf. [KP]. In the following lemma, we recall a version of the Kato-Ponce inequality from [KWZ].

Lemma 2.3 ([KWZ]). *Let $s \in (0,1)$. For $1 < q < \infty$ and $j \in \{1,2,3\}$ and for $f, g \in \mathcal{S}(\mathbb{R}^3)$, the inequality*

$$\|[\Lambda^s \partial_j, g]f\|_{L^q(\mathbb{R}^3)} \leq C\|f\|_{L^{q_1}(\mathbb{R}^3)}\|\Lambda^{1+s} g\|_{L^{\tilde{q}_1}(\mathbb{R}^3)} + C\|\Lambda^s f\|_{L^{q_2}(\mathbb{R}^3)}\|\Lambda g\|_{L^{\tilde{q}_2}(\mathbb{R}^3)}$$

holds, where $q_1, \tilde{q}_1, \tilde{q}_2 \in [q,\infty]$ and $q_2 \in [q,\infty)$ satisfy $1/q = 1/q_1 + 1/\tilde{q}_1 = 1/q_2 + 1/\tilde{q}_2$ and $C = C(q_1, \tilde{q}_1, \tilde{q}_2, q_2, s)$. In particular,

$$\|[\Lambda^s \partial_j, g]f\|_{L^2(\mathbb{R}^3)} \leq C\|f\|_{L^\infty(\mathbb{R}^3)}\|\Lambda^{1+s} g\|_{L^2(\mathbb{R}^3)} + C\|\Lambda^s f\|_{L^2(\mathbb{R}^3)}\|\Lambda g\|_{L^\infty(\mathbb{R}^3)}$$

for $f, g \in \mathcal{S}(\mathbb{R}^3)$.

3 Proof of Theorem 2.1

In this section, we prove Theorem 2.1. Next we establish the global existence and the persistence of regularity, while the uniqueness is shown in the next sectioni.

Proof of Theorem 2.1(existence). Assume that $\|u_0\|_{H^s}, \|\rho_0\|_{H^s} \lesssim 1$, where $s > 3/2$ is fixed. Since $s > 3/2$, we have $H^s(\mathbb{R}^3) \subseteq L^\infty(\mathbb{R}^3)$, and thus

$$\rho_0 \in L^{\bar{q}}, \qquad \bar{q} \in [2, \infty].$$

Using the $L^{\bar{q}}$ conservation property for the density equation (1.2), we get

$$\|\rho(t)\|_{L^{\bar{q}}} \le \|\rho_0\|_{L^{\bar{q}}} \lesssim 1, \qquad \bar{q} \in [2, \infty].$$

The H^1 energy inequality

$$\|u\|_{H^1}^2 + \int_0^t \|\nabla u\|_{H^1}^2 \, d\tau \le C(t), \qquad t \ge 0 \tag{3.1}$$

was obtained in [Y]. Next, we use the parabolic regularity to estimate $\int_0^t \|\nabla u\|_{L^\infty} \, d\tau$. Fix $t_0 > 0$ and let $\theta \colon \mathbb{R} \to [0, \infty)$ be a smooth non-decreasing function such that $\theta \equiv 0$ on $[0, t_0/2]$ and $\theta \equiv 1$ on $[t_0, \infty]$. Then we have

$$\partial_t(\theta(t)u) - \Delta(\theta(t)u) = -\mathcal{R} \times (\theta(t)u \times \omega) - \mathcal{R} \times \mathcal{R} \times (\theta(t)\rho e_3) + \theta'(t)u.$$

For any $t > 0$, we get

$$\begin{aligned} \left(\int_0^t \|\theta(s)\Delta u(s)\|_{L^4}^2 \, ds \right)^{1/2} &\le \left(\int_0^t \|\mathcal{R} \times (\theta u \times \omega)\|_{L^4}^2 \, ds \right)^{1/2} + \left(\int_0^t \|\mathcal{R} \times \mathcal{R} \times (\theta \rho e_3)\|_{L^4}^2 \, ds \right)^{1/2} \\ &\quad + \left(\int_0^t \|\theta'(s)u\|_{L^2}^2 \, ds \right)^{1/4} \\ &\lesssim \left(\int_0^t \|u \times \omega\|_{L^4}^2 \, ds \right)^{1/2} + \left(\int_0^t \|\rho\|_{L^4}^2 \, ds \right)^{1/2} + \left(\int_0^t \|u\|_{L^2}^2 \, ds \right)^{1/4} \\ &\lesssim \left(\int_0^t \|u\|_{L^8}^2 \|\nabla u\|_{L^8}^2 \, ds \right)^{1/2} + t^{1/2} + C(t). \end{aligned}$$

By the Gagliardo-Nirenberg inequality, we get

$$\begin{aligned} \left(\int_0^t \|\theta(s)\Delta u(s)\|_{L^4}^2 \, ds \right)^{1/2} &\lesssim \left(t^{3/8} \left(\int_0^t \|D^2 u\|_{L^2}^{5/8} \, ds \right) \right)^{1/2} + t^{1/2} + C(t) \\ &\lesssim t^{7/8} \left(\int_0^t \|D^2 u\|_{L^2}^2 \, ds \right)^{5/16} + t^{1/2} + C(t) \lesssim C(t), \end{aligned}$$

from where

$$\int_{t_0}^t \|\Delta u\|_{L^4}^2 \, ds \le C(t).$$

Thus, by interpolation, we have

$$\begin{aligned}\|\Lambda u\|_{L^\infty} &\lesssim \|\Lambda u\|_{L^4}^{1/4}\|\nabla\Lambda u\|_{L^4}^{3/4} \lesssim \|\nabla u\|_{L^4}^{1/4}\|\Delta u\|_{L^4}^{3/4} \\ &\lesssim \left(\|\nabla u\|_{L^2}^{1/4}\|\Delta u\|_{L^2}^{3/4}\right)^{1/4}\|\Delta u\|_{L^4}^{3/4} \lesssim C(t)\|\Delta u\|_{L^2}^{3/16}\|\Delta u\|_{L^4}^{3/4}.\end{aligned}$$

Therefore, we get

$$\int_0^t \|\Lambda u\|_{L^\infty}\,d\tau \lesssim C(t). \tag{3.2}$$

Next, for the evolution of $\|\Lambda^s u\|_{L^2}$, we apply the operator Λ^s to the equation (1.1), multiply by $\Lambda^s u$, and integrate the resulting equation obtaining

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^s u\|_{L^2}^2 + \|\nabla(\Lambda^s u)\|_{L^2}^2 = -\int \Lambda^s\left(\mathcal{R}\times(u\times\omega)\right)\cdot\Lambda^s u\,dx - \int \Lambda^s\left(\mathcal{R}\times\mathcal{R}\times(\rho e_3)\right)\cdot\Lambda^s u\,dx = J_1 + J_2.$$

We apply the Cauchy-Schwarz inequality and the fractional product rule to estimate

$$\begin{aligned}J_1 &\lesssim \|\Lambda^s(u\times\omega)\|_{L^2}\|\Lambda^s u\|_{L^2} \lesssim (\|\Lambda^s u\|_{L^2}\|\omega\|_{L^\infty} + \|\Lambda^s\omega\|_{L^2}\|u\|_{L^\infty})\|\Lambda^s u\|_{L^2} \\ &\leq C\|\omega\|_{L^\infty}\|\Lambda^s u\|_{L^2}^2 + \frac{1}{2}\|\nabla(\Lambda^s u)\|_{L^2}^2 + C\|u\|_{L^\infty}^2\|\Lambda^s u\|_{L^2}^2.\end{aligned}$$

For J_2, we apply the Cauchy-Schwarz inequality

$$J_2 \leq \|\Lambda^s\rho\|_{L^2}\|\Lambda^s u\|_{L^2}.$$

Thus, using the estimates for J_1 and J_2 above yields

$$\frac{d}{dt}\|\Lambda^s u\|_{L^2}^2 + \frac{1}{C}\|\nabla(\Lambda^s u)\|_{L^2}^2 \lesssim (\|\omega\|_{L^\infty} + \|u\|_{L^\infty}^2)\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s\rho\|_{L^2}\|\Lambda^s u\|_{L^2}. \tag{3.3}$$

Next, we consider higher order derivatives of the density ρ. We apply the operator Λ^s to the equation (1.2), multiply by $\Lambda^s\rho$, and integrate the resulting equation obtaining

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^s\rho\|_{L^2}^2 = -\int [\Lambda^s, u\cdot\nabla]\rho\Lambda^s\rho\,dx \leq \|[\Lambda^s, u\cdot\nabla]\rho\|_{L^2}\|\Lambda^s\rho\|_{L^2}. \tag{3.4}$$

By Lemma 2.3, we have

$$\|[\Lambda^s, u\cdot\nabla]\rho\|_{L^2} \lesssim \|\rho\|_{L^\infty}\|\Lambda^{s+1}u\|_{L^2} + \|\Lambda u\|_{L^\infty}\|\Lambda^s\rho\|_{L^2} \lesssim \|\nabla(\Lambda^s u)\|_{L^2} + \|\Lambda u\|_{L^\infty}\|\Lambda^s\rho\|_{L^2}.$$

Therefore,

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^s\rho\|_{L^2}^2 \leq \frac{1}{2}\|\nabla(\Lambda^s u)\|_{L^2}^2 + (1 + \|\Lambda u\|_{L^\infty})\|\Lambda^s\rho\|_{L^2}^2.$$

Finally, adding (3.3) and (3.4) yields

$$\begin{aligned}\frac{d}{dt}(\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s\rho\|_{L^2}^2) + \|\Lambda^{s+1}u\|_{L^2}^2 &\lesssim (\|\omega\|_{L^\infty} + \|u\|_{L^\infty}^2)\|\Lambda^s u\|_{L^2}^2 + (1 + \|\Lambda u\|_{L^\infty})\|\Lambda^s\rho\|_{L^2}^2 \\ &\quad + \|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s\rho\|_{L^2}^2.\end{aligned}$$

By the Sobolev embedding, (3.1), and (3.2), we get

$$\int_0^t \|u\|_{L^\infty}^2\,d\tau \lesssim \int_0^t \|u\|_{H^2}^2\,d\tau \leq C(t)$$

and

$$\int_0^t \|\Lambda u\|_{L^\infty}\, d\tau \le C(t).$$

We thus conclude the proof by applying the Gronwall inequality. □

4 Uniquenesss in $L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$

In this section, we address the uniqueness.

Proof of Theorem 2.1(uniqueness). Consider two solutions $(u^{(1)}, \rho^{(1)})$ and $(u^{(2)}, \rho^{(2)})$ of the system (1.1)–(1.3) and set

$$\begin{aligned} U &= u^{(1)} - u^{(2)} \\ \Theta &= \rho^{(1)} - \rho^{(2)} \\ \Omega &= \omega^{(1)} - \omega^{(2)} = \nabla \times u^{(1)} - \nabla \times u^{(2)}. \end{aligned}$$

Subtracting the equations for $(u^{(1)}, \rho^{(1)})$ and $(u^{(2)}, \rho^{(2)})$, we get

$$U_t - \Delta U + \mathcal{R} \times (U \times \omega^{(1)}) + \mathcal{R} \times (u^{(2)} \times \Omega) + \mathcal{R} \times \mathcal{R} \times (\Theta e_3) = 0 \tag{4.1}$$

$$\Theta_t + u^{(2)} \cdot \nabla\Theta + U \cdot \nabla\rho^{(1)} = 0. \tag{4.2}$$

We multiply (4.1) with U and integrate the resulting equation obtaining

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|U\|_{L^2}^2 + \|\nabla U\|_{L^2}^2 = &-\int \Big(\mathcal{R} \times (U \times \omega^{(1)})\Big) \cdot U\, dx - \int \Big(\mathcal{R} \times (u^{(2)} \times \Omega)\Big) \cdot U\, dx \\ &- \int (\mathcal{R} \times \mathcal{R} \times (\rho e_3)) \cdot U\, dx = I_1 + I_2 + I_3. \end{aligned}$$

For I_1, we apply Hölder's inequality

$$I_1 \le C\|U \times \omega^{(1)}\|_{L^2}\|U\|_{L^2} \le \|\omega^{(1)}\|_{L^\infty}\|U\|_{L^2}^2.$$

Similarly, for I_2 we have

$$\begin{aligned} I_2 &\le \|u^{(2)} \times \Omega\|_{L^2}\|U\|_{L^2} \le \|u^{(2)}\|_{L^\infty}\|\Omega\|_{L^2}\|U\|_{L^2} \le C\|u^{(2)}\|_{L^\infty}\|\nabla U\|_{L^2}\|U\|_{L^2} \\ &\le \frac{1}{2}\|\nabla U\|_{L^2}^2 + C\|u^{(2)}\|_{L^\infty}^2\|U\|_{L^2}^2. \end{aligned}$$

For I_3, we apply the Cauchy–Schwarz inequality

$$I_3 \le \|\Theta\|_{L^2}\|U\|_{L^2}.$$

Combining the estimates for I_1, I_2, and I_3 gives

$$\frac{1}{2}\frac{d}{dt}\|U\|_{L^2}^2 + \|\nabla U\|_{L^2}^2 \le \frac{1}{2}\|\nabla U\|_{L^2}^2 + (\|u^{(2)}\|_{L^\infty}^2 + \|\omega^{(1)}\|_{L^\infty})\|U\|_{L^2}^2 + \|\Theta\|_{L^2}\|U\|_{L^2}. \tag{4.3}$$

Next, for the evolution of $\|\Theta\|_{L^2(\mathbb{R}^3)}$, we multiply (4.2) with Θ and integrate the resulting equation yielding

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\Theta\|_{L^2}^2 &= -\int_{\mathbb{R}^3} U\cdot\nabla\rho^{(1)}\Theta\,dx \le \|U\|_{L^6}\|\nabla\rho^{(1)}\|_{L^3}\|\Theta\|_{L^2} \\
&\le \|\nabla U\|_{L^2}\|\nabla\rho^{(1)}\|_{L^3}\|\Theta\|_{L^2} \le \frac{1}{2}\|\nabla U\|_{L^2}^2 + C\|\nabla\rho^{(1)}\|_{L^3}^2\|\Theta\|_{L^2}^2.
\end{aligned} \tag{4.4}
$$

Adding (4.3) and (4.4), we obtain

$$
\frac{d}{dt}(\|U\|_{L^2}^2+\|\Theta\|_{L^2}^2) \le (\|u^{(2)}\|_{L^\infty}^2+\|\omega^{(1)}\|_{L^\infty})\|U\|_{L^2}^2 + C\|\Theta\|_{L^2}\|U\|_{L^2} + C\|\nabla\rho^{(1)}\|_{L^3}^2\|\Theta\|_{L^2}^2.
$$

Next, $\|\omega^{(1)}\|_{L^\infty}$ is integrable in time. Indeed, for any $T>0$ and $s>3/2$, by the Sobolev embedding we get

$$
\int_0^T \|\omega^{(1)}\|_{L^\infty}\,d\tau \le \int_0^T \|\nabla u^{(1)}\|_{L^\infty}\,d\tau < \infty.
$$

Finally, by the Sobolev embedding again we get

$$
\|\nabla\rho^{(1)}\|_{L^3(\mathbb{R}^3)} \le C\|\rho^{(1)}\|_{H^s(\mathbb{R}^3)} \le C.
$$

Since $U_0=\Theta_0=0$, by using the Gronwall inequality, we have the uniqueness. □

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