Almost sure existence of global weak solutions to the Boussinesq equations

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Abstract

In this paper, we show that after a suitable randomization of the initial data in negative-order Sobolev spaces $H^{-\alpha}$ with $0 < \alpha < 1/2$, there exist almost sure global weak solutions for the Boussinesq equations in $\mathbb{R}^d$ and $\mathbb{T}^d$, when $d = 2, 3$. Furthermore, we prove that the global weak solutions are unique in 2d.

1 Introduction

In this paper, we address the almost sure existence of global weak solutions to the Boussinesq equations in the whole space $\mathbb{R}^d$ and the tori $\mathbb{T}^d$ for $d = 2, 3$,

$$u_t - \Delta u + u \cdot \nabla \rho + \nabla \pi = \rho e_3 \tag{1.1}$$

$$\rho_t - \Delta \rho + u \cdot \nabla \rho = 0 \tag{1.2}$$

$$\nabla \cdot u = 0. \tag{1.3}$$

Here, u is the velocity and ρ is represents the density or temperature of the fluid which depends on the physical context. π denotes the pressure and $e_3 = (0, 0, 1)^T$. The Boussinesq system is an important physical model arising particularly in two situations. It is a model for the inhomogeneous Navier-Stokes system, which is derived from the full compressible NavierStokes system under the low Mach assumption. Under this scenario, u represents the velocity while ρ represents the variation of the density. In the second context, the Boussinesq system is also related to the RayleighBénard problem, in which case ρ represents the temperature.

Data in $\dot{H}^s$ with $s < s_c$ (super-critical regime) is rougher than the data of critical regularity. Intuitively, scaling is against well-posedness in this case. Ill-posedness in some cases can be circumvented by an appropriate probabilistic method in some probability space of initial data, in the other words, one may hope to establish almost sure local well-posedness with respect to certain probability random data space. This random data approach to well-posedness first appeared in the series of papers [B] of Bourgain in the context of studying the invariance of Gibbs measures associated to NLS on tori ($\mathbb{T}$ and $\mathbb{T}^2$). Later, Burq-Tzvetkov [BT1, BT2] obtained similar results in the context of the cubic nonlinear wave equation (NLW) on a three dimensional compact Riemannian manifold. The random data approach to well-posedness has also been pursued by many authors and applied to several nonlinear evolution equations on different manifolds. In the context of the incompressible Navier-Stokes equations, almost sure local well-posedness and in some instances almost sure global existence results in the context of the Navier-Stokes equations include: [DC, NPS, WY, WW].

Recently, Nahmod, Pavlović, and Staffilani [NPS] gave the first construction of almost sure global weak solutions for the Navier-Stokes equations with initial data in $H^{-\alpha}(\mathbb{T}^d)$, where $0 < \alpha < 1/2$ for $d = 2$ and $0 < \alpha < 1/4$ for $d = 3$ in the probabilistic point of view. In the context of the Navier–Stokes equations, the local in time well posedness for randomized initial data in $L^2(\mathbb{T}^3)$ was proven by Zhang and Fang [ZF] and by Deng and Cui [DC] using similar approach. In [NPS], by suitably randomizing the initial data u_0 and using the mild formulation $u = e^{t\Delta}u_0^\omega + w$, the authors singled out the linear evolution $e^{t\Delta}u_0^\omega$ and the difference equation for w was identified, where they showed that the energy of w is conserved. Later, J. Wang and K. Wang in [WW] extended the global existence results from the periodic domain to $\mathbb{R}^d$, for $d = 2, 3$, and improved the range of the parameter of the negative-order Sobolev spaces from $0 < \alpha < 1/4$ to $0 < \alpha < 1/2$ for $d = 3$. By using the approach in [NPS], L. Du and T. Zhang in [DZ] proved the almost sure global existence of weak solutions for the MHD equations in $\mathbb{T}^d$ and $\mathbb{R}^d$, for $d = 2, 3$, where a uniform bound for the energy of the nonlinear part of the solutions was also obtained.

In recent years, there has been extensive research on the Boussinesq equations. People have been studying the persistence of regularity and global existence since the seminal work of Chae [C] and of Hou and Li [HL], who proved the global existence of a unique solution. In [LLT], Lunasin et al established global existence and uniqueness in the low regularity space $H^1 \times L^2$. Kukavica and the first author of this paper addressed the persistence of regularity in $W^{s,q} \times W^{s,q}$ for the 2D fractional Boussinesq equations in [KW1] and the long time behavior of solutions in [KW2]. For other global results of solutions, see [ACW, CW, HK1, LLT, SW, W]; however, the analogous almost sure existence of global weak solutions is less studied. In fact, to the best knowledge of the authors, these are the first results addressing global existence and long time behavior of weak solutions to the Boussinesq equations with initial random data. We also address the long time behavior of the constructed weak solutions. More precisely, we study in which way the variations of the density affect the asymptotic behavior of the velocity field from a probabilistic point of view. The main tool is the Fourier splitting method introduced by Schonbek [S1, S2].

The paper is organized as follows. In Section 2, we introduce relevant notation and state the main results. Section 3 contains lemma on the estimates in terms of random data. Section 4 contains the energy estimates. In the final section, we give the proof for the long time behavior for the weak solution.

2 Notation and the main results

In this section, we introduce basic notations and state our main results. We first define the Leray projector $\mathbb{P}$

$$\mathbb{P} = I + \nabla(-\Delta)^{-1}\nabla\cdot$$

to be a bounded operator into divergence-free vector fields. The Leray projector $\mathbb{P}$ may also be defined via the Fourier transform

$$\widehat{(\mathbb{P}u)}_j(\xi) = \left(\delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2}\right) \hat{u}_k(\xi), \qquad j = 1, 2, 3.$$

We next introduce the construction of random initial data in the whole space $\mathbb{R}^d$ for $d \geq 1$, which was first introduced by Burq and Tzvetkov [BT1]. In $\mathbb{R}^d$, we divide the frequency space by using the Wiener decomposition. For $n \in \mathbb{Z}^d$, let Q_n be the unit cube $Q_n = n + (-\frac{1}{2}, \frac{1}{2}]^d$. Then we have

$$\mathbb{R}^d = \bigcup_n Q_n.$$

Note that $Q_n \cap Q_m = \emptyset$ if $m \neq n$ and $\sum_n \chi_{Q_n}(\xi) = 1$. Hence, we have the decomposition

$$f(x) = \sum_{n \in \mathbb{Z}^d} \mathcal{F}^{-1}(\chi_{Q_n} \hat{f}).$$

Define a nonnegative and even smooth function φ such that $\phi(\xi) = 1$ for $\xi \in (-\frac{1}{2}, \frac{1}{2})^d$ and $\phi(\xi) = 0$ for $\xi \in ([-1, 1]^d)^c$, and let

$$\varphi(\xi) = \frac{\phi(\xi)}{\sum_n \phi(\xi - n)}.$$

Note that $\sum_n \varphi(\xi - n) = 1$. Define

$$\varphi(D - n)f = \int_{\mathbb{R}^d} \hat{f}(\xi) \varphi(\xi - n) e^{i2\pi x \cdot \xi} \, d\xi.$$

Then f has a smooth version for the Wiener decomposition:

$$f(x) = \sum_{n \in \mathbb{Z}^d} \varphi(D - n) f.$$

For any real-valued function f, we obtain

$$\overline{\varphi(D + n) f} = \varphi(D - n) f$$

and $\sum_n \varphi(D - n) f$ is also a real-valued function.

In the $\mathbb{T}^d$ case, the frequencies of functions are in $\mathbb{Z}^d$, so we can divide the frequency space into the integer points. To keep the consistence of the notations, we denote the decomposition operator $\phi(D - n) f = \hat{f}(n) e^{i2\pi x \cdot n}$.

We now introduce the randomization of functions in the negative-order Sobolev spaces. We first introduced the randomization of elements in negative order Sobolev spaces $H^{-\alpha}$.

Definition 2.1. Let $(l_n(\omega))_{n \in \mathbb{Z}^d}$ be a sequence of real, 0-mean, independent random variables on a probability space (Ω, A, p) with associated sequence of distributions $(\mu_n)_{n \in \mathbb{Z}^d}$ so that there exists $c > 0$, for all $\gamma \in \mathbb{R}$ and for all $n \in \mathbb{Z}^d$ we have that

$$\left| \int_{-\infty}^{\infty} e^{\gamma x} \, d\mu_n(x) \right| \leq e^{c\gamma^2}. \tag{2.4}$$

For $f \in H^{-\alpha}(\mathbb{R}^d)$ or $f \in H^{-\alpha}(\mathbb{T}^d)$, we define the map from (Ω, A) to $H^{-\alpha}$ by

$$\omega \to f^\omega$$

where

$$f^\omega = \sum_{n \in \mathbb{Z}^d} l_n(\omega)\phi(D-n)f, \tag{2.5}$$

where $\phi(D-n)f$ is defined as before. We call such a map randomization.

The following are the main results of this paper.

Theorem 2.1 (Existence and uniqueness in 2D)**.** *Fix $T > 0$, $0 < \alpha \leq 1/2$. Let $u_0, \rho_0 \in \dot{H}^{-\alpha}(\mathbb{R}^2)$ or $H^{-\alpha}(\mathbb{T}^2)$ and $\nabla \cdot u_0 = 0$. We further suppose u_0 and ρ_0 are mean zero in the periodic case. Then there exists a set $\Sigma \subseteq \Omega$ of probability 1 such that for any $\omega \in \Sigma$ the initial value problem (1.1) − (1.3) with datum $(u_0^\omega, \rho_0^\omega)$ has a unique global weak solution in the sense of Definition 2.1 of the form*

$$u = g_1^\omega + v,$$

$$\rho = g_2^\omega + \theta,$$

and

$$(v, \theta) \in L^\infty([0,T]; L^2) \cap L^2([0,T]; \dot{H}^1),$$

where $g_1^\omega = e^{t\Delta}u_0^\omega$ and $g_2^\omega = e^{t\Delta}\rho_0^\omega$.

Theorem 2.2 (Existence in 3D)**.** *Fix $T > 0$, $0 < \alpha \leq 1/2$. Let $u_0, \rho_0 \in \dot{H}^{-\alpha}(\mathbb{R}^3)$ or $H^{-\alpha}(\mathbb{T}^3)$ and $\nabla \cdot u_0 = 0$. We further suppose u_0 and ρ_0 are mean zero in the periodic case. Then there exists a set $\Sigma \subseteq \Omega$ of probability 1 such that for any $\omega \in \Sigma$ the initial value problem (1.1) − (1.3) with datum $(u_0^\omega, \rho_0^\omega)$ has a global weak solution in the sense of Definition 2.1 of the form*

$$u = g_1^\omega + v,$$

$$\rho = g_2^\omega + \theta,$$

and

$$(v, \theta) \in L^\infty([0,T]; L^2) \cap L^2([0,T]; \dot{H}^1),$$

where $g_1^\omega = e^{t\Delta}u_0^\omega$ and $g_2^\omega = e^{t\Delta}\rho_0^\omega$.

3 A priori estimates on the random data

We now introduce the deterministic estimates for the random initial data and probabilistic estimates for the heat kernel in terms of random data. The following lemma is a standard large deviation property (see Lemma 3.1 in [BT1]) and it will be used to analyze the heat flow on the randomized data.

Lemma 3.1 (Lemma 3.1 in [BT1])**.** *Let $(l_r(\omega))_{r=1}^{\infty}$ be a sequence of real, 0-mean, independent random variables on a probability space $(\Omega, A, \mathbf{P})$ with associated sequence of distributions $(\mu_r)_{r=1}^{\infty}$. Assume that there exists $c > 0$ such that $\forall \gamma \in \mathbb{R}$, $\forall r \geq 1$ we have*

$$\left| \int_{-\infty}^{\infty} e^{\gamma x} \, d\mu_r(x) \right| \leq e^{c\gamma^2}.$$

Then there exists $\alpha > 0$ such that for every $\lambda > 0$, every sequence $(a_r)_{r=1}^{\infty} \in \ell^2$ of real numbers,

$$\mathbf{P}\left(\omega : \left| \sum_{r=1}^{\infty} a_r l_r(\omega) \right| > \lambda \right) \leq C \exp\left(-\frac{\alpha \lambda^2}{\|a_r\|_{\ell_r^2}^2} \right).$$

As a consequence, for every $q \geq 2$ and $(a_r^2)_{r=1}^{\infty} \in \ell^2$,

$$\left\| \sum_{r=1}^{\infty} a_r l_r(\omega) \right\|_{L^q(\Omega)} \leq C\sqrt{q}\, \|a_r\|_{\ell_r^2}.$$

We next recall another classical result for a sequence of real, mean-0, independent random variables.

Lemma 3.2. *Let $\{l_n(\omega)\}_{n \in \mathbb{Z}^d}$ be a sequence of real, mean-0, independent random variables satisfy Definition 2.1 on a probability space $(\Omega, A, \mathbf{P})$. Then given $\epsilon, \delta > 0$, there exists a subset $\Omega_\delta \subset \Omega$ satisfying $\mathbf{P}(\Omega_\delta^c) \lesssim e^{-\frac{1}{\delta^\epsilon}}$, such that for all $\omega \in \Omega_\delta$*

$$|l_n(\omega)| \lesssim \frac{1}{\delta^\epsilon} \log(\langle n \rangle + 1)$$

where $\langle n \rangle = \sqrt{|n|^2 + 1}$.

Proof. For each n and a small $\epsilon > 0$, we have a constant C,

$$\mathbb{E} e^{|l_n(\omega)|} \leq C.$$

Set $M = \frac{1}{\delta^\epsilon}$, and then we have

$$\mathbb{E} \left| \frac{e^{|l_n(\omega)|}}{e^M} \right| \leq C e^{-\frac{1}{\delta^\epsilon}}$$

Then we obtain,

$$C e^{-\frac{1}{\delta^\epsilon}} > \mathbb{E} \left| \frac{e^{|l_n(\omega)|}}{e^M} \right| \geq \sum_{j \in \mathbb{Z}^d} \mathbf{P}(e^{|l_j(\omega)|} \geq e^M \langle j \rangle^d) = \sum_{j \in \mathbb{Z}^d} \mathbf{P}(|l_j(\omega)| \geq \frac{1}{\delta^\epsilon} + d \log\langle j \rangle).$$

Excluding $\Omega_\delta^c := \cup_j \{|l_j(\omega)| \geq \frac{1}{\delta^\epsilon} + d \log\langle j \rangle\}$ from Ω, for all $\omega \in \Omega_\delta$, we have

$$|l_n(\omega)| \leq \frac{1}{\delta^\epsilon} + d \log\langle n \rangle \lesssim \frac{1}{\delta^\epsilon} \log(\langle n \rangle + 1), \text{for } n \in \mathbb{Z}^d.$$

with $\mathbf{P}(\Omega_\delta^c) < C e^{-\frac{1}{\delta^\epsilon}}$. ∎

Remark 3.3. For given $\epsilon > 0$ and arbitrary small $\gamma > 0$, it's easy to check the fact that

$$\mathbf{P}(\|f^\omega\|_{H^{-\alpha-\gamma}} > \frac{1}{\delta^\epsilon} \|f\|_{H^{-\alpha}}) \lesssim e^{\frac{1}{\delta^\epsilon}}$$

which implies almost surely $f^\omega \in H^{-\alpha-\gamma}$ for arbitrary small $\gamma > 0$.

Lemma 3.4. *For $0<\alpha<1$ and $k\in\mathbb{N}$. Given $\epsilon>0$ and arbitrary small $\gamma>0$. If $f\in\dot H^{-\alpha}(\mathbb{R}^d)$ or $f\in H^{-\alpha}(\mathbb{T}^d)$ of mean zero. Suppose f^ω is defined as (2.5), then there exists a subset $\Omega_\delta\subset\Omega$ satisfying $\mathbf{P}(\Omega_\delta^c)\lesssim e^{-\frac{1}{\delta^\epsilon}}$, for all $\omega\in\Omega_\delta$ we obtain that*

$$\|\nabla^k e^{t\Delta}f^\omega(t)\|_{L^2}\le\frac{1}{\delta^\epsilon}(1+t^{-\frac{\alpha+\gamma+k}{2}})\|f\|_{\dot H^{-\alpha}} \tag{3.6}$$

and

$$\|\nabla^k e^{t\Delta}f^\omega(t)\|_{L^\infty}\le\frac{1}{\delta^\epsilon}\max\{t^{-\frac12},t^{\frac{-k+\alpha+\gamma+d/2}{2}}\}\|f\|_{\dot H^{-\alpha}}. \tag{3.7}$$

Proof. Based on the deterministic properties of heat kernel (see Lemma 3.1 in [NPS]), we have

$$\|\nabla^k e^{t\Delta}f^\omega(t)\|_{L^2}\le(1+t^{-\frac{\alpha+k}{2}})\|f^\omega\|_{\dot H^{-\alpha-\gamma}} \tag{3.8}$$

and

$$\|\nabla^k e^{t\Delta}f^\omega(t)\|_{L^\infty}\le\max\{t^{-\frac12},t^{\frac{-k+\alpha+d/2}{2}}\}\|f^\omega\|_{\dot H^{-\alpha-\gamma}}. \tag{3.9}$$

Applying Remark 3.3 into (3.8) and (3.9), we have (3.6) and (3.7) after excluding a subset of probability $\lesssim e^{-\frac{1}{\delta^\epsilon}}$. □

Remark 3.5. Given $f\in H^{-\alpha}(\mathbb{T}^d)$ of mean zero, $\|f\|_{H^{-\alpha}(\mathbb{T}^d)}=(\sum_{k\in\mathbb{Z}^d,k\neq0}\langle k\rangle^{-\alpha}|\hat f(k)|^2)^{1/2}$ is comparable to $(\sum_{k\in\mathbb{Z}^d,k\neq0}|k|^{-\alpha}|\hat f(k)|^2)^{1/2}=\|f\|_{\dot H^{-\alpha}(\mathbb{T}^d)}$. So in the periodic case, it's equivalent to use $H^{-\alpha}$ and $\dot H^{-\alpha}$ when the functions are mean zero.

We use the deterministic properties of heat kernel but we still leave the linear evolution $e^{t\Delta}f^\omega(t)$ unbounded when t is near zero. This is also the reason why we can not construct the weak solution of negative regularity deterministically. In the following lemma, we exploit the randomness in the data f^ω and then we bound the linear evolution $e^{t\Delta}f^\omega(t)$ in the small interval around zero in the L^p sense.

Lemma 3.6. *For $p,q\ge2$, $0<\alpha<1$ with $\alpha p\le2$ and $\delta>0$. Given some $\epsilon<\frac1p-\frac\alpha2$, set*

$$E_{\delta,p,q}=\{\omega\in\Omega:\|e^{t\Delta}f^\omega\|_{L^p([0,\delta'],L^q)}>(\delta')^{\frac1p-\frac\alpha2-\epsilon}\|f\|_{H^{-\alpha}},\ \forall\delta'\in(0,\delta]\}.$$

Then we have

$$\mathbf{P}(E_{\delta,p,q})\lesssim e^{-\frac{1}{\delta^\epsilon}}.$$

Proof. By Minkowski's inequality and large deviation property (Lemma 3.1), for $r\ge p,q$ we can have the following bound (see Lemma 2.4 in [WW])

$$(\mathbb{E}\|e^{t\Delta}f^\omega\|^r_{L^p([0,\delta'],L^q)})^{1/r}\le C_{p,q}\sqrt{r}(\delta')^{\frac1p-\frac\alpha2}\|f\|_{H^{-\alpha}}. \tag{3.10}$$

By Chebyshev's inequality, we have

$$\mathbf{P}(\|e^{t\Delta}f^\omega\|_{L^p([0,\delta'],L^q)}>\lambda)\le C^r_{p,q}\lambda^{-r}r^{\frac r2}(\delta')^{\frac rp-\frac{r\alpha}{2}}\|f\|^r_{H^{-\alpha}}$$

for any $r\ge p,q$. When $\lambda/\left((\delta')^{\frac rp-\frac{r\alpha}{2}}\|f\|^r_{H^{-\alpha}}\right)\ge e^2$, we select $r=\lambda/\left((\delta')^{\frac rp-\frac{r\alpha}{2}}\|f\|^r_{H^{-\alpha}}\right)$. By $r^{-\frac r2}\le e^{-r^2}$ when $r\ge e^2$, we have

$$\mathbf{P}(\|e^{t\Delta}f^\omega\|_{L^p([0,\delta'],L^q)}>\lambda)\lesssim_{p,q}\exp\left(-\frac{\lambda^2}{\left((\delta')^{\frac rp-\frac{r\alpha}{2}}\|f\|^r_{H^{-\alpha}}\right)^2}\right).$$

When $\lambda/\left((\delta')^{\frac{r}{p}-\frac{r\alpha}{2}}\|f\|_{H^{-\alpha}}^r\right) < e^2$, we select $r = \max p, q$ (WLOG suppose $r = p$). It's easy to check that $C_{p,q}^r \lambda^{-p} p^{\frac{p}{2}} (\delta')^{1-\frac{p\alpha}{2}} \|f\|_{H^{-\alpha}}^p \lesssim_{p,q} \exp\left(-\frac{\lambda^2}{\left((\delta')^{\frac{r}{p}-\frac{r\alpha}{2}}\|f\|_{H^{-\alpha}}^r\right)^2}\right)$. So we have

$$\mathbf{P}(\|e^{t\Delta} f^\omega\|_{L^p([0,\delta'],L^q)} > \lambda) \lesssim_{p,q} \exp\left(-\frac{\lambda^2}{\left((\delta')^{\frac{r}{p}-\frac{r\alpha}{2}}\|f\|_{H^{-\alpha}}^r\right)^2}\right).$$

By choosing $\lambda = (\delta')^{\frac{1}{p}-\frac{\alpha}{2}-\epsilon}\|f\|_{H^{-\alpha}}$, we prove

$$\mathbf{P}(E_{\delta,\delta',p,q}) \lesssim e^{-\frac{1}{(\delta')^{2\epsilon}}}.$$

where

$$E_{\delta,\delta',p,q} = \{\omega \in \Omega : \|e^{t\Delta} f^\omega\|_{L^p([0,\delta'],L^q)} > (\delta'/2)^{\frac{1}{p}-\frac{\alpha}{2}-\epsilon}\|f\|_{H^{-\alpha}}, \text{ where } \delta' \in (0,\delta]\}.$$

By choosing $\delta' = \delta, \frac{\delta}{2}, \frac{\delta}{4}, \frac{\delta}{8}, ...$, we have $\mathbf{P}(\cup_{j=1}^\infty E_{\delta,2^{-j}\delta,p,q}) \leq \sum_{j=1}^\infty \mathbf{P}(E_{\delta,2^{-j}\delta,p,q}) \lesssim e^{-\frac{1}{\delta^\epsilon}}$. It is easy to check $E_{\delta,p,q} \subset \cup_{j=1}^\infty E_{\delta,2^{-j}\delta,p,q}$, so we also have (3.10). $\square$

4 Energy estimates for the Boussinesq system

In this section, we give energy estimates for the difference equation. We will use these *a priori* estimates to construct weak solutions. First, we set

$$u = g_1^\omega + v,$$

$$\rho = g_2^\omega + \theta.$$

It is equivalent to consider the new system

$$v_t - \Delta v + \mathbb{P}\nabla \cdot ((g_1^\omega + v) \otimes (g_1^\omega + v)) = \mathbb{P}((g_2^\omega + \theta)e_3), \tag{4.11}$$

$$\theta_t - \Delta\theta + \nabla \cdot ((g_1^\omega + v)(g_2^\omega + \theta)) = 0, \tag{4.12}$$

$$\nabla \cdot v = 0. \tag{4.13}$$

Now we define the energy for v and θ, respectively.

$$E_1(v,t) = \|v\|_{L^2}^2 + \int_0^t \|\nabla v\|_{L^2}^2 \, ds$$

and

$$E_2(\theta,t) = \|\theta\|_{L^2}^2 + \int_0^t \|\nabla \theta\|_{L^2}^2 \, ds.$$

The following theorem establishes energy bounds which will be used in constructing global weak solutions. The proofs of the whole space and the periodic space are similar, so we only present the proof of the whole space. Denote that $f = (u_0, \rho_0)$ and $\|f\|_{\dot{H}^{-\alpha}} = \|u_0\|_{\dot{H}^{-\alpha}} + \|\rho_0\|_{\dot{H}^{-\alpha}}$.

Theorem 4.1. *Fix $T > 0$ and $\alpha \in (0, \frac{1}{2})$. Given $0 < \epsilon < \frac{1}{4} - \frac{\alpha}{2}$ and $\gamma > 0$ can be arbitrarily small. Consider a function g_1 and g_2 satisfying the following properties, for $i = 1, 2$*

$$\|g_i\|_{L^2} \le \frac{1}{\delta^\epsilon}(1 + t^{-\frac{\alpha+\gamma+k}{2}})\|f\|_{\dot{H}^{-\alpha}}, \tag{4.14}$$

and

$$\|g_i\|_{L^4([0,\delta],L^4)} + \|g_i\|_{L^4([0,\delta],L^{4^+})} \le \delta^{\frac{1}{4}-\frac{\alpha}{2}-\epsilon}\|f\|_{\dot{H}^{-\alpha}}, \qquad \text{when } d = 2 \tag{4.15}$$

where $\frac{1}{4^+} = \frac{1}{4} - \gamma$ and

$$\|g_i\|_{L^3([0,\delta],L^9)} + \|g_i\|_{L^4([0,\delta],L^4)} \le \delta^{\frac{1}{4}-\frac{\alpha}{2}-\epsilon}\|f\|_{\dot{H}^{-\alpha}}, \qquad \text{when } d = 3 \tag{4.16}$$

where δ is small enough. Suppose $(v, \theta) \in L^\infty([0,T]; L^2(\mathbb{T}^2)) \cap L^2; \dot{H}^1)$ is a solution, then

$$E_1 + E_2 \le C(T, \alpha, \|f\|_{\dot{H}^{-\alpha}}).$$

Proof of Theorem 2.2. First, by multiplying by v and integrating the resulting equation for $t \in [0, \delta]$ we obtain the following equation:

$$\begin{aligned} E_1(v,t) = &-\int_0^t \int v \cdot \mathbb{P}\nabla \cdot (g_1 \otimes g_1)\, dxds - \int_0^t \int v \cdot \mathbb{P}\nabla \cdot (g_1^\omega \otimes v)\, dxds - \int_0^t \int v \cdot \mathbb{P}\nabla \cdot (v \otimes g_1)\, dxds \\ &- \int_0^t \int v \cdot \mathbb{P}\nabla \cdot (v \otimes v)\, dxds - \int_0^t \int v \cdot \mathbb{P}(g_2 + \theta)e_3\, dxds = \sum_{i=1}^5 I_i. \end{aligned}$$

By the divergence-free condition on v and g_1, we have

$$I_2 = I_4 = 0.$$

Therefore, it remains to estimate I_1, I_3 and I_5. For I_1, by Hölder inequality, the definition of E_1 and (4.15) (4.16), we have

$$I_1 \lesssim \|\nabla v\|_{L^2([0,t],L^2)}\|g_1\|^2_{L^4([0,t],L^4)} \lesssim \delta^{\frac{1}{2}-\alpha-2\epsilon}\|f\|^2_{\dot{H}^\alpha}E_1(t)^{\frac{1}{2}}.$$

For I_3, when $d = 2$, by Hölder inequality, (4.15) and the definition of E_1, we have that

$$I_3 \lesssim \|\nabla v\|_{L^2([0,t],L^2)}\|g_1\|_{L^4([0,t],L^{4^+})}\|v\|_{L^4([0,t],L^{4^-})} \lesssim \delta^{\frac{1}{4}-\frac{\alpha}{2}-\epsilon}\|f\|_{\dot{H}^\alpha}E_1(t)^{\frac{1}{2}}\|v\|_{L^4([0,t],L^{4^-})}, \tag{4.17}$$

where $\frac{1}{4^-} = \frac{1}{4} + \gamma$. For $\|v\|_{L^4([0,t],L^{4^-})}$, by L^p interpolation theory and Sobolev inequality, we have

$$\begin{aligned} \|v\|_{L^4([0,t],L^{4^-})} &\lesssim \left(\|v\|_{L^\infty([0,t],L^2)}\right)^{\frac{1}{2}} \left(\|v\|_{L^2([0,t],L^{\infty^-})}\right)^{\frac{1}{2}} \\ &\lesssim \left(\sup_{0\le s\le t} E_1(s)\right)^{\frac{1}{4}} \left(\|v\|_{L^2([0,t],H^1)}\right)^{\frac{1}{2}} \\ &\lesssim \left(\sup_{0\le s\le t} E_1(s)\right)^{\frac{1}{4}} \left(\|v\|_{L^2([0,t],L^2)} + \|\nabla v\|_{L^2([0,t],L^2)}\right)^{\frac{1}{2}} \\ &\lesssim \left(\sup_{0\le s\le t} E_1(s)\right)^{\frac{1}{4}} \left(t^{1/2} \sup_{0\le s\le t} E_1(s)^{\frac{1}{2}} + E_1(t)^{\frac{1}{2}}\right)^{\frac{1}{2}}, \end{aligned} \tag{4.18}$$

where $\frac{1}{\infty^-} = \gamma/2$. Combining (4.17) and (4.18), and taking $t = \delta$, we have that for $\delta < 1$

$$I_3 \lesssim \delta^{\frac{1}{4}-\frac{\alpha}{2}-\epsilon} \|f\|_{\dot{H}^{-\alpha}} \sup_{0\le s\le\delta} E_1(s).$$

For I_3, when $d = 3$, by Hölder inequality, (4.16) and the definition of E_1, we have that

$$I_3 \lesssim \|\nabla v\|_{L^2([0,t],L^2)} \|g_1\|_{L^3([0,t],L^9)} \|v\|_{L^6([0,t],L^{\frac{18}{7}})} \lesssim \delta^{\frac{1}{4}-\frac{\alpha}{2}-\epsilon} \|f\|_{\dot{H}^{\alpha}} E_1(t)^{\frac{1}{2}} \|v\|_{L^6([0,t],L^{\frac{18}{7}})}.$$

Based on the interpolation theory and Sobolev inequality, we have

$$\|v\|_{L^6([0,t],L^{\frac{18}{7}})} \lesssim \|v\|^{\frac{2}{3}}_{L^\infty([0,t],L^2)} \|v\|^{\frac{1}{3}}_{L^2([0,t],\dot{H}^1)} \lesssim \sup_{0\le s\le\delta} E_1(s)^{\frac{1}{2}} \tag{4.19}$$

Hence when $d = 3$, we have the same bound of I_3 as (4.19). For I_5, by Cauchy inequality and (4.14) we have that

$$\begin{aligned} I_5 &\lesssim \|v\|_{L^\infty([0,t],L^2)} \left(t\|\theta\|_{L^\infty([0,t],L^2)} + \|g_1\|_{L^1([0,t],L^2)}\right) \\ &\lesssim \delta \sup_{0\le s\le\delta} E_1(s)^{1/2} \sup_{0\le s\le\delta} E_2(s)^{1/2} + \delta^{1-\frac{\alpha+\gamma}{2}-\epsilon} \sup_{0\le s\le\delta} E_1(s)^{\frac{1}{2}}. \end{aligned} \tag{4.20}$$

For the estimate of E_2, First, we multiply by θ and integrate the resulting equation

$$\begin{aligned} E_2(\theta,t) &= -\int_0^t \int (\nabla\theta)\cdot(g_1g_2 + g_1\theta + vg_2 + v\theta)\,dxds \\ &= -\int_0^t \int (\nabla\theta)\cdot(g_1g_2) - \int_0^t \int (\nabla\theta)\cdot(vg_2)\,dxds = J_1 + J_2, \end{aligned}$$

where we use the fact that both v and g_1 are divergence-free. For J_1, we apply Hölder's inequality obtaining

$$J_1 \le \|\nabla\theta\|_{L^2([0,t],L^2)} \|g_1\|_{L^4([0,t],L^4)} \|g_2\|_{L^4([0,t],L^4)} \lesssim \delta^{\frac{1}{2}-\alpha-2\epsilon} \|f\|^2_{\dot{H}^\alpha} E_2(t)^{\frac{1}{2}}.$$

For J_2, when $d = 2$, applying Hölder's inequality, (4.15) and (4.18) we have

$$J_2 \lesssim \|\nabla\theta\|_{L^2([0,t],L^2)} \|g_2\|_{L^4([0,t],L^{4^+})} \|v\|_{L^4([0,t],L^{4^-})} \lesssim \delta^{\frac{1}{4}-\frac{\alpha}{2}-\epsilon} \|f\|_{\dot{H}^\alpha} \sup_{0\le s\le\delta} E_1(s)^{\frac{1}{2}} \sup_{0\le s\le\delta} E_2(s)^{\frac{1}{2}}.$$

For J_2, when $d = 3$, applying Hölder's inequality, (4.16) and (4.20) we have

$$J_2 \lesssim \|\nabla\theta\|_{L^2([0,t],L^2)} \|g_2\|_{L^3([0,t],L^9)} \|v\|_{L^6([0,t],L^{\frac{18}{7}})} \lesssim \delta^{\frac{1}{4}-\frac{\alpha}{2}-\epsilon} \|f\|_{\dot{H}^\alpha} \sup_{0\le s\le\delta} E_1(s)^{\frac{1}{2}} \sup_{0\le s\le\delta} E_2(s)^{\frac{1}{2}}.$$

Summarizing $\sum_{i=1}^5 I_i$ and $\sum_{i=1}^2 J_i$, when $t \in [0,\delta]$ we have the following bound

$$\begin{aligned} \sup_{0\le s\le\delta} (E_1(s) + E_2(s)) \lesssim{} & \delta^{\frac{1}{2}-\alpha-2\epsilon} \|f\|^2_{\dot{H}^\alpha} \left(\sup_{0\le s\le\delta} E_1(s)^{\frac{1}{2}} + E_2(s)^{\frac{1}{2}}\right) + \delta^{\frac{1}{4}-\frac{\alpha}{2}-\epsilon} \|f\|_{\dot{H}^{-\alpha}} \sup_{0\le s\le\delta} E_1(s) \\ & + \delta \sup_{0\le s\le\delta} E_1(s)^{1/2} \sup_{0\le s\le\delta} E_2(s)^{1/2} + \delta^{1-\frac{\alpha+\gamma}{2}-\epsilon} \sup_{0\le s\le\delta} E_1(s)^{\frac{1}{2}} \\ & + \delta^{\frac{1}{2}-\alpha-2\epsilon} \|f\|^2_{\dot{H}^\alpha} E_2(t)^{\frac{1}{2}} + \delta^{\frac{1}{4}-\frac{\alpha}{2}-\epsilon} \|f\|_{\dot{H}^\alpha} \sup_{0\le s\le\delta} E_1(s)^{\frac{1}{2}} \sup_{0\le s\le\delta} E_2(s)^{\frac{1}{2}}. \end{aligned} \tag{4.21}$$

Since $\alpha < \frac{1}{2} - 2\epsilon$, we could choose δ is small enough such that $\delta^{1-\frac{\alpha+\gamma}{2}-\epsilon} \ll 1$ and $\delta^{\frac{1}{4}-\frac{\alpha}{2}-\epsilon}\|f\|_{\dot{H}^{-\alpha}} \ll 1$. Then the continuity argument with (4.21) helps us obtain that

$$\sup_{0\le s\le\delta} (E_1(s) + E_2(s)) \le C(\alpha, \|f\|_{\dot{H}^{-\alpha}}).$$

Let us know consider $t \in [\delta, T]$. The previous energy of (v, θ) is bounded at $t = \delta$ which gives $\|v(\delta)\|_{L^2}$ and $\|\theta(\delta)\|_{L^2}$ are bounded by $C(\alpha, \|f\|_{\dot{H}^{-\alpha}})$. Back to (u, ρ) which is the solution of (1.1)-(1.3), we know that

$$\|u(\delta)\|_{L^2} \leq \|v(\delta)\|_{L^2} + \|g_1(\delta)\|_{L^2} \lesssim E_1(\delta)^{\frac{1}{2}} + \delta^{-\frac{\alpha+\gamma-\epsilon}{2}} \|f\|_{\dot{H}^{-\alpha}} \leq C(\alpha, \|f\|_{\dot{H}^{-\alpha}})$$

and

$$\|\rho(\delta)\|_{L^2} \leq \|\theta(\delta)\|_{L^2} + \|g_2(\delta)\|_{L^2} \lesssim E_2(\delta)^{\frac{1}{2}} + \delta^{-\frac{\alpha+\gamma-\epsilon}{2}} \|f\|_{\dot{H}^{-\alpha}} \leq C(\alpha, \|f\|_{\dot{H}^{-\alpha}}).$$

By the property of classical L^2 weak solution of (u, ρ), we have that for $t \in [\delta, T]$

$$E_1(u, t) + E_2(\rho, t) \leq C(T, \alpha, \|f\|_{\dot{H}^{-\alpha}}).$$

Hence for the energy of (v, θ) we have that for $t \in [\delta, T]$

$$\begin{aligned} E_1(v, t) + E_2(\theta, t) &\leq E_1(g_1, t) + E_1(u, t) + E_2(g_2, t) + E_2(\rho, t) \\ &\lesssim E_1(u, t) + E_2(\rho, t) + \delta^{-\frac{\alpha+\gamma+2\epsilon}{2}} \|f\|_{\dot{H}^{-\alpha}} + t\delta^{-\frac{\alpha+\gamma+1+2\epsilon}{2}} \|f\|_{\dot{H}^{-\alpha}} \\ &\lesssim C(T, \alpha, \|f\|_{\dot{H}^{-\alpha}}) \end{aligned}$$

□

5 Construction of the weak solutions to the difference equation of the Boussinesq system

In this section, we construct weak solutions to the initial value problem (1.1)–(1.3).

$$\begin{cases} v_t - \Delta v + \mathbb{P}\nabla \cdot ((g_1^\omega + v) \otimes (g_1^\omega + v)) - \mathbb{P}((g_2^\omega + \theta)e_3) = 0, \\ \theta_t - \Delta\theta + \nabla \cdot ((g_1^\omega + v)(g_2^\omega + \theta)) = 0, \\ \nabla \cdot v = 0, \quad v(x, 0) = 0, \quad \theta(x, 0) = 0. \end{cases} \tag{5.22}$$

Theorem 5.1. *Fix $T > 0$ and $\alpha \in (0, \frac{1}{2})$. Given $0 < \epsilon < \frac{1}{4} - \frac{\alpha}{2}$ and $\gamma > 0$ can be arbitrarily small. Consider a function g_1 and g_2 satisfying the following properties, for $i = 1, 2$*

$$\|g_i\|_{L^2} \leq \frac{1}{\delta^\epsilon}(1 + t^{-\frac{\alpha+\gamma}{2}})\|f\|_{\dot{H}^{-\alpha}}$$

and

$$\|g_i\|_{L^4([0,\delta], L^4)} + \|g_i\|_{L^4([0,\delta], L^{4^+})} \leq \delta^{\frac{1}{4} - \frac{\alpha}{2} - \epsilon} \|f\|_{\dot{H}^{-\alpha}}, \qquad \text{when } d = 2$$

where $\frac{1}{4^+} = \frac{1}{4} - \gamma$ and

$$\|g_i\|_{L^3([0,\delta], L^9)} + \|g_i\|_{L^4([0,\delta], L^4)} \leq \delta^{\frac{1}{4} - \frac{\alpha}{2} - \epsilon} \|f\|_{\dot{H}^{-\alpha}}, \qquad \text{when } d = 3$$

when δ is small enough. Then there exists a weak solution (v, θ) in $[0, T]$ for the initial value problem (5.22).

Proof. In the construction of weak solutions, we follow Galerkin approximations approach. We first construct the solutions (v^M, θ^M) (where $M > 1$) of finite dimensional approximation equations as follows

$$\begin{cases} v_t^M - \Delta v^M + P_M \mathbb{P} \nabla \cdot ((P_M g_1^\omega + v^M) \otimes (P_M g_1^\omega + v^M)) - P_M \mathbb{P}((P_M g_2^\omega + \theta^M) e_3) = 0, \\ \theta_t^M - \Delta \theta^M + P_M \nabla \cdot ((P_M g_1^\omega + v^M)(P_M g_2^\omega + \theta^M)) = 0, \\ \nabla \cdot v^M = 0, \quad v^M(x,0) = 0, \quad \theta^M(x,0) = 0. \end{cases} \tag{5.23}$$

Our plan is to obtain the local-in-time well-posedness of the finite approximation equations via the fixed point argument in the space

$$X_T = C([0,T], L_x^2) \cap L^2([0,T], \dot{H}_x^1).$$

Define

$$\Phi(v^M, \theta^M) = \int_0^t \Delta v^M \, dt - \int_0^t P_M \mathbb{P} \nabla \cdot ((P_M g_1^\omega + v^M) \otimes (P_M g_1^\omega + v^M)) - P_M \mathbb{P}((P_M g_2^\omega + \theta^M) e_3) \, dt$$

and

$$\Psi(v^M, \theta^M) = \int_0^t \Delta \theta^M \, dt - P_M \nabla \cdot ((P_M g_1^\omega + v^M)(P_M g_2^\omega + \theta^M)) \, dt.$$

It's easy to obtain the following estimates

$$\begin{aligned} \|\Phi(v^M, \theta^M)\|_{L_t^\infty L_x^2([0,T])} &\lesssim M^2 T \|v^M\|_{L_t^\infty L_x^2} + M^{1+\frac{d}{2}} T \|v^M\|_{L_t^\infty L_x^2}^2 \\ &+ M^{1+\frac{d}{2}+} T^{1-\frac{\alpha}{2}} \|v^M\|_{L_t^\infty L_x^2} + M T^{h(d)-2\gamma} \lambda^2 + T^{1-\frac{\alpha}{2}} \lambda + T \|\theta^M\|_{L_t^\infty L_x^2}. \end{aligned}$$

$$\begin{aligned} \|\Psi(v^M, \theta^M)\|_{L_t^\infty L_x^2([0,T])} &\lesssim M^2 T \|\theta^M\|_{L_t^\infty L_x^2} + M^{1+\frac{d}{2}} T \|v^M\|_{L_t^\infty L_x^2} \|\theta^M\|_{L_t^\infty L_x^2} \\ &+ M^{1+\frac{d}{2}+} T^{1-\frac{\alpha}{2}} (\|\theta^M\|_{L_t^\infty L_x^2} + \|v^M\|_{L_t^\infty L_x^2}) + M T^{h(d)-2\gamma} \lambda^2 + T^{1-\frac{\alpha}{2}} \lambda. \end{aligned}$$

And

$$\begin{aligned} \|\Phi(v^M, \theta^M)\|_{L_t^2 \dot{H}_x^1([0,T])} &\lesssim M^3 T^{\frac{3}{2}} \|v^M\|_{L_t^\infty L_x^2} + M^{2+\frac{d}{2}} T^{\frac{3}{2}} \|v^M\|_{L_t^\infty L_x^2}^2 \\ &+ M^{2+\frac{d}{2}+} T^{-\gamma+\theta(d)} \|v^M\|_{L_t^\infty L_x^2} + M^2 T^{\rho(d)-2\gamma} \lambda^2 + T^{1-\frac{\alpha}{2}} \lambda + T \|\theta^M\|_{L_t^\infty L_x^2}. \end{aligned}$$

$$\begin{aligned} \|\Psi(v^M, \theta^M)\|_{L_t^2 \dot{H}_x^1([0,T])} &\lesssim M^3 T^{\frac{3}{2}} \|\theta^M\|_{L_t^\infty L_x^2} + M^{2+\frac{d}{2}} T^{\frac{3}{2}} \|v^M\|_{L_t^\infty L_x^2} \|\theta^M\|_{L_t^\infty L_x^2} \\ &+ M^{2+\frac{d}{2}+} T^{-\gamma+\theta(d)} (\|\theta^M\|_{L_t^\infty L_x^2} + \|v^M\|_{L_t^\infty L_x^2}) + M^2 T^{\rho(d)-2\gamma} \lambda^2 + T^{1-\frac{\alpha}{2}} \lambda. \end{aligned}$$

Since $P_M g_i$ satisfies the same assumptions as g_i in Section 4, we can repeat the proof of Theorem 4.1 and obtain the same energy bounds given in Theorem 4.1 for finite dimensional approximation solutions (v^M, θ^M). As a consequence we can use an iteration argument to evolve the solution up to time T. By applying a standard compactness argument, together with the fact that $P_M g_i$ strongly converges to g_i for $i = 1, 2$, we obtains a weak solution (v, θ) to (5.22) on $[0, T]$. Since T is arbitrary large, we obtained a global weak solution. $\square$

6 Uniqueness in 2D

In this section, we give the proof of uniqueness of 2D global weak solutions.

Theorem 6.1. *Suppose g_1 and g_2 satisfy the decay properties in Theorem 4.1. Then, the weak solutions in $L^2([0,T];V)\cap L^\infty([0,T];H)$ are unique when $d=2$.*

Proof of Theorem 6.1. Suppose (v_1,θ_1) and (v_2,θ_2) are two solutions. Then, set

$$w = v_1 - v_2$$

$$z = \theta_1 - \theta_2.$$

Thus, we obtain the equation in terms of w and z

$$w_t - \Delta w + \mathbb{P}\nabla\cdot(g_1\otimes w) + \mathbb{P}\nabla\cdot(w\otimes g_1) + \mathbb{P}\nabla\cdot(v_1\otimes w) + \mathbb{P}\nabla\cdot(w\otimes v_2) = \mathbb{P}(ze_3)$$

and

$$z_t - \Delta z + w\cdot\nabla g_2 + w\cdot\nabla\theta_1 + g_1^\omega\cdot\nabla z + v_1\cdot\nabla z = 0. \tag{6.24}$$

Now we do the L^2 energy estimates given $w(0)=z(0)=0$. Take the L^2 inner product on (6.24) with w and we get that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 &\le \|w\|_{L^4}^2\|\nabla v_2\|_{L^2} + \|g_1\|_{L^4}\|w\|_{L^4}\|\nabla w\|_{L^2} + \|z\|_{L^2}\|w\|_{L^2}\\ &\le \|w\|_{L^4}^2\|\nabla v_2\|_{L^2} + \frac{1}{2}\|g_1\|_{L^4}^2\|w\|_{L^4}^2 + \frac{1}{2}\|\nabla w\|_{L^2}^2 + \frac{1}{2}\|z\|_{L^2}^2 + \frac{1}{2}\|w\|_{L^2}^2.\end{aligned}$$

Therefore, we have

$$\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2 \le C(\|\nabla v_2\|_{L^2}^4 + \|g_1\|_{L^4}^2)\|w\|_{L^4}^2 + \frac{1}{2}\|z\|_{L^2}^2.$$

Next, we consider the energy estimates for z by using Holder's inequality and Ladyzhenskaya inequality

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|z\|_{L^2}^2 + \|\nabla z\|_{L^2}^2 &= \int w\cdot\nabla z g_2\,dx - \int w\cdot\nabla\theta_1 z\,dx - \int g_1\cdot\nabla z\cdot z\,dx - \int v_2\cdot\nabla z\cdot z\,dx\\ &= \int w\cdot\nabla z g_2\,dx - \int w\cdot\nabla\theta_1 z\,dx\\ &\le \|w\|_{L^4}\|\nabla z\|_{L^2}\|g_2\|_{L^4} + \|w\|_{L^4}\|\nabla\theta_1\|_{L^2}\|z\|_{L^4}\\ &\le (\frac{1}{2}\|\nabla z\|_{L^2}^2 + \frac{1}{2}\|w\|_{L^4}^2\|g_2\|_{L^4}^2) + (\|w\|_{L^2}^{1/2}\|\nabla w\|_{L^2}^{1/2}\|\nabla\theta_1\|_{L^2}\|z\|_{L^2}^{1/2}\|\nabla z\|_{L^2}^{1/2})\\ &= M_1 + M_2.\end{aligned}$$

For M_1,

$$M_1 \le \frac{1}{2}\|\nabla z\|_{L^2}^2 + C\|w\|_{L^2}^2\|g_2\|_{L^4}^4 + \frac{1}{3}\|\nabla w\|_{L^2}^2.$$

For M_2,

$$\begin{aligned}M_2 &\le C\|\nabla\theta_1\|_{L^2}\|w\|_{L^2}\|z\|_{L^2} + C\|\nabla w\|_{L^2}\|\nabla z\|_{L^2}\\ &\quad C\|\nabla\theta_1\|_{L^2}^2\|w\|_{L^2}^2 + C\|z\|_{L^2}^2 + \frac{1}{3}\|\nabla w\|_{L^2}^2 + \frac{1}{3}\|\nabla z\|_{L^2}.\end{aligned}$$

Combining the estimates above gives the uniqueness of the solutions. □

7 Proof of main theorems

We find the solution (u, ρ) by

$$u = g_1^\omega + v,$$

$$\rho = g_2^\omega + \theta,$$

where $g_1^\omega = e^{t\Delta} u_0^\omega$ and $g_2^\omega = e^{t\Delta}\rho_0^\omega$. Then we consider the corresponding system of (v, θ): (4.11)-(4.13).

Proof of Theorem 2.1 and 2.2. In Theorem 4.1, Theorem 5.1 and Theorem 6.1, we show that main theorems (Theorem 2.1) and (Theorem 2.2) are true when g_1^ω and g_2^ω satisfy the following conditions for $i = 1, 2$:

$$\|g_i^\omega\|_{L^2} \le \frac{1}{\delta^\epsilon}(1 + t^{-\frac{\alpha+\gamma+k}{2}})\|f\|_{\dot{H}^{-\alpha}}, \tag{7.1}$$

and for all $\delta' \le \delta$

$$\|g_i^\omega\|_{L^4([0,\delta'],L^4)} + \|g_i^\omega\|_{L^4([0,\delta'],L^{4^+})} \le (\delta')^{\frac{1}{4}-\frac{\alpha}{2}-\epsilon}\|f\|_{\dot{H}^{-\alpha}}, \qquad \text{when } d = 2, \tag{7.2}$$

where $\frac{1}{4^+} = \frac{1}{4} - \gamma$ and

$$\|g_i^\omega\|_{L^3([0,\delta'],L^9)} + \|g_i^\omega\|_{L^4([0,\delta'],L^4)} \le (\delta')^{\frac{1}{4}-\frac{\alpha}{2}-\epsilon}\|f\|_{\dot{H}^{-\alpha}}, \qquad \text{when } d = 3. \tag{7.3}$$

Define $\Omega_\delta^{(1)} = \{\omega \in \Omega : g_1^\omega, g_2^\omega \text{ satifies } (7.1)\}$ and $\Omega_\delta^{(2)} = \{\omega \in \Omega : g_1^\omega, g_2^\omega \text{ satifies } (7.2)\&(7.3)\}$. It is easy to see that for any $0 < \delta_1 < \delta_2$, $\Omega_{\delta_2}^{(1)} \subset \Omega_{\delta_1}^{(1)}$ and $\Omega_{\delta_2}^{(2)} \subset \Omega_{\delta_1}^{(2)}$. Suppose $\omega \in \Omega_{good} = (\cup_{\delta>0}\Omega_\delta^{(1)}) \cap (\cup_{\delta>0}\Omega_\delta^{(2)})$, for the initial data u_0^ω and ρ_0^ω we can solve the system (1.1)-(1.3) on $[0, T]$. It remains to show $\mathbf{P}(\Omega_{good}) = 1$. First we have

$$\mathbf{P}(\Omega_{good}) = 1 - \mathbf{P}\left(\cap_{\delta>0}(\Omega_\delta^{(1)})^c \cup \cap_{\delta>0}(\Omega_\delta^{(2)})^c\right).$$

By Lemma 3.4 and Lemma 3.6, we know that $\mathbf{P}((\Omega_\delta^{(1)})^c) \lesssim e^{-\frac{1}{\delta^\epsilon}}$ and $\mathbf{P}((\Omega_\delta^{(2)})^c) \lesssim e^{-\frac{1}{\delta^\epsilon}}$ so we have

$$\mathbf{P}\left(\cap_{\delta>0}(\Omega_\delta^{(1)})^c \cup \cap_{\delta>0}\Omega_\delta^{(2)})^c\right) \le \lim_{\delta\to 0}\mathbf{P}((\Omega_\delta^{(1)})^c) + \mathbf{P}((\Omega_\delta^{(2)})^c) \lesssim \lim_{\delta\to 0} e^{-\frac{1}{\delta^\epsilon}} = 0$$

which shows $\mathbf{P}(\Omega_{good}) = 1$. $\square$

8 Long time behavior of the weak solution

In this section, we address the long time behavior of the weak solution (u, ρ). We give the growth and decay rates for u and ρ, respectively.

Theorem 8.1 (Long time behavior of the weak solution)**.** *Let (v, θ) be the weak solution provided the bounds for the energy:*

$$E_1(v, t) = \|v\|_{L^2}^2 + \int_0^t \|\nabla v\|_{L^2}^2\, ds \le C + Ct$$

and

$$E_2(\theta, t) = \|\theta\|_{L^2}^2 + \int_0^t \|\nabla\theta\|_{L^2}^2\, ds \le \|\theta_0\|_{L^2}^2.$$

We further assume that $\theta_0 \in L^1 \cap L^2$. Then,

$$\|\rho\|_{L^2} \le Ct^{-\alpha/2} \tag{8.4}$$

and

$$\|u\|_{L^2} \le Ct^{1-\alpha/2}. \tag{8.5}$$

Proof of Theorem 8.1 . Suppose $d = 3$ (in fact the following proof also works for $d \ge 3$ provided the bounded energy). We know that θ satisfies the following perturbed density equation: By Lemma 3.2, we have $E \le C$. Therefore, by the Sobolev embedding $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, we have

$$\frac{d}{dt}\|\theta\|_{L^2}^2 \le -2\|\nabla\theta\|_{L^2}^2 \le -2\|\theta\|_{L^6}^2.$$

By interpolation we obtain

$$\|\theta\|_{L^2} \le C\|\theta\|_{L^1}^{2/5}\|\theta\|_{L^6}^{3/5} \le C\|\theta_0\|_{L^1}^{2/5}\|\theta\|_{L^6}^{3/5} \le C\|\theta\|_{L^6}^{3/5},$$

Thus,

$$\frac{d}{dt}\|\theta\|_{L^2}^2 \le -C(\|\theta\|_{L^2}^2)^{5/3}.$$

Thus, after integrating we obtain

$$\|\theta\|_{L^2} \le Ct^{-3/4}.$$

Combining the estimates above, we obtain

$$\|\rho\|_{L^2} \le \|g_2^\omega\|_{L^2} + \|\theta\|_{L^2} \lesssim t^{-\alpha/2} + Ct^{-3/4} \lesssim t^{-\alpha/2}.$$

Next, we address the growth rate for u. We multiply by u on equation (1.1) and integrate the resulting equation obtaining

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 = \int \rho e_2 u\,dx \le \|\rho\|_{L^2}\|u\|_{L^2},$$

where we used the divergence-free condition of u. Thus,

$$\frac{d}{dt}\|u\|_{L^2} \le \|\rho\|_{L^2}.$$

Therefore, we get

$$\|u\|_{L^2} \lesssim t^{1-\alpha/2}.$$

The proof is complete. $\square$

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