

A NOTE ON METRIC-MEASURE SPACES SUPPORTING POINCARÉ INEQUALITIES

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Dedicated to Professor Vladimir Maz'ya on the occasion of his 80th birthday.

ABSTRACT. Using a method of Korobenko, Maldonado and Rios we show a new characterization of doubling metric-measure spaces supporting Poincaré inequalities without assuming a priori that the measure is doubling.

Non-smooth functions have played a key role in analysis since the nineteenth century. One fundamental development in this vein came with the introduction of Sobolev spaces, which turned out to be a key tool in studying nonlinear partial differential equations and calculus of variations. Although classically Sobolev functions themselves were not smooth, they were defined on smooth objects such as domains in the Euclidean space or, more generally, Riemannian manifolds. By the late 1970s it became well recognized that several results in real analysis required little structure from the underlying ambient space, and could be generalized to non-smooth settings, such as to the so-called spaces of homogeneous type. The latter spaces are (quasi)metric spaces equipped with a doubling Borel measure (see [9, 10]). In fact, maximal functions, Hardy spaces, functions of bounded mean oscillation, and singular integrals of Calderón-Zygmund-type all continue to have a fruitful theory in the context of spaces of homogeneous type. However, this rich theory was, in a sense, only zeroth-order analysis given that no derivatives were involved. The study of first-order analysis with suitable generalizations of derivatives, a fundamental theorem of calculus, and Sobolev spaces, in the setting of spaces of homogeneous type, was initiated in the 1990s. This area, known as analysis on metric spaces, has since grown into a multifaceted theory which continues to play an important role in many areas of contemporary mathematics. For an introduction to the subject we recommend [2, 3, 4, 5, 8, 12, 13, 19, 20, 23].

One of the main objects of study in analysis on metric spaces are so called spaces supporting Poincaré inequalities introduced in [19]. To define this notion, recall that a *metric-measure space* (X, d, μ) is a metric space (X, d) with a Borel measure μ such that $0 < \mu(B(x, r)) < \infty$ for all $x \in X$ and all $r \in (0, \infty)$, where $B(x, r)$ denotes the (open) metric ball with center x and radius r , i.e., $B(x, r) := \{y \in X : d(x, y) < r\}$. If the measure μ is *doubling*, that is, if there exists a finite constant $C > 0$ such that $\mu(2B) \leq C\mu(B)$ for all balls $B \subseteq X$, then we call (X, d, μ) a *doubling metric-measure space*. The notation τB denotes the dilation of a ball B by a factor $\tau \in (0, \infty)$, i.e., $\tau B := B(x, \tau r)$. A Borel

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function $g : X \rightarrow [0, \infty]$ is said to be an *upper gradient* of another Borel function $u : X \rightarrow \mathbb{R}$ if

$$|u(x) - u(y)| \leq \int_{\gamma_{xy}} g \, ds, \quad (1)$$

holds for each $x, y \in X$ and all rectifiable curves γ_{xy} joining x, y . Finally, a metric-measure space (X, d, μ) is said to *support a p -Poincaré inequality*, $p \in [1, \infty)$, if there exist constants $C \in (0, \infty)$ and $\sigma \in [1, \infty)$ such that

$$\int_B |u - u_B| \, d\mu \leq Cr \left(\int_{\sigma B} g^p \, d\mu \right)^{1/p}, \quad (2)$$

whenever B is a ball of radius $r \in (0, \infty)$, $u \in L^1_{\text{loc}}(X, \mu)$, and $g : X \rightarrow [0, \infty]$ is an upper gradient of u . Here and in what follows the barred integral and f_E stand for the integral average:

$$f_E = \int_E f \, d\mu = \frac{1}{\mu(E)} \int_E f \, d\mu,$$

where E is a μ -measurable set of positive measure. To be consistent with the definition of the upper gradient, in what follows we will always assume that functions $u \in L^1_{\text{loc}}(X, \mu)$ are everywhere finite Borel representatives. The above definitions of the upper gradient and spaces supporting Poincaré inequalities are due to Heinonen and Koskela in [19] (see also [20] for a more detailed exposition).

It was proved in [14] and [13, Theorem 5.1] that if a doubling metric-measure space supports a Poincaré inequality, then the p -Poincaré inequality self-improves in the sense that for some $q \in (p, \infty)$ and $C' \in (0, \infty)$, there holds

$$\left(\int_B |u - u_B|^q \, d\mu \right)^{1/q} \leq C' r \left(\int_{5\sigma B} g^p \, d\mu \right)^{1/p}, \quad (3)$$

whenever B is a ball of radius $r \in (0, \infty)$, $u \in L^1_{\text{loc}}(X, \mu)$, and $g : X \rightarrow [0, \infty]$ is an upper gradient of u .

The purpose of this note is to show that the family of inequalities in (3) on a metric measure space imply that the underlying measure is doubling, and thus providing a characterization of doubling metric-measure spaces supporting Poincaré inequalities without assuming a priori that the measure is doubling, see Theorem 1, below. This result is a minor refinement of a beautiful result in [22], where it was proved that in a related context, a family of weak Sobolev inequalities imply that the measure is doubling. However, the authors considered Sobolev inequalities where the balls had the same radius on both sides, and such a condition is stronger than the one in (3). Moreover, they did not address the important applications to Sobolev spaces supporting Poincaré inequalities.

While the proof presented below is almost the same as the one in [22], it is important to provide details: the proof employs an infinite iteration of Sobolev inequalities and since now we have balls of different size on both sides, it is not obvious without checking details that this will not cause estimates to blow up. This paper should be regarded as a supplement to the work of [22] and an advertisement of their work. Different, but related iterative

arguments to the one presented below were used in [7, 11, 15, 16, 17] in the proofs that a Sobolev inequality implies a measure density condition. Other applications of a method developed in [22] are given in [1, 21].

We now state the main result of this note.

Theorem 1. *Let (X, d, μ) be a metric-measure space and fix $p \in [1, \infty)$. Then the following two statements are equivalent.*

- (a) *The measure μ is doubling and the space (X, d, μ) supports a p -Poincaré inequality.*
- (b) *There exist $q \in (p, \infty)$, $C_P \in [1, \infty)$, and $\sigma \in [1, \infty)$ such that*

$$\left(\int_B |u - u_B|^q d\mu \right)^{1/q} \leq C_P r \left(\int_{\sigma B} g^p d\mu \right)^{1/p}, \quad (4)$$

whenever B is a ball of radius $r \in (0, \infty)$, $u \in L^1_{\text{loc}}(X, \mu)$, and $g : X \rightarrow [0, \infty]$ is an upper gradient of u .

Remark 2. We could assume that (4) holds with $C_P \in (0, \infty)$, but the estimates presented below are more elegant if $C_P \geq 1$. Clearly, if (4) holds with a constant strictly greater than zero, then we can increase it to a constant greater than or equal to 1.

A positive locally integrable function $0 < w \in L^1_{\text{loc}}(\mathbb{R}^n)$ defines an absolutely continuous measure $d\mu = w(x) dx$ with the weight w . A class of the so called p -admissible weights plays a fundamental role in the nonlinear potential theory [18]. To make the presentation brief, we will not recall the definition of a p -admissible weight, but we refer the reader to [18] for details. As an immediate consequence of Theorem 1 and [14, Theorem 2] we obtain a new characterization of p -admissible weights. A variant of this result has also been proved in [22].

Corollary 3. *A function $0 < w \in L^1_{\text{loc}}(\mathbb{R}^n)$ is a p -admissible weight for some $1 < p < \infty$, if and only if there exist $q \in (p, \infty)$, $C \in [1, \infty)$ and $\sigma \in [1, \infty)$ such that*

$$\left(\int_B |u - u_B|^q d\mu \right)^{1/q} \leq C r \left(\int_{\sigma B} |\nabla u|^p d\mu \right)^{1/p}$$

whenever $B \subset \mathbb{R}^n$ is a ball of radius $r \in (0, \infty)$, $u \in C^\infty(\sigma B)$ and $d\mu = w dx$.

Example 4. The following example from [6, Example 6.2] shows a metric-measure space with a non-doubling measure that supports the (p, p) -Poincaré inequality (5) for all $1 \leq p < \infty$. This shows that Theorem 1 is sharp in the sense that $q > p$ in (4) cannot be replaced by $q = p$. Let $X = [0, \infty)$ be equipped with the Euclidean metric $d(x, y) = |x - y|$ and the measure $d\mu = w(x) dx$, where $w(x) = \min\{1, x^{-1}\}$. First observe that w is not doubling. Indeed, for $r > 1$, $\mu(B(2r, r)) = \mu((r, 3r)) = \int_r^{3r} x^{-1} dx = \log 3$, but $\mu(2B(2r, r)) = \mu((0, 4r)) > \int_1^{4r} x^{-1} dx = \log(4r) \rightarrow \infty$ as $r \rightarrow \infty$. It remains to show the (p, p) -Poincaré inequality for $1 \leq p < \infty$.

Any ball $B = B(x, r)$ is an interval (a, b) or $[a, b) = [0, b)$ with $r \leq b - a \leq 2r$. Since the function w is non-increasing, for any $x \in [a, b)$ we have

$$\int_x^b w(t) dt \leq (b - x)w(x) \leq 2rw(x).$$

This observation and Hölder's inequality yield

$$\begin{aligned} \frac{1}{\mu(B)} \int_B |u - u(a)|^p d\mu &\leq \frac{1}{\mu(B)} \int_a^b \left| \int_a^t g(x) dx \right|^p w(t) dt \\ &\leq \frac{(2r)^{p-1}}{\mu(B)} \int_a^b \left(\int_a^t g(x)^p dx \right) w(t) dt \\ &= \frac{(2r)^{p-1}}{\mu(B)} \int_a^b \left(\int_x^b w(t) dt \right) g(x)^p dx \\ &\leq \frac{(2r)^p}{\mu(B)} \int_a^b g(x)^p w(x) dx \end{aligned}$$

so

$$\left(\int_B |u - u_B|^p d\mu \right)^{1/p} \leq 2 \left(\int_B |u - u(a)|^p d\mu \right)^{1/p} \leq 4r \left(\int_B g^p d\mu \right)^{1/p}. \quad (5)$$

Note that since the measure μ is not doubling, inequality (5) cannot hold with any exponent $q > p$ on the left hand side as otherwise we would arrive to a contradiction with Theorem 1.

Proof of Theorem 1. The implication $(a) \Rightarrow (b)$ follows immediately from [14, Theorem 1]. Note however, that the constant σ in (4) might be larger than that in the p -Poincaré inequality (see (3)). Thus we will focus on proving that (a) follows from (b) . To this end, suppose that X satisfies the condition displayed in (4). Making use of Hölder's inequality and the fact that $1 \leq p < q$, we may conclude that the (q, p) -Poincaré inequality in (4) implies that the space (X, d, μ) supports a p -Poincaré inequality (see (2)).

There remains to show that the condition in (4) forces the measure μ to be doubling. Fix a ball $B := B(x, r)$, $x \in X$, $r \in (0, \infty)$, and observe that specializing (4) to the case when B is replaced by $2\sigma B$ yields

$$\left(\int_{2\sigma B} |u - u_{2\sigma B}|^q d\mu \right)^{1/q} \leq 2\sigma r C_P \left(\int_{2\sigma^2 B} g^p d\mu \right)^{1/p}, \quad (6)$$

whenever $u \in L^1_{\text{loc}}(X, \mu)$ and $g : X \rightarrow [0, \infty]$ is an upper gradient of u . Since $p \geq 1$, it follows from (6) and Hölder's inequality that,

$$\begin{aligned} \left(\int_{2\sigma B} |u|^q d\mu \right)^{1/q} &\leq \left(\int_{2\sigma B} |u - u_{2\sigma B}|^q d\mu \right)^{1/q} + |u_{2\sigma B}| \\ &\leq 2\sigma r C_P \left(\int_{2\sigma^2 B} g^p d\mu \right)^{1/p} + \left(\int_{2\sigma B} |u|^p d\mu \right)^{1/p}. \end{aligned} \quad (7)$$

We now define a collection of functions $\{u_j\}_{j \in \mathbb{N}}$ as follows: for each fixed $j \in \mathbb{N}$, let $r_j := (2^{-j-1} + 2^{-1})r$ and set $B_j := B(x, r_j)$. Then

$$\frac{1}{2}r < r_{j+1} < r_j \leq \frac{3}{4}r, \quad \forall j \in \mathbb{N}. \quad (8)$$

For each $j \in \mathbb{N}$, let $u_j : X \rightarrow \mathbb{R}$ be the function defined by setting for each $y \in X$,

$$u_j(y) := \begin{cases} 1 & \text{if } y \in B_{j+1}, \\ \frac{r_j - d(x, y)}{r_j - r_{j+1}} & \text{if } y \in B_j \setminus B_{j+1}, \\ 0 & \text{if } y \in X \setminus B_j. \end{cases} \quad (9)$$

Noting that $(r_j - r_{j+1})^{-1} = 2^{j+2}r^{-1}$, a straightforward computation will show that u_j is $2^{j+2}r^{-1}$ -Lipschitz on X and that the function $g_j := 2^{j+2}r^{-1}\chi_{B_j}$ is an upper gradient of u_j , where χ_{B_j} denotes the characteristic function of the set B_j . In particular, we have that $u_j \in L^1_{\text{loc}}(X, \mu)$ and that the functions u_j and g_j satisfy (7). Observe that for each fixed $j \in \mathbb{N}$, we have (keeping in mind $\sigma \geq 1$)

$$2\sigma r C_P \left(\int_{2\sigma^2 B} g_j^p d\mu \right)^{1/p} = \sigma C_P 2^{j+3} \left(\frac{\mu(B_j)}{\mu(2\sigma^2 B)} \right)^{1/p} \leq \sigma C_P 2^{j+3} \left(\frac{\mu(B_j)}{\mu(2\sigma B)} \right)^{1/p} \quad (10)$$

and

$$\left(\int_{2\sigma B} |u_j|^p d\mu \right)^{1/p} \leq \left(\frac{\mu(B_j)}{\mu(2\sigma B)} \right)^{1/p}. \quad (11)$$

Moreover,

$$\left(\int_{2\sigma B} |u_j|^q d\mu \right)^{1/q} \geq \left(\frac{\mu(B_{j+1})}{\mu(2\sigma B)} \right)^{1/q}. \quad (12)$$

In concert, (10)-(12) and the extreme most sides of the inequality in (7), give

$$\left(\frac{\mu(B_{j+1})}{\mu(2\sigma B)} \right)^{1/q} \leq \sigma C_P 2^{j+4} \left(\frac{\mu(B_j)}{\mu(2\sigma B)} \right)^{1/p}, \quad \forall j \in \mathbb{N}. \quad (13)$$

Therefore

$$\mu(B_{j+1})^{1/q} \leq \sigma C_P 2^{j+4} \frac{\mu(B_j)^{1/p}}{\mu(2\sigma B)^{(q-p)/pq}}, \quad \forall j \in \mathbb{N}. \quad (14)$$

With $\alpha := q/p \in (1, \infty)$ we raise both sides of the inequality in (14) to the power p/α^{j-1} in order to obtain

$$\mu(B_{j+1})^{1/\alpha^j} \leq 2^{p(j+4)/\alpha^{j-1}} \left(\frac{\sigma C_P}{\mu(2\sigma B)^{(q-p)/pq}} \right)^{p/\alpha^{j-1}} \mu(B_j)^{1/\alpha^{j-1}}, \quad \forall j \in \mathbb{N}. \quad (15)$$

If we let $P_j := \mu(B_j)^{1/\alpha^{j-1}}$, then the inequality in (15) becomes

$$P_{j+1} \leq 2^{p(j+4)/\alpha^{j-1}} \left(\frac{\sigma C_P}{\mu(2\sigma B)^{(q-p)/pq}} \right)^{p/\alpha^{j-1}} P_j, \quad \forall j \in \mathbb{N}, \quad (16)$$

which, together with an inductive argument and the fact that $P_1 \leq \mu(B)$, implies

$$\begin{aligned} P_{j+1} &\leq P_1 \prod_{k=1}^j \left[2^{p(k+4)/\alpha^{k-1}} \left(\frac{\sigma C_P}{\mu(2\sigma B)^{(q-p)/pq}} \right)^{p/\alpha^{k-1}} \right] \\ &\leq \mu(B) \prod_{k=1}^j \left[2^{p(k+4)/\alpha^{k-1}} \left(\frac{\sigma C_P}{\mu(2\sigma B)^{(q-p)/pq}} \right)^{p/\alpha^{k-1}} \right], \quad \forall j \in \mathbb{N}. \end{aligned} \quad (17)$$

We claim that the product in (17) converges as $j \rightarrow \infty$. Indeed, observe that

$$\begin{aligned} \prod_{k=1}^{\infty} \left(\frac{\sigma C_P}{\mu(2\sigma B)^{(q-p)/pq}} \right)^{p/\alpha^{k-1}} &= \left(\frac{\sigma C_P}{\mu(2\sigma B)^{(q-p)/pq}} \right)^{p \sum_{k=1}^{\infty} \alpha^{1-k}} \\ &= \left(\frac{\sigma C_P}{\mu(2\sigma B)^{(q-p)/pq}} \right)^{\frac{p\alpha}{\alpha-1}} = \left(\frac{\sigma C_P}{\mu(2\sigma B)^{(q-p)/pq}} \right)^{\frac{pq}{q-p}}, \end{aligned} \quad (18)$$

and

$$\prod_{k=1}^{\infty} (2^{p(k+4)})^{1/\alpha^{k-1}} = 2^{\sum_{k=1}^{\infty} p(k+4)\alpha^{1-k}} =: A(p, q) \in (0, \infty). \quad (19)$$

On the other hand, it follows from (8) that

$$0 < \mu(2^{-1}B)^{1/\alpha^{j-1}} \leq P_j = \mu(B_j)^{1/\alpha^{j-1}} \leq \mu(B)^{1/\alpha^{j-1}} < \infty, \quad (20)$$

which, in turn, further implies $\lim_{j \rightarrow \infty} P_j = 1$. Consequently, passing to the limit in (17) yields

$$1 \leq \mu(B) \frac{(\sigma C_P)^{pq/(q-p)}}{\mu(2\sigma B)} A(p, q). \quad (21)$$

Hence,

$$\mu(2\sigma B) \leq (\sigma C_P)^{pq/(q-p)} A(p, q) \mu(B). \quad (22)$$

Since $\sigma \geq 1$, it follows that μ is doubling. This finishes the proof of the second implication and, in turn, the proof of the theorem. $\square$

Remark 5. In the proof of the $(b) \Rightarrow (a)$ in Theorem 1, one can compute the constant $A(p, q)$ appearing in (19) by observing that (keeping in mind $\alpha = q/p$),

$$\begin{aligned} \sum_{k=1}^{\infty} p(k+4)\alpha^{1-k} &= p \sum_{k=1}^{\infty} \frac{k}{\alpha^{k-1}} + 4p \sum_{k=1}^{\infty} \frac{1}{\alpha^{k-1}} \\ &= \frac{p}{(1 - 1/\alpha)^2} + \frac{4p\alpha}{\alpha - 1} = \frac{pq^2}{(q-p)^2} + \frac{4pq}{q-p}. \end{aligned} \quad (23)$$

Therefore,

$$A(p, q) = 2^{\frac{pq^2}{(q-p)^2} + \frac{4pq}{q-p}}.$$

Hence, condition (4) implies that measure μ satisfies the following doubling condition:

$$\mu(2B) \leq \left(\sigma C_P 2^{\frac{q}{(q-p)+4}} \right)^{pq/(q-p)} \mu(B) \quad \text{for all balls } B \subseteq X.$$

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