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Abstract
In this paper, we study the option pricing problem for the conditional Asian option that appears as a recent market product, offering a cheaper and new alternative to the regular Asian option. We develop the new characteristics of short-maturity asymptotic for the prices of the conditional Asian option provided that the underlying asset follows a local volatility model. The asymptotics for out-of-the-money and at-the-money using fixed strike conditional Asian options are presented, respectively, which provide the linear approximation description of call/put option price. Moreover, the approximating solution for the corresponding variational problem under the well-known Black–Scholes model is also given. The theoretical results derived in the paper are practically relevant and numerical experiments are shown to validate the theoretical outcomes of the paper.

Keywords Conditional Asian options · Short maturity · Local volatility · Large deviations · Variational problem

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1 Introduction

An Asian option is a particular type of option contract, for which the payoff is determined by the average of underlying price over some preset period of time. Different from European or American options, Asian option not only involves its relative cost but also reduces the risk of market manipulation of the underlying instrument at maturity due to its averaging feature.

For the short maturity European option, due to its practical importance, there have been extensive studies on asymptotics of the option prices as well as the resultant volatility in the literature; see, e.g., [8,12,19] for the local volatility models and [6,16,24] for the stochastic volatility models. Unlike the European option, the Asian option is typically less analytically tractable, including its corresponding Black–Scholes model. This explains why in the financial industry, Asian option is often quoted by price rather than by implied volatility. Recently, there are some study appeared to address the short maturity (regular) Asian option problem in literature in order to effectively capture the characteristics of the option price [25,27,29].

To further reduce the volatility in the payoffs, in recent years, a financial company, Banque Nationale de Paris (BNP) and Paribas introduced a variation of regular Asian option, called conditional Asian option, under which the average of asset prices is only based on prices that are above certain threshold [14]. Mathematically, the payoff for call option and put option from the continuous conditional Asian option can be described by the following quantities respectively

$$\left(\frac{\int_0^T S_t \mathbb{1}_{\{S_t>b\}}dt}{\int_0^T \mathbb{1}_{\{S_t>b\}}dt} - K\right)^+ \quad \text{and} \quad \left(K - \frac{\int_0^T S_t \mathbb{1}_{\{S_t>b\}}dt}{\int_0^T \mathbb{1}_{\{S_t>b\}}dt}\right)^+, \tag{1}$$

where $(x)^+ = \max\{x, 0\}$, K is the strike, the threshold level is denoted by a given constant $b > 0$, and $\mathbb{1}_A$ represents an indicator function which equals 1 if A is true and 0 otherwise.

In this paper, we assume that the stock price follows a local volatility model dynamics:

$$dS_t = (r - q)S_t dt + \sigma(S_t)S_t dW_t, \quad S_0 > 0, \tag{2}$$

where W_t is a standard Brownian motion, $r \geq 0$ stands for the risk-free rate, $q \geq 0$ is the continuous dividend yield, $\sigma(\cdot)$ describes the local volatility, and the log-stock price process $X_t = \log S_t$ is characterized by

$$dX_t = \left(r - q - \frac{1}{2}\sigma^2\left(e^{X_t}\right)\right)dt + \sigma\left(e^{X_t}\right)dW_t.$$

Technically, we assume that the local volatility function $\sigma(\cdot)$ satisfies

$$0 < \underline{\sigma} \leq \sigma(\cdot) \leq \bar{\sigma} < \infty, \tag{3}$$

$$|\sigma(e^x) - \sigma(e^y)| \leq M|x - y|^\alpha, \tag{4}$$

with some fixed $M, \alpha > 0$ for any x, y, and $0 < \underline{\sigma} < \bar{\sigma} < \infty$ are some given constants.

The price of the conditional Asian call and put options with maturity T and strike K are given, respectively, by

$$C(T) := e^{-rT}\mathbb{E}\left[\left(\frac{\int_0^T S_t \mathbb{1}_{\{S_t>b\}}dt}{\int_0^T \mathbb{1}_{\{S_t>b\}}dt} - K\right)^+\right], \tag{5}$$

$$P(T) := e^{-rT}\mathbb{E}\left[\left(K - \frac{\int_0^T S_t \mathbb{1}_{\{S_t>b\}}dt}{\int_0^T \mathbb{1}_{\{S_t>b\}}dt}\right)^+\right], \tag{6}$$

where $C(T)$ and $P(T)$ indicate the dependence on the maturity T.

For *regular* Asian option, there are some studies appeared in the mathematical finance community, and two main approaches exist in current literature for determining the call/put option price. The first one is to use the relationship between the distributional property of the time-integral resulted from the geometric Brownian motion and Bessel processes, and it is consistent with the pricing under the corresponding Black–Scholes model (see, e.g., [7,23,26]). The exact closed-form solutions usually are not available. Geman and Yor [20] first introduce the Laplace transform of the corresponding arithmetic Asian option price in terms of Kummer confluent hypergeometric function. Along with the approach based on Laplace transform, some recent development is to characterize the Laplace transforms of Asian option prices as solutions to a related functional equation, see [4,10], in order to gain computational improvement. Other effort is to utilize the Markov chain approximation in pricing and hedging path-dependent options, see [5,9,21,22] etc., aiming at a further improvement of the estimation. The second approach is to use PDE modelling, e.g., see [33,34]. The resultant PDE either can be solved numerically or can be used to derive analytic approximation by using asymptotic expansion methods and the implicit numerical solutions are known to be unconditionally stable, which requires quite extensive effort. There are many representative works including but not limited to [2,17,35,36]. Particularly the last paper by Vecer gives the Black–Scholes representation for all price evolution models.

For the more complicated *conditional* Asian option, there is little study available except for a recent work by Feng and Volkmer [14], in which they propose the analytic approach to compute the price and delta of conditional Asian options by using inverse Laplace transform for the corresponding Black–Scholes model. However, practitioners have to price these products by numerical simulations under the proposed framework. It is expected that the corresponding numerical approach will be challenging, since the inherent problem appears to be harder than the one in regular Asian option. Different from their framework, in this paper, we develop a new approach by using large deviation theory to characterize the short maturity estimation for the conditional Asian option, which provides an alternative approach to the study of conditional Asian option and offers some new mathematical insight for this problem, which to the best of our knowledge, it is not available in current literature.

Large deviation theory (see Sect. 3.1) is a quite effective tool for asymptotic analysis and has been widely used in various applications. It provides a natural framework for estimation, such as for approximating the exponentially small probabilities associated with the behaviour of a diffusion process over a short period of time, as shown in this paper. In the study of financial mathematics, it is often used in the computation of small-maturity, out-of-the-money (OTM) call/put option prices or the probability of reaching a default level in a short period of time. In addition, it also has been applied to the study of local and stochastic volatility models [1,3]. Recently, Pirjol and Zhu [27] study the short maturity (regular) Asian option problem by using large deviation approach. The advantage for using large deviation approach results from the fact that it allows us to use the contraction technique for small time arithmetic average of the diffusion process, and hence one can rigorously obtain the asymptotic behavior for the out-of-the-money regular Asian call/put options. The asymptotic exponent can be given as the rate function (see Sect. 3.1) from the large deviation principle, which itself can be formulated as a variational problem but appears to be not straightforward. Recently, Pirjol and Zhu [28] derive the asymptotics for the price of regular Asian options with discrete-time averaging for its Black–Scholes model, with both fixed and floating strike. Most recently, researches appeared in [25,29] study regular Asian option for the CEV model as well as forward start regular Asian option in local volatility model respectively.

When $S_0 < K$, the call option is out-of-the-money and $C(T) \to 0$ as the maturity time T is becoming shorter and when $S_0 > K$, the put option is said to be out-of-the-money and $P(T)$ becomes small when the time variable T is small. When $S_0 = K$, i.e., at-the-money (ATM), both $C(T)$ and $P(T)$ tend to be small as $T \to 0$. In practice, it is desirable to estimate (5) and (6) in order to determine the call/put option price for a given maturity T. T usually is typically e.g. 3 months, or 6 months, or a year, which is much shorter comparing to the underlying economic cycle. For Black–Scholes models it remains to be true for the corresponding $\sigma^2 T$. In the context of local volatility models, the rate function involved in the large deviation estimates is given in terms of a distance function, which in general cannot be calculated by a closed form, hence a suitable approximation is practically desirable [13,15]. Due to the complexity of (5) and (6), it would be very difficult, if it is not impossible, to compute $C(T)$ and $P(T)$ directly. For the short maturity case, the study of first-order approximation of $C(T)$ and $P(T)$ becomes necessary, and such a study will not only provide a meaningful approach but also a practical first-order estimation of the call/put option price.

In this paper, we focus on the option pricing problem for *conditional* Asian option by applying the large deviation theory, which appears to be a new development. We first use large deviation technique to derive the short maturity asymptotics for the conditional Asian call/put option price, provided that the underlying asset follows a local volatility model. In particular, we present asymptotics for the cases of out-of-the-money by considering the fixed strike conditional Asian options. It is worthy of mentioning here that the proposed approach given in [27] for regular Asian option is not directly applicable here in terms of resolving the corresponding variational problem due to the complexity of conditional Asian option. Thus we use different analytic approach to deal with the Black–Scholes model. In addition, the asymptotics for ATM short maturity conditional Asian option are also presented.
The main contributions of this paper can be illustrated as follows:

1. We establish the short maturity asymptotics for conditional Asian call/put option price (Theorem 1), which does not exist in current literature. The key is to derive a corresponding large deviation principle for the ratio:

$$\frac{\int_0^T S_t \mathbb{1}_{\{S_t > b\}} dt}{\int_0^T \mathbb{1}_{\{S_t > b\}} dt}$$

   by the contraction principle (see Sect. 3.1) as well as mollifier function approximation.
2. We extend the Black–Scholes model for conditional Asian options studied in [14] to the local volatility model.
3. We generalize the regular Asian option case appeared in [27] to the conditional Asian option case under the well-known Black–Scholes model by a new approach, and provide the explicit expression of the corresponding rate function that serves as a dominating term for the short maturity case.

The paper is organized as follows. In Sect. 2, we present asymptotic behaviors for out-of-the-money and ATM conditional Asian options in a local volatility model for short maturity. Some numerical experiments are conducted to show the consistency with the obtained theoretical outcomes. The complete solution of the corresponding variation problem for the Black–Scholes model is given. All detailed proofs of the main results are given in Sect. 3. The paper ends with concluding remarks in Sect. 4.

2 Statement of main results

Recall that the stock price follows a local volatility model:

$$dS_t = (r - q)S_t dt + \sigma(S_t)S_t dW_t, \quad S_0 > 0, \tag{7}$$

where W_t is a standard Brownian motion, $r \geq 0$ is the risk-free rate, $q \geq 0$ is the continuous dividend yield, and $\sigma(\cdot)$ is the local volatility satisfying (3) and (4). We are interested in the short maturity case, i.e., the characteristics of $C(T)$ and $P(T)$ when T is small.

2.1 Out-of-the-money conditional Asian options

Following the standard setting, see, e.g. [27], we denote the expectation of the averaged asset price in the risk-neutral measure as

$$A(T) := \mathbb{E}\left[\frac{\int_0^T S_t \mathbb{1}_{\{S_t > b\}} dt}{\int_0^T \mathbb{1}_{\{S_t > b\}} dt}\right]. \tag{8}$$

When $K > A(T)$, the call Asian option is out-of-the-money and $C(T) \to 0$ as $T \to 0$. When $A(T) > K$, the put conditional Asian option is out-of-the-money and $P(T) \to 0$ as $T \to 0$.

One can easily see that

$$\mathbb{E}\left[\min_{0\leq t\leq T} S_t\right] \leq A(T) = \mathbb{E}\left[\frac{\int_0^T S_t \mathbb{1}_{\{S_t > b\}} dt}{\int_0^T \mathbb{1}_{\{S_t > b\}} dt}\right] \leq \mathbb{E}\left[\max_{0\leq t\leq T} S_t\right],$$

thus as $T \to 0$, $A(T) \to S_0$. Therefore, for the small maturity regime, the conditional Asian call option is out-of-the-money if and only if $K > S_0$, etc. And for the rest of the paper, the conditional Asian call option is said to be out-of-the-money (resp., in-the-money) if $K > S_0$ (resp., $K < S_0$), and the conditional Asian put option is said to be out-of-the-money (resp., in-the-money) if $K < S_0$ (resp., $K > S_0$), and it is referred to be ATM if $K = S_0$.

2.2 Short maturity out-of-the-money conditional Asian options

We will use large deviations theory to compute the leading-order approximation at $T \to 0$ for the price of the out-of-the-money conditional Asian options.

Theorem 1 *Assume that* (3) *and* (4) *both hold.*

(i) *For the out-of-the-money case of conditional Asian call options, i.e., $K > S_0$, we have*

$$C(T) = e^{-\frac{1}{T}\inf_{x\geq K}\mathcal{I}_b(x,S_0)+o(\frac{1}{T})} \quad \text{as } T \to 0. \tag{9}$$

(ii) *For the out-of-the-money case of conditional Asian put options, i.e., $K < S_0$, we have*

$$P(T) = e^{-\frac{1}{T}\inf_{x\leq K}\mathcal{I}_b(x,S_0)+o(\frac{1}{T})} \quad \text{as } T \to 0. \tag{10}$$

Here $o(\cdot)$ represents infinitesimal of the higher order of its variable, and for any $S_0, K > 0$, the function $\mathcal{I}_b(\cdot, \cdot)$ is defined as

$$\mathcal{I}_b(K, S_0) := \inf_{\substack{\frac{\int_0^1 e^{g(t)} \mathbb{1}_{\{g(t)>\log b\}} dt}{\int_0^1 \mathbb{1}_{\{g(t)>\log b\}} dt} = K \\ g(0)=\log S_0,\, g(t)\in\mathcal{AC}[0,1]}} \frac{1}{2} \int_0^1 \left(\frac{g'(t)}{\sigma\left(e^{g(t)}\right)} \right)^2 dt, \tag{11}$$

where $\mathcal{AC}[0, 1]$ is the space of absolutely continuous functions defined on $[0, 1]$.

Remark 1 When $\sigma(\cdot)$ is a constant, i.e., $\sigma(\cdot) = \sigma$, then we have $\inf_{x\geq K} \mathcal{I}_b(x, S_0) = \mathcal{I}_b(K, S_0)$ because of the fact that $\mathcal{I}_b(K, S_0)$ is increasing in K for $K > S_0$ and decreasing in K for $K < S_0$; see Proposition 2 for more details.

2.3 At-the-money conditional Asian options

When $K = S_0$, the conditional Asian call and put options are ATM. We have the following result.

Theorem 2 *Assume that* (3) *and* (4) *both hold.*

(i) *When $K = S_0 > b$, as $T \to 0$, the conditional Asian call option is characterized by*

$$C(T) = \sigma(S_0)S_0 \frac{\sqrt{T}}{\sqrt{6\pi}} + O(T). \tag{12}$$

(ii) *When $K = S_0 < b$, as $T \to 0$, the conditional Asian put option is characterized by*

$$P(T) = \sigma(S_0)S_0 \frac{\sqrt{T}}{\sqrt{6\pi}} + O(T). \tag{13}$$

Here $O(\cdot)$ represents infinitesimal of the same order of its variable.

Remark 2 Comparing to Theorem 2 with Theorem 1, one can see that the out-of-the-money conditional Asian option with short maturity is governed by the rare events (due to large deviations), while the ATM conditional Asian option with short maturity is fluctuated by typical events (Gaussian fluctuations), which is consistent with the results appeared in [27] for the regular Asian option case. It is not clear to us, at this point, if $K = S_0 \leq b$ in case (i) [or $K = S_0 \geq b$ in case (ii)] of Theorem 2 belongs to the ATM regime.

2.4 Variational problem for Black–Scholes model

We present in this section the solution of the variational problem for the short-time asymptotics of the out-of-the-money conditional Asian options given by Theorem 1. In the Black–Scholes model the volatility is constant $\sigma(\cdot) = \sigma$. The expression for the rate function $\mathcal{I}_b(K, S_0)$ is obtained explicitly.

Proposition 1 *For Black–Scholes model, the rate function $\mathcal{I}_b(K, S_0)$ appearing in Theorem 1 is given by*

$$\mathcal{I}_b(K,S_0)=\begin{cases}\frac{\left(\log\frac{b}{S_0}+\sqrt{G^+(f_1)F^+(f_1)}\right)^2}{2\sigma^2}, & S_0<b<K \text{ and } S_0<K<b.\\ \frac{\left(\log\frac{b}{S_0}-\sqrt{G^+(f_1)F^+(f_1)}\right)^2}{2\sigma^2}, & b<S_0<K.\\ \frac{\left(\log\frac{b}{S_0}-\sqrt{G^-(g_1)F^-(g_1)}\right)^2}{2\sigma^2}, & K<b<S_0 \text{ and } b<K<S_0.\\ \frac{\left(\log\frac{b}{S_0}+\sqrt{G^-(g_1)F^-(g_1)}\right)^2}{2\sigma^2}, & K<S_0<b.\end{cases} \tag{14}$$

The details of above expression are further given below:

(i) *For $S_0 < b < K$. The value $f_1(\geq \log\frac{K}{S_0})$ is the solution of the following algebraic equation*

$$e^{f_1}-\frac{K}{S_0}=G^+(f_1)/F^+(f_1) \tag{15}$$

where

$$G^+(f_1)=\int_{\log\frac{b}{S_0}}^{f_1}\sqrt{e^{f_1}-e^y}\,dy,\quad F^+(f_1)=\int_{\log\frac{b}{S_0}}^{f_1}\frac{1}{\sqrt{e^{f_1}-e^y}}dy.$$

(ii) *For $K < b < S_0$. The value $g_1(\leq \log\frac{K}{S_0})$ is the solution of the following algebraic equation*

$$\frac{K}{S_0}-e^{g_1}=G^-(g_1)/F^-(g_1) \tag{16}$$

where

$$G^-(g_1)=\int_{g_1}^{\log\frac{b}{S_0}}\sqrt{e^y-e^{g_1}}\,dy,\quad F^-(g_1)=\int_{g_1}^{\log\frac{b}{S_0}}\frac{1}{\sqrt{e^y-e^{g_1}}}dy.$$

(iii) *For $S_0 < K < b$. The value $f_1(\geq \log\frac{b}{S_0})$ is the solution of (15).*
(iv) *For $K < S_0 < b$. The value $g_1(\leq \log\frac{K}{S_0})$ is the solution of (16).*
(v) *For $b < S_0 < K$. The value $f_1(\geq \log\frac{K}{S_0})$ is the solution of (15).*
(vi) *For $b < K < S_0$. The value $g_1(\leq \log\frac{b}{S_0})$ is the solution of (16).*

Here a further study reveals more detailed properties of the rate function $\mathcal{I}_b(K, S_0)$. In particular, we will show that for Black–Scholes model, the rate function $\mathcal{I}_b(K, S_0)$ is continuous in K and it is increasing in K for $K > S_0$ and decreasing in K for $K < S_0$, which is based on an alternative representation of the rate function. Also, it is not difficult to see from the proof (given in next section) that Proposition 1 remains to be true either $b = S_0$ or $b = K$ but not these three variables are identical.

Proposition 2 *The rate function $\mathcal{I}_b(K, S_0)$ can be expressed as*

$$\mathcal{I}_b(K,S_0)=\begin{cases}\dfrac{\left(\log\frac{b}{S_0}+\sqrt{\inf\limits_{\varphi<K/S_0}\frac{\left(\int_{\varphi}^{b/S_0}\frac{\sqrt{z-\varphi}}{z}dz\right)^2}{\frac{K}{S_0}-\varphi}}\right)^2}{2\sigma^2}, & K<S_0<b.\\ \dfrac{\left(\log\frac{b}{S_0}-\sqrt{\inf\limits_{\varphi<b/S_0}\frac{\left(\int_{\varphi}^{b/S_0}\frac{\sqrt{z-\varphi}}{z}dz\right)^2}{\frac{K}{S_0}-\varphi}}\right)^2}{2\sigma^2}, & b<K<S_0.\\ \dfrac{\left(\log\frac{b}{S_0}-\sqrt{\inf\limits_{\varphi<K/S_0}\frac{\left(\int_{\varphi}^{b/S_0}\frac{\sqrt{z-\varphi}}{z}dz\right)^2}{\frac{K}{S_0}-\varphi}}\right)^2}{2\sigma^2}, & K<b<S_0.\\ \dfrac{\left(\log\frac{b}{S_0}+\sqrt{\inf\limits_{\varphi>K/S_0}\frac{\left(\int_{b/S_0}^{\varphi}\frac{\sqrt{\varphi-z}}{z}dz\right)^2}{\varphi-\frac{K}{S_0}}}\right)^2}{2\sigma^2}, & S_0<b<K.\\ \dfrac{\left(\log\frac{b}{S_0}+\sqrt{\inf\limits_{\varphi>b/S_0}\frac{\left(\int_{b/S_0}^{\varphi}\frac{\sqrt{\varphi-z}}{z}dz\right)^2}{\varphi-\frac{K}{S_0}}}\right)^2}{2\sigma^2}, & S_0<K<b.\\ \dfrac{\left(\log\frac{b}{S_0}-\sqrt{\inf\limits_{\varphi>K/S_0}\frac{\left(\int_{b/S_0}^{\varphi}\frac{\sqrt{\varphi-z}}{z}dz\right)^2}{\varphi-\frac{K}{S_0}}}\right)^2}{2\sigma^2}, & b<S_0<K.\end{cases} \tag{17}$$

Remark 3 In practice, the trading volume of out-of-the-money usually is much higher than the one of in-the-money. Hence this renders us to focus on out-of-the-money and at-the-money cases. It is not clear at this point if the asymptotics for short maturity in-the-money conditional Asian call/put options can be obtained under the Black–Scholes model. Again, Proposition 2 holds for either $b = S_0$ or $b = K$ but not $b = S_0 = K$.

2.5 Implied volatility and numerical tests

The Black–Scholes *implied volatility* is defined as the $\sigma_{implied}$ that uniquely solves

$$C(K, S_0, T) = C^{BS}(K, S_0, \sigma_{implied}, T), \tag{18}$$

where $C(K, S_0, T) = e^{-rT}\mathbb{E}\left[\left(\frac{\int_0^T S_t \mathbb{1}_{\{S_t>b\}}dt}{\int_0^T \mathbb{1}_{\{S_t>b\}}dt} - K\right)^+\right]$. Equivalently, it is the $\sigma_{implied}$ that uniquely solves

$$P(K, S_0, T) = P^{BS}(K, S_0, \sigma_{implied}, T), \tag{19}$$

Here we write $C(K, S_0, T)$ and $P(K, S_0, T)$ instead of $C(T)$ and $P(T)$ to highlight the dependence on K and S_0.

One can define an equivalent Black–Scholes volatility of a conditional Asian option as that value of the volatility for which the Black–Scholes price of a European(Vanilla) option with maturity T and underlying value $A(T)$ reproduces the price of the conditional Asian option with the same maturity T. We will denote this volatility as $\Sigma_{LN}(K, S_0, T)$. And we know that

$$C^{BS}(K, S_0, \sigma, T) := B_0(F_0\Phi(d_+) - K\Phi(d_-)), \quad d_\pm = \frac{-k}{\sigma\sqrt{T}} \pm \frac{\sigma\sqrt{T}}{2},$$

and

$$P^{BS}(K, S_0, \sigma, T) := B_0(K\Phi(-d_-) - F_0\Phi(-d_+)).$$

and Φ is the normal cumulative distribution function.

We have thus

$$\begin{aligned} C(K, S_0, T) &= C^{BS}(K, S_0, \Sigma_{LN}(K, S_0, T), T) = e^{-rT}\left[A(T)\Phi(d_+) - K\Phi(d_-)\right], \\ P(K, S_0, T) &= P^{BS}(K, S_0, \Sigma_{LN}(K, S_0, T), T) = e^{-rT}\left[K\Phi(-d_-) - A(T)\Phi(-d_+)\right], \end{aligned} \tag{20}$$

where $A(T)$ is given in (8).

The short maturity asymptotics for out-of-the-money conditional Asian options given in Theorem 1 gives the following short-time asymptotics for the equivalent volatilities of the conditional Asian options in the local volatility model (2). For simplicity, we prove only the $r = q = 0$ case. We expect the same result holds for general r and q.

Proposition 3 *Assume $r = q = 0$ and (3) and (4) both hold. The short-time limit $T \to 0$ of the Black–Scholes equivalent volatility of an out-of-the money conditional Asian option is given by*

$$\lim_{T\to 0} \Sigma_{LN}^2(K, S_0, T) = \frac{1}{2}\frac{\log^2\left(\frac{K}{S_0}\right)}{\mathcal{I}_b(K, S_0)} \tag{21}$$

where $\mathcal{I}_b(K, S_0)$ is given in Proposition 2.

Proof The proof is similar with Proposition 17 in [27], so we omit here. □

2.6 Numerical experiments

Next we present some numerical tests for the short-maturity asymptotic results of conditional Asian options obtained in this paper. Using (17)–(21) one can obtain the predicted value of out-of-money call conditional Asian option price directly. We consider next a few numerical tests of the asymptotic pricing formulas, on the example of the conditional Asian options in the Black–Scholes model. One first test assumes the model parameters

$$r = q = 0, \quad S_0 = 100, \quad \sigma = 30\%. \tag{22}$$

In Table 1 we show the prices of out-of-money call conditional Asian options with maturities $T = 0.1, 0.25, 0.5$ years and condition $b = 120$, comparing with the results of a Monte Carlo calculation versus the asymptotic results C_{as} given in (20). The Monte Carlo simulation was performed with $N = 10^5$ paths, and the time line was discretized with $n = 800$ time steps.

In Table 2 we show the prices of out-of-money call conditional Asian options with maturities $T = 0.1, 0.25, 0.5$ years and condition $b = 50$, comparing with the results of a Monte Carlo calculation versus the asymptotic results C_{as} given in (20). The Monte Carlo simulation was performed with $N = 10^5$ paths, and the time line was discretized with $n = 800$ time steps.

Table 1 When $K < S_0 < b$

K	T = 0.1		T = 0.25		T = 0.5	
	MC (n = 800)	C_{as}(K)	MC (n = 800)	C_{as}(K)	MC (n = 800)	C_{as}(K)
75	26.0482	26.0482	29.7356	29.7356	34.0873	34.0873
80	21.0474	21.0474	24.6857	24.6857	29.0158	29.0158
85	16.0444	16.0444	19.7675	19.7675	24.0777	24.0777
90	11.0380	11.0380	14.7427	14.7427	19.7140	19.1740
95	6.0479	6.0479	9.7380	9.7380	14.0966	14.0966

Numerical results for call conditional Asian options with maturity $T = 0.1, 0.25, 0.5$ and condition $b = 120$ and parameters (22) obtain by Monte Carlo simulation in Black–Scholes model, compared with the asymptotic results C_{as}(K) given by (20)

Table 2 When $b < K < S_0$

K	T = 0.1		T = 0.25		T = 0.5	
	MC (n = 800)	C_{as}(K)	MC (n = 800)	C_{as} (K)	MC (n = 800)	C_{as}(K)
55	44.9880	44.9880	44.9913	44.9913	45.0523	45.0542
60	39.9923	39.9923	39.9954	39.9954	40.0218	40.0235
65	34.9744	34.9744	35.0046	35.0046	35.0312	35.0327
70	30.0057	30.0057	29.9732	29.9732	30.0149	30.0149
75	25.0072	25.0072	25.0029	25.0024	25.0528	25.0539

Numerical results for call conditional Asian options with maturity $T = 0.1, 0.25, 0.5$ and condition $b = 50$ and parameters (22) obtain by Monte Carlo simulation in Black–Scholes model, compared with the asymptotic results C_{as}(K) given by (20)

Table 3 When $K < b < S_0$

K	T = 0.1		T = 0.25		T = 0.5	
	MC (n = 800)	C_{as}(K)	MC (n = 800)	C_{as}(K)	MC (n = 800)	C_{as}(K)
60	40.5456	40.5456	41.8220	41.8220	43.5552	43.5552
65	35.5251	35.5251	36.8597	36.8597	38.5359	38.5365
70	30.5653	30.5653	31.8197	31.8189	33.5222	33.5295
75	25.5524	25.5524	26.8048	26.8079	28.5319	28.6435
80	20.5279	20.5285	21.8170	21.8584	23.5878	23.8386

Numerical results for call conditional Asian options with maturity $T = 0.1, 0.25, 0.5$ and condition $b = 90$ and parameters (22) obtain by Monte Carlo simulation in Black–Scholes model, compared against the asymptotic results C_{as}(K) given by (20)

In Table 3 we show the prices of out-of-money call conditional Asian options with maturities $T = 0.1, 0.25, 0.5$ years and condition $b = 90$, comparing the results of a Monte Carlo calculation against the asymptotic results C_{as} given in (20). The Monte Carlo simulation was performed with $N = 10^5$ paths, and the time line was discretized with $n = 800$ time steps.

Remark 4 The numerical tests presented show that the short maturity asymptotic results of this paper appear to provide a quite accurate approximation for conditional Asian options prices with maturities in those cases.

3 Proofs of the main results

3.1 Preliminary

Before we proceed, recall that a sequence $(P_\epsilon)_{\epsilon\in\mathbb{R}_+}$ of probability measures on a topological space X satisfies the large deviation principle (LDP in short) with rate function $I : X \to \mathbb{R}$ if I is non-negative, lower semi-continuous and for any measurable set A, we have

$$\liminf_{\epsilon\to 0} \epsilon \log P_\epsilon(A) \geq -\inf_{x\in A^\circ} I(x),$$
$$\limsup_{\epsilon\to 0} \epsilon \log P_\epsilon(A) \leq -\inf_{x\in\bar{A}} I(x). \tag{23}$$

Here, A° is the interior of A and $\bar{A}$ is its closure. More details about large deviations and its applications can be found in classical literature, e.g., [11,31,32].

The contraction principle plays a key role in our approach. The existence and uniqueness of strong solutions for local volatility model is also important in our estimate. For the purpose of completeness, we present these results below.

Theorem 3 (Contraction principle in [11]). *If P_ϵ satisfies a large deviation principle on X with rate function $I(x)$ and $F : X \mapsto Y$ is a continuous map, then the probability measures $Q_\epsilon := P_\epsilon F^{-1}$ satisfies a large deviation principle on Y with rate function*

$$J(y) = \inf_{x:F(x)=y} I(x). \tag{24}$$

Remark 5 Compared to the standard comparing method, there is an extension of contraction principle (see Prop 4.1 in Chapter 4 in [37]), $F : X \mapsto Y$ being a continuous map can be extended to being an a.e. continuous map.

Theorem 4 (Existence and uniqueness) *Assume that the function $\sigma(s)s$ and $\sigma(s)$ are uniformly Lipschitz, i.e., there exist $\alpha, \beta > 0$, such that for any $x, y \geq 0$,*

$$|\sigma(x)x - \sigma(y)y| \leq \alpha|x-y|, \quad |\sigma(x) - \sigma(y)| \leq \beta|x-y|. \tag{25}$$

Then there exists a unique solution S_t of the SDE (2) under assumption (3). S_t has continuous paths, moreover for any $p \geq 2$,

$$\mathbb{E}\left[\sup_{0\leq t\leq T} S_t^p\right] \leq C(1+S_0^p) \tag{26}$$

where C is a constant depending only on p and some positive power of T.

Proof According to Theorem 1.1 of Chapter 5 [18], we only need to verify that our assumption meets the requirement of this theorem. First it is not difficult to see that

$$S_t = S_0 + \int_0^t (r-q)S_u du + \int_0^t \sigma(S_u)S_u dW_u. \tag{27}$$

From (27) one can obtain the following estimate:

$$\sup_{0\leq t\leq T} S_t^p \leq 2^{p-1}\left[\left(S_0 + \int_0^T |r-q|S_u du\right)^p + \sup_{0\leq t\leq T}\left|\int_0^T \sigma S_u dW_u\right|^p\right]$$
$$\leq 2^{2(p-1)}S_0^p + 2^{2(p-1)}\left[\int_0^T |r-q|S_u du\right]^p + 2^{p-1}\sup_{0\leq t\leq T}\left|\int_0^t \sigma(S_u)S_u dW_u\right|^p. \tag{28}$$

Under assumption (3), by Burkholder–Davis–Gundy inequality and Jensen's inequality, for any $p \geq 2$, we have

$$\mathbb{E}\left\{\sup_{0\leq t\leq T}\left|\int_0^t \sigma(S_u)S_u dW_u\right|^p\right\} \leq C_p\mathbb{E}\left|\int_0^T \sigma^2(S_u)S_u^2 du\right|^{\frac{p}{2}} \leq \bar{C}_p\int_0^T \mathbb{E}[S_u^p]du.$$

With the same approach in Theorem 1.1 of Ch5 in [18], we can get the following $\mathbb{E}[S_t^p] \leq C(1+S_0^p)e^{Ct}$, where C is constant depending on p. By taking the expectation on both side of (28), we then have $\mathbb{E}\Big[\sup_{0\leq t\leq T} S_t^p\Big] \leq C(1+S_0^p)$, where C depends on p and some positive power of T. □

3.2 Proofs of the main results

Proof of Theorem 1 For the purpose of clarity, the proof will be shown by subsequent steps below:

Step 1 (i) Let us first claim the following the relation between $C(T)$ and S_t as shown below:

$$\lim_{T\to 0} T\log C(T) = \lim_{T\to 0} T\log \mathbb{P}\left(\frac{\int_0^T S_t \mathbb{1}_{\{S_t>b\}}dt}{\int_0^T \mathbb{1}_{\{S_t>b\}}dt} \geq K\right). \tag{29}$$

By Hölder inequality, for any $\frac{1}{p}+\frac{1}{s}=1$, $p, s>1$, we have

$$\begin{aligned} C(T) &\leq e^{-rT}\mathbb{E}\left[\left|\frac{\int_0^T S_t \mathbb{1}_{\{S_t>b\}}dt}{\int_0^T \mathbb{1}_{\{S_t>b\}}dt} - K\right| \mathbb{1}_{\left\{\frac{\int_0^T S_t \mathbb{1}_{\{S_t>b\}}dt}{\int_0^T \mathbb{1}_{\{S_t>b\}}dt}\geq K\right\}}\right] \\ &\leq e^{-rT}\mathbb{E}\left[\left|\frac{\int_0^T S_t \mathbb{1}_{\{S_t>b\}}dt}{\int_0^T \mathbb{1}_{\{S_t>b\}}dt} - K\right|^p\right]^{\frac{1}{p}} \mathbb{P}\left(\frac{\int_0^T S_t \mathbb{1}_{\{S_t>b\}}dt}{\int_0^T \mathbb{1}_{\{S_t>b\}}dt}\geq K\right)^{\frac{1}{s}} \end{aligned}$$

Assume that $p \geq 2$. Note that for $p \geq 2$, $x \mapsto x^p$ is a convex function for $x \geq 0$, and by Jensen's inequality, $\left(\frac{x+y}{2}\right)^p \leq \frac{x^p+y^p}{2}$ for any $x, y \geq 0$, we thus have

$$\begin{aligned} \mathbb{E}\left[\left|\frac{\int_0^T S_t \mathbb{1}_{\{S_t>b\}}dt}{\int_0^T \mathbb{1}_{\{S_t>b\}}dt} - K\right|^p\right] &\leq \mathbb{E}\left[\left(\frac{\int_0^T S_t \mathbb{1}_{\{S_t>b\}}dt}{\int_0^T \mathbb{1}_{\{S_t>b\}}dt} + K\right)^p\right] \\ &\leq 2^{p-1}\left[\mathbb{E}\left[\left(\frac{\int_0^T S_t \mathbb{1}_{\{S_t>b\}}dt}{\int_0^T \mathbb{1}_{\{S_t>b\}}dt}\right)^p\right] + K^p\right] \\ &\leq 2^{p-1}\left[\mathbb{E}\left[\max_{0\leq t\leq T} S_t^p\right] + K^p\right] \text{ (by Theorem 4)} \\ &\leq 2^{p-1}\left[C(1+S_0^p) + K^p\right], \end{aligned}$$

which yields

$$\limsup_{T\to 0} T\log C(T) \leq \limsup_{T\to 0} \frac{T}{s}\log \mathbb{P}\left(\frac{\int_0^T S_t \mathbb{1}_{\{S_t>b\}}dt}{\int_0^T \mathbb{1}_{\{S_t>b\}}dt} \geq K\right). \tag{30}$$

Notice that above estimate holds for any $1 < s < 2$, we have the upper bound.

On the other hand, for any $\epsilon > 0$,

$$C(T) \geq e^{-rT}\mathbb{E}\left[\left(\frac{\int_0^T S_t \mathbb{1}_{\{S_t>b\}}dt}{\int_0^T \mathbb{1}_{\{S_t>b\}}dt} - K\right)\mathbb{1}_{\left\{\frac{\int_0^T S_t \mathbb{1}_{\{S_t>b\}}dt}{\int_0^T \mathbb{1}_{\{S_t>b\}}dt}\geq K+\epsilon\right\}}\right] \tag{31}$$

$$\geq e^{-rT}\epsilon\mathbb{P}\left(\frac{\int_0^T S_t \mathbb{1}_{\{S_t>b\}}dt}{\int_0^T \mathbb{1}_{\{S_t>b\}}dt} \geq K+\epsilon\right), \tag{32}$$

which implies that

$$\liminf_{T\to 0} T\log C(T) \geq \liminf_{T\to 0} T\log\mathbb{P}\left(\frac{\int_0^T S_t \mathbb{1}_{\{S_t>b\}}dt}{\int_0^T \mathbb{1}_{\{S_t>b\}}dt} \geq K+\epsilon\right). \tag{33}$$

Since it holds for any $\epsilon > 0$, we thus obtain the lower bound.

(ii) We conclude by proving the analogous relation to (29) for conditional Asian put options, that is, for out-of-the-money put options, $S_0 > K$, we will show that

$$\lim_{T\to 0} T\log P(T) = \lim_{T\to 0} T\log\mathbb{P}\left(\frac{\int_0^T S_t \mathbb{1}_{\{S_t>b\}}dt}{\int_0^T \mathbb{1}_{\{S_t>b\}}dt} \leq K\right).$$

By Hölder inequality, for any $\frac{1}{p}+\frac{1}{s}=1$, $p, s > 1$, we have

$$\begin{aligned}
P(T) &\leq e^{-rT}\mathbb{E}\left[\left(K-\frac{\int_0^T S_t \mathbb{1}_{\{S_t>b\}}dt}{\int_0^T \mathbb{1}_{\{S_t>b\}}dt}\right)^+\mathbb{1}_{\left\{\frac{\int_0^T S_t \mathbb{1}_{\{S_t>b\}}dt}{\int_0^T \mathbb{1}_{\{S_t>b\}}dt}\leq K\right\}}\right]\\
&\leq e^{-rT}\mathbb{E}\left[\left(\left(K-\frac{\int_0^T S_t \mathbb{1}_{\{S_t>b\}}dt}{\int_0^T \mathbb{1}_{\{S_t>b\}}dt}\right)^+\right)^p\right]^{\frac{1}{p}}\mathbb{P}\left(\frac{\int_0^T S_t \mathbb{1}_{\{S_t>b\}}dt}{\int_0^T \mathbb{1}_{\{S_t>b\}}dt}\leq K\right)^{\frac{1}{s}}\\
&\leq e^{-rT}K\mathbb{P}\left(\frac{\int_0^T S_t \mathbb{1}_{\{S_t>b\}}dt}{\int_0^T \mathbb{1}_{\{S_t>b\}}dt}\leq K\right)^{\frac{1}{s}}.
\end{aligned}$$

Thus, $\limsup_{T\to 0} T\log P(T) \leq \limsup_{T\to 0}\frac{T}{s}\log\mathbb{P}\left(\frac{\int_0^T S_t \mathbb{1}_{\{S_t>b\}}dt}{\int_0^T \mathbb{1}_{\{S_t>b\}}dt}\leq K\right)$. Since it holds for any $s > 1$, we proved the upper bound. For the lower bound, for any sufficiently small $\epsilon > 0$,

$$\begin{aligned}
P(T) &\geq e^{-rT}\mathbb{E}\left[\left(K-\frac{\int_0^T S_t \mathbb{1}_{\{S_t>b\}}dt}{\int_0^T \mathbb{1}_{\{S_t>b\}}dt}\right)\mathbb{1}_{\left\{\frac{\int_0^T S_t \mathbb{1}_{\{S_t>b\}}dt}{\int_0^T \mathbb{1}_{\{S_t>b\}}dt}+\epsilon\leq K\right\}}\right]\\
&\geq e^{-rT}\epsilon\mathbb{P}\left(\frac{\int_0^T S_t \mathbb{1}_{\{S_t>b\}}dt}{\int_0^T \mathbb{1}_{\{S_t>b\}}dt}+\epsilon\leq K\right),
\end{aligned}$$

which implies that $\liminf_{T\to 0} T\log P(T) \geq \limsup_{T\to 0} T\log\mathbb{P}\left(\frac{\int_0^T S_t \mathbb{1}_{\{S_t>b\}}dt}{\int_0^T \mathbb{1}_{\{S_t>b\}}dt}+\epsilon\leq K\right)$. By letting $\epsilon \downarrow 0$, we proved the lower bound.

Step 2 Next we compute the following limit:

$$\lim_{T\to 0} T\log\mathbb{P}\left(\frac{\int_0^T S_t \mathbb{1}_{\{S_t>b\}}dt}{\int_0^T \mathbb{1}_{\{S_t>b\}}dt}\geq K\right).$$

Let two mollifier functions $\bar{h}_b^\epsilon(x)$ and $\underline{h}_b^\epsilon(x)$ such that they satisfy $\underline{h}_b^\epsilon(X_t) \le \mathbb{1}_{\{X_t > \log b\}} \le \bar{h}_b^\epsilon(X_t)$, and $\lim_{\epsilon\to 0} \bar{h}_b^\epsilon(x) = \lim_{\epsilon\to 0} \underline{h}_b^\epsilon(x) = \mathbb{1}_{\{x>\log b\}}$, respectively, then we have

$$\begin{aligned}\mathbb{P}\left(\frac{\int_0^1 e^{X_{tT}}\underline{h}_b^\epsilon(X_{tT})dt}{\int_0^1 \bar{h}_b^\epsilon(X_{tT})dt} \ge K\right) &= \mathbb{P}\left(\frac{\int_0^T e^{X_t}\underline{h}_b^\epsilon(X_t)dt}{\int_0^T \bar{h}_b^\epsilon(X_t)dt} \ge K\right)\\ &\le \mathbb{P}\left(\frac{\int_0^T e^{X_t}\mathbb{1}_{\{X_t>\log b\}}dt}{\int_0^T \mathbb{1}_{\{X_t>\log b\}}dt} \ge K\right) \le \mathbb{P}\left(\frac{\int_0^T e^{X_t}\bar{h}_b^\epsilon(X_t)dt}{\int_0^T \underline{h}_b^\epsilon(X_t)dt} \ge K\right)\\ &= \mathbb{P}\left(\frac{\int_0^1 e^{X_{tT}}\bar{h}_b^\epsilon(X_{tT})dt}{\int_0^1 \underline{h}_b^\epsilon(X_{tT})dt} \ge K\right)\end{aligned}$$

If $S_t = e^{X_t}$ is governed by (2), then according to the large deviations theory for small time diffusion processes [30], with assumptions (3) and (4), $\mathbb{P}(X_{\cdot T} \in \cdot)$ satisfies a sample path large deviation principle on $L^\infty[0,1]$ and the corresponding rate function is given by

$$I(g) = \frac{1}{2}\int_0^1 \left(\frac{g'(t)}{\sigma\left(e^{g(t)}\right)}\right)^2 dt. \tag{34}$$

Notice that the map $g \mapsto \frac{\int_0^1 e^g \bar{h}_b^\epsilon(g)dt}{\int_0^1 \underline{h}_b^\epsilon(g)dt}$ is a.e. continuous and $g \mapsto \frac{\int_0^1 e^g \underline{h}_b^\epsilon(g)dt}{\int_0^1 \bar{h}_b^\epsilon(g)dt}$ is continuous from $L^\infty[0,1]$ to R_+, hence, the extension of contraction principle (Theorem 3) implies that both $\mathbb{P}\left(\frac{\int_0^1 e^{X_{tT}}\bar{h}_b^\epsilon(X_{tT})dt}{\int_0^1 \underline{h}_b^\epsilon(X_{tT})dt} \in \cdot\right)$ and $\mathbb{P}\left(\frac{\int_0^1 e^{X_{tT}}\underline{h}_b^\epsilon(X_{tT})dt}{\int_0^1 \bar{h}_b^\epsilon(X_{tT})dt} \in \cdot\right)$ satisfy the large deviation principles with the rate functions, respectively,

$$\bar{\mathcal{I}}_b^\epsilon(x, S_0) := \inf_{\substack{\frac{\int_0^1 e^g \bar{h}_b^\epsilon(g)dt}{\int_0^1 \underline{h}_b^\epsilon(g)dt}=x\\ g(0)=\log S_0,\, g\in\mathcal{AC}[0,1]}} \frac{1}{2}\int_0^1 \left(\frac{g'(t)}{\sigma\left(e^{g(t)}\right)}\right)^2 dt,$$

$$\underline{\mathcal{I}}_b^\epsilon(x, S_0) := \inf_{\substack{\frac{\int_0^1 e^g \underline{h}_b^\epsilon(g)dt}{\int_0^1 \bar{h}_b^\epsilon(g)dt}=x\\ g(0)=\log S_0,\, g\in\mathcal{AC}[0,1]}} \frac{1}{2}\int_0^1 \left(\frac{g'(t)}{\sigma\left(e^{g(t)}\right)}\right)^2 dt,$$

Hence, one can obtain

$$\limsup_{T\to 0} T\log\mathbb{P}\left(\frac{\int_0^T S_t\mathbb{1}_{\{S_t>b\}}dt}{\int_0^T \mathbb{1}_{\{S_t>b\}}dt} \ge K\right) \le -\inf_{x\ge K}\bar{\mathcal{I}}_b^\epsilon(x, S_0),$$
$$\liminf_{T\to 0} T\log\mathbb{P}\left(\frac{\int_0^T S_t\mathbb{1}_{\{S_t>b\}}dt}{\int_0^T \mathbb{1}_{\{S_t>b\}}dt} \ge K\right) \ge -\inf_{x\ge K}\underline{\mathcal{I}}_b^\epsilon(x, S_0).$$

Thus as $\epsilon \to 0$, for out-of-the money call and put options, i.e., $S_0 < K$, we have

$$\lim_{T\to 0} T\log\mathbb{P}\left(\frac{\int_0^T S_t\mathbb{1}_{\{S_t>b\}}dt}{\int_0^T \mathbb{1}_{\{S_t>b\}}dt} \ge K\right) = -\inf_{x\ge K}\mathcal{I}_b(x, S_0), \tag{35}$$

and

$$\lim_{T\to 0} T\log\mathbb{P}\left(\frac{\int_0^T S_t\mathbb{1}_{\{S_t>b\}}dt}{\int_0^T \mathbb{1}_{\{S_t>b\}}dt} \le K\right) = -\inf_{x\le K}\mathcal{I}_b(x, S_0), \tag{36}$$

where

$$\mathcal{I}_b(x, S_0) = \inf_{\substack{\frac{\int_0^1 e^{g(t)}\mathbb{1}_{\{g(t)>\log b\}}dt}{\int_0^1 \mathbb{1}_{\{g(t)>\log b\}}dt}=x \\ g(0)=\log S_0, g\in\mathcal{AC}[0,1]}} \frac{1}{2}\int_0^1 \left(\frac{g'(t)}{\sigma\left(e^{g(t)}\right)}\right)^2 dt, \tag{37}$$

which completes the proof. □

Proof of Theorem 2 (i) For the ATM call option, namely, $K = S_0$, we have

$$C(T) = e^{-rT}\mathbb{E}\left[\left(\frac{\int_0^T e^{(r-q)t}Y_t\mathbb{1}_{\{Y_t>e^{-(r-q)t}b\}}dt}{\int_0^T \mathbb{1}_{\{Y_t>e^{-(r-q)t}b\}}dt} - S_0\right)^+\right] \tag{38}$$

where $Y_t := \frac{S_t}{e^{(r-q)t}}$ is a martingale and satisfies the the following SDE

$$dY_t = \sigma(Y_t e^{(r-q)t})Y_t dW_t, \quad Y_0 = S_0.$$

Step 1 First, let $a_t = e^{-(r-q)t}b$, we will show that as $T \to 0$, we have the following estimate:

$$\left|\mathbb{E}\left[\left(\frac{\int_0^T e^{(r-q)t}Y_t\mathbb{1}_{\{Y_t>a_t\}}dt}{\int_0^T \mathbb{1}_{\{Y_t>a_t\}}dt} - S_0\right)^+\right] - \mathbb{E}\left[\left(\frac{\int_0^T Y_t\mathbb{1}_{\{Y_t>a_t\}}dt}{\int_0^T \mathbb{1}_{\{Y_t>a_t\}}dt} - S_0\right)^+\right]\right| = O(T). \tag{39}$$

A direct estimation yields

$$\begin{aligned}
&\left|\mathbb{E}\left[\left(\frac{\int_0^T e^{(r-q)t}Y_t\mathbb{1}_{\{Y_t>a_t\}}dt}{\int_0^T \mathbb{1}_{\{Y_t>a_t\}}dt} - S_0\right)^+\right] - \mathbb{E}\left[\left(\frac{\int_0^T Y_t\mathbb{1}_{\{Y_t>a_t\}}dt}{\int_0^T \mathbb{1}_{\{Y_t>a_t\}}dt} - S_0\right)^+\right]\right| \\
&\le \mathbb{E}\left[\left|\left(\frac{\int_0^T e^{(r-q)t}Y_t\mathbb{1}_{\{Y_t>a_t\}}dt}{\int_0^T \mathbb{1}_{\{Y_t>a_t\}}dt} - S_0\right)^+ - \left(\frac{\int_0^T Y_t\mathbb{1}_{\{Y_t>a_t\}}dt}{\int_0^T \mathbb{1}_{\{Y_t>a_t\}}dt} - S_0\right)^+\right|\right] \\
&\le \mathbb{E}\left[\frac{\int_0^T |e^{(r-q)t}-1|Y_t\mathbb{1}_{\{Y_t>a_t\}}dt}{\int_0^T \mathbb{1}_{\{Y_t>a_t\}}dt}\right] \le \mathbb{E}\left[\max_{0\le t\le T}\left(|e^{(r-q)t}-1|Y_t\right)\right] \\
&\le \max_{0\le t\le T}|e^{(r-q)t}-1|\cdot\mathbb{E}\left[\max_{0\le t\le T}Y_t\right] \text{ (by Davis' inequality)} \\
&\le C|e^{(r-q)T}-1|\mathbb{E}\left[\sqrt{\langle Y, Y\rangle(T)}\right] \\
&= C|e^{(r-q)T}-1|\mathbb{E}\left[\sqrt{\int_0^T \sigma^2(Y_t e^{(r-q)t})Y_t^2 dt}\right] \text{ (by Jensen's inequality)} \\
&\le C|e^{(r-q)T}-1|\sqrt{\bar{\sigma}^2 T\mathbb{E}\left[\max_{0\le t\le T}Y_t^2\right]} \text{ (by Theorem 4)} \\
&\le C_1|e^{(r-q)T}-1|\bar{\sigma}\sqrt{T}\sqrt{1+S_0^2},
\end{aligned}$$

where C is constant and C_1 depends only on a positive power of T. Hence, we proved (39).

Next, let us define $\hat{Y}_t$, which satisfies the SDE

$$d\hat{Y}_t = \sigma(S_0)S_0 dW_t, \quad \hat{Y}_0 = S_0.$$

According to the proof of Theorem 6 in [27], one can have

$$\mathbb{E}\left[\max_{0\le t\le T}|Y_t - \hat{Y}_t|\right] = O(T) \quad \text{as } T \to 0. \tag{40}$$

Step 2 Next we show that when $T \to 0$, we have

$$\left|\mathbb{E}\left[\left(\frac{\int_0^T Y_t \mathbb{1}_{\{Y_t>a\}}dt}{\int_0^T \mathbb{1}_{\{Y_t>a\}}dt} - S_0\right)^+\right] - \mathbb{E}\left[\left(\frac{\int_0^T \hat{Y}_t \mathbb{1}_{\{\hat{Y}_t>a\}}dt}{\int_0^T \mathbb{1}_{\{\hat{Y}_t>a\}}dt} - S_0\right)^+\right]\right| = O(T). \tag{41}$$

Let us write $\mu(dt) = \frac{\mathbb{1}_{\{Y_t>a\}}dt}{\int_0^T \mathbb{1}_{\{Y_t>a\}}dt}$ and $\hat{\mu}(dt) = \frac{\mathbb{1}_{\{\hat{Y}_t>a\}}dt}{\int_0^T \mathbb{1}_{\{\hat{Y}_t>a\}}dt}$, one can easily see that

$$\begin{aligned}
&\left|\mathbb{E}\left[\left(\frac{\int_0^T Y_t \mathbb{1}_{\{Y_t>a\}}dt}{\int_0^T \mathbb{1}_{\{Y_t>a\}}dt} - S_0\right)^+\right] - \mathbb{E}\left[\left(\frac{\int_0^T \hat{Y}_t \mathbb{1}_{\{\hat{Y}_t>a\}}dt}{\int_0^T \mathbb{1}_{\{\hat{Y}_t>a\}}dt} - S_0\right)^+\right]\right| \\
&\le \mathbb{E}\left[\left|\int_0^T Y_t\mu(dt) - \int_0^T \hat{Y}_t\hat{\mu}(dt)\right|\right] \\
&\le \mathbb{E}\left[\left|\int_0^T Y_t\mu(dt) - \frac{1}{T}\int_0^T Y_t dt\right|\right] + \mathbb{E}\left[\left|\frac{1}{T}\int_0^T Y_t dt - \frac{1}{T}\int_0^T \hat{Y}_t dt\right|\right] \\
&\quad + \mathbb{E}\left[\left|\frac{1}{T}\int_0^T \hat{Y}_t dt - \int_0^T \hat{Y}_t\hat{\mu}(dt)\right|\right] \\
&= (E_1) + (E_2) + (E_3).
\end{aligned}$$

For (E_1), since we know that when $S_0 > b$, $\frac{\mathbb{1}_{\{Y_t>a\}}}{\int_0^T \mathbb{1}_{\{Y_t>a\}}dt} - \frac{1}{T} \sim O(T)$ a.e. as $T \to 0$,

$$\begin{aligned}
(E_1) &\le \mathbb{E}\left[\int_0^T Y_t \left|\frac{\mathbb{1}_{\{Y_t>a\}}}{\int_0^T \mathbb{1}_{\{Y_t>a\}}dt} - \frac{1}{T}\right| dt\right] \\
&\le \mathbb{E}\left[\max_{0\le t\le T} Y_t \cdot \int_0^T \left|\frac{\mathbb{1}_{\{Y_t>a\}}}{\int_0^T \mathbb{1}_{\{Y_t>a\}}dt} - \frac{1}{T}\right| dt\right] \\
&\le \sqrt{\mathbb{E}\left[\max_{0\le t\le T} Y_t^2\right]} \cdot \sqrt{\mathbb{E}\left[\left(\int_0^T \left|\frac{\mathbb{1}_{\{Y_t>a\}}}{\int_0^T \mathbb{1}_{\{Y_t>a\}}dt} - \frac{1}{T}\right| dt\right)^2\right]} \\
&\le C\sqrt{1+S_0^2} \cdot \sqrt{\int_0^T \mathbb{E}\left[\left(\frac{\mathbb{1}_{\{Y_t>a\}}}{\int_0^T \mathbb{1}_{\{Y_t>a\}}dt} - \frac{1}{T}\right)^2\right] dt}\sqrt{T} \sim O(T).
\end{aligned}$$

The second square root in the last line can be controled by dominated convergence theory. The estimate of (E_3) is the similar.
For (E_2), it is not difficult to see that

$$(E_2) \le \mathbb{E}\left[\max_{0\le t\le T}|Y_t - \hat{Y}_t|\right].$$

Hence, the estimate (41) holds.

Step 3 Finally, we need to compute $\mathbb{E}\Big[\Big(\frac{\int_0^T \hat{Y}_t \mathbb{1}_{\{\hat{Y}_t>a\}}dt}{\int_0^T \mathbb{1}_{\{\hat{Y}_t>a\}}dt} - S_0\Big)^+\Big]$. Note that $\hat{Y}_t = S_0 + \sigma(S_0)S_0 W_t$. Thus one can write $d_t = \frac{e^{-(r-q)t}b - S_0}{\sigma(S_0)S_0}$, when $S_0 > b$, we obtain

$$\begin{aligned}\mathbb{E}\left[\left(\frac{\int_0^T \hat{Y}_t \mathbb{1}_{\{\hat{Y}_t>a_t\}}dt}{\int_0^T \mathbb{1}_{\{\hat{Y}_t>a_t\}}dt} - S_0\right)^+\right] &= \sigma(S_0)S_0\mathbb{E}\left[\left(\frac{\int_0^T W_t \mathbb{1}_{\{W_t>d_t\}}dt}{\int_0^T \mathbb{1}_{\{W_t>d_t\}}dt}\right)^+\right] \\ &= \sigma(S_0)S_0\frac{1}{T}\mathbb{E}\left[\left(\int_0^T W_t dt\right)^+\right] + O(T) \text{ (as } T\to 0) \\ &= \sigma(S_0)S_0\frac{\sqrt{T}}{\sqrt{6\pi}} + O(T) \end{aligned} \tag{42}$$

Therefore, from (39),(41), (42), we can conclude that $C(T) = \sigma(S_0)S_0\frac{\sqrt{T}}{\sqrt{6\pi}} + O(T)$.

(ii) For the ATM put option, the proof is similar to (i) and is omitted here.

□

Proof of Proposition 1 Let $\sigma(S) = \sigma$. For any given g in the domain of the variational problem (11), we denote $c = \sup\{x : x \in (0, 1) \text{ and } g(x) = \log b\}$. Let $g(t) = f(t) + \log S_0$. It is straightforward to see $f(0) = 0$ and $f(c) = \log\frac{b}{S_0}$. Actually c has been defined as the latest time function f hits the value $\log\frac{b}{S_0}$. Since in the variation problem (11), we would like to minimizing the quantity $\int_0^1(f'(t))^2dt$, without of generality, we may assume that f is either increasing or decreasing in $(c, 1)$. For simplicity, we just prove the (i) and (ii) case, another four cases are similar.

(i) When $S_0 < b < K$, the variation problem (11) is equivalent to minimizing the quantity

$$\int_0^1 (f'(t))^2dt \tag{43}$$

subjected to the following constraints

$$f(0) = 0, \quad f(c) = \log\frac{b}{S_0} > 0, \quad \int_c^1 e^{f(t)}dt = \frac{K}{S_0}(1-c).$$

We shall characterize $\inf_f \int_0^1(f'(t))^2dt$ by the following two terms:

$$\int_0^c (f'(t))^2dt \quad \text{and} \quad \int_c^1 (f'(t))^2dt,$$

respectively.

Step 1 First, we analyze the term $\int_0^c(f'(t))^2dt$ where f satisfies the condition $f(0) = 0$, $f(c) = \log\frac{b}{S_0}$. If f minimizes the quantity $\int_0^c(f'(t))^2dt$, then for any function h which vanishes at 0 and c we have

$$\int_0^c ((f+h)'(t))^2dt \geq \int_0^c f'(t)^2dt,$$

which implies that for any h, we should arrive at $\int_0^c f'(t)h'(t)dt = 0$. By using integration by parts and the fact that h vanishes on the boundary 0 and c, we get

$\int_0^c f''(t)h(t)dt = 0$, which implies that f is linear in $(0, c)$. Therefore

$$f = \frac{\log \frac{b}{S_0}}{c} x \quad \text{in} \quad (0, c).$$

Step 2 In this step we focus on $\int_c^1 (f'(t))^2 dt$ subject to the conditions

$$f(c) = \log \frac{b}{S_0}, \quad \int_c^1 e^{f(t)} dt = \frac{K}{S_0}(1 - c).$$

one can see that given the optimal function f subjected to the above constraints, it requires that

$$\frac{1}{2} \int_c^1 ((f + sh)'(t))^2 dt - \lambda \left(\int_c^1 e^{(f+sh)(t)} dt - \frac{K}{S_0}(1 - c) \right)$$

has a critical point at $s = 0$, where λ is the Lagrangian multiplier. It implies that f verifies the following two-point boundary value problem:

$$\begin{cases} f''(t) + \lambda e^{f(t)} = 0, \\ f(c) = \log \frac{b}{S_0}, \quad f'(1) = 0. \end{cases} \tag{44}$$

Multiplying the above equation (44) by f' and integrating by parts, one can see

$$\frac{1}{2} f'(t)^2 + \lambda e^{f(t)} = \text{constant} = \lambda e^{f(1)},$$

which implies that

$$\frac{1}{2} \int_c^1 f'(t)^2 dt = \lambda \left(e^{f(1)} - \frac{K}{S_0} \right) (1 - c).$$

On the other hand, it is not difficult to see that

$$f'(x) = \sqrt{2\lambda(e^{f(1)} - e^{f(x)})}, \quad x \in (c, 1).$$

To proceed, we introduce two functions

$$G^+(f_1) = \int_{\log \frac{b}{S_0}}^{f_1} \sqrt{e^{f_1} - e^y} dy, \quad F^+(f_1) = \int_{\log \frac{b}{S_0}}^{f_1} \frac{1}{\sqrt{e^{f_1} - e^y}} dy,$$

where $f_1 = f(1)$. Then we have

$$\frac{1}{2} \int_c^1 (f')^2 dt = \frac{1}{2} \int_{\log \frac{b}{S_0}}^{f_1} f' df = \sqrt{\frac{\lambda}{2}} \int_{\log \frac{b}{S_0}}^{f_1} \sqrt{e^{f_1} - e^y} \mathrm{d}y = \sqrt{\frac{\lambda}{2}} G^+(f_1),$$

and

$$1 - c = \int_c^1 dt = \int_{\log \frac{b}{S_0}}^{f_1} \frac{1}{f'} df = \sqrt{\frac{1}{2\lambda}} \int_{\log \frac{b}{S_0}}^{f_1} \frac{1}{\sqrt{e^{f_1} - e^y}} \mathrm{d}y = \sqrt{\frac{1}{2\lambda}} F^+(f_1).$$

Combining above two equations, we have

$$\min_{\substack{f \in \mathcal{AC}[c,1], f(c) = \log \frac{b}{S_0} > 0, \\ \int_c^1 e^{f(t)} dt = \frac{K}{S_0}(1-c).}} \int_c^1 (f')^2 dt = G^+(f_1) F^+(f_1) / (1 - c), \tag{45}$$

where f_1 satisfies

$$e^{f_1} - \frac{K}{S_0} = G^+(f_1)/F^+(f_1). \tag{46}$$

Step 3 By plugging (45) into the integration formula (43), we have

$$\min_{\substack{f\in\mathcal{AC}[0,1],\int_c^1 e^{f(t)}dt=\frac{K}{S_0}(1-c)\\ f(0)=0,f(c)=\log\frac{b}{S_0}>0.}} \int_0^1 (f')^2 dt = \frac{\left(\log\frac{b}{S_0}\right)^2}{c} + \frac{G^+(f_1)F^+(f_1)}{1-c}.$$

Then we can view it as a function depend on c only, after taking derivative we have

$$-\frac{\left(\log\frac{b}{S_0}\right)^2}{c^2} + \frac{G^+(f_1)F^+(f_1)}{(1-c)^2} = 0. \tag{47}$$

The two real roots are given by (47)

$$c_\pm = \frac{-\left(\log\frac{b}{S_0}\right)^2 \pm \sqrt{\left(\log\frac{b}{S_0}\right)^2 G^+(f_1)F^+(f_1)}}{G^+(f_1)F^+(f_1) - \left(\log\frac{b}{S_0}\right)^2}.$$

And one can easily verify that $c_+ = \frac{\log\frac{b}{S_0}}{\log\frac{b}{S_0}+\sqrt{G^+(f_1)F^+(f_1)}} \in [0,1]$. Hence, we have

$$\min_{\substack{f\in\mathcal{AC}[0,1],\int_c^1 e^{f(t)}dt=\frac{K}{S_0}(1-c)\\ f(0)=0,f(c)=\log\frac{b}{S_0}>0.}} \int_0^1 (f')^2 dt = \left(\log\frac{b}{S_0} + \sqrt{G^+(f_1)F^+(f_1)}\right)^2.$$

(ii) For $S_0 > b > K$, the variation problem (11) is to minimize the quantity

$$\int_0^1 (f'(t))^2 dt \tag{48}$$

subjected to the following condition

$$f(0) = 0, \quad f(c) = \log\frac{b}{S_0} < 0, \quad \int_c^1 e^{f(t)}dt = \frac{K}{S_0}(1-c).$$

Due to $f(c) = \log\frac{b}{S_0} < 0$, use the similar approach for case (i), we have

$$\min_{\substack{f\in\mathcal{AC}[c,1],f(c)=\log\frac{b}{S_0}<0,\\ \int_c^1 e^{f(t)}dt=\frac{K}{S_0}(1-c).}} \int_c^1 (f')^2 dt = G^-(g_1)F^-(g_1)/(1-c), \tag{49}$$

where

$$G^-(g_1) = \int_{g_1}^{\log\frac{b}{S_0}} \sqrt{e^y - e^{g_1}}\,dy, \quad F^-(g_1) = \int_{g_1}^{\log\frac{b}{S_0}} \frac{1}{\sqrt{e^y - e^{g_1}}}dy.$$

and $g_1 = f(1) \le 0$ satisfies

$$\frac{K}{S_0} - e^{g_1} = G^-(g_1)/F^-(g_1). \tag{50}$$

Then we can obtain that optimal $c = \frac{\log \frac{b}{S_0}}{\log \frac{b}{S_0} - \sqrt{G^-(g_1)F^-(g_1)}} \in [0, 1]$ and then

$$\min_{\substack{f \in \mathcal{AC}[0,1], \int_c^1 e^{f(t)}dt = \frac{K}{S_0}(1-c) \\ f(0)=0, f(c)=\log \frac{b}{S_0} < 0.}} \int_0^1 (f')^2 dt = \left(\log \frac{b}{S_0} - \sqrt{G^-(g_1)F^-(g_1)}\right)^2 .$$

Therefore, the proof of is completed. □

Proof of Proposition 2 From Proposition 1, we know that the rate function $\mathcal{I}_b(K, S_0)$ is given by

$$\mathcal{I}_b(K, S_0) = \begin{cases} \frac{\left(\log \frac{b}{S_0} + \sqrt{G^+(f_1)F^+(f_1)}\right)^2}{2\sigma^2}, & S_0 < b < K \text{ and } S_0 < K < b. \\ \frac{\left(\log \frac{b}{S_0} - \sqrt{G^+(f_1)F^+(f_1)}\right)^2}{2\sigma^2}, & b < S_0 < K. \\ \frac{\left(\log \frac{b}{S_0} - \sqrt{G^-(g_1)F^-(g_1)}\right)^2}{2\sigma^2}, & K < b < S_0 \text{ and } b < K < S_0. \\ \frac{\left(\log \frac{b}{S_0} + \sqrt{G^-(g_1)F^-(g_1)}\right)^2}{2\sigma^2}, & K < S_0 < b. \end{cases} \tag{51}$$

It is straightforward to see that the monotonicity in K of rate function $\mathcal{I}_b(K, S_0)$ is the same as $\sqrt{G^+(f_1)F^+(f_1)}$ and $\sqrt{G^-(g_1)F^-(g_1)}$ respectively. Using the similar proof given by Proposition 9 in [27], we know that

Case I When $K > S_0$.

(1) For $S_0 < b < K$.

$$G^+(f_1)F^+(f_1) = \inf_{\varphi > K/S_0} \frac{\left(\int_{b/S_0}^{\varphi} \frac{\sqrt{\varphi - z}}{z} dz\right)^2}{\varphi - \frac{K}{S_0}}. \tag{52}$$

(2) For $S_0 < K < b$.

$$G^+(f_1)F^+(f_1) = \inf_{\varphi > b/S_0} \frac{\left(\int_{b/S_0}^{\varphi} \frac{\sqrt{\varphi - z}}{z} dz\right)^2}{\varphi - \frac{K}{S_0}}. \tag{53}$$

(3) For $b < S_0 < K$.

$$G^+(f_1)F^+(f_1) = \inf_{\varphi > K/S_0} \frac{\left(\int_{b/S_0}^{\varphi} \frac{\sqrt{\varphi - z}}{z} dz\right)^2}{\varphi - \frac{K}{S_0}}. \tag{54}$$

Case II When $K < S_0$.

(1) For $b < K < S_0$.

$$G^-(g_1)F^-(g_1) = \inf_{\varphi < b/S_0} \frac{\left(\int_{\varphi}^{b/S_0} \frac{\sqrt{z - \varphi}}{z} dz\right)^2}{\frac{K}{S_0} - \varphi}. \tag{55}$$

(2) For $K < b < S_0$.

$$G^-(g_1)F^-(g_1) = \inf_{\varphi < K/S_0} \frac{\left(\int_{\varphi}^{b/S_0} \frac{\sqrt{z - \varphi}}{z} dz\right)^2}{\frac{K}{S_0} - \varphi}. \tag{56}$$

(3) For $K < S_0 < b$.

$$G^{-}(g_1)F^{-}(g_1) = \inf_{\varphi < K/S_0} \frac{\left(\int_{\varphi}^{b/S_0} \frac{\sqrt{z-\varphi}}{z} dz\right)^2}{\frac{K}{S_0} - \varphi}. \tag{57}$$

□

4 Concluding remarks

The conditional Asian option is a recent market product, offering a cheaper and new alternative to the regular Asian option. But it is more challenging to study than the regular Asian option. Similar to the short maturity European option, the study of short maturity Asian option plays an important role in today's financial market in terms of risk control. The conditional Asian option, aiming at reducing the volatility and offering cheaper prices, is more difficult to be characterized through its price due to its inherent complexity. In this paper, by using the large deviation theory, we establish the short maturity asymptotics for the conditional Asian option under a local volatility model, allowing us to approximate the dominated term directly. Furthermore, we develop a new approach for conditional Asian options under the Black–Scholes model through the explicit expression of the corresponding rate function. The obtained theoretical results render us to be able to obtain suitable approximations for the conditional Asian options instead of relying on Monte Carlo simulations or other numerical methods, which is one of the main contributions of the paper.

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