

Sensitivity analysis of linear continuous-time feedback systems subject to control and measurement noise: An information-theoretic approach[☆]

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ABSTRACT

Sensitivity of linear continuous-time SISO feedback systems, subject to control and measurement noise, is analyzed by deriving the lower bounds of Bode-like integrals via an information-theoretic approach. Bode integrals of four different sensitivity-like functions are employed to gauge the performance limitations of feedback systems. When the signals of the control system are stationary Gaussian, these four different Bode-like integrals can be represented as the differences between mutual information rates. These mutual information rates and hence the corresponding Bode-like integrals are proven to be bounded below by the unstable poles and zeros of the plant model, if the signals of the control system are wide-sense stationary.

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1. Introduction

Stabilization of systems subject to external disturbances and achieving desired level of performance have been the objective of feedback synthesis since its inception [1–6]. With the visible progress of information technologies and their applications to feedback control systems over the last two decades, a great deal of attention has been given to understanding the fundamental limitations of closed-loop systems in the presence of communication channels, [7–11]. The main contribution of these papers was to explore the performance limitations of stochastic systems in the presence of limited information. While [7–9,12] investigated the Bode-like integrals for discrete-time systems by using Kolmogorov's entropy-rate equality [13], the results in [10] put forward an approach to explore the continuous-time systems by resorting to mutual information rates. In [8–10], sensitivity-like functions were introduced to define the Bode-like integrals, which can be regarded as a generalization of the classical Bode integrals from the deterministic LTI systems to stochastic nonlinear systems. Previous results based on complex analysis for deterministic LTI systems have shown that the lower bounds of

the Bode integrals are determined by the unstable poles and zeros of plant models [1,14–17]. Seminal results on this topic were also reported in [18–22].

Performance limitations of stochastic systems in the presence of limited information were analyzed through the sensitivity-like functions [8–10,23–25], which are defined by the power spectral densities (PSDs) of signals. Taking an information-theoretic approach was the key to get the Bode integrals extended to stochastic nonlinear systems. Unlike the frequency-domain approach, which explicitly depends on the input–output relationship of the feedback system (transfer function), the focus of the information-theoretic approach is on the signals. The lower bound for sensitivity Bode-like integral in stochastic continuous-time systems was first reported in [10]. This result can be applied to systems with nonlinear controllers, which is an improvement upon the prior results based on the frequency-domain approaches [1,3,14–17,19,26]. Nevertheless, to the best of authors' knowledge, the lower bounds for complementary sensitivity, load disturbance sensitivity, and noise sensitivity Bode-like integrals have rarely been investigated in stochastic continuous-time single-input-single-output (SISO) systems. While the lower bounds of these four Bode-like integrals were studied in [9] for discrete-time feedback systems, it is still challenging to derive their counterpart results for continuous-time systems. The unboundedness of continuous-time sensitivity-like functions in high frequencies [26] as well as the challenges in information-theoretic representations of weighted Bode-like integrals have been the main obstacles on this path.

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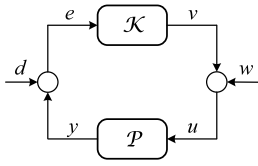


Fig. 1. A SISO feedback system.

In this paper, a comprehensive sensitivity analysis of stochastic continuous-time SISO feedback systems subject to control and measurement noise is investigated by deriving the lower bounds for four different Bode-like integral via an information-theoretic approach. Among these Bode-like integrals, load disturbance sensitivity and noise sensitivity Bode-like integrals have seldom been studied for continuous-time systems, and more detailed results and interpretations on complementary sensitivity Bode-like integral are presented in this paper compared with [27]. In order to constrain the values of noise sensitivity-like and complementary sensitivity-like functions in high frequencies, the corresponding Bode-like integrals are defined with a weight function, $1/\omega^2$. A frequency transformation and the lower bound for sensitivity Bode-like integral derived in [10] are utilized to derive the lower bounds for these weighted Bode-like integrals. Meanwhile, when seeking the lower bounds of Bode-like integrals for continuous-time systems, since Kolmogorov's entropy-rate equality, which was widely employed for discrete-time systems [8,9,24], is no longer applicable, we resort to a seminal lemma on mutual information rates [28, p. 181]. With this lemma, we prove that when signals are stationary Gaussian, the lower bounds of Bode-like integrals can be represented as the differences between mutual information rates. Furthermore, when signals are only wide-sense stationary, we show that the lower bounds of these mutual information rates and hence the lower bounds of Bode-like integrals are determined by the unstable poles and zeros of the plant model. We also give the relationship between Bode integrals and Bode-like integrals for stochastic linear continuous-time SISO systems, which complements previous investigations for discrete-time systems [8,9].

The paper is organized as follows: Section 2 introduces preliminary results and defines the Bode-like integrals for continuous-time systems; Section 3 derives the lower bounds for these Bode-like integrals in terms of mutual information rates; Section 4 shows that these lower bounds are further bounded below by the unstable poles and zeros of the plant model; and Section 5 concludes the paper.

Notations. The notations used throughout this paper are defined as follows. $x(t)$ represents a continuous-time stochastic process with $x_{t_1}^{t_2}$ indicating a sample path on an interval $[t_1, t_2] \subset \mathbb{R}^+$ and $x^t := x_0^t$. $x(k)$ denotes a discrete-time stochastic process with x_m^n indicating the segment $\{x(k)\}_{k=m}^n$, $m < n \in \mathbb{N}$ and $x_0^n := x^n$. $x^{(\delta)}$ denotes the discrete-time process obtained from sampling of $x(t)$ with an interval $\delta > 0$ with $x_i^{(\delta)} = x^{(\delta)}(i) := x(t_0 + i\delta)$, $i \in \mathbb{N}$. We also use the notations $\mathbb{E}[\cdot]$ for expectation, $h(x)$ for Shannon differential entropy, $I(\cdot; \cdot)$ for mutual information, and $I_\infty(\cdot; \cdot)$ for mutual information rate. The logarithmic function $\log(\cdot)$ in this paper assumes the basis e by default. For a matrix M , $|M|$ denotes its determinant. Complex variable $s = j\omega$.

2. Preliminaries and problem formulation

Consider a stochastic continuous-time SISO feedback system shown in Fig. 1, where $\mathcal{P}$ is the plant model, $\mathcal{K}$ denotes the causal feedback control mapping, and $d \in \mathbb{R}$ and $w \in \mathbb{R}$ respectively represent the

noise over measurement and control channels. In classical control theory, with zero initial condition, the linear plant $\mathcal{P}$ can be described by the following transfer function

$$G(s) = c \cdot \frac{\prod_{i=1}^m (s - z_i)}{\prod_{i=1}^n (s - p_i)}, \quad (1)$$

where $c \in \mathbb{R}$, $m \leq n$ so that the system is causal, and z_i and p_i denote the zeros and poles of plant $\mathcal{P}$, respectively. When the control mapping $\mathcal{K}$ is linear, we use $C(s)$ to denote its transfer function, which has a similar form as (1). In this scenario, the four important transfer functions for sensitivity analysis are

$$\begin{aligned} T_{uw}(j\omega) &= \frac{1}{1 + G(s)C(s)}, & T_{yw}(j\omega) &= \frac{G(s)}{1 + G(s)C(s)}, \\ T_{ud}(j\omega) &= \frac{C(s)}{1 + G(s)C(s)}, & T_{yd}(j\omega) &= \frac{G(s)C(s)}{1 + G(s)C(s)}, \end{aligned} \quad (2)$$

where T_{uw} , T_{yw} , T_{ud} and T_{yd} respectively denote the sensitivity, load disturbance sensitivity, noise sensitivity, and complementary sensitivity functions and are referred to as *Gang of Four* in [29]. For brevity, we also use the notation $L(s) = G(s)C(s)$ in the following context. The integral of sensitivity function $T_{uw}(j\omega)$ over all frequencies, $\frac{1}{2\pi} \int_{-\infty}^{\infty} \log |T_{uw}(j\omega)| d\omega$, is known as the Bode's integral [1,19], which is a critical index characterizing the performance limitations of feedback systems subject to noise. However, simply replacing the $T_{uw}(j\omega)$ in preceding integral with the other three sensitivity functions will not give us a new index as significant as the Bode's integral, due to the unboundedness of their logarithmic functions when $s \rightarrow \infty$ and the absence of a trade-off equality, similar to $T_{uw}(s) + T_{yd}(s) = 1$, between $T_{yw}(j\omega)$ and $T_{ud}(j\omega)$. A compromised solution dealing with these difficulties is to multiply the logarithmic integrands by a weight function, such as $1/\omega^2$ [17] or a Poisson-type kernel function [19].

In stochastic setting, the linear plant model $\mathcal{P}$ in Fig. 1 can be described by the following state-space model

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx, \end{cases} \quad (3)$$

where x is the state vector, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are the input and output of $\mathcal{P}$. Throughout this paper, we assume that the initial states x_0 of stochastic systems are unknown but have finite entropy. The initial state x_0 , measurement noise d , and control noise w in Fig. 1 are assumed to be mutually independent and zero-mean Gaussian. A specific discussion on different initial conditions for deterministic systems and stochastic systems is given in [9]. Derived from the sensitivity functions (2), the sensitivity-like functions in terms of signals' PSDs were adopted by later researchers to analyze the sensitivity properties of stochastic nonlinear systems via information-theoretic methods [9,10,23]:

$$\begin{aligned} T_{uw}(\omega) &= \sqrt{\frac{\phi_u(\omega)}{\phi_w(\omega)}}, & T_{yw}(\omega) &= \sqrt{\frac{\phi_y(\omega)}{\phi_w(\omega)}}, \\ T_{ud}(\omega) &= \sqrt{\frac{\phi_u(\omega)}{\phi_d(\omega)}}, & T_{yd}(\omega) &= \sqrt{\frac{\phi_y(\omega)}{\phi_d(\omega)}}, \end{aligned} \quad (4)$$

where each pair of signals is stationary and stationary correlated, $\phi_u(\omega)$ denotes the PSD of stationary signal $u(t)$ with

$$\phi_u(\omega) = \int_{-\infty}^{\infty} r_u(\tau) \cdot e^{-j\omega\tau} d\tau. \quad (5)$$

$r_u(\tau) = r_{uu}(t + \tau, t)$ is the auto-covariance of signal u . $r_{uw}(\tau) = r_{uw}(t + \tau, t) = \text{Cov}[u(t + \tau), w(t)]$ denotes the covariance between signals u and w , and ϕ_{uw} is defined by replacing $r_u(\tau)$ with $r_{uw}(\tau)$ in (5). In order to differentiate the sensitivity functions and sensitivity-like functions, the former were defined with argument

$j\omega$ in (2), while the latter were defined with argument ω in (4). Inspired by the weight function adopted in [17] when defining the complementary sensitivity Bode integral, in this paper we are interested in seeking the lower bounds for the Bode-like integrals defined as follows

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} \log T_{uw}(\omega) d\omega, & \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} \log T_{yw}(\omega) d\omega, \\ \frac{1}{2\pi} \int_{-\infty}^{\infty} \log T_{ud}(\omega) \frac{d\omega}{\omega^2}, & \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} \log T_{yd}(\omega) \frac{d\omega}{\omega^2}. \end{aligned} \quad (6)$$

Remark 1. In this paper, the integrals of logarithmic sensitivity functions defined in (2) are referred to as Bode integrals, while the integrals of logarithmic sensitivity-like functions defined in (4) are named as Bode-like integrals. As Lemma 2 in this paper will reveal, the Bode integrals and the Bode-like integrals are equal when signals are stationary Gaussian. Hence, according to [17] on complementary sensitivity Bode integral, the weight function $1/\omega^2$ should also refine the boundedness of the complementary sensitivity Bode-like integral defined in (6). Meanwhile, the definitions of load disturbance sensitivity and noise sensitivity Bode-like integrals are respectively inherited from the definitions of sensitivity and complementary sensitivity Bode-like integrals, which preserves some nice properties when deriving the lower bounds. Nonetheless, the absence of a trade-off identity, similar to $T_{uw} + T_{yd} = 1$, between T_{yw} and T_{ud} and the involvement of weight function $1/\omega^2$ generate additional challenges and constraints.

Before we proceed to discuss the lower bounds of Bode-like integrals, some basic definitions, properties, and lemmas are given in the remaining part of this section. Motivated by the frequency inversion adopted in [17], consider the following frequency transformation

$$\tilde{s} = j\tilde{\omega} = (j\omega)^{-1} = s^{-1}, \quad (7)$$

where the frequencies satisfy $\tilde{\omega} = -(\omega)^{-1}$. Applying (7) to the plant model $G(s)$ in (1), we refer to the resulting system as the *auxiliary system* and write its transfer function as

$$\tilde{G}(\tilde{s}) = c \cdot \tilde{s}^{n-m} \cdot \frac{\prod_{i=1}^m (1 - z_i \cdot \tilde{s})}{\prod_{i=1}^n (1 - p_i \cdot \tilde{s})} = G(\tilde{s}^{-1}) = G(s). \quad (8)$$

The Laplace transforms of signal u in the original system and signal $\tilde{u}$ in the auxiliary system satisfy $U(s) = U(\tilde{s}^{-1}) = \int_{-\infty}^{\infty} u(\tau) \cdot e^{-\tau \tilde{s}^{-1}} d\tau = \tilde{U}(\tilde{s})$, where signal u can be replaced by other signals shown in Fig. 1. When the control mapping $\mathcal{K}$ is linear, we use the notation $\tilde{L}(\tilde{s}) = \tilde{G}(\tilde{s})\tilde{C}(\tilde{s})$, with $\tilde{C}(\tilde{s})$ being the frequency inversion of $C(s)$. Compared with the frequency domain of s , it is more convenient to represent and compute the Bode-like integrals weighted by $1/\omega^2$ in the domain of $\tilde{s}$. However, since the auxiliary system (8) is not realizable when $n - m > 0$ or some $p_i = 0$, we consider the inverse system $(\tilde{\mathcal{P}}^{-1}, \tilde{\mathcal{K}}^{-1})$ in (9) obtained by swapping the input and output of auxiliary system.

$$\begin{aligned} \tilde{G}^{-1}(\tilde{s}) &= \frac{1}{c} \cdot \frac{\prod_{i=1}^n (1 - \tilde{s} \cdot p_i)}{\tilde{s}^{n-m} \cdot \prod_{i=1}^m (1 - \tilde{s} \cdot z_i)} \\ &= \frac{\prod_{i=1}^n (-p_i)}{c \cdot \prod_{i=1}^m (-z_i)} \cdot \frac{\prod_{i=1}^n (\tilde{s} - p_i^{-1})}{\tilde{s}^{n-m} \cdot \prod_{i=1}^m (\tilde{s} - z_i^{-1})}. \end{aligned} \quad (9)$$

When $p_i = 0$, the corresponding $(\tilde{s} - p_i^{-1})$ term vanishes, and the plant model (9) should always be proper with respect to $\tilde{s}$. The signals and models of inverse system satisfy $\tilde{E}(\tilde{s}) = \tilde{G}^{-1}(\tilde{s})\tilde{V}(\tilde{s})$ and $\tilde{U}(\tilde{s}) = \tilde{C}^{-1}(\tilde{s})\tilde{Y}(\tilde{s})$, where $\tilde{C}^{-1}(\tilde{s})$ is obtained by following the same steps as $\tilde{G}^{-1}(\tilde{s})$ in (8) and (9). A minimal realization of $\tilde{G}^{-1}(\tilde{s})$ is described by

$$\begin{cases} \dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}\tilde{v}, \\ \tilde{e} = \tilde{C}\tilde{x}. \end{cases} \quad (10)$$

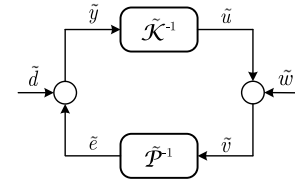


Fig. 2. Inverse system.

A block diagram of the inverse system is given in Fig. 2.

Some definitions and properties employed in this paper are stated below, while more information on these topics can be found in [10,11,13,30–32].

Definition 1 (Wide Sense Stationary). A second order random process $x(t)$ is called wide sense stationary, if $\mathbb{E}[x(t)] = \mathbb{E}[x(t+v)]$ and $\text{Cov}[x(t), x(t+\tau)] = \text{Cov}[x(v), x(v+\tau)]$.

Definition 2 (Mean-Square Stability). A closed loop system is said to be mean-square stable, if the state $x(t)$ satisfies $\sup_{t \geq 0} \mathbb{E}[x^T(t)x(t)] < \infty$.

Definition 3 (Class $\mathbb{F}$ Function: See [10] or [28, p. 182]). We define class $\mathbb{F}$ function in the following way: $\mathbb{F} = \{l : l(\omega) = p(\omega)(1 - \phi(\omega)), l(\omega) \in \mathbb{C}, \omega \in \mathbb{R}\}$, where $p(\cdot)$ is rational and $\phi(\cdot)$ is a measurable function, such that $0 \leq \phi \leq 1$ for all $\omega \in \mathbb{R}$ and $\int_{\mathbb{R}} |\log(1 - \phi(\omega))| d\omega < \infty$.

Definition 4 (Shannon Differential Entropy: See [13]). For a continuous random variable x with density $f(x)$, its Shannon differential entropy is defined as $h(x) = -\int_S f(x) \log f(x) dx$, where S is the support set of the random variable.

Definition 5 (Mutual Information & Mutual Information Rate: See [31]). The mutual information between two random variables x and y is defined as $I(x; y) = \int \log[d\mu(x, y)/d(\mu(x) \times \mu(y))] d\mu(x, y)$, where $d\mu(x, y)/d(\mu(x) \times \mu(y))$ is the Radon–Nikodym derivative of the joint probability measure $\mu(x, y)$ with respect to the product probability measure $\mu(x) \times \mu(y)$. The mutual information rate is defined as $I_\infty(x; y) = \lim_{t \rightarrow \infty} I(x^t; y^t)/t$.

Property 1 (See [10]). For two continuous-time random processes x^t and y^t , the mutual information between $x_{t_1}^{t_2}$ and $y_{t_1}^{t_2}$, $0 \leq t_1 < t_2 < \infty$, can be obtained as $I(x_{t_1}^{t_2}; y_{t_1}^{t_2}) = \lim_{k \rightarrow \infty} I(x_0^{(\delta(k))}, \dots, x_k^{(\delta(k))}; y_0^{(\delta(k))}, \dots, y_k^{(\delta(k))})$ for any fixed t_1 and t_2 with $x_i^{(\delta(k))} = x(t_1 + i\delta(k))$ and $\delta(k) = (t_2 - t_1)/(k + 1)$.

Property 2 (See [11,13,31]). For a pair of random variables x and y , we have $h(x|y) = h(x+g(y)|y)$ and $I(x; y) = h(x) - h(x|y) = h(y) - h(y|x) = I(y; x)$, where $g(\cdot)$ is a measurable function.

Property 3 (See [30, p. 660]). For continuous random variables, $x_1, \dots, x_n$, if the transformation $y_i = g_i(x_1, \dots, x_n)$ has a unique inverse, then $h(y_1, \dots, y_n|z_1, z_2, \dots, z_m) = h(x_1, \dots, x_n|z_1, z_2, \dots, z_m) + \mathbb{E}[\log |J(x_1, \dots, x_n)|]$, where $J(x_1, \dots, x_n)$ is the Jacobian matrix of the transformation and expectation is taken with respect to the conditional distribution.

Property 4 (Maximum Entropy: See [10,31]). For a random vector $x \in \mathbb{R}^n$ with covariance matrix Σ_x , we have $h(x) \leq h(x_G) = 1/2 \cdot \log((2\pi e)^n \cdot |\Sigma_x|)$, where x_G is Gaussian with the same covariance as x .

Property 5 (See [10] or [28, p. 181]). Suppose that two one-dimensional continuous-time random processes $x(t)$ and $y(t)$ form a stationary Gaussian process (x, y) , and ϕ_x and ϕ_y belong to class $\mathbb{F}$. Then, $I_\infty(x; y) = -(4\pi)^{-1} \int_{-\infty}^{\infty} \log\{1 - |\phi_{xy}(\omega)|^2 / [\phi_x(\omega)\phi_y(\omega)]\} d\omega$.

The following lemmas present a preliminary result on the sensitivity Bode-like integral and the relationship between Bode integrals and Bode-like integrals for linear continuous-time systems.

Lemma 1 (See Theorem 4.8 in [10]). When a continuous-time SISO feedback system, shown as Fig. 1, is mean-square stable, we have $I_\infty(u; v) - I_\infty(w; v) \geq \sum_{p_i \in \mathcal{UP}} p_i$, where $\mathcal{UP}$ denotes the set of unstable poles in plant $\mathcal{P}$. If (u, v) and (w, v) form stationary Gaussian processes, we then have $\frac{1}{2\pi} \int_{-\infty}^{\infty} \log |T_{uw}(\omega)| d\omega \geq \sum_{\lambda \in \mathcal{UP}} p_i$.

Lemma 2. When the plant model $\mathcal{P}$ and the controller $\mathcal{K}$ are linear, and the control noise $w(t)$ is wide sense stationary, the Bode integrals and the Bode-like integrals satisfy

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \log T_{uw}(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \log |T_{uw}(j\omega)| d\omega, \quad (11a)$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \log T_{yw}(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \log |T_{yw}(j\omega)| d\omega. \quad (11b)$$

When the measurement noise $d(t)$ is wide sense stationary, the Bode integrals and the Bode-like integrals satisfy

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \log T_{ud}(\omega) \frac{d\omega}{\omega^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \log |T_{ud}(j\omega)| \frac{d\omega}{\omega^2}, \quad (11c)$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \log T_{yd}(\omega) \frac{d\omega}{\omega^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \log |T_{yd}(j\omega)| \frac{d\omega}{\omega^2}. \quad (11d)$$

Proof. The proof of Lemma 2 is given in Appendix A. $\square$

Remark 2. Since Lemma 2 implies that the magnitudes of Bode integrals equal the magnitudes of the corresponding Bode-like integrals when disturbances are wide sense stationary, it is reasonable to infer that the lower bounds of Bode-like integrals defined in (6) should be identical to the lower bounds of Bode integrals, despite the difference of initial conditions, [9], and some of these lower bounds for Bode integrals have been derived in [1,17,19]. Moreover, when signals are not stationary and the Bode-like integrals are not defined, we can still resort to mutual information rates to describe the performance limitations of stochastic continuous-time systems as in [10,23].

In the following sections, we first discuss the information-theoretic representations of the Bode-like integrals defined in (6), and then we derive the lower bounds for these performance limitations.

3. Information-theoretic representations of Bode-like integrals

When signals in feedback systems are stationary Gaussian, we show that the Bode-like integrals (6) can be bounded below by the difference between two mutual information rates. This information-theoretic representation of Bode-like integrals not only enables to derive the lower bounds of Bode-like integrals with tools from information theory, but provides with an alternative metric to measure the performance limitations of stochastic continuous-time systems when Bode-like integrals are undefined, i.e. the stationary assumption fails to hold. An assumption on the sample path of stochastic system (3) is stated before we show the information-theoretic representation for sensitivity and load disturbance sensitivity Bode-like integrals in Theorem 1.

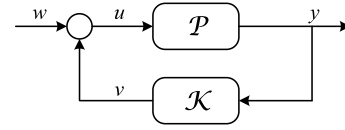


Fig. 3. General feedback system with control noise.

Assumption 1. In the limit of $k \rightarrow \infty$ or $\delta(k) \rightarrow 0$, the input sequence $[u^{(\delta(k))}]^i$ converges to a continuous-time stochastic process $u(t)$, and the output $y_i^{(\delta(k))}$ of stochastic system (3) at time $i\delta(k)$ is an injective function of control inputs $[u^{(\delta(k))}]^i$ and initial condition x_0 , i.e. $\lim_{k \rightarrow \infty} y_i^{(\delta(k))} = \lim_{k \rightarrow \infty} g_i([u^{(\delta(k))}]^i, x_0)$.

Remark 3. This assumption creates a condition such that Property 3 can be applied to continuous-time stochastic processes after proper discretizations. The premise of convergent input sequence casts a constraint on the discretization in Property 1 and was previously adopted in [33]. Some accurate approximations of $y_i^{(\delta(k))}$ from $[u^{(\delta(k))}]^i$ and x_0 as $\delta(k) \rightarrow 0$ have been investigated in [34].

Theorem 1. For a stochastic linear continuous-time SISO feedback system subject to control noise w , when (u, v) and (w, v) form stationary processes, ϕ_u , ϕ_v , and $\phi_w \in \mathbb{F}$, and w is a stationary Gaussian process, the sensitivity Bode-like integral satisfies

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \log T_{uw}(\omega) d\omega \geq I_\infty(u; v) - I_\infty(w; v), \quad (12)$$

and the load disturbance sensitivity Bode-like integral satisfies

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} \log T_{yw}(\omega) d\omega &\geq I_\infty(y; v) - I_\infty(w; v) \\ &+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \log |G(j\omega)| d\omega. \end{aligned} \quad (13)$$

Proof. The block diagram of a continuous-time SISO feedback system subject to control noise $w(t)$ is illustrated in Fig. 3.

The first inequality in Theorem 1 has been proved as Theorem 4.8 in [10]. Thus, only the proof of inequality (13) is provided here. When y_G , u_G , and v_G respectively denote the Gaussian processes with the same covariances as the random processes y , u , and v , the following inequality holds

$$\begin{aligned} &I(y^t; v^t) - I(w^t; v^t) \\ &\stackrel{(a)}{=} \lim_{k \rightarrow \infty} \{I([y^{(\delta(k))}]^k; [v^{(\delta(k))}]^k) - I([w^{(\delta(k))}]^k; [v^{(\delta(k))}]^k)\} \\ &\stackrel{(b)}{=} \lim_{k \rightarrow \infty} \left\{ h([y^{(\delta(k))}]^k) - h([u^{(\delta(k))}]^k | [v^{(\delta(k))}]^k) \right. \\ &\quad - \mathbb{E}[\log |J_{[u^{(\delta(k))}]^k}([y^{(\delta(k))}]^k)|] \\ &\quad \left. - h([w^{(\delta(k))}]^k) + h([w^{(\delta(k))}]^k | [v^{(\delta(k))}]^k) \right\} \\ &\stackrel{(c)}{\leq} \lim_{k \rightarrow \infty} \left\{ h([y_G^{(\delta(k))}]^k) - h([w^{(\delta(k))}]^k) - \mathbb{E}[\log |J_{[u^{(\delta(k))}]^k}([y^{(\delta(k))}]^k)|] \right\} \\ &\stackrel{(d)}{=} \lim_{k \rightarrow \infty} \left\{ h([y_G^{(\delta(k))}]^k) - h([u_G^{(\delta(k))}]^k | [v_G^{(\delta(k))}]^k) \right. \\ &\quad - \mathbb{E}[\log |J_{[u_G^{(\delta(k))}]^k}([y_G^{(\delta(k))}]^k)|] \\ &\quad \left. - h([w^{(\delta(k))}]^k) + h([w^{(\delta(k))}]^k | [v_G^{(\delta(k))}]^k) \right\} \\ &\stackrel{(e)}{=} I(y_G^t; v_G^t) - I(w^t; v_G^t). \end{aligned} \quad (14)$$

(a) follows from Property 1; (b) is ensured by Properties 2, 3, and Assumption 1, with $J_{[u^{(\delta(k))}]^k}([y^{(\delta(k))}]^k)$ being the Jacobian matrix of $[y^{(\delta(k))}]^k$ with respect to $[u^{(\delta(k))}]^k$, and the one-step delay is omitted here for brevity [8,10]; (c) employs Property 4

and identity $h([u^{(\delta(k))}]^k | [v^{(\delta(k))}]^k) = h([w^{(\delta(k))}]^k - [v^{(\delta(k))}]^k | [v^{(\delta(k))}]^k) = h([w^{(\delta(k))}]^k | [v^{(\delta(k))}]^k)$ derived by [Property 2](#); (d) uses the identities $h([u_G^{(\delta(k))}]^k | [v_G^{(\delta(k))}]^k) = h([w^{(\delta(k))}]^k | [v_G^{(\delta(k))}]^k)$ derived by [Property 2](#) and $|[u^{(\delta(k))}]^k| = |[u_G^{(\delta(k))}]^k| + |[y_G^{(\delta(k))}]^k|$; and (e) follows the same arguments in steps (a) and (b). Dividing both sides of (14) by t and letting t go to infinity, when the limits exist, we have the following relationship on mutual information rates: $I_\infty(y; v) - I_\infty(w; v) \leq I_\infty(y_G; v_G) - I_\infty(w; v_G)$, where equality holds when $y(t)$ is Gaussian. Since (y_G, v_G) and (w, v_G) form stationary Gaussian processes, when ϕ_{y_G}, ϕ_{v_G} , and $\phi_w \in \mathbb{F}$, subject to [Property 5](#), we have the following inequality on load disturbance sensitivity Bode-like integral:

$$\begin{aligned} & I_\infty(y; v) - I_\infty(w; v) \\ & \leq I_\infty(y_G; v_G) - I_\infty(w; v_G) \\ & \stackrel{(a)}{=} -\frac{1}{4\pi} \int_{-\infty}^{\infty} \log \left(1 - \frac{\phi_{y_G v_G}(\omega) \phi_{v_G y_G}(\omega)}{\phi_{y_G}(\omega) \phi_{v_G}(\omega)} \right) d\omega \\ & \quad + \frac{1}{4\pi} \int_{-\infty}^{\infty} \log \left(1 - \frac{\phi_{w v_G}(\omega) \phi_{v_G w}(\omega)}{\phi_w(\omega) \phi_{v_G}(\omega)} \right) d\omega \\ & \stackrel{(b)}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} \log T_{yw}(\omega) d\omega \\ & \quad + \frac{1}{4\pi} \int_{-\infty}^{\infty} \log \frac{\phi_{u_G}(\omega) \phi_{v_G}(\omega) - \phi_{u_G v_G}(\omega) \phi_{v_G u_G}(\omega)}{\phi_{y_G}(\omega) \phi_{v_G}(\omega) - \phi_{y_G v_G}(\omega) \phi_{v_G y_G}(\omega)} d\omega \quad (15) \\ & \stackrel{(c)}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} \log T_{yw}(\omega) d\omega - \frac{1}{2\pi} \int_{-\infty}^{\infty} \log |G(j\omega)| d\omega. \end{aligned}$$

(a) is obtained by applying [Property 5](#) to $I_\infty(y_G, v_G)$ and $I_\infty(w, v_G)$; (b) employs identities, $\phi_w = \phi_{v_G} + \phi_{v_G u_G} + \phi_{u_G v_G} + \phi_{u_G}$, $\phi_{w v_G} = \phi_{u_G v_G} + \phi_{v_G}$, and $\phi_{v_G w} = \phi_{v_G v_G} + \phi_{v_G}$, which can be derived by the definition of PSD in (5) and identity $w = v_G + u_G$; and (c) follows from $\phi_{u_G} = G^{-1}(s)G^{-1}(-s)\phi_{y_G}$, $\phi_{u_G v_G} = G^{-1}(-s)\phi_{y_G v_G}$, and $\phi_{v_G u_G} = G^{-1}(s)\phi_{v_G y_G}$, which can be inferred from the proof of [Lemma 2](#) in [Appendix A](#). Inequality (15) readily implies inequality (13) in [Theorem 1](#). This completes the proof. $\square$

Remark 4. According to [Definition 3](#), all rational functions belong to class $\mathbb{F}$ function. A presumption of derivation in (15) is that ϕ_{y_G}, ϕ_w , and $\phi_{v_G} \in \mathbb{F}$, which requires the Fourier transforms of auto-covariance r_{y_G}, r_w and r_{v_G} be rational and was formerly presumed in [10]. Meanwhile, the employment of [Property 3](#) in (14) requires $y = g(u)$ in [Fig. 3](#) be injective, which simplifies the technical development while delivering the essential conceptual message and was previously assumed also in [24,25].

The following theorem gives the information-theoretic representations of noise sensitivity and complementary sensitivity Bode-like integrals in the presence of measurement noise d .

Theorem 1'. For the general continuous-time feedback control system, when $(\tilde{y}, \tilde{e})$ and $(\tilde{d}, \tilde{e})$ form stationary processes, $\phi_{\tilde{y}}, \phi_{\tilde{e}}$, and $\phi_{\tilde{d}} \in \mathbb{F}$, and $\tilde{d}$ is a stationary Gaussian process, the complementary sensitivity and noise sensitivity Bode-like integrals satisfy

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \log T_{yd}(\omega) \frac{d\omega}{\omega^2} \geq I_\infty(\tilde{y}; \tilde{e}) - I_\infty(\tilde{d}; \tilde{e}), \quad (16)$$

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} \log T_{ud}(\omega) \frac{d\omega}{\omega^2} & \geq I_\infty(\tilde{u}; \tilde{e}) - I_\infty(\tilde{d}; \tilde{e}) \\ & - \frac{1}{2\pi} \int_{-\infty}^{\infty} \log |G(j\omega)| \frac{d\omega}{\omega^2}. \end{aligned} \quad (17)$$

Proof. The inverse system subject to measurement noise is shown in [Fig. 4](#).

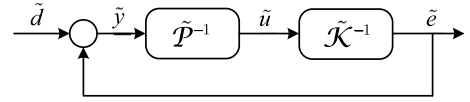


Fig. 4. Inverse system with measurement noise.

First, we consider the complementary sensitivity Bode-like integral. When $\tilde{y}_G$ and $\tilde{e}_G$ respectively denote the Gaussian processes with the same covariance as the random processes $\tilde{y}$ and $\tilde{e}$, the following inequality can be established:

$$\begin{aligned} & I(\tilde{y}^t; \tilde{e}^t) - I(\tilde{d}^t; \tilde{e}^t) \\ & \stackrel{(a)}{=} \lim_{k \rightarrow \infty} \left\{ h([\tilde{y}^{(\delta(k))}]^k) - h([\tilde{y}^{(\delta(k))}]^k | [\tilde{e}^{(\delta(k))}]^k) \right. \\ & \quad \left. - h([\tilde{d}^{(\delta(k))}]^k) + h([\tilde{d}^{(\delta(k))}]^k | [\tilde{e}^{(\delta(k))}]^k) \right\} \\ & \stackrel{(b)}{\leq} \lim_{k \rightarrow \infty} \left\{ h([\tilde{y}_G^{(\delta(k))}]^k) - h([\tilde{y}_G^{(\delta(k))}]^k | [\tilde{e}_G^{(\delta(k))}]^k) - h([\tilde{d}^{(\delta(k))}]^k) \right. \\ & \quad \left. + h([\tilde{d}^{(\delta(k))}]^k | [\tilde{e}_G^{(\delta(k))}]^k) \right\} \quad (18) \\ & \stackrel{(c)}{=} I(\tilde{y}_G; \tilde{e}_G) - I(\tilde{d}; \tilde{e}_G). \end{aligned}$$

(a) is obtained by [Properties 1](#) and [2](#); (b) employs [Property 4](#), identities $h([\tilde{y}^{(\delta(k))}]^k | [\tilde{e}^{(\delta(k))}]^k) = h([\tilde{d}^{(\delta(k))}]^k - [\tilde{e}^{(\delta(k))}]^k | [\tilde{e}^{(\delta(k))}]^k) = h([\tilde{d}^{(\delta(k))}]^k | [\tilde{e}^{(\delta(k))}]^k)$ and $h([\tilde{y}_G^{(\delta(k))}]^k | [\tilde{e}_G^{(\delta(k))}]^k) = h([\tilde{d}^{(\delta(k))}]^k | [\tilde{e}_G^{(\delta(k))}]^k)$ derived by [Property 2](#), and equality in (b) holds when $(\tilde{y}, \tilde{e})$ and $(\tilde{d}, \tilde{e})$ are Gaussian; and (c) follows from [Properties 1](#) and [2](#). Dividing both sides of (18) by t and letting t go to infinity, when the limits exist, we readily have $I_\infty(\tilde{y}; \tilde{e}) - I_\infty(\tilde{d}; \tilde{e}) \leq I_\infty(\tilde{y}_G; \tilde{e}_G) - I_\infty(\tilde{d}; \tilde{e}_G)$. When $(\tilde{y}_G, \tilde{e}_G)$ and $(\tilde{d}, \tilde{e}_G)$ are stationary Gaussian processes with $\phi_{\tilde{y}_G}, \phi_{\tilde{e}_G}$, and $\phi_{\tilde{d}} \in \mathbb{F}$, by [Property 5](#), an inequality between mutual information rates and complementary sensitivity Bode-like integral can be established:

$$\begin{aligned} & I_\infty(\tilde{y}; \tilde{e}) - I_\infty(\tilde{d}; \tilde{e}) \\ & \leq I_\infty(\tilde{y}_G; \tilde{e}_G) - I_\infty(\tilde{d}; \tilde{e}_G) \\ & \stackrel{(a)}{=} -\frac{1}{4\pi} \int_{-\infty}^{\infty} \log \left(1 - \frac{\phi_{\tilde{y}_G \tilde{e}_G}(\tilde{\omega}) \phi_{\tilde{e}_G \tilde{y}_G}(\tilde{\omega})}{\phi_{\tilde{y}_G}(\tilde{\omega}) \phi_{\tilde{e}_G}(\tilde{\omega})} \right) d\tilde{\omega} \\ & \quad + \frac{1}{4\pi} \int_{-\infty}^{\infty} \log \left(1 - \frac{\phi_{\tilde{d} \tilde{e}_G}(\tilde{\omega}) \phi_{\tilde{e}_G \tilde{d}}(\tilde{\omega})}{\phi_{\tilde{d}}(\tilde{\omega}) \phi_{\tilde{e}_G}(\tilde{\omega})} \right) d\tilde{\omega} \\ & \stackrel{(b)}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} \log T_{\tilde{y}\tilde{d}}(\tilde{\omega}) d\tilde{\omega} \\ & \quad + \frac{1}{4\pi} \int_{-\infty}^{\infty} \log \frac{\phi_{\tilde{y}_G}(\tilde{\omega}) \phi_{\tilde{e}_G}(\tilde{\omega}) - \phi_{\tilde{y}_G \tilde{e}_G}(\tilde{\omega}) \phi_{\tilde{e}_G \tilde{y}_G}(\tilde{\omega})}{\phi_{\tilde{y}_G}(\tilde{\omega}) \phi_{\tilde{e}_G}(\tilde{\omega}) - \phi_{\tilde{y}_G \tilde{e}_G}(\tilde{\omega}) \phi_{\tilde{e}_G \tilde{y}_G}(\tilde{\omega})} d\tilde{\omega} \quad (19) \\ & \stackrel{(c)}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} \log T_{yd}(\omega) \frac{d\omega}{\omega^2}. \end{aligned}$$

(a) is the application of [Property 5](#); (b) follows from identity $\tilde{d} = \tilde{y}_G + \tilde{e}_G$, which implies $\phi_{\tilde{d}} = \phi_{\tilde{y}_G} + \phi_{\tilde{y}_G \tilde{e}_G} + \phi_{\tilde{e}_G \tilde{y}_G} + \phi_{\tilde{e}_G}$, $\phi_{\tilde{d} \tilde{e}_G} = \phi_{\tilde{y}_G \tilde{e}_G} + \phi_{\tilde{e}_G}$, and $\phi_{\tilde{e}_G \tilde{d}} = \phi_{\tilde{e}_G \tilde{y}_G} + \phi_{\tilde{e}_G}$; and (c) can be implied by the frequency transformation (7). Inequality (16) in [Theorem 1'](#) can then be readily implied from (19).

Next, we consider the noise sensitivity Bode-like integral. Since the steps for deriving inequality (17) can be inferred from the preceding proofs, only some critical steps are presented. Similar to the derivations in (14), for noise sensitivity Bode-like integral, we have $I_\infty(\tilde{u}_G; \tilde{e}_G) - I_\infty(\tilde{d}; \tilde{e}_G) \geq I_\infty(\tilde{u}; \tilde{e}) - I_\infty(\tilde{d}; \tilde{e})$, where $\tilde{u}_G$ and $\tilde{e}_G$ denote the Gaussian processes with the same covariances as $\tilde{u}$ and $\tilde{e}$, respectively. With the facts that $(\tilde{u}_G, \tilde{e}_G)$ and $(\tilde{d}, \tilde{e}_G)$ are stationary Gaussian processes, when $\phi_{\tilde{u}_G}, \phi_{\tilde{e}_G}$, and $\phi_{\tilde{d}} \in \mathbb{F}$,

we have:

$$\begin{aligned}
 & I_\infty(\tilde{u}; \tilde{e}) - I_\infty(\tilde{d}; \tilde{e}) \\
 & \leq I_\infty(\tilde{u}_G; \tilde{e}_G) - I_\infty(\tilde{d}; \tilde{e}_G) \\
 & \stackrel{(a)}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} \log T_{\tilde{u}\tilde{d}}(\tilde{\omega}) d\tilde{\omega} \\
 & \quad + \frac{1}{4\pi} \int_{-\infty}^{\infty} \log \frac{\phi_{\tilde{y}_G}(\tilde{\omega})\phi_{\tilde{e}_G}(\tilde{\omega}) - \phi_{\tilde{y}_G\tilde{e}_G}(\tilde{\omega})\phi_{\tilde{e}_G\tilde{y}_G}(\tilde{\omega})}{\phi_{\tilde{u}_G}(\tilde{\omega})\phi_{\tilde{e}_G}(\tilde{\omega}) - \phi_{\tilde{u}_G\tilde{e}_G}(\tilde{\omega})\phi_{\tilde{e}_G\tilde{u}_G}(\tilde{\omega})} d\tilde{\omega} \quad (20) \\
 & \stackrel{(b)}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} \log T_{ud}(\omega) \frac{d\omega}{\omega^2} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \log |G(j\omega)| \frac{d\omega}{\omega^2}
 \end{aligned}$$

(a) employs [Property 5](#) and identity $\tilde{d} = \tilde{y}_G + \tilde{e}_G$; and (b) follows from (7), (8), and the identities, $\phi_{\tilde{u}_G} = \tilde{G}^{-1}(\tilde{s})\tilde{G}^{-1}(-\tilde{s})\phi_{\tilde{y}_G}$, $\phi_{\tilde{u}_G\tilde{e}_G} = \tilde{G}^{-1}(-\tilde{s})\phi_{\tilde{y}_G\tilde{e}_G}$, and $\phi_{\tilde{e}_G\tilde{u}_G} = \tilde{G}^{-1}(\tilde{s})\phi_{\tilde{e}_G\tilde{y}_G}$, which can be inferred from the proof of [Lemma 2](#) in [Appendix A](#). Inequality (17) can then be readily implied from (20). This completes the proof. $\square$

Remark 5. The terms $(2\pi)^{-1} \int_{-\infty}^{\infty} \log |G(j\omega)| d\omega$ in (13) and $(2\pi)^{-1} \int_{-\infty}^{\infty} \log |G(j\omega)|/\omega^2 d\omega$ in (17) are constants once a linear plant model $G(j\omega)$ is given, thus the lower bounds in [Theorems 1](#) and [1'](#) are invariant of the choice of control mapping. Nevertheless, these two integrals are not always bounded for arbitrary $G(j\omega)$. Some numerical examples are presented in [Appendix B](#), and heuristic analysis on the boundedness of these types of integrals is available in [35,36].

With the information-theoretic representations of Bode-like integrals given in [Theorems 1](#) and [1'](#), we can now establish the lower bounds of Bode-like integrals by deriving the lower bounds of their information-theoretic representations with tools and preliminary results from information theory.

4. Lower bounds of performance limitations

Compared with the definitions of Bode-like integrals, which require the stationary Gaussian condition, the existence of their information-theoretic representations is less restrictive, as it only requires the stationary condition. In this section, we derive the lower bounds for these information-theoretic representations, and hence the lower bounds for the corresponding Bode-like integrals in a stochastic continuous-time setting. For systems subject to control noise, we have the following result.

Theorem 2. When the linear continuous-time SISO system shown in [Fig. 3](#) is mean-square stable, and the control noise $w(t)$ is stationary Gaussian, we have

$$I_\infty(u; v) - I_\infty(w; v) \geq \sum_{p_i \in \mathcal{UP}} p_i, \quad (21)$$

$$I_\infty(y; v) - I_\infty(w; v) \geq \sum_{p_i \in \mathcal{UP}} p_i, \quad (22)$$

where $\mathcal{UP}$ denotes the set of unstable poles in plant $\mathcal{P}$.

Proof. The first inequality (21) in [Theorem 2](#) can be readily implied from [Lemma 1](#). In order to derive the lower bound in inequality (22), consider the feedback configuration illustrated in [Fig. 3](#) and the following equality

$$\begin{aligned}
 I(u^t; v^t) & \stackrel{(a)}{=} \lim_{k \rightarrow \infty} \left\{ h([u^{(\delta(k))}]^k) - h([u^{(\delta(k))}]^k | [v^{(\delta(k))}]^k) \right\} \\
 & \stackrel{(b)}{=} \lim_{k \rightarrow \infty} \left\{ h([y^{(\delta(k))}]^k) - \mathbb{E}[\log |J_{[u^{(\delta(k))}]^k}([y^{(\delta(k))}]^k)|] \right. \\
 & \quad \left. - h([y^{(\delta(k))}]^k | [v^{(\delta(k))}]^k) \right\} \quad (23)
 \end{aligned}$$

$$\begin{aligned}
 & + \mathbb{E}[\log |J_{[u^{(\delta(k))}]^k}([y^{(\delta(k))}]^k)|] \Big\} \\
 & \stackrel{(c)}{=} I(y^t; v^t).
 \end{aligned}$$

(a) follows from [Properties 1](#) and [2](#); (b) follows from the identities $h([u^{(\delta(k))}]^k) = h([y^{(\delta(k))}]^k) - \mathbb{E}[\log |J_{[u^{(\delta(k))}]^k}([y^{(\delta(k))}]^k)|]$ and $h([u^{(\delta(k))}]^k | [v^{(\delta(k))}]^k) = h([y^{(\delta(k))}]^k | [v^{(\delta(k))}]^k) - \mathbb{E}[\log |J_{[u^{(\delta(k))}]^k}([y^{(\delta(k))}]^k)|]$ derived by [Assumption 1](#) and [Property 3](#); and (c) follows from [Properties 1](#) and [2](#). The derivation in (23) can also be verified by data-processing inequality [13]. Dividing both sides of (23) by t and letting t go to infinity gives $I_\infty(u; v) = I_\infty(y; v)$, which combining with (21) gives (22) in [Theorem 2](#). This completes the proof. $\square$

The following theorem derives the lower bounds of the information-theoretic representations for the feedback systems subject to measurement noise described in [Fig. 4](#).

Theorem 2'. When the continuous-time SISO feedback system shown in [Fig. 4](#) is mean-square stable and the inverse frequency measurement noise $\tilde{d}(t)$ is stationary Gaussian, we have

$$I_\infty(\tilde{y}; \tilde{e}) - I_\infty(\tilde{d}; \tilde{e}) \geq \sum_{z_i \in \mathcal{UZ}} \frac{1}{z_i}, \quad (24)$$

$$I_\infty(\tilde{u}; \tilde{e}) - I_\infty(\tilde{d}; \tilde{e}) \geq \sum_{z_i \in \mathcal{UZ}} \frac{1}{z_i}, \quad (25)$$

where $\mathcal{UZ}$ denotes the set of nonminimum phase zeros of plant $\mathcal{P}$.

Proof. Consider the inverse system shown in [Fig. 4](#). Applying [Lemma 1](#) and noticing that the poles of inverse plant model $\tilde{\mathcal{P}}^{-1}$ defined in (9) are relocated at $s = 1/z_i$, the first inequality (24) can be implied. In order to derive inequality (25), consider the following equality

$$\begin{aligned}
 I(\tilde{y}^t; \tilde{e}^t) & \stackrel{(a)}{=} \lim_{k \rightarrow \infty} \left\{ h([\tilde{y}^{(\delta(k))}]^k) - h([\tilde{y}^{(\delta(k))}]^k | [\tilde{e}^{(\delta(k))}]^k) \right\} \\
 & \stackrel{(b)}{=} \lim_{k \rightarrow \infty} \left\{ h([\tilde{u}^{(\delta(k))}]^k) - \mathbb{E}[\log |J_{[\tilde{y}^{(\delta(k))}]^k}([\tilde{u}^{(\delta(k))}]^k)|] \right. \\
 & \quad \left. - h([\tilde{u}^{(\delta(k))}]^k | [\tilde{e}^{(\delta(k))}]^k) \right. \\
 & \quad \left. + \mathbb{E}[\log |J_{[\tilde{y}^{(\delta(k))}]^k}([\tilde{u}^{(\delta(k))}]^k)|] \right\} \\
 & \stackrel{(c)}{=} I(\tilde{u}; \tilde{e}).
 \end{aligned} \quad (26)$$

(a) follows from [Properties 1](#) and [2](#); (b) adopts the assumption that $\tilde{u} = \tilde{g}^{-1}(\tilde{y})$ in [Fig. 4](#) is injective and [Property 3](#), which imply $h([\tilde{y}^{(\delta(k))}]^k) = h([\tilde{u}^{(\delta(k))}]^k) - \mathbb{E}[\log |J_{[\tilde{y}^{(\delta(k))}]^k}([\tilde{u}^{(\delta(k))}]^k)|]$ and $h([\tilde{y}^{(\delta(k))}]^k | [\tilde{e}^{(\delta(k))}]^k) = h([\tilde{u}^{(\delta(k))}]^k | [\tilde{e}^{(\delta(k))}]^k) - \mathbb{E}[\log |J_{[\tilde{y}^{(\delta(k))}]^k}([\tilde{u}^{(\delta(k))}]^k)|]$, where $J_{[\tilde{y}^{(\delta(k))}]^k}([\tilde{u}^{(\delta(k))}]^k)$ is the Jacobian matrix of vector $[\tilde{u}^{(\delta(k))}]^k$ with respect to vector $[\tilde{y}^{(\delta(k))}]^k$; and (c) follows from [Properties 1](#) and [2](#). Dividing both sides of (26) by t and letting t go to infinity, we have $I_\infty(\tilde{y}; \tilde{e}) = I_\infty(\tilde{u}; \tilde{e})$, which combining with (24) gives inequality (25). This completes the proof. $\square$

With all the preceding theorems, the following corollary gives the lower bounds of the Bode-like integrals in stochastic continuous-time systems.

Corollary 3. For a stochastic continuous-time SISO feedback control system that is mean-square stable, when (u, v) and (w, v) form stationary processes, ϕ_u , ϕ_v , and $\phi_w \in \mathbb{F}$, and w is a stationary Gaussian process, the sensitivity and the load disturbance sensitivity Bode-like integrals satisfy

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \log T_{uw}(\omega) d\omega \geq \sum_{p_i \in \mathcal{UP}} p_i, \quad (27)$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \log T_{yw}(\omega) d\omega \geq \sum_{p_i \in \mathcal{UP}} p_i + \frac{1}{2\pi} \int_{-\infty}^{\infty} \log |G(j\omega)| d\omega. \quad (28)$$

When $(\tilde{y}, \tilde{e})$ and $(\tilde{d}, \tilde{e})$ form stationary processes, $\phi_{\tilde{y}}$, $\phi_{\tilde{e}}$, and $\phi_{\tilde{d}} \in \mathbb{F}$, and $\tilde{d}$ is a stationary Gaussian process, the complementary sensitivity and noise sensitivity Bode-like integrals satisfy

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \log T_{yd}(\omega) \frac{d\omega}{\omega^2} \geq \sum_{z_i \in \mathcal{U}_Z} \frac{1}{z_i}, \quad (29)$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \log T_{ud}(\omega) \frac{d\omega}{\omega^2} \geq \sum_{z_i \in \mathcal{U}_Z} \frac{1}{z_i} - \frac{1}{2\pi} \int_{-\infty}^{\infty} \log |G(j\omega)| \frac{d\omega}{\omega^2}. \quad (30)$$

Proof. Corollary 3 can be implied by applying Theorems 2 and 2' to Theorems 1 and 1', respectively. $\square$

Numerical examples are presented in Appendix B to show the validness as well as some limitations of the lower bounds presented in Corollary 3.

Remark 6. From the proof of Lemma 2 given in Appendix A, identities $T_{yw}(\omega) = G(s)T_{uw}(\omega)$ and $T_{yd}(\omega) = G(s)T_{ud}(\omega)$ can be inferred. Once we admit the inequalities given in (27) and (29), we can also retrieve inequalities (28) and (30) by substituting the two aforementioned identities into (27) and (29), respectively. Meanwhile, since a nice property similar to $T_{uw} + T_{yd} = 1$ does not exist between T_{yw} and T_{ud} , the boundedness of right-hand side terms in (28) and (30) is more difficult to guarantee, which is explained with a few examples in Appendix B. Lastly, the relaxations of injective assumptions adopted in (14), (23) and (26), the discretization requirements in Assumption 1, and the stationary Gaussian condition on the inverse frequency signals when studying the weighted Bode-like integrals are still some interesting problems that deserve further investigations.

We notice that since the pioneering papers [8,23], information-theoretic approaches have been widely employed to seek the lower bounds for Bode-like integrals. Among these papers, the lower bounds for sensitivity Bode-like integral and complementary sensitivity Bode-like integral have been investigated in [8, 23,25] for discrete-time SISO systems. Lower bounds for load disturbance sensitivity Bode-like integral and noise sensitivity Bode-like integral have been studied in [9] with discrete-time MIMO systems. For continuous-time systems, [10] has defined the lower bound for sensitivity Bode-like integral, and lower bounds for continuous-time complementary sensitivity, load disturbance sensitivity, and noise sensitivity Bode-like integrals have been discussed in this paper. Bode-like integrals for continuous-time systems with non-Gaussian disturbance, non-LTI models, MIMO configuration, switching logics, or distributed agents are some open topics for future investigation, which may benefit from this paper and also the preliminary results on discrete-time systems [11,20,21,24].

5. Conclusion

In this paper, we investigated the performance limitations of linear continuous-time SISO control systems subject to control and measurement noise via an information-theoretic approach. Bode integrals of four different sensitivity-like functions were defined, and the relationship between Bode integrals and Bode-like integrals were established for stochastic continuous-time systems. The information-theoretic representations of Bode-like integrals were derived, and the lower bounds of these representations and hence the Bode-like integrals were established in terms of the unstable zeros and poles of plant model. Some open problems and challenges are discussed towards the end, and the hope is that more innovative results can be put forward to expand the frontier in this direction.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Proof of Lemma 2

Proof of Lemma 2. Part of the proof in this appendix relies on the results given in [32]. We first consider the scenario in the presence of control noise $w(t)$ and start with Bode-like integrals. When both the plant model $\mathcal{P}$ and control mapping $\mathcal{K}$ in Fig. 1 are linear, using $v(t) = (c * (g * u))(t) = \int_0^\infty c(\theta) \left[\int_0^\infty g(\eta) u(t - \theta - \eta) d\eta \right] d\theta$, we define $L(s) = G(s)C(s) = \int_0^\infty (g * c)(t) \cdot e^{-j\omega t} dt = \int_0^\infty l(t) \cdot e^{-j\omega t} dt$. Since $w(t) = u(t) + v(t)$, the PSD function $\phi_w(\omega)$ defined as (5) satisfies

$$\phi_w(\omega) = \phi_u(\omega) + \phi_{uw}(\omega) + \phi_{vu}(\omega) + \phi_v(\omega). \quad (31)$$

When $w(t)$, $v(t)$, and $u(t)$ are wide sense stationary, with $v(t) = \int_0^\infty l(\sigma') \cdot u(t - \sigma') d\sigma'$ and $\tau = \sigma - t$, the covariances r_u , r_{uw} , r_{vu} and r_v in (31) respectively satisfy $r_u(\sigma, t) = \text{Cov}[u(t + \tau), u(t)] = r_u(\tau)$, $r_{uw}(\sigma, t) = \int_0^\infty l(\sigma') \cdot \text{Cov}[u(\sigma), u(t - \sigma')] d\sigma' = \int_0^\infty l(\sigma') \cdot r_u(\sigma' + \tau) d\sigma' = r_{uw}(\tau)$, $r_{vu}(\sigma, t) = \int_0^\infty l(\sigma') \cdot \text{Cov}[u(\sigma - \sigma'), u(t)] d\sigma' = \int_0^\infty l(\sigma') \cdot r_u(-\sigma' + \tau) d\sigma' = r_{vu}(\tau)$, and $r_v(\sigma, t) = \int_0^\infty \int_0^\infty l(\sigma') \cdot l(t') \cdot \text{Cov}[u(\sigma - \sigma'), u(t - t')] d\sigma' dt' = \int_0^\infty \int_0^\infty l(\sigma') \cdot l(t') \cdot r_u(\tau - \sigma' + t') d\sigma' dt' = r_v(\tau)$. Hence, the spectral density functions ϕ_{uw} , ϕ_{vu} , and ϕ_v in (31) respectively satisfy

$$\begin{aligned} \phi_{uw}(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} r_{uw}(\tau) \cdot e^{-j\omega\tau} d\tau \\ &= \frac{1}{2\pi} \int_0^\infty e^{j\omega\sigma'} \cdot l(\sigma') \int_{-\infty}^{\infty} e^{-j\omega(\tau + \sigma')} \cdot r_u(\tau + \sigma') d\tau d\sigma' \\ &= L(-j\omega) \cdot \phi_u(\omega), \end{aligned} \quad (32)$$

$$\begin{aligned} \phi_{vu}(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} r_{vu}(\tau) \cdot e^{-j\omega\tau} d\tau \\ &= \frac{1}{2\pi} \int_0^\infty e^{-j\omega\sigma'} \cdot l(\sigma') \int_{-\infty}^{\infty} e^{-j\omega(\tau - \sigma')} \cdot r_u(\tau - \sigma') d\tau d\sigma' \\ &= L(j\omega) \cdot \phi_u(\omega), \end{aligned} \quad (33)$$

$$\begin{aligned} \phi_v(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} r_v(\tau) \cdot e^{-j\omega\tau} d\tau \\ &= \frac{1}{2\pi} \int_0^\infty e^{j\omega t'} \cdot l(t') \int_0^\infty e^{-j\omega\sigma'} \\ &\quad \cdot l(\sigma') \int_{-\infty}^{\infty} e^{-j\omega(\tau - \sigma' + t')} \cdot r_u(\tau - \sigma' + t') d\tau d\sigma' dt' \\ &= L(-j\omega) \cdot L(j\omega) \cdot \phi_u(\omega). \end{aligned} \quad (34)$$

Substituting (31)–(34) into the sensitivity-like function $T_{uw}(\omega)$ defined in (4), we can rewrite the sensitivity-like function as $T_{uw}(\omega) = [\phi_u(\omega)/\phi_w(\omega)]^{1/2} = \{\phi_u(\omega)/[\phi_u(\omega) + \phi_{uw}(\omega) + \phi_{vu}(\omega) + \phi_v(\omega)]\}^{1/2} = \{\phi_u(\omega)/[1 + L(-j\omega) \cdot (1 + L(j\omega)) \cdot \phi_u(\omega)]\}^{1/2}$. When $\phi_u(\omega) \neq 0$, with $L(s) = G(s) \cdot C(s)$, we have $T_{uw}(\omega) = \sqrt{T_{uw}(-s) \cdot T_{uw}(s)}$. Since $T_{uw}(-s) = \bar{T}_{uw}(s)$, where $\bar{T}_{uw}(s)$ is the complex conjugate of $T_{uw}(s)$, the equality (11a) in Lemma 2 can be retrieved from $(2\pi)^{-1} \cdot \int_{-\infty}^{\infty} \log T_{uw}(\omega) d\omega = (4\pi)^{-1} \int_{-\infty}^{\infty} \log [\bar{T}_{uw}(j\omega) \cdot T_{uw}(j\omega)] d\omega = (4\pi)^{-1} \int_{-\infty}^{\infty} \log |T_{uw}(j\omega)|^2 d\omega = (2\pi)^{-1} \int_{-\infty}^{\infty} \log |T_{uw}(j\omega)| d\omega$.

Since $y(t) = g * u(t) = \int_0^\infty g(\theta) \cdot u(t - \theta) d\theta$, the auto-covariance of signal y satisfies $r_y(\sigma, t) = \int_0^\infty \int_0^\infty g(\sigma') \cdot g(t') \cdot \text{Cov}[u(\sigma - \sigma'), u(t - t')] d\sigma' dt' = \int_0^\infty \int_0^\infty g(\sigma') \cdot g(t') \cdot r_u(\tau - \sigma' + t') d\sigma' dt' = r_y(\tau)$. Hence, the PSD of the stationary signal y is

$$\phi_y(\omega) = \frac{1}{2\pi} \int_0^\infty r_y(\tau) \cdot e^{-j\omega\tau} d\tau$$

$$\begin{aligned}
&= \frac{1}{2\pi} \int_0^\infty e^{j\omega t'} \cdot g(t') \int_0^\infty e^{-j\omega \sigma'} \cdot g(\sigma') \int_{-\infty}^\infty e^{-j\omega(\tau - \sigma' + t')} \\
&\quad \cdot r_u(\tau - \sigma' + t') d\tau d\sigma' dt' \\
&= G(-j\omega) \cdot G(j\omega) \cdot \phi_u(\omega).
\end{aligned} \tag{35}$$

Substituting (31)–(35) into the load disturbance sensitivity-like function $T_{yw}(\omega)$ defined in (4), we can rewrite the load disturbance sensitivity-like function as follows $T_{yw}(\omega) = [\phi_y(\omega)/\phi_w(\omega)]^{1/2} = \{\phi_y(\omega)/[\phi_u(\omega) + \phi_{uv}(\omega) + \phi_v(\omega)]\}^{1/2} = \{[G(-j\omega) \cdot G(j\omega) \cdot \phi_u(\omega)]/[(1 + L(-j\omega)) \cdot (1 + L(j\omega)) \cdot \phi_u(\omega)]\}^{1/2}$. When $\phi_u(\omega) \neq 0$, it follows that $T_{yw}(\omega) = \sqrt{T_{yw}(-s) \cdot T_{yw}(s)}$. Since $T_{yw}(-j\omega) = \bar{T}_{yw}(j\omega)$, where $\bar{T}_{yw}(j\omega)$ is the complex conjugate of $T_{yw}(j\omega)$, the equality (11b) in Lemma 2 can be retrieved from $(2\pi)^{-1} \int_{-\infty}^\infty \log T_{yw}(\omega) d\omega = (4\pi)^{-1} \int_{-\infty}^\infty \log[\bar{T}_{yw}(j\omega) \cdot T_{yw}(j\omega)] d\omega = (4\pi)^{-1} \int_{-\infty}^\infty \log |T_{yw}(j\omega)|^2 d\omega = (2\pi)^{-1} \int_{-\infty}^\infty \log |T_{yw}(j\omega)| d\omega$. The steps for deriving (11c) and (11d) are similar to the preceding derivations; hence, only abbreviated steps are given below.

Next, we consider the scenario in the presence of measurement noise $d(t)$. When both the plant model $\mathcal{P}$ and the control mapping $\mathcal{K}$ are linear, we have $y(t) = (c * (g * e))(t) = \int_0^\infty c(\theta) [\int_0^\infty g(\eta) \cdot e(t - \theta - \eta) d\eta] d\theta = \int_0^\infty l(\sigma') \cdot e(t - \sigma') d\sigma'$. Since $d(t) = e(t) + y(t)$, similar to the result presented in (31), we have $\phi_d(\omega) = \phi_e(\omega) + \phi_{ey}(\omega) + \phi_{ye}(\omega) + \phi_y(\omega)$. When disturbance $d(t)$ is zero-mean stationary, with $y(t) = \int_0^\infty l(\sigma') \cdot e(t - \sigma') d\sigma'$, $u(t) = c * e(t) = \int_0^\infty c(\theta) \cdot e(t - \theta) d\theta$, and $\tau = \sigma - t$, the covariances r_e , r_{ey} , r_{ye} , r_y , and r_u satisfy $r_e(\sigma, t) = r_e(\sigma - t) = r_e(\tau)$, $r_{ey}(\sigma, t) = \int_0^\infty l(\sigma') \cdot r_e(\sigma' + \tau) d\sigma' = r_{ey}(\tau)$, $r_{ye}(\sigma, t) = \int_0^\infty l(\sigma') \cdot r_e(-\sigma' + \tau) d\sigma' = r_{ye}(\tau)$, $r_y(\sigma, t) = \int_0^\infty \int_0^\infty l(\sigma') \cdot l(t') \cdot r_e(\tau - \sigma' + t') d\sigma' dt' = r_y(\tau)$, and $r_u(\sigma, t) = \int_0^\infty \int_0^\infty c(\sigma') \cdot c(t') \cdot r_e(t - \sigma' + t') d\sigma' dt' = r_u(\tau)$. Hence, similar to (34), the PSDs $\phi_u(\omega)$, $\phi_y(\omega)$, $\phi_{ey}(\omega)$, and $\phi_{ye}(\omega)$ satisfy $\phi_u(\omega) = C(-j\omega) \cdot C(j\omega) \cdot \phi_e(\omega)$, $\phi_y(\omega) = L(-j\omega) \cdot L(j\omega) \cdot \phi_e(\omega)$, $\phi_{ey}(\omega) = L(-j\omega) \cdot \phi_e(\omega)$, and $\phi_{ye}(\omega) = L(j\omega) \cdot \phi_e(\omega)$. Then when $\phi_e(\omega) \neq 0$, we have the following relationship $T_{ud}(\omega) = [\phi_u(\omega)/\phi_d(\omega)]^{1/2} = \{[C(-j\omega) \cdot C(j\omega) \cdot \phi_e(\omega)]/[(1 + L(-j\omega)) \cdot (1 + L(j\omega)) \cdot \phi_e(\omega)]\}^{1/2} = |T_{ud}(j\omega)|$, and $T_{yd}(\omega) = [\phi_y(\omega)/\phi_d(\omega)]^{1/2} = \{[L(-j\omega) \cdot L(j\omega) \cdot \phi_e(\omega)]/[(1 + L(-j\omega)) \cdot (1 + L(j\omega)) \cdot \phi_e(\omega)]\}^{1/2} = |T_{yd}(j\omega)|$, which readily imply (11c) and (11d) in Lemma 2. This completes the proof. $\square$

Appendix B. Numerical examples

Since there exist substantial examples on inequalities (27) and (29) [10,14,15,17,27,35], only examples verifying (28) and (30) are presented for brevity. For inequality (28) on load disturbance sensitivity Bode-like integral, consider a plant model $G_1(s) = (s + 0.2)/(s - 0.1)$ with an unstable pole at $s = 0.1$ and stabilized by $C_1(s) = (s + 0.1)/[s(s - 0.2)(s + 1)]$ and $C_2(s) = 1/[s \cdot (s + 1)]$. For both control mappings, the lower bound on the RHS of (28) is 0.15. While the LHS of (28) with $C_1(s)$ is 0.35, the LHS of (28) with $C_2(s)$ is 0.15, which implies that the plant model $G_1(s)$ stabilized by control mapping $C_2(s)$ has better property for load disturbance sensitivity. For inequality (30) on noise sensitivity Bode-like integral, consider a plant model $G_2(s) = 1/(s + 1)$ stabilized respectively by $C_3(s) = -1 \times 10^{-4} \cdot (s - 1)(s + 0.05)/s^2$ with a nonminimum phase zero at $s = 1$ and $C_4(s) = -1 \times 10^{-4} \cdot (s - 2)(s + 0.05)/s^2$ with a nonminimum phase zero at $s = 2$. While for the plant model $G_2(s)$ stabilized by $C_3(s)$, both the LHS and RHS values of (30) are 1.5, these values are 1 for system with $C_4(s)$, which indicates that the plant model $G_2(s)$ stabilized by $C_4(s)$ has a better noise sensitivity property. However, as we mentioned in Remarks 1 and 6, the integrals in (28) and (30) are not guaranteed to be bounded for all systems.

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