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Multilinear PageRank: Uniqueness, error bound and perturbation analysis ☆

Wen Li [a], Dongdong Liu [b,*], Seak-Weng Vong [c], Mingqing Xiao [d]

[a] School of Mathematical Sciences, South China Normal University, Guangzhou, 510631, China
[b] School of Applied Mathematics, Guangdong University of Technology, Guangzhou, 510006, China
[c] Department of Mathematics, University of Macau, Macau, China
[d] Department of Mathematics, Southern Illinois University Carbondale, Carbondale, IL 62901, USA

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ABSTRACT

In this paper, we revisit the multilinear PageRank problem. Under the framework of tensor, we establish several new and tighter uniqueness conditions for the multilinear PageRank vector. Meanwhile, a refined error bound for the inverse iteration as well as the new perturbation bounds under different norms, which improve the existing ones in the current literature, are developed with feasible computations. Several numerical examples are given to validate the significant effectiveness of the proposed bounds.

1. Introduction

Higher-order Markov chains play important roles in analyzing a variety of stochastic (probabilistic) processes over time in multi-dimensional spaces. Recently, Li and Ng [9] proposed an approximate tensor model for the higher-order Markov chain, which has recently brought much attention (e.g., see [1], [4], [6], and [12]) due to the broad applications under this formulation. One of the important applications is the study of the multilinear PageRank, which was first proposed by Gleich, Lim, and Yu in [6]. The approximate model for the higher-order Markov chain is given as follows.

$$\mathbf{x} = \mathcal{P}\mathbf{x}^{m-1}, \mathbf{x} \geq 0, \|\mathbf{x}\|_1 = 1, \tag{1.1}$$

where $\mathbf{x} = (x_i)$ is called a stochastic vector (or probability distribution vector) and $\mathcal{P}$ is an order-m dimension n stochastic tensor (or called transition probability tensor), i.e.,

$$p_{i_1,i_2,\cdots,i_m} \geq 0\ , \quad \sum_{i_1 \in \langle n \rangle} p_{i_1,i_2,\cdots,i_m} = 1. \tag{1.2}$$

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* Corresponding author.
E-mail address: ddliu@gdut.edu.cn (D. Liu).

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The model (1.1) can be used to formulate the multilinear PageRank problem:

$$\mathbf{x} = \alpha \mathcal{P} \mathbf{x}^{m-1} + (1-\alpha)\mathbf{v}, \tag{1.3}$$

or equivalently

$$\mathbf{x} = \alpha \mathcal{P}_{(1)} (\overbrace{\mathbf{x} \otimes \cdots \otimes \mathbf{x}}^{m-1}) + (1-\alpha)\mathbf{v}, \tag{1.4}$$

where $\mathbf{x}$ is called a multilinear PageRank vector, $\otimes$ is the standard Kronecker product and $\mathcal{P}_{(1)}$ is an n-by-n^{m-1} stochastic matrix of the flattened tensor along the first index (see [6] for more detail). For example, let $\mathcal{P} = (p_{i,j,k})_{i,j,k=1}^n$ be an order-3 dimension n tensor. Then we have

$$\mathcal{P}_{(1)} = \begin{bmatrix} p_{1,1,1} & \cdots & p_{1,n,1} & p_{1,1,2} & \cdots & p_{1,n,2} & \cdots & p_{1,1,n} & \cdots & p_{1,n,n} \\ p_{2,1,1} & \cdots & p_{2,n,1} & p_{2,1,2} & \cdots & p_{2,n,2} & \cdots & p_{2,1,n} & \cdots & p_{2,n,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{n,1,1} & \cdots & p_{n,n,1} & p_{n,1,2} & \cdots & p_{n,n,2} & \cdots & p_{n,1,n} & \cdots & p_{n,n,n} \end{bmatrix}.$$

The multilinear PageRank vector is a nonnegative, stochastic solution of the system of polynomial equations (1.3) or (1.4).

It is noted that the multilinear PageRank model (1.3) reduces to the higher-order Markov chain model (1.1) when we take $\alpha = 1$ (also see [7] for a general case). Generally, the model (1.3) can be also rewritten equivalently as (1.1):

$$\mathbf{x} = \mathcal{P}_{\mathbf{v}} \mathbf{x}^{m-1}, \mathcal{P}_{\mathbf{v}} = \alpha \mathcal{P} + (1-\alpha)\mathcal{V}, (\mathcal{V})_{i,i_2,\cdots,i_m} = v_i. \tag{1.5}$$

Therefore, the multilinear PageRank model (1.3) can be transformed to the higher-order Markov chain model (1.1), and the condition for the uniqueness of a probability distribution vector is obtained recently (see, e.g., [4,5,9]). In [6], Gleich et al. presented a simplified uniqueness condition, i.e., if $\alpha < \frac{1}{m-1}$, the multilinear PageRank vector in (1.3) is unique. This condition is independent of the tensor $\mathcal{P}$, and is rather simple to be verified. However, it is too strong and appears to be over-simplified since $\alpha \approx 1$ has been excluded. It is well-known that the model still performs well when α tends to 1. Therefore this motivates us to seek some more general but tighter conditions. Most recently, Li et al. [11] proposed the following uniqueness condition:

$$\alpha < \frac{1}{\max_s \sum_{k=2}^m \max_{i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m \in \langle n \rangle} \sum_{i=1}^n |p_{i,i_1,\cdots,i_{k-1},s,i_{k+1},\cdots,i_m} - \sigma_i|}, \tag{1.6}$$

where $\sigma_i, i = 1, \cdots, n$ are parameters. Clearly, when we take those $\sigma_i, i = 1, \cdots, n$ to be all zero, the uniqueness condition (1.6) can be reduced to $\alpha < \frac{1}{m-1}$. A large number of numerical examples randomly generated showed that the condition (1.6) is less restrictive than $\alpha < \frac{1}{m-1}$. However, the improvement is not significant when the given tensor is very sparse. Thus it is desirable to find a better uniqueness condition in order to obtain an adequate choice of the parameter α, which is one of the motivations of this paper.

In [6], algorithms such as the fixed-point method, the shifted fixed-point method, the nonlinear inner-outer iteration, the inverse iteration, and the Newton iteration are discussed, and error bounds for these algorithms are provided. Notice that the uniqueness conditions are also convergence conditions for these algorithms. In this paper, we focus on the improvement of the convergence condition and the error bounds for the inverse iteration by a new approach. The proposed idea is also applicable for other algorithms in a similar way.

Another issue is to seek the 1-norm perturbation bounds for the multilinear PageRank vector, which is an open problem and is stated in the concluding remarks given by [11]. Theoretically, one could make use of the perturbation bound of limiting probability vector (see [10]) to compute the multilinear PageRank vector. However, such an approach may not be computationally feasible due to the involvement of the limit. Hence, we develop some perturbation bounds in terms of 1-norm as well as ∞-norm, which appear to be quite effective in our numerical experiments.

The main contributions of this paper can be summarized as follows:

1. We establish several new and tighter uniqueness conditions for the multilinear PageRank problem, which can provide the better approaches in solving this problem numerically.
2. We improve the error bound for the inverse iteration algorithm for solving the multilinear PageRank problem.
3. Further, under our framework, we conduct the perturbation analysis for the multilinear PageRank vector with new upper bounds provided.
4. Extensive numerical experiments, along with various practical problems, are provided and the outcomes significantly outperform the existing ones in most cases.

The rest of this paper is organized as follows. In Section 2, we discuss the uniqueness of the multilinear PageRank vector. Some uniqueness conditions of the multilinear PageRank vector are established. Also, we show that our new uniqueness conditions are more effective than the existing ones given in [6,11]. In Section 3, we discuss the convergence condition and the error bounds for the inverse iteration, and the new estimations appear to be sharper than the one in [6]. In Section 4, we develop the perturbation bounds based on the 1-norm and the ∞-norm for the multilinear PageRank vector. Numerical examples are given in Section 5 to illustrate the effectiveness of our theoretical results. The paper ends with the concluding remarks in Section 6.

2. The uniqueness conditions

2.1. The uniqueness conditions based on the new parameter method

Definition 2.1. Let $\mathcal{A}$ be an order-m dimension n tensor, $\mathbf{x}$ be an n dimensional vector, and x_i denote the i-th entry of $\mathbf{x}$, $\mathcal{A}\mathbf{x}^{m-r}$ to be an order-r, dimension n tensor whose entry is given by

$$(\mathcal{A}\mathbf{x}^{m-r})_{i,i_2,\cdots,i_r} = \sum_{i_{r+1},\cdots,i_m\in\langle n\rangle} a_{i,i_2,\cdots,i_r,i_{r+1},\cdots,i_m} x_{i_{r+1}}\cdots x_{i_m}. \tag{2.1}$$

If $r=2$, (2.1) reduces to a matrix:

$$(\mathcal{A}\mathbf{x}^{m-2})_{ij} = \sum_{i_3,\cdots,i_m=1}^{n} a_{i,j,i_3,\cdots,i_m} x_{i_3}\cdots x_{i_m}, \quad i,j=1,2,\cdots,n. \tag{2.2}$$

Furthermore, if $r=1$, (2.1) reduces to Qi's definition [17]:

$$(\mathcal{A}\mathbf{x}^{m-1})_{i} = \sum_{i_2,\cdots,i_m=1}^{n} a_{i,i_2,\cdots,i_m} x_{i_2}\cdots x_{i_m}, \quad i=1,2,\cdots,n.$$

In particular, $\mathcal{A}\mathbf{x}^{m-1}$ can be written as a form of matrix-vector product:

$$\mathcal{A}\mathbf{x}^{m-1} = \mathcal{A}\mathbf{x}^{m-2}\mathbf{x}.$$

Definition 2.2. Let $\mathcal{A}$ be an order-m dimension n tensor, both $\mathbf{x}$ and $\mathbf{y}$ be n dimensional vectors. We define

$$\mathcal{A}(\mathbf{x}^{m-r}-\mathbf{y}^{m-r}) \equiv \mathcal{A}\mathbf{x}^{m-r} - \mathcal{A}\mathbf{y}^{m-r}. \tag{2.3}$$

Especially, $r=1$ and $r=2$, (2.3) follows

$$\mathcal{A}(\mathbf{x}^{m-1}-\mathbf{y}^{m-1}) \equiv \mathcal{A}\mathbf{x}^{m-1} - \mathcal{A}\mathbf{y}^{m-1},$$

and

$$\mathcal{A}(\mathbf{x}^{m-2}-\mathbf{y}^{m-2}) \equiv \mathcal{A}\mathbf{x}^{m-2} - \mathcal{A}\mathbf{y}^{m-2}.$$

Definition 2.3. Let $\mathcal{A}$ be an order-m dimension n tensor, both $\mathbf{x}$ and $\mathbf{y}$ be n dimensional vectors. We define $A_{xy}^{(k)}$ to be an $n\times n$ matrix whose the (i,j)-entry is given by

$$(A_{xy}^{(k)})_{i,j} = \sum_{i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m\in\langle n\rangle} a_{i,i_2,\cdots i_{k-1},j,i_{k+1},\cdots,i_m} x_{i_2}\cdots x_{i_{k-1}} y_{i_{k+1}}\cdots y_{i_m}. \tag{2.4}$$

The following lemmas will be used subsequently throughout this paper. The proof of the first lemma is analogical to Lemma 1 of [11].

Lemma 2.1. *Let both $\mathbf{x}$ and $\mathbf{y}$ be n dimensional vectors, and let $\mathcal{A}$ be an order-m dimension n tensor. Then we have*

$$\mathcal{A}(\mathbf{x}^{m-1}-\mathbf{y}^{m-1}) = \sum_{k=2}^{m} \mathcal{A}\mathbf{x}^{k-2}\Delta\mathbf{x}\mathbf{y}^{m-k}, \tag{2.5}$$

where $\Delta\mathbf{x}=\mathbf{x}-\mathbf{y}$.

Lemma 2.2. *Under the same assumptions as in Lemma 2.1, we have*

$$\mathcal{A}\mathbf{x}^{k-2}\Delta\mathbf{x}\mathbf{y}^{m-k} = A_{xy}^{(k)}\Delta\mathbf{x}. \tag{2.6}$$

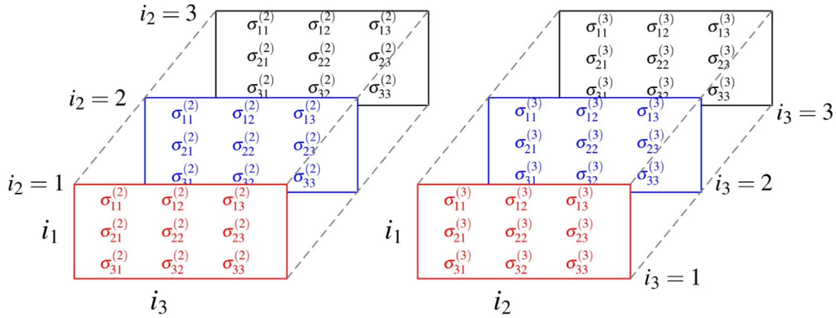

Fig. 2.1. The parameter tensors $\mathcal{J}^{(2)}$ (left) and $\mathcal{J}^{(3)}$ (right).

Proof. Let Δx_i be i-entry of $\Delta\mathbf{x}$. By Definition 2.3 we have for any i

$$\begin{aligned}
&(\mathcal{A}\mathbf{x}^{k-2}\Delta\mathbf{x}\mathbf{y}^{m-k})_i\\
&= \sum_{i_2,\cdots,i_m\in\langle n\rangle} a_{i,i_2,\cdots i_{k-1},i_k,i_{k+1},\cdots,i_m} x_{i_2}\cdots x_{i_{k-1}}\Delta x_{i_k} y_{i_{k+1}}\cdots y_{i_m}\\
&= \sum_{i_k=1}^{n}\Big[\sum_{i_2,\cdots,i_{k-1},i_k,i_{k+1},\cdots,i_m\in\langle n\rangle} a_{i,i_2,\cdots i_{k-1},i_k,i_{k+1},\cdots,i_m} x_{i_2}\cdots x_{i_{k-1}} y_{i_{k+1}}\cdots y_{i_m}\Big]\Delta x_{i_k}\\
&= (A_{xy}^{(k)}\Delta\mathbf{x})_i,
\end{aligned}$$

which proves (2.6). □

Lemma 2.3. *Let both* $\mathbf{x}$ *and* $\mathbf{y}$ *be* n *dimensional stochastic vectors, and let* $\mathcal{J}^{(k)}$ $(k = 2,\cdots,m)$ *be order-*m *dimension* n *tensors given by*

$$(\mathcal{J}^{(k)})_{i_1,\cdots,i_k,\cdots,i_m} = \sigma^{(k)}_{i_1,i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m}, \quad \forall i_k \in \langle n\rangle,$$

where $\sigma^{(k)}_{i_1,i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m} \in \mathbb{R}$ *for any* $i_l \in \langle n\rangle$, $l = 1, 2,\cdots,k-1,k+1,\cdots,i_m$ *and* $k = 2, 3,\cdots,m$. *Then*

$$J_{xy}^{(k)}\Delta\mathbf{x} = 0. \tag{2.7}$$

Proof. By Lemma 2.2 we have for given k and any $i \in \langle n\rangle$

$$\begin{aligned}
(J_{xy}^{(k)}\Delta\mathbf{x})_i &= (\mathcal{J}^{(k)}\mathbf{x}^{k-2}\Delta\mathbf{x}\mathbf{y}^{m-k})_i\\
&= \sum_{i_k=1}^{n}\sum_{i_2,\cdots,i_{k-1},i_k,i_{k+1},\cdots,i_m\in\langle n\rangle} \sigma^{(k)}_{i,i_2,\cdots i_{k-1},i_{k+1},\cdots,i_m} x_{i_2}\cdots x_{i_{k-1}}\Delta x_{i_k} y_{i_{k+1}}\cdots y_{i_m}\\
&= \sum_{i_2,\cdots,i_{k-1},i_k,i_{k+1},\cdots,i_m\in\langle n\rangle} \sigma^{(k)}_{i,i_2,\cdots i_{k-1},i_{k+1},\cdots,i_m} x_{i_2}\cdots x_{i_{k-1}} y_{i_{k+1}}\cdots y_{i_m}\sum_{i_k=1}^{n}\Delta x_{i_k}\\
&= 0,
\end{aligned}$$

which proves (2.7). □

Remark 2.1. The structure of tensors $\mathcal{J}^{(k)}(k = 2, 3)$, for example, with $m = 3$ and $n = 3$, can be illustrated by Fig. 2.1.

Lemma 2.4. *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be* n *dimension stochastic vectors, and let* $\mathcal{A}$ *be an order-*m *dimension* n *tensor. Then we have*

$$\mathcal{A}(\mathbf{x}^{m-1} - \mathbf{y}^{m-1}) = \sum_{k=2}^{m}(A_{xy}^{(k)} - J_{xy}^{(k)})\Delta\mathbf{x}. \tag{2.8}$$

Proof. The formula (2.8) is a direct result of Lemmas 2.1, 2.2 and 2.3. □

Let $\mathcal{J}^{(k)}(k = 2, 3,\cdots,m)$ be defined in Lemma 2.3.

Lemma 2.5. *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be n dimensional stochastic vectors, and let* $\mathcal{P}$ *be an order-m dimension n stochastic tensor. For any* $\mathcal{J}^{(k)} (k = 2, 3, \cdots, m)$*, we have*

$$||\mathcal{P}(\mathbf{x}^{m-1} - \mathbf{y}^{m-1})||_1 \le \mu(\mathcal{J}^{(2)}, \cdots, \mathcal{J}^{(m)})||\Delta\mathbf{x}||_1 \tag{2.9}$$

and

$$||\mathcal{P}(\mathbf{x}^{m-1} - \mathbf{y}^{m-1})||_\infty \le \nu(\mathcal{J}^{(2)}, \cdots, \mathcal{J}^{(m)})||\Delta\mathbf{x}||_\infty, \tag{2.10}$$

where

$$\mu(\mathcal{J}^{(2)}, \cdots, \mathcal{J}^{(m)}) = \max_{i_k \in \langle n \rangle} \sum_{k=2}^{m} \max_{i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m \in \langle n \rangle} \sum_{i_1=1}^{n} |p_{i_1,i_2,\cdots,i_k,\cdots,i_m} - \sigma^{(k)}_{i_1,i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m}|, \tag{2.11}$$

and

$$\nu(\mathcal{J}^{(2)}, \cdots, \mathcal{J}^{(m)}) = \max_{i_1 \in \langle n \rangle} \sum_{k=2}^{m} \max_{i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m \in \langle n \rangle} \sum_{i_k=1}^{n} |p_{i_1,i_2,\cdots,i_k,\cdots,i_m} - \sigma^{(k)}_{i_1,i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m}|. \tag{2.12}$$

Proof. By Definition 2.3 and Lemma 2.4, we have

$$\begin{aligned}
\|\mathcal{P}(\mathbf{x}^{m-1} - \mathbf{y}^{m-1})\|_1 &= \|\sum_{k=2}^{m}(P^{(k)}_{xy} - J^{(k)}_{xy})\Delta\mathbf{x}\|_1 \\
&= \sum_{i=1}^{n} |\sum_{k=2}^{m} \sum_{i_2,\cdots,i_m \in \langle n \rangle} (p_{i,i_2,\cdots,i_m} - \sigma^{(k)}_{i,i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m}) x_{i_2} \cdots x_{i_{k-1}} \Delta\mathbf{x}_{i_k} y_{i_{k+1}} \cdots y_{i_m}| \\
&\le \sum_{k=2}^{m} \sum_{i_2,\cdots,i_m \in \langle n \rangle} \sum_{i=1}^{n} |p_{i,i_2,\cdots,i_m} - \sigma^{(k)}_{i,i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m}| x_{i_2} \cdots x_{i_{k-1}} y_{i_{k+1}} \cdots y_{i_m} |\Delta\mathbf{x}_{i_k}| \\
&\le \sum_{k=2}^{m} \sum_{i_k = j \in \langle n \rangle} \max_{i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m \in \langle n \rangle} \sum_{i=1}^{n} |p_{i,i_2,\cdots,j,\cdots,i_m} - \sigma^{(k)}_{i,i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m}||\Delta\mathbf{x}_j| \\
&\equiv \sum_{k=2}^{m} \sum_{j \in \langle n \rangle} \tau^{(k)}_j |\Delta\mathbf{x}_j|,
\end{aligned} \tag{2.13}$$

where $\tau^{(k)}_j = \max_{i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m \in \langle n \rangle} \sum_{i=1}^{n} |p_{i,i_2,\cdots,j,\cdots,i_m} - \sigma^{(k)}_{i,i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m}|$. Since

$$\begin{aligned}
\sum_{k=2}^{m} \sum_{j \in \langle n \rangle} \tau^{(k)}_j |\Delta\mathbf{x}_j| &= \sum_{i_k = j \in \langle n \rangle} (\tau^{(2)}_j + \cdots + \tau^{(m)}_j)|\Delta\mathbf{x}_j| \\
&= (\tau^{(2)}_1 + \cdots + \tau^{(m)}_1)|\Delta\mathbf{x}_1| + \cdots + (\tau^{(2)}_n + \cdots + \tau^{(m)}_n)|\Delta\mathbf{x}_n| \\
&\le \max_{j \in \langle n \rangle} \sum_{k=2}^{m} \tau^{(k)}_j \|\Delta\mathbf{x}\|_1,
\end{aligned} \tag{2.14}$$

this gives (2.9).

On the other hand, set $s = \arg\max_{i \in \langle n \rangle} |(\mathcal{P}\mathbf{x}^{m-1} - \mathcal{P}\mathbf{y}^{m-1})_i|$. Then by Lemma 2.4, we have

$$\begin{aligned}
\|\mathcal{P}(\mathbf{x}^{m-1} - \mathbf{y}^{m-1})\|_\infty &= |(\mathcal{P}\mathbf{x}^{m-1} - \mathcal{P}\mathbf{y}^{m-1})_s| \\
&= |\sum_{k=2}^{m} \sum_{i_2,\cdots,i_m \in \langle n \rangle} (p_{s,i_2,\cdots,i_m} - \sigma^{(k)}_{i,i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m}) x_{i_2} \cdots x_{i_{k-1}} \Delta\mathbf{x}_{i_k} y_{i_{k+1}} \cdots y_{i_m}| \\
&\le \sum_{k=2}^{m} \sum_{i_2,\cdots,i_m \in \langle n \rangle} |p_{s,i_2,\cdots,i_m} - \sigma^{(k)}_{s,i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m}| x_{i_2} \cdots x_{i_{k-1}} y_{i_{k+1}} \cdots y_{i_m} |\Delta\mathbf{x}_{i_k}| \\
&\le \sum_{k=2}^{m} \max_{i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m \in \langle n \rangle} \sum_{i_k=1}^{n} |p_{s,i_2,\cdots,i_k,\cdots,i_m} - \sigma^{(k)}_{s,i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m}||\Delta\mathbf{x}_{i_k}|
\end{aligned}$$

$$\leq \max_{i_1} \sum_{k=2}^{m} \max_{i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m \in \langle n \rangle} \sum_{i_k=1}^{n} |p_{i_1,i_2,\cdots,i_k,\cdots,i_m} - \sigma^{(k)}_{i_1,i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m}| \|\Delta \mathbf{x}\|_\infty,$$

which yields (2.10). □

Next, we present the main result as follows.

Theorem 2.1. *Let $\mathcal{P}$ be an order-m stochastic tensor, $\mathbf{v}$ be a stochastic vector. Then the multilinear PageRank problem (1.3) has the unique solution if*

$$\alpha < \frac{1}{\min\{\mu, \nu\}}, \tag{2.15}$$

where $\mu = \min_{\mathcal{J}^{(k)},k=2,3,\cdots,m} \mu(\mathcal{J}^{(2)},\cdots,\mathcal{J}^{(m)})$ *and* $\nu = \min_{\mathcal{J}^{(k)},k=2,3,\cdots,m} \nu(\mathcal{J}^{(2)},\cdots,\mathcal{J}^{(m)})$.

Proof. Assume there exist both stochastic vectors $\mathbf{x}$ and $\mathbf{y}$ with $\mathbf{y} \neq \mathbf{x}$ which satisfy the multilinear PageRank problem (1.3). Then we have

$$\Delta \mathbf{x} = \mathbf{x} - \mathbf{y} = \alpha(\mathcal{P}\mathbf{x}^{m-1} - \mathcal{P}\mathbf{y}^{m-1}). \tag{2.16}$$

Taking $\mathcal{J}_\mu^{(k)} = \arg \min_{\mathcal{J}^{(k)},k=2,3,\cdots,m} \mu(\mathcal{J}^{(2)},\cdots,\mathcal{J}^{(m)})$ and $\mathcal{J}_\nu^{(k)} = \arg \min_{\mathcal{J}^{(k)},k=2,3,\cdots,m} \nu(\mathcal{J}^{(2)},\cdots,\mathcal{J}^{(m)})$ for $k = 2, 3, ..., m$, by Lemma 2.5, it is easy to get

$$||\Delta \mathbf{x}||_1 = \alpha||\mathcal{P}(\mathbf{x}^{m-1} - \mathbf{y}^{m-1})||_1 \leq \alpha\mu(\mathcal{J}_\mu^{(2)},\cdots,\mathcal{J}_\mu^{(m)})||\Delta \mathbf{x}||_1 = \alpha\mu||\Delta \mathbf{x}||_1 \tag{2.17}$$

and

$$||\Delta \mathbf{x}||_\infty = \alpha||\mathcal{P}(\mathbf{x}^{m-1} - \mathbf{y}^{m-1})||_\infty \leq \alpha\nu(\mathcal{J}_\nu^{(2)},\cdots,\mathcal{J}_\nu^{(m)})||\Delta \mathbf{x}||_\infty = \alpha\nu||\Delta \mathbf{x}||_\infty, \tag{2.18}$$

which contradicts to the assumption (2.15). □

For simplicity, we give the following corollary.

Corollary 2.1. *Under the same assumptions as in Theorem 2.1, for any $\mathcal{J}^{(k)}$ $(k = 2, 3, \cdots, m)$, the multilinear PageRank model (1.3) has the unique solution provided*

$$\alpha < \frac{1}{\min_{s\in\{1,k\}} \sum_{k=2}^{m} \max_{i_s,i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m\in\langle n\rangle} \sum_{i_s=1}^{n} |p_{i_1,\cdots,i_s,\cdots,i_m} - \sigma^{(k)}_{i_1,\cdots,i_{k-1},i_{k+1},\cdots,i_m}|}. \tag{2.19}$$

Proof. Since for any $s \in \langle n \rangle$

$$\min\{\mu, \nu\} \leq \min_{s\in\{1,k\}} \sum_{k=2}^{m} \max_{i_s,i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m\in\langle n\rangle} \sum_{i_s=1}^{n} |p_{i_1,\cdots,i_s,\cdots,i_m} - \sigma^{(k)}_{i_1,\cdots,i_{k-1},i_{k+1},\cdots,i_m}|.$$

It follows from Theorem 2.1 that the multilinear PageRank model (1.3) has the unique solution provided (2.19) holds. □

Remark 2.2. For a given set $\{\mathcal{J}^{(2)},\cdots,\mathcal{J}^{(m)}\}$, it is easy to see

$$\alpha < \frac{1}{\min\{\mu(\mathcal{J}^{(2)},\cdots,\mathcal{J}^{(m)}), \nu(\mathcal{J}^{(2)},\cdots,\mathcal{J}^{(m)})\}} \leq \frac{1}{\min\{\mu, \nu\}}.$$

Thus, one can get different uniqueness conditions of the solution for the multilinear PageRank model (1.3) via choosing the parameter tensors $\mathcal{J}^{(2)},\cdots,\mathcal{J}^{(m)}$. In next subsection, we will provide algorithms to choose these $\mathcal{J}^{(k)}(k = 2, 3, \cdots, m)$.

2.2. Choices of the parameters

In general, it is difficult to compute μ and ν directly. Next we give some computable bounds that can simplify the approach. Let $\mathbf{e}$ be a dimension n vector with all entries being 1. The following Algorithms 2.1 and 2.2 are proposed for choosing the parameter $\mathcal{J}^{(k)}, k = 2, 3, \cdots, m$ in $\mu(\mathcal{J}^{(2)},\cdots,\mathcal{J}^{(m)})$ and $\nu(\mathcal{J}^{(2)},\cdots,\mathcal{J}^{(m)})$ respectively.

Algorithm 2.1 Computing $\mathcal{J}^{(k)}(k=2,3,\cdots,m)$ for $\mu(\mathcal{J}^{(2)},\cdots,\mathcal{J}^{(m)})$.

Input $\mathcal{P}$.
for $k=2,3,\cdots,m$ **do**
  for $i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m\in\langle n\rangle$ **do**
    Let $A=(a_{i_1 i_k})$ with $a_{i_1 i_k}=p_{i_1,i_2,\cdots,i_k,\cdots,i_m}$. Solve the nonlinear optimization problems:

$$\widehat{\mathbf{y}}=\arg\min_{\mathbf{y}}||A-\mathbf{y}\mathbf{e}^T||_1. \tag{2.20}$$

    Update $\sigma^{(k)}_{i_1,\cdots,i_{k-1},i_{k+1},\cdots,i_m}=\widehat{y}_{i_1}$.
  end for
end for
Compute $\widehat{\mu}_1=\mu(\mathcal{J}^{(2)},\cdots,\mathcal{J}^{(m)})$.
Output $\widehat{\mu}_1$.

Algorithm 2.2 Computing $\mathcal{J}^{(k)}(k=2,3,\cdots,m)$ for $\nu(\mathcal{J}^{(2)},\cdots,\mathcal{J}^{(m)})$.

Input $\mathcal{P}$.
for $k=2,3,\cdots,m$ **do**
  for $i_1,\cdots,i_{k-1},i_{k+1},\cdots,i_m\in\langle n\rangle$ **do**
    Let $\mathbf{b}=(b_{i_k})$ with $b_{i_k}=p_{i_1,i_2,\cdots,i_k,\cdots,i_m}$. Solve the nonlinear optimization problems:

$$\widehat{t}=\arg\min_{t}||\mathbf{b}-t\mathbf{e}||_1. \tag{2.21}$$

    Update $\sigma^{(k)}_{i_1,\cdots,i_{k-1},i_{k+1},\cdots,i_m}=\widehat{t}$.
  end for
end for
Compute $\widehat{\nu}_1=\nu(\mathcal{J}^{(2)},\cdots,\mathcal{J}^{(m)})$.
Output $\widehat{\nu}_1$.

Remark 2.3. It is easy to show that the optimization problem (2.20) is equivalent to the following minimax constraint problem:

$$\arg\min_{\mathbf{y}}\max_{j}\sum_{i=1}^{n}\sqrt{(a_{ij}-y_i)^2}. \tag{2.22}$$

It is a typical optimization problem that can be computed by the function *fminimax* in Matlab.

Furthermore, it can be seen that (2.21) is equivalent to

$$\arg\min_{t}\sum_{i=1}^{n}|t-b_i|, \tag{2.23}$$

which has the optimal solution:

$$\widehat{t}=\begin{cases}[c_{\frac{n}{2}},c_{\frac{n}{2}+1}], & n \text{ is even},\\ c_{\frac{n+1}{2}}, & n \text{ is odd},\end{cases}$$

where $\mathbf{c}=(c_i)$ is a vector obtained from $\mathbf{b}$ by rearranging its entries in ascending order, i.e., $\mathbf{c}=(b_{i_1},b_{i_2},\cdots,b_{i_n})^T$ with $b_{i_1}\le b_{i_2}\le\cdots\le b_{i_n}$.

Remark 2.4. Since $\mu\le\widehat{\mu}_1$ and $\nu\le\widehat{\nu}_1$, by the proof of Theorem 2.1, the multilinear PageRank model (1.3) has the unique solution if

$$\alpha<\frac{1}{\min\{\widehat{\mu}_1,\widehat{\nu}_1\}}. \tag{2.24}$$

However, if m or n is large, it is expensive to compute $\widehat{\mu}_1$ and $\widehat{\nu}_1$ by Algorithms 2.1 and 2.2 directly. Therefore, we also proposed some feasible methods to generate the parameter tensors $\mathcal{J}^{(k)}(k=2,\cdots,m)$, which leads to the more computable bounds.

Remark 2.5. We may consider some special $\mathcal{J}^{(k)}(k=2,3,\cdots,m)$.

- Taking $\sigma^{(k)}_{i,i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m}=\sigma_i$, then we have

$$\sum_{k=2}^{m}\max_{i_2,\cdots,i_m\in\langle n\rangle}\sum_{i=1}^{n}|p_{i,i_2,\cdots,i_k,\cdots,i_m}-\sigma^{(k)}_{i,i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m}|=(m-1)\max_{i_2,\cdots,i_m\in\langle n\rangle}\sum_{i=1}^{n}|p_{i,i_2,\cdots,i_m}-\sigma_i|,$$

which is the formula (8) of [11].
- Taking $\sigma^{(k)}_{i,i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m}=\min_{i_k}p_{i,i_2,\cdots,i_k,\cdots,i_m}$, then

$$\begin{aligned}\sum_{i=1}^{n}|p_{i,i_2,\cdots,i_k,\cdots,i_m}-\sigma^{(k)}_{i,i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m}|&=\sum_{i=1}^{n}(p_{i,i_2,\cdots,i_k,\cdots,i_m}-\min_{i_k}p_{i,i_2,\cdots,i_k,\cdots,i_m})\\&=1-\sum_{i=1}^{n}\min_{i_k}p_{i,i_2,\cdots,i_k,\cdots,i_m}.\end{aligned}$$

By Theorem 2.1, the uniqueness can be guaranteed provided

$$\alpha<\frac{1}{m-1-\sum_{k=2}^{m}\min_{i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m\in\langle n\rangle}\sum_{i=1}^{n}\min_{i_k}p_{i,i_2,\cdots,i_k,\cdots,i_m}}.\tag{2.25}$$

It is also noted that $\min_{i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m\in\langle n\rangle}\sum_{i=1}^{n}\min_{i_k}p_{i,i_2,\cdots,i_k,\cdots,i_m}\geq\sum_{i=1}^{n}\min_{i_2,\cdots,i_m}p_{i,i_2,\cdots,i_m}$. This shows that the uniqueness condition (2.25) is always better than the estimate (11) in [11].
- Taking $\sigma^{(k)}_{i,i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m}=\max_{i_k}p_{i,i_2,\cdots,i_k,\cdots,i_m}$, then

$$\begin{aligned}\sum_{i=1}^{n}|p_{i,i_2,\cdots,i_k,\cdots,i_m}-\sigma^{(k)}_{i,i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m}|&=\sum_{i=1}^{n}(\max_{i_k}p_{i,i_2,\cdots,i_k,\cdots,i_m}-p_{i,i_2,\cdots,i_k,\cdots,i_m})\\&=\sum_{i=1}^{n}\max_{i_k}p_{i,i_2,\cdots,i_k,\cdots,i_m}-1.\end{aligned}$$

By Theorem 2.1, the solution of (1.3) is unique if

$$\alpha<\frac{1}{\sum_{k=2}^{m}(\max_{i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m\in\langle n\rangle}\sum_{i=1}^{n}\max_{i_k}p_{i,i_2,\cdots,i_k,\cdots,i_m}-1)}.\tag{2.26}$$

This improves the estimate (12) appeared in [11].

Hence our new results are always better than the corresponding ones in [6] and [11] respectively.

Remark 2.6. Now let us give a simple example to compare the proposed conditions (2.25) and (2.26) with the ones in (9), (11) and (12) of [11]. Let an order-3 dimension 2 stochastic tensor $\mathcal{P}$ with its 1-unfolding $\mathcal{P}_{(1)}$ be given by

$$\mathcal{P}_{(1)}=\begin{bmatrix}1-s_1 & 1-s_2 & 1-s_3 & 1-s_4\\ s_1 & s_2 & s_3 & s_4\end{bmatrix},$$

where $1\geq s_1\geq s_2\geq s_3\geq s_4\geq\frac{1}{2}$ and $s_1-s_2\leq s_3-s_4$. Then both the uniqueness conditions (2.25) and (2.26) are

$$\alpha<\frac{1}{s_2+s_3-2s_4},$$

which is better than the condition $\alpha<\frac{1}{2(s_1-s_4)}$ given by (9), (11) and (12) of [11].

2.3. The uniqueness conditions based on the optimal set method and the parameter method

Let $\mathbb{S}$ denote a proper subset of $\langle n\rangle$ and $\mathbb{S}'$ be its complementary set in $\langle n\rangle$. Next, combining our results with those in [9] gives the following lemma.

Lemma 2.6. *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be solutions of* (1.1)*, and let* $\mathcal{P}$ *be an order-m dimension n stochastic tensor. Then we have*

$$||\mathcal{P}(\mathbf{x}^{m-1}-\mathbf{y}^{m-1})||_1\leq\omega||\Delta\mathbf{x}||_1,\tag{2.27}$$

where

$$\omega = m-1-\min_{\mathbb{S}\subset\langle n\rangle}\sum_{k=2}^{m}(\min_{i_k\in\mathbb{S}}\min_{i_2,\cdots,i_{k-1},i_{k+1}\cdots,i_m\in\langle n\rangle}\sum_{i\in\mathbb{S}'}p_{i,i_2,\cdots,i_m}+\min_{i_k\in\mathbb{S}'}\min_{i_2,\cdots,i_{k-1},i_{k+1}\cdots,i_m\in\langle n\rangle}\sum_{i\in\mathbb{S}}p_{i,i_2,\cdots,i_m}). \tag{2.28}$$

Proof. If $\mathbf{x}=\mathbf{y}$, the inequality holds. Assume $\Delta x\neq 0$. Let $\mathbb{V}=\{i|\Delta x_i\geq 0\}$. Then $\mathbb{V}\neq\emptyset$ and $\mathbb{V}\subset\langle n\rangle$. Let $\mathbb{V}'=\langle n\rangle/\mathbb{V}$. Then $\mathbb{V}'\neq\emptyset$. Thus,

$$\begin{aligned}\sum_{i\in\mathbb{V}}\Delta x_i &= \sum_{k=2}^{m}\sum_{i_2,\cdots,i_m\in\langle n\rangle}\sum_{i\in\mathbb{V}}(p_{i,i_2,\cdots,i_m}-\sigma^{(k)}_{i,i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m})x_{i_2}\cdots x_{i_{k-1}}y_{i_{k+1}}\cdots y_{i_m}\Delta\mathbf{x}_{i_k}\\ &=\sum_{k=2}^{m}\sum_{i_k\in\mathbb{V}}\sum_{i_2,\cdots,i_{k-1},i_{k+1}\cdots,i_m\in\langle n\rangle}\sum_{i\in\mathbb{V}}(p_{i,i_2,\cdots,i_m}-\sigma^{(k)}_{i,i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m})x_{i_2}\cdots x_{i_{k-1}}y_{i_{k+1}}\cdots y_{i_m}\Delta\mathbf{x}_{i_k}\\ &\quad+\sum_{k=2}^{m}\sum_{i_k\in\mathbb{V}'}\sum_{i_2,\cdots,i_{k-1},i_{k+1}\cdots,i_m\in\langle n\rangle}\sum_{i\in\mathbb{V}}(p_{i,i_2,\cdots,i_m}-\sigma^{(k)}_{i,i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m})x_{i_2}\cdots x_{i_{k-1}}y_{i_{k+1}}\cdots y_{i_m}\Delta\mathbf{x}_{i_k}.\end{aligned} \tag{2.29}$$

Now taking $\sigma^{(k)}_{i,i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m}$ such that

$$\sum_{i\in\mathbb{V}}\sigma^{(k)}_{i,i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m}=\min_{i_k\in\mathbb{V}'}\sum_{i\in\mathbb{V}}p_{i,i_2,\cdots,i_m}.$$

Then by (2.29) we have

$$\begin{aligned}\sum_{i\in\mathbb{V}}\Delta x_i &\leq [\sum_{k=2}^{m}\max_{i_k\in\mathbb{V}}\max_{i_2,\cdots,i_{k-1},i_{k+1}\cdots,i_m\in\langle n\rangle}\sum_{i\in\mathbb{V}}p_{i,i_2,\cdots,i_m}-\sum_{k=2}^{m}\min_{i_2,\cdots,i_{k-1},i_{k+1}\cdots,i_m\in\langle n\rangle}\min_{i_k\in\mathbb{V}'}\sum_{i\in\mathbb{V}}p_{i,i_2,\cdots,i_m}]\sum_{i_k\in\mathbb{V}}\Delta\mathbf{x}_{i_k}\\ &=\sum_{k=2}^{m}[1-\min_{i_k\in\mathbb{V}}\min_{i_2,\cdots,i_{k-1},i_{k+1}\cdots,i_m\in\langle n\rangle}\sum_{i\in\mathbb{V}'}p_{i,i_2,\cdots,i_m}-\min_{i_k\in\mathbb{V}'}\min_{i_2,\cdots,i_{k-1},i_{k+1}\cdots,i_m\in\langle n\rangle}\sum_{i\in\mathbb{V}}p_{i,i_2,\cdots,i_m}]\sum_{i\in\mathbb{V}}\Delta\mathbf{x}_i\\ &\leq\omega\sum_{i\in\mathbb{V}}\Delta\mathbf{x}_i,\end{aligned}$$

combining with $\sum_{i\in\mathbb{V}}\Delta\mathbf{x}_i=\frac{1}{2}||\Delta\mathbf{x}||_1$ arrives at (2.27). □

Next, we give a new uniqueness condition for the multilinear PageRank model (1.3).

Theorem 2.2. *Under the same assumptions as given in Theorem 2.1, the multilinear PageRank model (1.3) has the unique solution if*

$$\alpha<\frac{1}{\omega}, \tag{2.30}$$

where ω is given by (2.28).

Proof. Assume that $\mathbf{x}$ and $\mathbf{y}$ are two solutions of (1.3) with $\mathbf{x}\neq\mathbf{y}$. Then we have

$$\Delta\mathbf{x}=\mathbf{x}-\mathbf{y}=\alpha(\mathcal{P}\mathbf{x}^{m-1}-\mathcal{P}\mathbf{y}^{m-1}). \tag{2.31}$$

By (2.27) and (2.30) we have

$$||\Delta\mathbf{x}||_1=||\alpha\mathcal{P}(\mathbf{x}^{m-1}-\mathbf{y}^{m-1})||_1\leq\omega\alpha||\Delta\mathbf{x}||_1<||\Delta\mathbf{x}||_1, \tag{2.32}$$

which is a contradiction. This proves theorem. □

Remark 2.7. For $m=3$, Fasino and Tudisco [5] also presented a uniqueness conditions

$$\alpha<\frac{1}{\mathcal{T}(\mathcal{P})}, \tag{2.33}$$

where

$$\mathcal{T}(\mathcal{P}) = \frac{1}{2} \max_{j,k_1,k_2 \in \langle n \rangle} \sum_i |p_{i,j,k_1} - p_{i,j,k_2} + p_{i,k_1,j} - p_{i,k_2,j}|.$$

It is difficult to compare the bounds (2.30) and (2.33) in theory. The following examples show that one bound can't always be better than another.

(1) For the first example, taking

$$\mathcal{P}(:,:,1) = \begin{pmatrix} 0.0459 & 0.4597 & 0.5329 \\ 0.8608 & 0.0346 & 0.2047 \\ 0.0933 & 0.5057 & 0.2624 \end{pmatrix}, \mathcal{P}(:,:,2) = \begin{pmatrix} 0.5963 & 0.2943 & 0.5501 \\ 0.2841 & 0.3786 & 0.0518 \\ 0.1196 & 0.3271 & 0.3981 \end{pmatrix},$$

$$\mathcal{P}(:,:,3) = \begin{pmatrix} 0.3891 & 0.3908 & 0.5364 \\ 0.4823 & 0.3108 & 0.1482 \\ 0.1286 & 0.2984 & 0.3154 \end{pmatrix},$$

we get $\omega = 0.6709 < \mathcal{T}(\mathcal{P}) = 1.4030$.

(2) For the second example, taking

$$\mathcal{P}(:,:,1) = \begin{pmatrix} 0.5073 & 0.2795 & 0.3617 \\ 0.4567 & 0.5804 & 0.5441 \\ 0.0361 & 0.1402 & 0.0943 \end{pmatrix}, \mathcal{P}(:,:,2) = \begin{pmatrix} 0.3442 & 0.4090 & 0.4581 \\ 0.3429 & 0.4978 & 0.1162 \\ 0.3129 & 0.0933 & 0.4257 \end{pmatrix},$$

$$\mathcal{P}(:,:,3) = \begin{pmatrix} 0.2166 & 0.5042 & 0.1572 \\ 0.3750 & 0.0598 & 0.3705 \\ 0.4083 & 0.4359 & 0.4723 \end{pmatrix},$$

we get $\omega = 0.9021 > \mathcal{T}(\mathcal{P}) = 0.8195$.

By (2.28), it is easy to see that $\omega \le (m-1)[1 - \min_{\mathbb{S}}(\min_{i_2,\cdots,i_m \in \langle n \rangle} \sum_{i \in \mathbb{S}'} p_{i,i_2,\cdots,i_m} + \min_{i_2,\cdots,i_m \in \langle n \rangle} \sum_{i \in \mathbb{S}} p_{i,i_2,\cdots,i_m})]$. Hence, we have the following corollary.

Corollary 2.2. *Under the same assumptions as in Theorem 2.1, the multilinear PageRank model (1.3) has the unique solution if*

$$\alpha < \frac{1}{\eta_1}, \tag{2.34}$$

where $\eta_1 = (m-1)[1 - \min_{\mathbb{S} \subset \langle n \rangle}(\min_{i_2,\cdots,i_m \in \langle n \rangle} \sum_{i \in \mathbb{S}'} p_{i,i_2,\cdots,i_m} + \min_{i_2,\cdots,i_m \in \langle n \rangle} \sum_{i \in \mathbb{S}} p_{i,i_2,\cdots,i_m})]$.

Note that $\sum_{i \in \mathbb{S}} p_{i,i_2,\cdots,i_m} + \sum_{i \in \mathbb{S}'} p_{i,i_2,\cdots,i_m} = 1$. Then the following corollary follows from (2.34) directly.

Corollary 2.3. *Under the same assumptions as in Theorem 2.1, the multilinear PageRank model (1.3) has the unique solution if*

$$\alpha < \frac{1}{\eta_2}, \tag{2.35}$$

or

$$\alpha < \frac{2}{\eta_3}, \tag{2.36}$$

where

$$\eta_2 = (m-1) \max_{\mathbb{S} \subset \langle n \rangle} [\max_{i_2,\cdots,i_m \in \langle n \rangle} \sum_{i \in \mathbb{S}'} p_{i,i_2,\cdots,i_m} + \max_{i_2,\cdots,i_m \in \langle n \rangle} \sum_{i \in \mathbb{S}} p_{i,i_2,\cdots,i_m} - 1]$$

and

$$\eta_3 = (m-1) \max_{\mathbb{S} \subset \langle n \rangle} [\max_{i_2,\cdots,i_m \in \langle n \rangle} \sum_{i \in \mathbb{S}'} p_{i,i_2,\cdots,i_m} + \max_{i_2,\cdots,i_m \in \langle n \rangle} \sum_{i \in \mathbb{S}} p_{i,i_2,\cdots,i_m} - \min_{i_2,\cdots,i_m \in \langle n \rangle} \sum_{i \in \mathbb{S}'} p_{i,i_2,\cdots,i_m} - \min_{i_2,\cdots,i_m \in \langle n \rangle} \sum_{i \in \mathbb{S}} p_{i,i_2,\cdots,i_m}].$$

3. Convergence analysis for the inverse iteration algorithm

Let $\mathcal{J}^{(k)}(k=3,\cdots,m)$ be given by Lemma 2.3. Firstly, we give the following lemma which will be used in the sequel.

Lemma 3.1. *Let* $\mathbf{x}$, $\mathbf{y}$ *and* $\mathbf{z}$ *be* n *dimension stochastic vectors, and let* $\mathcal{P}$ *be an order-*m *dimension* n *stochastic tensor. For any set* $\mathcal{J}^{(k)}(k=3,\cdots,m)$*, we have*

$$||\mathcal{P}(\mathbf{x}^{m-2}-\mathbf{y}^{m-2})\mathbf{z}||_1 \le \overline{\mu}(\mathcal{J}^{(3)},\cdots,\mathcal{J}^{(m)})||\Delta\mathbf{x}||_1,$$

where

$$\overline{\mu}(\mathcal{J}^{(3)},\cdots,\mathcal{J}^{(m)}) = \max_{i_2,i_k\in\langle n\rangle}\sum_{k=3}^{m}\max_{i_3,\cdots,i_{k-1},i_{k+1},\cdots,i_m\in\langle n\rangle}\sum_{i_1=1}^{n}|p_{i_1,i_2,\cdots,i_k,\cdots,i_m}-\sigma^{(k)}_{i_1,i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m}|,$$

Proof. Clearly, we have

$$\begin{aligned}||\mathcal{P}(\mathbf{x}^{m-2}-\mathbf{y}^{m-2})\mathbf{z}||_1 &\le ||\mathcal{P}(\mathbf{x}^{m-2}-\mathbf{y}^{m-2})||_1||\mathbf{z}||_1\\ &= ||\mathcal{P}(\mathbf{x}^{m-2}-\mathbf{y}^{m-2})||_1\\ &= \max_{i_2\in\langle n\rangle}\sum_{i=1}^{n}|\sum_{k=3}^{m}\sum_{i_3,\cdots,i_m\in\langle n\rangle}(p_{i,i_2,\cdots,i_m}-\sigma^{(k)}_{i,i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m})x_{i_3}\cdots x_{i_{k-1}}\Delta\mathbf{x}_{i_k}y_{i_{k+1}}\cdots y_{i_m}|\\ &\le \max_{i_2\in\langle n\rangle}\sum_{i=1}^{n}\sum_{k=3}^{m}\sum_{i_3,\cdots,i_m\in\langle n\rangle}|p_{i,i_2,\cdots,i_m}-\sigma^{(k)}_{i,i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m}|x_{i_3}\cdots x_{i_{k-1}}|\Delta\mathbf{x}_{i_k}|y_{i_{k+1}}\cdots y_{i_m}.\end{aligned}$$

By the same technique as (2.13) and (2.14) we can get the deserved inequality. □

In [6] the authors proposed an inverse iteration for solving (1.3):

$$\mathbf{x}_k = \alpha\mathcal{P}\mathbf{x}_{k-1}^{m-2}\mathbf{x}_k + (1-\alpha)\mathbf{v}. \tag{3.1}$$

The k-th error bound for the inverse iteration was given as follows:

Theorem 3.1. *[6] Let* $\mathcal{P}$ *be an order-*m *stochastic tensor, let* $\mathbf{v}$ *and* $\mathbf{x}_0$ *be stochastic vectors,* $\mathbf{x}_k$ *be generated by the inverse iterative (3.1). If* $\alpha < 1/(m-1)$*, then the multilinear PageRank model (1.3) has the unique solution* $\mathbf{x}$ *and*

$$\|\mathbf{x}-\mathbf{x}_k\|_1 \le \epsilon^k\|\mathbf{x}-\mathbf{x}_0\|_1, \tag{3.2}$$

where $\epsilon = \frac{(m-2)\alpha}{1-\alpha}$.

Next we further discuss convergence for the inverse iteration algorithm. Let $\mathbf{x}$ be a solution of (1.3). Then we can rewrite (1.3) as follows:

$$\mathbf{x} = \alpha\mathcal{P}\mathbf{x}^{m-2}\mathbf{x} + (1-\alpha)\mathbf{v}.$$

Let $\Delta\mathbf{x}_k = \mathbf{x}-\mathbf{x}_k$ and Δx_i be the i-th entry of $\Delta\mathbf{x}_k$. We have

$$\begin{aligned}\Delta\mathbf{x}_k &= \alpha[\mathcal{P}\mathbf{x}^{m-2}\mathbf{x}-\mathcal{P}\mathbf{x}_{k-1}^{m-2}\mathbf{x}_k]\\ &= \alpha[\mathcal{P}\mathbf{x}^{m-2}\mathbf{x}-\mathcal{P}\mathbf{x}^{m-2}\mathbf{x}_k+\mathcal{P}\mathbf{x}^{m-2}\mathbf{x}_k-\mathcal{P}\mathbf{x}_{k-1}^{m-2}\mathbf{x}_k]\\ &= \alpha[\mathcal{P}\mathbf{x}^{m-2}\Delta\mathbf{x}_k+\mathcal{P}[(\mathbf{x}^{m-2}-\mathbf{x}_{k-1}^{m-2})\mathbf{x}_k].\end{aligned} \tag{3.3}$$

Let $\mathcal{J}^2 = (\sigma^{(2)}_{i_1,i_3,\cdots,i_m})$ defined in Lemma 2.3. By an analogous proof to Lemma 2.5, we have

$$\begin{aligned}||\mathcal{P}\mathbf{x}^{m-2}\Delta\mathbf{x}_k||_1 &= \sum_{i_1=1}^{n}|\sum_{i_2,\cdots,i_m\in\langle n\rangle}(p_{i_1,i_2,\cdots,i_m}-\sigma^{(2)}_{i_1,i_3,\cdots,i_m})\Delta x_{i_2}x_{i_3}\cdots x_{i_m}|\\ &\le \sum_{i_1=1}^{n}\sum_{i_2,\cdots,i_m\in\langle n\rangle}|p_{i_1,i_2,\cdots,i_m}-\sigma^{(2)}_{i_1,i_3,\cdots,i_m}||\Delta x_{i_2}|x_{i_3}\cdots x_{i_m}\\ &\le \max_{i_2,\cdots,i_m\in\langle n\rangle}\sum_{i_1=1}^{n}|p_{i_1,i_2,\cdots,i_m}-\sigma^{(2)}_{i_1,i_3,\cdots,i_m}|||\Delta\mathbf{x}_k||_1.\end{aligned} \tag{3.4}$$

Algorithm 3.1 Computing $\mathcal{J}^{(k)}(k=3,\cdots,m)$ for $\overline{\mu}(\mathcal{J}^{(3)},\cdots,\mathcal{J}^{(m)})$.

Input $\mathcal{P}$.
for $k=3,\cdots,m$ **do**
  for $i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m\in\langle n\rangle$ **do**
    Let $A=(a_{i_1i_k})$ with $a_{i_1i_k}=p_{i_1,i_2,\cdots,i_k,\cdots,i_m}$. Get and solve the nonlinear optimization problem (2.20) and obtain the solution $\widehat{\mathbf{y}}$.
    Update $\sigma^{(k)}_{i_1,\cdots,i_{k-1},i_{k+1},\cdots,i_m}=\widehat{y}_{i_1}$.
  end for
end for
Compute $\widehat{\mu}_2=\overline{\mu}(\mathcal{J}^{(3)},\cdots,\mathcal{J}^{(m)})$.
Output $\widehat{\mu}_2$.

By taking $\sigma^{(2)}_{i_1,i_3,\cdots,i_m}=\min_{i_2}p_{i_1,i_2,i_3\cdots,i_m}$ and $\sigma^{(2)}_{i_1,i_3\cdots,i_m}=\max_{i_2}p_{i_1,i_2,i_3,\cdots,i_m}$ in (3.4) respectively, it is easy to see

$$||\mathcal{P}\mathbf{x}^{m-2}\Delta\mathbf{x}_k||_1\le\overline{\gamma}||\Delta\mathbf{x}_k||_1 \tag{3.5}$$

and

$$||\mathcal{P}\mathbf{x}^{m-2}\Delta\mathbf{x}_k||_1\le\check{\gamma}||\Delta\mathbf{x}_k||_1, \tag{3.6}$$

where

$$\overline{\gamma}=1-\min_{i_3,\cdots,i_m\in\langle n\rangle}\sum_{i=1}^{n}\min_{i_2\in\langle n\rangle}p_{i,i_2,\cdots,i_m}\ \text{and}\ \check{\gamma}=\max_{i_3,\cdots,i_m\in\langle n\rangle}\sum_{i=1}^{n}\max_{i_2\in\langle n\rangle}p_{i,i_2,\cdots,i_m}-1.$$

Furthermore, by Lemma 3.1, we have

$$||\alpha\mathcal{P}(\mathbf{x}^{m-2}-\mathbf{x}^{m-2}_{k-1})\mathbf{x}_k||_1\le\alpha\overline{\mu}(\mathcal{J}^{(3)},\cdots,\mathcal{J}^{(m)})||\Delta\mathbf{x}_{k-1}||_1.$$

Let $\gamma=\min\{\overline{\gamma},\check{\gamma}\}$. It is easy to check that $0<\gamma\le 1$. By (3.3)-(3.6), we obtain

$$\|\Delta\mathbf{x}_k\|_1\le\frac{\alpha\overline{\mu}(\mathcal{J}^{(3)},\cdots,\mathcal{J}^{(m)})}{1-\alpha\gamma}\|\Delta\mathbf{x}_{k-1}\|_1.$$

Hence we have the convergence theorem for the inverse iteration as follows:

Theorem 3.2. *Let $\mathcal{P}$ be an order-m stochastic tensor, let $\mathbf{v}$ and $\mathbf{x}_0$ be stochastic vectors, and let α satisfy (2.15). The inverse iteration algorithm (3.1) converges to the unique solution $\mathbf{x}$ of the multilinear PageRank problem* (1.3) *and*

$$\|\Delta\mathbf{x}_k\|_1\le\epsilon^k_{\mathcal{J}}\|\Delta\mathbf{x}_0\|_1, \tag{3.7}$$

where $\overline{\mu}=\min_{\mathcal{J}^{(k)},k=3,\cdots,m}\overline{\mu}(\mathcal{J}^{(3)},\cdots,\mathcal{J}^{(m)})$, $\epsilon_{\mathcal{J}}=\frac{\alpha\overline{\mu}}{1-\alpha\gamma}$.

We next revise Algorithm 2.1 by choosing some special $\mathcal{J}^{(k)}(k=3,\cdots,m)$ and obtain an upper bound of $\overline{\mu}$. From Algorithm 3.1, the following corollary can be derived.

Corollary 3.1. *Under the same assumption as in Theorem 3.2, we have*

$$\|\Delta\mathbf{x}_k\|_1\le\widehat{\epsilon}^k||\mathbf{x}_0\|_1, \tag{3.8}$$

where $\widehat{\epsilon}=\frac{\alpha\widehat{\mu}_2}{1-\alpha\gamma}$.

Note that

$$\overline{\mu}(\mathcal{J}^{(3)},\cdots,\mathcal{J}^{(m)})\le\max_{i_k\in\langle n\rangle}\sum_{k=3}^{m}\max_{i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m\in\langle n\rangle}\sum_{i_1=1}^{n}|p_{i_1,i_2,\cdots,i_k,\cdots,i_m}-\sigma^{(k)}_{i_1,i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m}|.$$

Then we take $\sigma^{(k)}_{i_1,\cdots,i_{k-1},i_{k+1},\cdots,i_m}=\min_{i_k}p_{i_1,\cdots,i_k,\cdots,i_m}$ and $\sigma^{(k)}_{i_1,\cdots,i_{k-1},i_{k+1},\cdots,i_m}=\max_{i_k}p_{i_1,\cdots,i_k,\cdots,i_m}$. Thus we have the following corollary.

Corollary 3.2. *Under the same assumption as in Theorem 3.2 we have*

$$\|\Delta \mathbf{x}_k\|_1 \leq (\min\{\overline{\epsilon}, \check{\epsilon}\})^k \|\Delta \mathbf{x}_0\|_1 \tag{3.9}$$

where

$$\overline{\epsilon} = \frac{\alpha \sum_{k=3}^{m}(1 - \min_{i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m \in \langle n\rangle} \sum_{i=1}^{n} \min_{i_k} p_{i,i_2,\cdots,i_k,\cdots,i_m})}{1-\alpha\gamma}$$

and

$$\check{\epsilon} = \frac{\alpha \sum_{k=3}^{m}(\max_{i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m \in \langle n\rangle} \sum_{i=1}^{n} \max_{i_k} p_{i,i_2,\cdots,i_k,\cdots,i_m} - 1)}{1-\alpha\gamma}.$$

Remark 3.1. Since

$$\sum_{k=3}^{m}(1 - \min_{i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m \in \langle n\rangle} \sum_{i=1}^{n} \min_{i_k} p_{i,i_2,\cdots,i_k,\cdots,i_m}) \leq m-2$$

and

$$1 - \alpha\gamma \geq 1 - \alpha,$$

the new proposed error bounds (3.9) are sharper than the one in (3.2).

Fasino and Tudisco gave an error analysis of an alternate higher-order power method for a second order Markov chain (see Theorem 6.3 in [5]). By an analogous technique to the one in [5], the error analysis for the inverse iteration method (3.1) for multilinear PageRank is also given as follows.

Corollary 3.3. *Let $\mathcal{P}$ be an order-3 stochastic tensor, and let $\mathbf{v}$ and $\mathbf{x}_0$ be stochastic vectors. If $\alpha(\mathcal{T}_L(\mathcal{P}) + \mathcal{T}_R(\mathcal{P})) < 1$, the inverse iteration algorithm (3.1) converges to the unique solution $\mathbf{x}$ of the multilinear PageRank problem (1.3) and*

$$\|\Delta \mathbf{x}_k\|_1 \leq \epsilon_{\mathcal{T}}^k \|\Delta \mathbf{x}_0\|_1, \tag{3.10}$$

where $\epsilon_{\mathcal{T}} = \frac{\alpha \mathcal{T}_R(\mathcal{P})}{1-\alpha\mathcal{T}_L(\mathcal{P})}$, $\mathcal{T}_L(\mathcal{P}) = \frac{1}{2}\max_{j,k_1,k_2} \sum_{i=1}^{n} |p_{i,j,k_1} - p_{i,j,k_2}|$ and $\mathcal{T}_R(\mathcal{P}) = \frac{1}{2}\max_{j_1,j_2,k} \sum_{i=1}^{n} |p_{i,j_1,k} - p_{i,j_2,k}|$.

Remark 3.2. As the same remarks as in Remark 2.7, we can't compare the bound (3.10) with (3.8)-(3.9) in theory. However, we will give some numerical examples in Subsection 5.2 to show these bounds.

4. Perturbation analysis

Perturbation analysis plays a critical role in numerical analysis, which studies the variation of solutions for a given problem when the data is perturbed. Alternatively, here we consider perturbation bounds for the multilinear PageRank problem (1.3).

The perturbed multilinear PageRank is given as follows:

$$\tilde{\mathbf{x}} = \alpha\tilde{\mathcal{P}}\tilde{\mathbf{x}}^{m-1} + (1-\alpha)\mathbf{v}, \tag{4.1}$$

where $\tilde{\mathbf{x}}$ is a perturbed stochastic vector of $\mathbf{x}$, and let $\Delta\mathbf{x} = \mathbf{x} - \tilde{\mathbf{x}}$ and $\Delta\mathcal{P} = \mathcal{P} - \tilde{\mathcal{P}}$. Then by (1.3) and (4.1) we have

$$\begin{aligned}\Delta\mathbf{x} &= \alpha(\mathcal{P}\mathbf{x}^{m-1} - \tilde{\mathcal{P}}\tilde{\mathbf{x}}^{m-1}) \\ &= \alpha(\mathcal{P}\mathbf{x}^{m-1} - \mathcal{P}\tilde{\mathbf{x}}^{m-1} + \mathcal{P}\tilde{\mathbf{x}}^{m-1} - \tilde{\mathcal{P}}\tilde{\mathbf{x}}^{m-1}) \\ &= \alpha[\mathcal{P}(\mathbf{x}^{m-1} - \tilde{\mathbf{x}}^{m-1}) + \Delta\mathcal{P}\tilde{\mathbf{x}}^{m-1}].\end{aligned} \tag{4.2}$$

By (4.2) and Lemma 2.4 we have

$$\Delta\mathbf{x} = \alpha[\sum_{k=2}^{m}(P_{x\tilde{x}}^{(k)} - J_{x\tilde{x}}^{(k)})\Delta\mathbf{x} + \Delta\mathcal{P}\tilde{\mathbf{x}}^{m-1}],$$

and thus we obtain

$$[I - \alpha \sum_{k=2}^{m} (P_{x\tilde{x}}^{(k)} - J_{x\tilde{x}}^{(k)})]\Delta \mathbf{x} = \alpha \Delta \mathcal{P} \tilde{\mathbf{x}}^{m-1}. \tag{4.3}$$

Under the uniqueness condition (2.15) one can show that the matrix $I - \alpha \sum_{k=2}^{m} (A_{x\tilde{x}}^{(k)} - J_{x\tilde{x}}^{(k)})$ is nonsingular. In fact, we only need to prove $\alpha \| \sum_{k=2}^{m} (P_{x\tilde{x}}^{(k)} - J_{x\tilde{x}}^{(k)})\|_1 < 1$, this inequality can be guaranteed by the uniqueness condition (2.15).

Therefore by (4.3) we have

$$\Delta \mathbf{x} = \alpha [I - \alpha \sum_{k=2}^{m} (P_{x\tilde{x}}^{(k)} - J_{x\tilde{x}}^{(k)})]^{-1} \Delta \mathcal{P} \tilde{\mathbf{x}}^{m-1}. \tag{4.4}$$

Furthermore, one can get that $I - \alpha \sum_{k=2}^{m} |P_{x\tilde{x}}^{(k)} - J_{x\tilde{x}}^{(k)}|$ is a nonsingular M-matrix. Thus we have

$$[I - \alpha \sum_{k=2}^{m} |P_{x\tilde{x}}^{(k)} - J_{x\tilde{x}}^{(k)}|]^{-1} \geq 0. \tag{4.5}$$

Combining (4.4) and (4.5) together gives

$$|\Delta \mathbf{x}| \leq \alpha [I - \alpha \sum_{k=2}^{m} |P_{x\tilde{x}}^{(k)} - J_{x\tilde{x}}^{(k)}|]^{-1} |\Delta \mathcal{P} \tilde{\mathbf{x}}^{m-1}|. \tag{4.6}$$

For an order-m dimension n tensor $\mathcal{A}$, the 1-norm and ∞-norm of a tensor are defined as follows.

$$||\mathcal{A}||_1 = \max_{i_2, i_3, \cdots, i_m \in \langle n \rangle} \sum_{i_1=1}^{n} |a_{i_1, i_2, \cdots, i_m}|,$$

and

$$||\mathcal{A}||_\infty = \max_{i_1 \in \langle n \rangle} \sum_{i_2, i_3, \cdots, i_m \in \langle n \rangle} |a_{i_1, i_2, \cdots, i_m}|.$$

Next we present the perturbation bound for the multilinear PageRank vectors.

Theorem 4.1. *Let $\mathcal{P}$ and $\tilde{\mathcal{P}} = \mathcal{P} + \Delta \mathcal{P}$ be order-m stochastic tensors and $\Delta \mathcal{P}$ be a perturbation of $\mathcal{P}$.*

(1) *If α satisfies (2.15), for any solution $\tilde{\mathbf{x}}$ of (4.1) the perturbation inequality (4.6) holds and the solution $\tilde{\mathbf{x}}$ of (4.1) satisfies*

$$\|\Delta \mathbf{x}\|_1 \leq \frac{\alpha \|\Delta \mathcal{P}\|_1}{1 - \alpha \mu}, \tag{4.7}$$

$$\|\Delta \mathbf{x}\|_\infty \leq \frac{\alpha \|\Delta \mathcal{P}\|_\infty}{1 - \alpha \nu} \tag{4.8}$$

and

$$\|\Delta \mathbf{x}\|_\infty \leq \alpha \|(I - \alpha P_\sigma)^{-1}\|_\infty \|\Delta \mathcal{P}\|_\infty, \tag{4.9}$$

where P_σ is an $n \times n$ matrix with

$$(P_\sigma)_{i,j} = \sum_{k=2}^{m} \min_{i_2, \cdots, i_{k-1}, i_{k+1}, \cdots, i_m \in \langle n \rangle} |p_{i, i_2, \cdots, i_{k-1}, j, i_{k+1}, \cdots, i_m} - \sigma_{i, i_2, \cdots, i_{k-1}, i_{k+1}, \cdots, i_m}^{(k)}|. \tag{4.10}$$

(2) *If α satisfies (2.30), then for a solution $\tilde{\mathbf{x}}$ of (4.1) we have*

$$\|\Delta \mathbf{x}\|_1 \leq \frac{\alpha \|\Delta \mathcal{P}\|_1}{1 - \alpha \omega}. \tag{4.11}$$

Proof. By (4.2), Lemma 2.5, and Lemma 2.6, we have

$$||\Delta \mathbf{x}||_1 \leq \alpha \mu ||\Delta \mathbf{x}||_1 + ||\Delta \mathcal{P}||_1,$$

$$||\Delta \mathbf{x}||_\infty \leq \alpha \nu ||\Delta \mathbf{x}||_\infty + ||\Delta \mathcal{P}||_\infty,$$

and

$$||\Delta\mathbf{x}||_1 \le \alpha\omega||\Delta\mathbf{x}||_1 + ||\Delta\mathcal{P}||_1.$$

If α satisfies (2.15) (or (2.30)), then (4.7) and (4.8) (or (4.11)) hold.

By (4.6), it is easy to check

$$\|\Delta\mathbf{x}\|_\infty \le \alpha\|[I-\alpha\sum_{k=2}^{m}|P_{x\tilde{x}}^{(k)} - J_{x\tilde{x}}^{(k)}|]^{-1}\|_\infty\|\Delta\mathcal{P}\tilde{\mathbf{x}}^{m-1}\|_\infty. \tag{4.12}$$

Since

$$|P_{x\tilde{x}}^{(k)} - J_{x\tilde{x}}^{(k)}|_{i,j} \le \max_{i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m\in\langle n\rangle}|p_{i,i_2,\cdots,i_{k-1},j,i_{k+1},\cdots,i_m} - \sigma^{(k)}_{i,i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m}|,$$

we have

$$I-\alpha\sum_{k=2}^{m}|P_{x\tilde{x}}^{(k)} - J_{x\tilde{x}}^{(k)}| \ge I-\alpha P_\sigma,$$

where P_σ is given by (4.10). By the uniqueness condition it is known that $I-\alpha P_\sigma$ is a nonsingular M-matrix. Moreover (e.g., see [3]),

$$(I-\alpha\sum_{k=2}^{m}|P_{x\tilde{x}}^{(k)} - J_{x\tilde{x}}^{(k)}|)^{-1} \le (I-\alpha P_\sigma)^{-1}. \tag{4.13}$$

Thus by (4.12) and (4.13) we have

$$\|\Delta\mathbf{x}\|_\infty \le \|(I-\alpha P_\sigma)^{-1}\|_\infty\|\Delta\mathcal{P}\tilde{\mathbf{x}}^{m-1}\|_\infty. \tag{4.14}$$

It is noted that

$$\|\Delta\mathcal{P}\tilde{\mathbf{x}}^{m-1}\|_\infty \le \|\Delta\mathcal{P}\|_\infty.$$

Hence by (4.14) we have

$$\|\Delta\mathbf{x}\|_\infty \le \|(I-\alpha P_\sigma)^{-1}\|_\infty\|\Delta\mathcal{P}\|_\infty.$$

This proves (4.9). $\square$

By taking the different set $\mathcal{J}^{(k)}(k=2,3,\cdots,m)$, we give the following corollary.

Corollary 4.1. *Let $\mathcal{P}$ and $\tilde{\mathcal{P}} = \mathcal{P}+\Delta\mathcal{P}$ be order-m stochastic tensors and $\Delta\mathcal{P}$ be a perturbation of $\mathcal{P}$.*

(1) *If α satisfies (2.24), the solution $\tilde{\mathbf{x}}$ of (4.1) satisfies*

$$\|\Delta\mathbf{x}\|_1 \le \frac{\alpha\|\Delta\mathcal{P}\|_1}{1-\alpha\widehat{\mu}_1}, \tag{4.15}$$

and

$$\|\Delta\mathbf{x}\|_\infty \le \frac{\alpha\|\Delta\mathcal{P}\|_\infty}{1-\alpha\widehat{\nu}_1}. \tag{4.16}$$

(2) *If α satisfies (2.25), the solution $\tilde{\mathbf{x}}$ of (4.1) satisfies*

$$\|\Delta\mathbf{x}\|_1 \le \frac{\alpha\|\Delta\mathcal{P}\|_1}{1-\alpha\sum_{k=2}^{m}(1-\min_{i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m\in\langle n\rangle}\sum_{i=1}^{n}\min_{i_k}p_{i,i_2,\cdots,i_k,\cdots,i_m})}. \tag{4.17}$$

(3) *If α satisfies (2.26), the solution $\tilde{\mathbf{x}}$ of (4.1) satisfies*

$$\|\Delta\mathbf{x}\|_1 \le \frac{\alpha\|\Delta\mathcal{P}\|_1}{1-\alpha\sum_{k=2}^{m}(\max_{i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m\in\langle n\rangle}\sum_{i=1}^{n}\max_{i_k}p_{i,i_2,\cdots,i_k,\cdots,i_m}-1)}. \tag{4.18}$$

(4) *If α satisfies (2.15), for any solution $\tilde{\mathbf{x}}$ of (4.1) the perturbation inequality (4.6) holds and the solution $\tilde{\mathbf{x}}$ of (4.1) satisfies*

$$\|\Delta\mathbf{x}\|_\infty \le \alpha\|(I-\alpha P_{\min})^{-1}\|_\infty\|\Delta\mathcal{P}\|_\infty \tag{4.19}$$

and

$$\|\Delta\mathbf{x}\|_\infty \le \alpha\|(I-\alpha P_{\max})^{-1}\|_\infty\|\Delta\mathcal{P}\|_\infty, \tag{4.20}$$

where P_σ is an $n\times n$ matrix with

$$(P_{\min})_{i,j} = \sum_{k=2}^{m} \min_{i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m\in\langle n\rangle}(p_{i,i_2,\cdots,i_{k-1},j,i_{k+1},\cdots,i_m} - \min_{i_k} p_{i,i_2,\cdots,i_{k-1},i_k,i_{k+1},\cdots,i_m}) \tag{4.21}$$

and

$$(P_{\max})_{i,j} = \sum_{k=2}^{m} \min_{i_2,\cdots,i_{k-1},i_{k+1},\cdots,i_m\in\langle n\rangle}(\max_{i_k} p_{i,i_2,\cdots,i_{k-1},i_k,i_{k+1},\cdots,i_m} - p_{i,i_2,\cdots,i_{k-1},j,i_{k+1},\cdots,i_m}). \tag{4.22}$$

Remark 4.1. For a stochastic tensor $\mathcal{P}$, let

$$\delta_m(\mathcal{P}) = \min_{S\subset\langle n\rangle}\{\min_{i_2,\cdots,i_m\in\langle n\rangle}\sum_{i\in S'} p_{i,i_2,\cdots,i_m} + \min_{i_2,\cdots,i_m\in\langle n\rangle}\sum_{i\in S} p_{i,i_2,\cdots,i_m}\}.$$

In [10], Li, Cui and Ng proposed a 1-norm perturbation bound for the limiting distribution of the higher-order Markov chain. Since the multilinear PageRank problem (1.3) can be rewritten as (1.5), the following perturbation bound can be followed from the perturbation bound in [10]

$$||\Delta\mathbf{x}||_1 \le \frac{||\Delta\mathcal{P}_{\mathbf{v}}||_1}{(m-1)\delta_m(\mathcal{P}_{\mathbf{v}})+2-m}, \tag{4.23}$$

where $\tilde{\mathcal{P}}_{\mathbf{v}} = \alpha\tilde{\mathcal{P}} + (1-\alpha)\mathcal{V}$ and $\Delta\mathcal{P}_{\mathbf{v}} = \mathcal{P}_{\mathbf{v}} - \tilde{\mathcal{P}}_{\mathbf{v}}$. Since

$$\begin{aligned}
\delta_m(\mathcal{P}_{\mathbf{v}}) &= \delta_m(\alpha\mathcal{P}+(1-\alpha)\mathcal{V}) \\
&= \min_{S\subset\langle n\rangle}\{\min_{i_2,\cdots,i_m\in\langle n\rangle}\sum_{i\in S'}[\alpha p_{i,i_2,\cdots,i_m}+(1-\alpha)v_i] + \min_{i_2,\cdots,i_m\in\langle n\rangle}\sum_{i\in S}[\alpha p_{i,i_2,\cdots,i_m}+(1-\alpha)v_i]\} \\
&= \min_{S\subset\langle n\rangle}\{\min_{i_2,\cdots,i_m\in\langle n\rangle}\sum_{i\in S'}\alpha p_{i,i_2,\cdots,i_m} + \min_{i_2,\cdots,i_m\in\langle n\rangle}\sum_{i\in S}\alpha p_{i,i_2,\cdots,i_m}\} + (1-\alpha)\sum_{i=1}^{n} v_i \\
&= \alpha\delta_m(\mathcal{P}) + 1 - \alpha,
\end{aligned}$$

and

$$||\Delta\mathcal{P}_{\mathbf{v}}||_1 = ||\alpha\mathcal{P}+(1-\alpha)\mathcal{V}-\alpha\tilde{\mathcal{P}}-(1-\alpha)\mathcal{V}||_1 = \alpha||\Delta\mathcal{P}||_1,$$

the bound (4.23) can be formulated as

$$||\Delta\mathbf{x}||_1 \le \frac{\alpha||\Delta\mathcal{P}||_1}{1-\alpha(m-1)(1-\delta_m(\mathcal{P}))}. \tag{4.24}$$

Remark 4.2. As shown as in Theorem 5.7 in [5], Fasino and Tudisco also proposed a 1-norm perturbation bound for the limiting distribution of the second order Markov chain. By analogous proofs to Theorem 5.7 in [5] and to (4.24), we can obtain the following bound (4.25).

If $\alpha\mathcal{T}(\mathcal{P}) < 1$, then the multilinear PageRank vector $\mathbf{x}$ is unique, and for any solution $\tilde{\mathbf{x}}$ of (4.1), we have

$$\|\Delta\mathbf{x}\|_1 \le \frac{\alpha||\Delta\mathcal{P}||_1}{1-\alpha\mathcal{T}(\mathcal{P})}, \tag{4.25}$$

where $\mathcal{T}(\mathcal{P})$ is defined in Remark 2.7. In order to compare these perturbation bounds we will give some numerical examples in Subsection 5.3.

5. Numerical examples

In this section, we show some numerical experiments to illustrate the effectiveness of the proposed theoretical results. All tests are conducted by MATLAB R2014a with a desktop computer (Dell optiplex 3020) which have the following configuration: Intel(R) Core(TM) i7-2600 CPU 3.40 GHz and 16.00G RAM.

Example 5.1. The first five examples (i)-(v) (e.g., see [9], [11], [12], [18]) are derived from the real world problems. Examples (i)-(iii) come from the DNA sequence data in [19] and [2]. Example (iv) is from the inter-personal relationship's data. Example (v) comes from the occupational mobility of physicist's data.

(i)

$$\mathcal{P}(:,:,1)=\begin{pmatrix}0.6000 & 0.4083 & 0.4935\\ 0.2000 & 0.2568 & 0.2426\\ 0.2000 & 0.3349 & 0.2639\end{pmatrix},\ \mathcal{P}(:,:,2)=\begin{pmatrix}0.5217 & 0.3300 & 0.4152\\ 0.2232 & 0.2800 & 0.2658\\ 0.2551 & 0.3900 & 0.3190\end{pmatrix},$$

$$\mathcal{P}(:,:,3)=\begin{pmatrix}0.5565 & 0.3648 & 0.4500\\ 0.2174 & 0.2742 & 0.2600\\ 0.2261 & 0.3610 & 0.2900\end{pmatrix};$$

(ii)

$$\mathcal{P}(:,:,1)=\begin{pmatrix}0.5200 & 0.2986 & 0.4462\\ 0.2700 & 0.3930 & 0.3192\\ 0.2100 & 0.3084 & 0.2346\end{pmatrix},\ \mathcal{P}(:,:,2)=\begin{pmatrix}0.6514 & 0.4300 & 0.5766\\ 0.1970 & 0.3200 & 0.2462\\ 0.1516 & 0.2500 & 0.1762\end{pmatrix},$$

$$\mathcal{P}(:,:,3)=\begin{pmatrix}0.5638 & 0.3424 & 0.4900\\ 0.2408 & 0.3638 & 0.2900\\ 0.1954 & 0.2938 & 0.2200\end{pmatrix};$$

(iii)

$$\mathcal{P}(:,:,1)=\begin{pmatrix}0.2091 & 0.2834 & 0.2194 & 0.1830\\ 0.3371 & 0.3997 & 0.3219 & 0.3377\\ 0.3265 & 0.0560 & 0.3119 & 0.2961\\ 0.1723 & 0.2608 & 0.1468 & 0.1832\end{pmatrix},\ \mathcal{P}(:,:,2)=\begin{pmatrix}0.1952 & 0.2695 & 0.2055 & 0.1690\\ 0.3336 & 0.3962 & 0.3184 & 0.3342\\ 0.2954 & 0.0249 & 0.2808 & 0.2650\\ 0.1758 & 0.3094 & 0.1953 & 0.2318\end{pmatrix},$$

$$\mathcal{P}(:,:,3)=\begin{pmatrix}0.3145 & 0.3887 & 0.3248 & 0.2883\\ 0.0603 & 0.1203 & 0.0451 & 0.0609\\ 0.3960 & 0.1255 & 0.3814 & 0.3656\\ 0.2293 & 0.3628 & 0.2487 & 0.2852\end{pmatrix},\ \mathcal{P}(:,:,4)=\begin{pmatrix}0.1685 & 0.2429 & 0.1789 & 0.1425\\ 0.3553 & 0.4180 & 0.3402 & 0.3559\\ 0.3189 & 0.0484 & 0.3043 & 0.2885\\ 0.1571 & 0.2907 & 0.1766 & 0.2131\end{pmatrix};$$

(iv)

$$\mathcal{P}(:,:,1)=\begin{pmatrix}0.5810 & 0.2432 & 0.1429\\ 0 & 0.4109 & 0.0701\\ 0.4190 & 0.3459 & 0.7870\end{pmatrix},\quad \mathcal{P}(:,:,2)=\begin{pmatrix}0.4708 & 0.1330 & 0.0327\\ 0.1341 & 0.5450 & 0.2042\\ 0.3951 & 0.3220 & 0.7631\end{pmatrix},$$

$$\mathcal{P}(:,:,3)=\begin{pmatrix}0.4381 & 0.1003 & 0\\ 0.0229 & 0.4338 & 0.0930\\ 0.5390 & 0.4659 & 0.9070\end{pmatrix};$$

(v)

$$\mathcal{P}(:,:,1)=\begin{pmatrix}0.9000 & 0.3340 & 0.3106\\ 0.0690 & 0.6108 & 0.0754\\ 0.0310 & 0.0552 & 0.6140\end{pmatrix},\quad \mathcal{P}(:,:,2)=\begin{pmatrix}0.6700 & 0.1040 & 0.0805\\ 0.2892 & 0.8310 & 0.2956\\ 0.0408 & 0.0650 & 0.6239\end{pmatrix},$$

$$\mathcal{P}(:,:,3)=\begin{pmatrix}0.6604 & 0.0945 & 0.0710\\ 0.0716 & 0.6133 & 0.0780\\ 0.2680 & 0.2922 & 0.8501\end{pmatrix};$$

Example 5.2. By the function *rand* of MATLAB, some order-3 dimension n stochastic tensors are generated.

We also give the numerical analysis for sparse tensors. The following two types of sparse tensors are used for testing the corresponding bounds.

Example 5.3. This example is given by Test 4 of [11] (or see [12]). Let Γ be a directed graph with node set $\mathbb{V}=\{1,2,\ldots n\}$. Then (i,j) is said to be an arc of Γ. Let $\mathbb{P}$ be the subset of $\mathbb{V}$ for which arbitrary two different nodes i, j, (i,j) is an arc in Γ, and let $\mathbb{D}$ be the set of all dangling nodes in $\mathbb{V}$, i.e., for any $i\in\mathbb{D}$, both (i,j) and (j,i) are not arcs of Γ for any $j\in\mathbb{V}$ (e.g., see [6], [8], [14], [13], [15] and [16]). By n_p $(n_p\geq 2)$ we denote the number of nodes in the subset $\mathbb{P}$. Suppose that a directed graph Γ is given by $\mathbb{V}=\mathbb{D}\bigcup\mathbb{P}$. Then one can construct a corresponding nonnegative tensor $\mathcal{A}$ as follows:

$$a_{i_1,i_2,\cdots,i_m}=\begin{cases}\upsilon_{i_1,i_2,\cdots,i_m}, & i_1\neq i_2,\, i_k\neq i_{k+1},\, i_1,\ i_k\in\mathbb{P},\ k=2,\cdots,m-1,\\ 0, & i_1=i_2\ (or\ i_1\in\mathbb{D}),\ i_k\neq i_{k+1},\ i_k\in\mathbb{P}, k=2,\cdots,m-1,\\ \frac{1}{n}, & else,\end{cases}$$

Table 5.1
Compare the proposed uniqueness conditions with the existing ones for (iv)-(v) in Example 5.1.

Example	$1/\widehat{\mu}_1$	$1/\widehat{\nu}_1$	(2.25)	(2.26)	(2.30)	(2.34)	(2.35)	(2.36)	(9) in [11]	(11) in [11]	(12) in [11]	$1/\mathcal{T}(\mathcal{P})$ in [5]
(iv)	**1.709402**	**1.709402**	**1.474926**	0.968054	0.854701	0.854701	0.854701	0.854701	0.584454	0.737463	0.484027	**1.709401**
(v)	**1.206127**	**1.206127**	**1.206127**	0.632071	0.603136	0.603136	0.603136	0.603136	0.414766	0.603136	0.316056	**1.206273**

Table 5.2
Compare the proposed uniqueness conditions with the existing ones by Example 5.2.

n	$1/\widehat{\mu}_1$	$1/\widehat{\nu}_1$	(2.25)	(2.26)	(2.30)	(2.34)	(2.35)	(2.36)	(9) in [11]	(11) in [11]	(12) in [11]	$1/\mathcal{T}(\mathcal{P})$ in [5]
3	0.743089	0.743089	0.724631	0.587861	**0.935437**	0.635534	0.635534	0.635534	0.505332	0.562953	0.458411	0.743089
5	0.722758	0.685285	0.624098	0.475379	**1.051038**	0.644482	0.644482	0.644482	0.474819	0.587186	0.39855	0.917655
8	0.808686	0.833864	0.588759	0.559018	**0.969624**	0.67851	0.67851	0.67851	0.454491	0.512925	0.40801	0.917098
10	0.751058	0.75722	0.563651	0.443124	**0.86204**	0.726348	0.726348	0.726348	0.387248	0.508496	0.312689	0.805905
12	0.751859	0.784676	0.55218	0.460356	0.804882	0.737056	0.737056	0.737056	0.404842	0.509015	0.336064	**0.806575**

Table 5.3
Compare the proposed uniqueness conditions with the existing ones by Example 5.2.

n	$1/\widehat{\nu}_1$	(2.25)	(2.26)	(9) in [11]	(11) in [11]	(12) in [11]
10	**0.712019**	0.595809	0.457731	0.403076	0.519549	0.329262
100	**0.849598**	0.507505	0.444854	0.427650	0.500106	0.373533
200	**0.876944**	0.504208	0.451571	0.440914	0.500023	0.394302
300	**0.891712**	0.502825	0.457447	0.448843	0.500012	0.407174
500	**0.910065**	0.501740	0.462457	0.456448	0.500004	0.419873
800	**0.921467**	0.501115	0.469030	0.464485	0.500002	0.433679
1000	**0.929432**	0.500911	0.471235	0.466742	0.500001	0.437632

Table 5.4
Compare the proposed uniqueness conditions with the existing ones by Example 5.3.

n_p	$1/\widehat{\nu}_1$	(2.25)	(2.26)	(9) in [11]	(11) in [11]	(12) in [11]
80	**0.586560**	0.504905	0.286232	0.333812	0.5	0.250539
100	**0.651050**	0.503989	0.302507	0.348820	0.5	0.267837
150	**0.796860**	0.502803	0.362419	0.389783	0.5	0.319381
180	**0.859840**	0.502278	0.408183	0.417020	0.5	0.357662

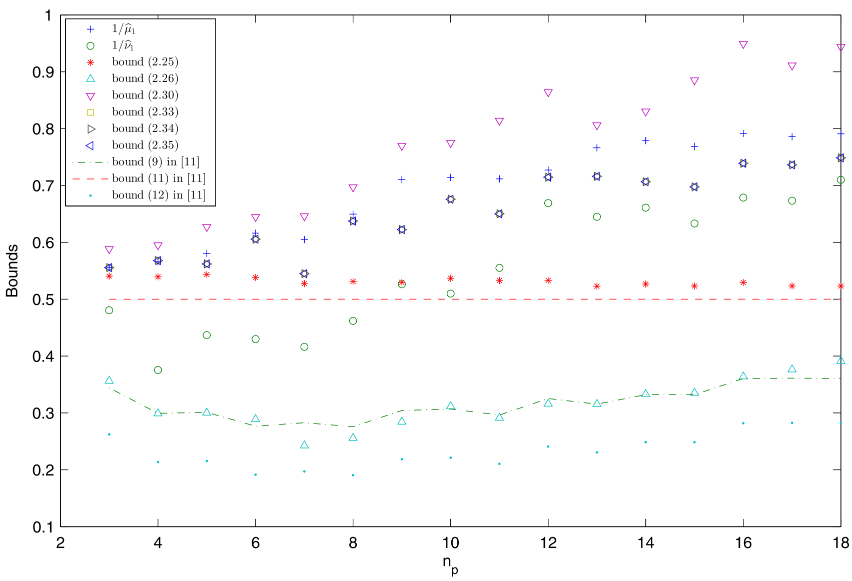

Fig. 5.1. Compare the proposed uniqueness conditions with the existing ones in [11].

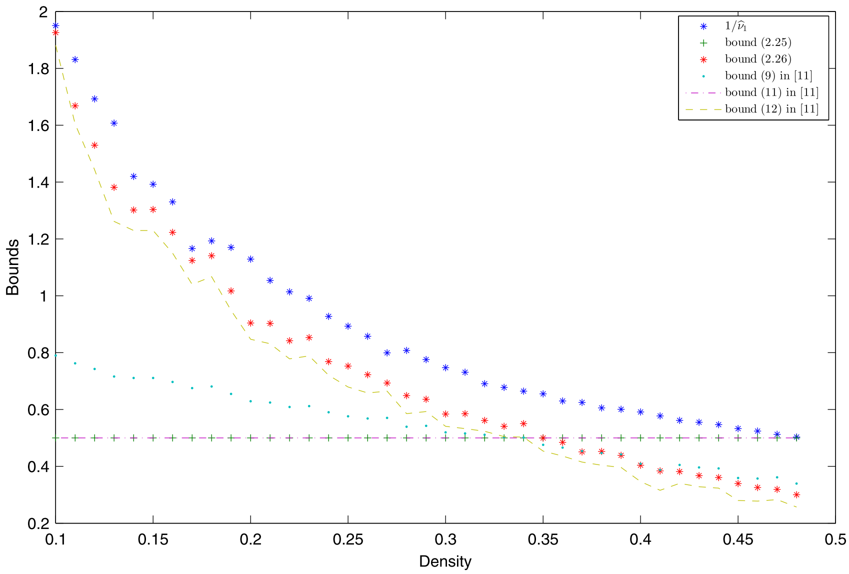

Fig. 5.2. Compare the proposed uniqueness conditions with the existing ones in [11].

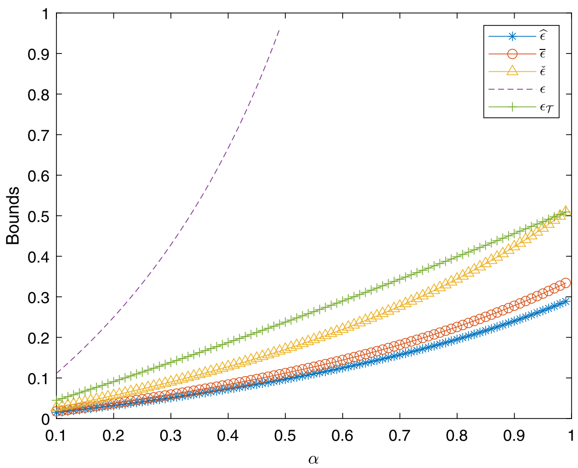

Fig. 5.3. Compare the proposed error bound with (3.2) in [6] and (3.10) by (iv) in Example 5.1.

where $\upsilon_{i_1,i_2,\cdots,i_m} \in (0,1)$, and then normalizing the entries with $p_{i_1,i_2,\cdots,i_m} = \frac{a_{i_1,i_2,\cdots,i_m}}{\sum_{i_1=1}^{n} a_{i_1,i_2,\cdots,i_m}}$ generates a stochastic tensor $\mathcal{P} = (p_{i_1,i_2,\cdots,i_m})$. It is noted that in the following numerical test, $\upsilon_{i_1,i_2,\cdots,i_m}$ is taken in $(0,1)$ randomly and independently.

Example 5.4. Let ϱ denote the density of zero entries for a tensor. If $\varrho = 0$ or $\varrho = 1$, $\mathcal{P}$ is a positive tensor or a zero tensor respectively. Test sparse tensors with different ϱ are generated by the function *randsample* of MATLAB.

5.1. Uniqueness conditions

Firstly, we compare the proposed uniqueness conditions with the existing ones in [6] and [11]. For (i)-(iii) in Example 5.1, the uniqueness of the solution for the multilinear PageRank model (1.3) has been guaranteed if $\alpha \in (0,1)$, but not for examples (iv) and (v). Therefore, we only report the numerical results in Table 5.1 for these two examples. It is shown that the proposed uniqueness conditions and the result in [5] are much better than other ones, and both the conditions (2.24) and (2.25) show that these two examples also have the unique solution for arbitrary α in $(0,1)$.

We also test the numerical results by Example 5.2. For a large dimension n, we only report the numerical results of the conditions $1/\widehat{\nu}$, (2.25) and (2.26). These results are reported in Tables 5.2 and 5.3. In Tables 5.2 and 5.1, we can find the proposed bounds and the bound given in [5] are sharper, but neither is the best.

In Example 5.3, we take $n = 20$, and $n_p = 3, 4, \ldots 18$, respectively, and compute the proposed uniqueness conditions and the existing ones, which are showed in Fig. 5.1. It is clear that the condition $1/\widehat{\mu}_1$ performs the best. The proposed

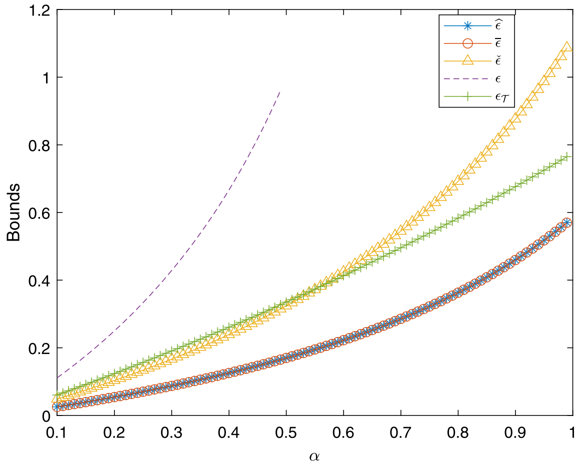

Fig. 5.4. Compare the proposed error bound with (3.2) in [6] and (3.10) by (v) in Example 5.1.

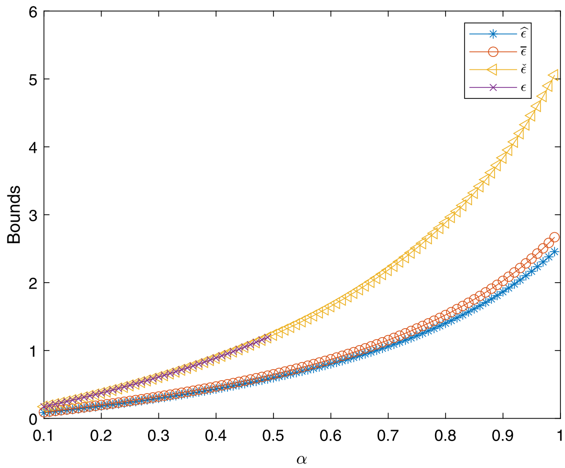

Fig. 5.5. Compare the proposed error bound with (3.2) in [6] by Example 5.3 if $n_p = 5$.

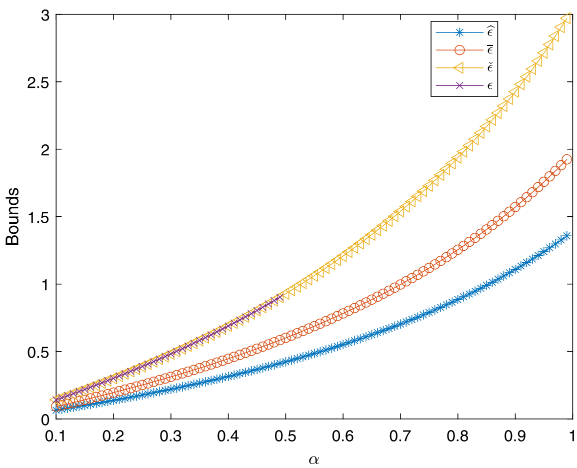

Fig. 5.6. Compare the proposed error bound with (3.2) in [6] by Example 5.3 if $n_p = 10$.

Table 5.5
Perturbation bounds on the 1-norm for (i)-(v) in Example 5.1 with different α.

Example	α	bound (4.11)	bound (4.15)	bound (4.17)	bound (4.18)	bound (4.24)
(i)	0.80	1.31E-06	**9.50E-07**	**9.50E-07**	**9.50E-07**	1.31E-06
	0.90	1.63E-06	**1.11E-06**	**1.11E-06**	**1.11E-06**	1.63E-06
	0.95	1.82E-06	**1.19E-06**	**1.19E-06**	**1.19E-06**	1.82E-06
	0.99	1.98E-06	**1.26E-06**	**1.26E-06**	**1.26E-06**	1.98E-06
	0.999	2.02E-06	**1.27E-06**	**1.27E-06**	**1.27E-06**	2.02E-06
(ii)	0.80	1.10E-06	**8.79E-07**	**8.79E-07**	**8.79E-07**	1.45E-06
	0.90	1.36E-06	**1.04E-06**	**1.04E-06**	**1.04E-06**	1.95E-06
	0.95	1.51E-06	**1.13E-06**	**1.13E-06**	**1.13E-06**	2.27E-06
	0.99	1.64E-06	**1.20E-06**	**1.20E-06**	**1.20E-06**	2.59E-06
	0.999	1.68E-06	**1.22E-06**	**1.22E-06**	**1.22E-06**	2.67E-06
(iii)	0.80	2.15E-06	**1.30E-06**	**1.51E-06**	**1.30E-06**	2.24E-06
	0.90	3.22E-06	**1.64E-06**	1.98E-06	**1.64E-06**	3.44E-06
	0.95	4.08E-06	**1.83E-06**	2.28E-06	**1.83E-06**	4.44E-06
	0.99	5.07E-06	**2.01E-06**	2.55E-06	**2.01E-06**	5.63E-06
	0.999	5.35E-06	**2.05E-06**	2.62E-06	**2.05E-06**	5.98E-06
(iv)	0.80	1.91E-05	**2.30E-06**	2.67E-06	7.04E-06	1.91E-05
	0.90	-	**2.90E-06**	3.52E-06	1.95E-05	-
	0.95	-	**3.26E-06**	4.08E-06	7.78E-05	-
	0.99	-	**3.59E-06**	4.60E-06	-	-
	0.999	-	**3.67E-06**	4.73E-06	-	-
(v)	0.80	-	**3.36E-06**	**3.36E-06**	-	-
	0.90	-	**5.02E-06**	**5.02E-06**	-	-
	0.95	-	**6.33E-06**	**6.33E-06**	-	-
	0.99	-	**7.82E-06**	**7.82E-06**	-	-
	0.999	-	**8.24E-06**	**8.24E-06**	-	-

Table 5.6
Perturbation bounds on the ∞-norm for (i)-(v) with different α.

Example	α	bound (4.16)	bound (4.19)	bound (4.20)
(i)	0.80	**1.84E-06**	1.96E-06	2.01E-06
	0.90	**2.14E-06**	2.31E-06	2.35E-06
	0.95	**2.31E-06**	2.49E-06	2.54E-06
	0.99	**2.44E-06**	2.65E-06	2.69E-06
	0.999	**2.47E-06**	2.68E-06	2.72E-06
(ii)	0.80	**2.13E-06**	2.30E-06	2.37E-06
	0.90	**2.52E-06**	2.74E-06	2.83E-06
	0.95	**2.74E-06**	2.98E-06	3.08E-06
	0.99	**2.91E-06**	3.18E-06	3.29E-06
	0.999	**2.95E-06**	3.23E-06	3.34E-06
(iii)	0.80	**3.36E-06**	5.26E-06	3.92E-06
	0.90	**4.02E-06**	6.75E-06	4.79E-06
	0.95	**4.37E-06**	7.65E-06	5.28E-06
	0.99	**4.68E-06**	8.46E-06	5.70E-06
	0.999	**4.75E-06**	8.65E-06	5.80E-06
(iv)	0.80	**3.91E-06**	4.73E-06	1.03E-05
	0.90	**4.94E-06**	6.28E-06	2.15E-05
	0.95	**5.56E-06**	7.27E-06	4.00E-05
	0.99	**6.12E-06**	8.22E-06	1.06E-04
	0.999	**6.25E-06**	8.45E-06	1.63E-04
(v)	0.80	**6.80E-06**	7.27E-06	9.47E-06
	0.90	1.01E-05	1.11E-05	**6.56E-06**
	0.95	1.28E-05	1.43E-05	**5.82E-06**
	0.99	1.58E-05	1.80E-05	**5.37E-06**
	0.999	1.66E-05	1.91E-05	**5.29E-06**

conditions are all better than the existing ones except $1/\widehat{\nu}_1$ if $n_p \leq 8$. We also report the numerical results in Table 5.4 for $n = 200$ and $n_p = 80, 100, 150$ and 180. The condition $1/\widehat{\nu}_1$ relatively performs well.

For Example 5.4, we generate order-3 dimension 100 tensors with the density ϱ from 0.1 to 0.48 at the interval of 0.01 respectively. The numerical results are shown in Fig. 5.2. From Table 5.4 and Fig. 5.2, it is seen that the condition $1/\widehat{\nu}_1$ always performs better than other ones.

Table 5.7
Perturbation bounds on the 1-norm via Example 5.2.

n	α	bound (4.11)	bound (4.15)	bound (4.17)	bound (4.18)	bound (4.24)	bound (4.25)
3	0.45	**7.24E-07**	8.69E-07	9.91E-07	8.87E-07	9.06E-07	8.69E-07
	0.75	**2.15E-06**	4.22E-06	1.05E-05	4.68E-06	5.28E-06	4.22E-06
	0.80	**2.63E-06**	6.61E-06	1.09E-04	7.83E-06	9.63E-06	6.61E-06
5	0.45	1.01E-06	1.01E-06	1.30E-06	1.74E-06	1.10E-06	**8.81E-07**
	0.75	7.26E-06	6.90E-06	-	-	1.72E-05	**3.48E-06**
	0.80	1.72E-05	1.53E-05	-	-	-	**4.82E-06**
8	0.45	**1.09E-06**	1.19E-06	2.39E-06	4.32E-06	1.22E-06	1.11E-06
	0.75	**8.86E-06**	2.71E-05	-	-	9.46E-05	1.08E-05
	0.80	**2.68E-05**	-	-	-	-	6.00E-05
10	0.45	**1.05E-06**	1.26E-06	2.81E-06	4.51E-06	1.46E-06	1.11E-06
	0.75	**6.18E-06**	2.428E-04	-	-	-	8.85E-06
	0.80	**1.14E-05**	-	-	-	-	2.57E-05
12	0.45	**1.12E-06**	1.34E-06	2.91E-06	-	1.48E-06	1.18E-06
	0.75	**8.25E-06**	-	-	-	-	1.28E-05
	0.80	**2.03E-05**	-	-	-	-	1.681E-04

Table 5.8
Perturbation bounds on the ∞-norm for Example 5.2.

n	α	bound (4.16)	bound (4.19)	bound (4.20)
3	0.45	1.51E-06	**1.18E-06**	9.96E-07
	0.75	4.89E-06	**2.53E-06**	1.82E-06
	0.80	6.19E-06	**2.83E-06**	1.97E-06
5	0.45	4.41E-06	**2.00E-06**	2.07E-06
	0.75	3.78E-03	**3.64E-06**	3.88E-06
	0.80	-	**3.94E-06**	4.22E-06
8	0.45	7.45E-06	**3.02E-06**	4.28E-06
	0.75	-	**5.52E-06**	9.73E-06
	0.80	-	**5.97E-06**	1.09E-05
10	0.45	9.18E-06	**4.09E-06**	6.50E-06
	0.75	-	**7.53E-06**	1.57E-05
	0.80	-	**8.17E-06**	1.78E-05
12	0.45	9.23E-06	**4.19E-06**	4.97E-06
	0.75	2.45E-04	**7.42E-06**	9.84E-06
	0.80	-	**8.00E-06**	1.08E-05

5.2. Error bounds

In this subsection, we compare the error bound based on the proposed uniqueness conditions with the existing bound ϵ in Theorem 3.1. We give numerical analysis for Example 5.1 (iv) and (v). Taking the value of α form 0.1 to 0.99 in the interval of 0.01, we drew Figs. 5.3 and 5.4. It can be seen that the proposed error bounds $\widehat{\epsilon}$ in Corollary 3.1 and $\overline{\epsilon}$, $\check{\epsilon}$ in Corollary 3.2 are always better than (3.2) in [6] and the proposed error bounds $\widehat{\epsilon}$ and $\overline{\epsilon}$ are always sharper than the bound $\epsilon_{\mathcal{T}}$ in Corollary 3.3.

Taking $n = 12$ and $n_p = 5$ and 10 in Example 5.3, we also drew Figs. 5.5 and 5.6 to show the error bound for these two examples. It is seen that the proposed error bounds are sharper than the existing bounds except the bound (3.9).

5.3. Perturbation bounds

Next, we give numerical experiments for the proposed perturbation bounds. For a given stochastic tensor $\mathcal{P}$, let $\mathcal{S}$ be randomly generated by Matlab such that

$$\sum_i s_{i,i_2,\cdots,i_m} = 0, \forall i, i_2, \cdots, i_m \in \langle n \rangle \tag{5.1}$$

with $s_{i,i_2,\cdots,i_m} \in (-1, 1)$. Therefor $\tilde{\mathcal{P}} = \mathcal{P} + \theta\mathcal{S}$ is still a stochastic tensor if θ is small enough. Let $\Delta\mathcal{P} = \theta\mathcal{S}$, where θ is a real number.

Firstly, for Example 5.1, taking $\theta = 10^{-6}$ and $\alpha = 0.80, 0.90, 0.95, 0.99$ and 0.999 respectively, the proposed bounds are computed and listed in Tables 5.5 and 5.6. The proposed bounds on the ∞-norm perform well in all cases. For the bounds with the 1-norm, (4.11), (4.18) and (4.24) fail if α is close to 1. The bounds (4.15) and (4.17) are always valid for these examples.

Table 5.9
Perturbation bounds on the 1-norm for Example 5.3.

n	α	bound (4.11)	bound (4.15)	bound (4.17)	bound (4.18)	bound (4.24)
12	0.25	**5.23E-07**	5.60E-07	6.94E-07	1.72E-06	5.72E-07
	0.45	**1.42E-06**	1.73E-06	4.30E-06	-	1.85E-06
	0.55	**2.32E-06**	3.29E-06	-	-	3.75E-06
	0.65	**4.14E-06**	8.79E-06	-	-	1.31E-05
	0.75	**9.80E-06**	-	-	-	-
15	0.25	**4.79E-07**	5.13E-07	6.70E-07	1.25E-06	5.62E-07
	0.45	**1.23E-06**	1.48E-06	4.55E-06	-	1.97E-06
	0.55	**1.90E-06**	2.58E-06	-	-	4.56E-06
	0.65	**3.06E-06**	5.35E-06	-	-	5.26E-05
	0.75	**5.56E-06**	2.49E-05	-	-	-
	0.85	**1.48E-05**	-	-	-	-
18	0.25	**3.74E-07**	4.02E-07	5.39E-07	7.48E-07	4.18E-07
	0.45	**9.20E-07**	1.11E-06	3.69E-06	-	1.24E-06
	0.55	**1.38E-06**	1.85E-06	-	-	2.23E-06
	0.65	**2.10E-06**	3.44E-06	-	-	5.02E-06
	0.75	**3.40E-06**	9.31E-06	-	-	6.08E-05
	0.85	**6.49E-06**	-	-	-	-
	0.95	**2.29E-05**	-	-	-	-
	0.99	**1.74E-04**	-	-	-	-

Table 5.10
Perturbation bounds on the ∞-norm for Example 5.3.

n	α	bound (4.16)	bound (4.19)	bound (4.20)
12	0.25	6.04E-06	**4.22E-06**	5.75E-06
	0.45	2.12E-05	**8.42E-06**	1.37E-05
	0.55	4.91E-05	**1.09E-05**	1.90E-05
	0.65	5.76E-04	**1.36E-05**	2.52E-05
	0.75	-	**1.67E-05**	3.26E-05
	0.85	-	**2.01E-05**	4.12E-05
	0.95	-	**2.41E-05**	5.11E-05
	0.99	-	**2.58E-05**	5.54E-05
	0.999	-	**2.62E-05**	5.64E-05
15	0.25	6.41E-06	**4.20E-06**	4.76E-06
	0.45	2.38E-05	**8.05E-06**	9.90E-06
	0.55	6.18E-05	**1.02E-05**	1.30E-05
	0.65	-	**1.24E-05**	1.64E-05
	0.75	-	**1.48E-05**	2.01E-05
	0.85	-	**1.74E-05**	2.43E-05
	0.95	-	**2.02E-05**	2.88E-05
	0.99	-	**2.13E-05**	3.07E-05
	0.999	-	**2.16E-05**	3.12E-05
18	0.25	6.69E-06	**4.61E-06**	5.25E-06
	0.45	2.04E-05	**8.57E-06**	1.07E-05
	0.55	3.80E-05	**1.06E-05**	1.40E-05
	0.65	9.53E-05	**1.28E-05**	1.75E-05
	0.75	-	**1.50E-05**	2.14E-05
	0.85	-	**1.73E-05**	2.57E-05
	0.95	-	**1.97E-05**	3.05E-05
	0.99	-	**2.07E-05**	3.25E-05
	0.999	-	**2.09E-05**	3.29E-05

We also generate some stochastic tensors such as in Example 5.2. The perturbation stochastic tensors are also produced via (5.1) with $\theta = 10^{-6}$. The corresponding numerical results are reported in Tables 5.7 and 5.8. Among all estimates, the bound (4.11) or (4.25) perform best with the 1-norm for these examples. For the ∞-norm case, the bound (4.19) is sharper than bounds (4.16) and (4.20).

Analogously, taking $n_p = 12, 15$ and 18, we give the numerical perturbation bounds for Example 5.3, respectively. The corresponding bounds are listed in Tables 5.9 and 5.10 which is seen that the bound (4.11) perform best with the 1-norm for these examples and the numerical results for the ∞-norm case are consistent with the ones appeared in Example 5.2.

6. Concluding remarks

The solution for the multilinear PageRank problem (1.3) is characterized by the corresponding PageRank vector. However, the verification of the uniqueness of the PageRank vector is cumbersome and often associates with the high computational cost. Recent studies suggest to develop some computable bounds as the uniqueness indicators. Current available results in the literature appear to be not tight enough that may prevent us from getting the desirable solutions. In this paper, we develop several uniqueness conditions of the solution for the multilinear PageRank problem with feasible computations, which improve the existing ones appeared in [11] and [6]. In addition, the error bound for the inverse iteration is given and is more suitable than the one given in [6]. Moreover, the perturbation bounds for the multilinear PageRank vector in different norms are derived, which characterizes the sensitivity under our framework.

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