Synchronization of Spatiotemporal Irregular Wave Propagation Via Boundary Coupling

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Wave dynamics reflect a broad spectrum of natural phenomena and are often characterized by wave equation such as in the development of meta-devices used to steer wave propagation. Modeling synchronization of wave dynamics is critical in various applications such as in communications and neuroscience. In this paper, we study the synchronization problem for oscillations governed by wave equation with nonlinear (van der Pol type) boundary conditions through a single boundary coupling. The dynamics of the master system is self-excited and presents sensitive and rapid oscillations. With the only signal received at one end of the boundary, by constructing a mathematical model, we show the existence of a slave system that can be synchronized with the master system via the study of wave reflections on the boundary to recover the actual wave dynamics. The coupling gain, which represents the strength of the connection between the master system and the slave system, has been identified. The obtained result can be also viewed as an observer construction when the measurable output is only on the boundary. Numerical simulations are provided to demonstrate the effectiveness of the theoretical outcomes. [DOI: 10.1115/1.4044923]

Keywords: vibrations, wave equation, synchronization, nonlinear van der Pol type boundary condition

1 Introduction

Synchronization is a rich phenomenon and a multidisciplinary discipline with broad range applications such as in physics, telecommunication, and neuroscience (see, e.g., Refs. [1–4] and references therein). There are many results for synchronizing non-spatiotemporal systems in the literature (e.g., see Ref. [5] and references therein), however, synchronizing spatiotemporal systems remains to be challenging and few results are available in the current literature. This is mainly due to the complexity of the spatiotemporal system (such as partial differential equations (PDEs)) as well as the restriction of available signals for the construction of a desirable slave system. In particular, for the chaotic systems which usually associate with high frequency (HF) oscillations, even two identical systems starting from slightly different initial conditions would evolve in time in an unsynchronized manner (e.g., see Ref. [6]) due to the weak stability [7]. In general, it is quite challenging to synchronize a spatiotemporal system when its state is only accessible in a finite number of locations such as measurement output from the boundary, which results in the so-called weak coupling (see Refs. [8–10] and references therein).

In this paper, we consider the synchronization problem of vibrations governed by the wave equation associated with nonlinear boundary condition in the form of

$$\begin{cases} w_{tt} - w_{xx} = 0, \quad x \in (0,1), \quad t > 0, \\ w_x(0,t) = -\eta w_t(0,t), \quad \eta \neq 1, \quad t > 0, \\ w_x(1,t) = \alpha w_t(1,t) - \beta w_t^3(1,t), \quad 0 < \alpha < 1,\ \beta \geq 0, \quad t > 0, \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \quad 0 \leq x \leq 1 \end{cases} \tag{1.1}$$

where α, β, and η are given real constants. When $\eta = 1$, the system (1.1) is not well-posed. This is mainly because in such a case, the initial conditions will coincide with one of the characteristic directions that leads to being indistinguishable with each other. More specifically, the general solution can be expressed as $w(x,t) = F(x-t) + G(x+t)$, that is, $x \pm t$ are characteristics of the system. If we plug it into the boundary condition at $x = 0$, one can see that

$$F'(-t) + G'(t) = -\eta(-F'(-t) + G'(t))$$

Then, $\eta = 1$ would lead to that $F'(-t)$ can be arbitrary while $G'(t) \equiv 0$. Hence, in such a case, the solution either does not exist or cannot be uniquely determined. This is true even for weak solutions, and detailed discussions can be found in Ref. [11]. Thus, throughout this paper, we assume $\eta \neq 1$. The wave equation itself is linear and represents the infinite-dimensional harmonic oscillator. The right-handed side boundary condition (at $x = 1$) is nonlinear when $\beta \neq 0$, which is usually called a *van der Pol* type boundary condition (see, e.g., Refs. [11–17]). The left-handed side boundary condition (at $x = 0$) is linear, where $\eta > 0$ indicates that energy is being injected into the system at $x = 0$. If we denote the total energy as

$$E(t) = \frac{1}{2}\int_0^1 |\nabla w(x,t)|^2 dx = \frac{1}{2}\int_0^1 \left[w_x^2(x,t) + w_t^2(x,t)\right] dx$$

and assume that Eq. (1.1) admits a classical solution (i.e., w has second continuous derivatives with respect to t and x, satisfying the system (1.1)), then by applying the boundary conditions, we have

$$\frac{d}{dt}E(t) = \eta w_t^2(0,t) + w_t^2(1,t)\left[\alpha - \beta w_t^2(1,t)\right]$$

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Thus, if $\eta > 0$, the system (1.1) has a self-excited mechanism that supplies energy to the system itself, which induces persistent (and irregular) vibrations [12,14]. Due to the energy expression, as long as $\beta > 0$(which is a more interesting case since the problem is nonlinear), it is not difficult to show that $w(\cdot, t)$ is bounded in $H^1(0,1)$ norm [18,19].

The existence and uniqueness of the classical solution of Eq. (1.1) can be found in Refs. [12] and [13]. Furthermore, the system (1.1) has a classical solution w if the initial data satisfy

$$w_0 \in \mathcal{C}_0^2([0,1]), \quad w_1 \in \mathcal{C}_0^1([0,1]) \tag{1.2}$$

where

$$\begin{aligned}\mathcal{C}_0^k([0,1]) = \{f \in \mathcal{C}^k([0,1])\,|\,f^{(i)}(0) \\ = f^{(i)}(1) = 0,\ 0 \le i \le k\},\ i = 0,1, k = 1,2\end{aligned} \tag{1.3}$$

$f^{(0)} := f, f^{(i)}$ stands for the ith derivative for $i \neq 0$, and $||f||_{\mathcal{C}^0} = \max_{[0,1]}|f(x)|$. The weak solution as well as its numerical approximation is discussed in Ref. [11]. When $\beta = 0$, discussions on the stabilization and the reconstruction of initial state of Eq. (1.1) with similar boundary conditions can be found in Refs. [20–22].

The PDE system (1.1) represents a broad spectrum of mathematical models in real applications and has received considerable attention since it exhibits many interesting and complicated dynamical phenomena, such as limit cycles and chaotic behavior of (w_t, w_x) when the parameters α, β, and η assume certain values [12,14]. Different from dynamics of a system of ordinary differential equations (ODEs), this is a simple and useful infinite-dimensional model for the study of spatiotemporal behaviors as time develops. For instance, the propagation of acoustic waves in a long pipe satisfies the linear wave equation

$$\frac{\partial^2 w(x,t)}{\partial t^2} - \frac{\partial^2 w(x,t)}{\partial x^2} = 0$$

Its general solution is the d'Alembert solution

$$w(x,t) = F(x-t) + G(x+t)$$

where F and G are arbitrary functions. This solution describes a superposition of two traveling waves with arbitrary profiles, one propagating with unit speed to the left and the other with unit speed to the right. The boundary conditions appeared in Eq. (1.1) can create irregular acoustical vibrations [12,14,15]. This type of vibrations, for example, can be generated by noise signals radiated from underwater vehicles, and there are intensive research for the properties of acoustical vibrations in the current literature (see, e.g., Refs. [23–25] and references therein). Hence, the study of synchronization of this type of vibration is not only important but also may lead to a better understanding of the dynamics of acoustic systems.

More specifically, for instance, in the development of metadevices that are common to be used to steer wave propagation, the traditional models in 1D virtual space in the absence of body forces can be written as

$$\begin{aligned}\sigma(x,t) &= c\frac{\partial w}{\partial x}(x,t) \\ \frac{\partial \sigma}{\partial x}(x,t) &= \rho\frac{\partial^2 w}{\partial t^2}(x,t)\end{aligned}$$

where x is the virtual space coordinate, t denotes time, w is the displacement along the coordinate axis, ρ is the longitudinal stress, c is the stiffness, and σ is the mass density. When c and ρ are constants, it models the wave traveling along homogeneous media. A key issue for the effectiveness of long-distance communication is the frequency. HF for the range of radiofrequency electromagnetic waves (radio waves) between 3 and 30 MHz, is suitable to be used for long-distance communication and is used by international shortwave broadcasting stations, aviation communication, government time stations, weather stations, amateur radio, and citizens band services, among other uses (see, e.g., Ref. [26]). The boundary condition setting of Eq. (1.1) can produce sustainable high frequencies within a range value of parameters α and β (see the discussion below). Here, the "sustainable" means that there is no external energy required to do so. Therefore, synchronization in such cases is practically useful in order to recover the true signals (for the receivers) for the purpose of a physical realization in practices.

For most systems associated with spatiotemporal time-dependent variables, in practice only certain selected points in the interior or the boundary of the spatial domain are accessible for sensing, and thus, the linking of two systems (between the master and the slave) usually is quite restrictive. Thus, this leads to a significant challenge in the construction of a suitable slave system from the theoretical point of view to achieve a desirable synchronization for both systems.

In this paper, we consider the case in which the only available signal is given by

$$y(t) = \begin{bmatrix} w(0,t) \\ w_t(0,t) \end{bmatrix}, \quad t \ge 0 \tag{1.4}$$

that is, only the signal on the boundary $x = 0$ can be used as an input for a slave system, which is often seen in applications. Our goal is to seek a slave (or responding) system via input signal (1.4) to synchronize

(1) the gradient (w_x, w_t) of Eq. (1.1) and
(2) the gradient and the state (w, w_x, w_t) of Eq. (1.1).

According to Ref. [14], assuming $\beta > 0$ and $0 < \alpha < 1$, if one defines

$$\eta_0 = \frac{3\sqrt{3} - (1+\alpha)}{3\sqrt{3} + (1+\alpha)} \tag{1.5}$$

and either $\eta_0 \le \eta < 1$ or $1 < \eta \le \eta_0^{-1}$, the gradient of Eq. (1.1) presents chaotic spatiotemporal behaviors that reflect the complexity of the system dynamics, including high frequency oscillation as the time develops. Here, the characterization of chaos is to use the total variation $V_I(f)$ of a function f on an interval I, which is defined to be the supremum of all sums

$$\sum_{k=1}^{m} |f(x_k) - f(x_{k-1})|$$

with respect to all partitions $\{x_k\}$ on I. Chen et al. [27,28] show that both $V_I(w_t(\cdot, t))$ and $V_I(w_x(\cdot, t))$ on a given interval I (spatial variable in I) grow exponentially with respect to t, thus they appear to be chaotic in the sense of Li-Yorke's definition.

By making use of the Riemann invariant approach, the chaotic dynamics of (w_t, w_x) can be generated by an iterated map with respect to time t (for more discussion, see Ref. [11], e.g., let $\alpha = 0.5$, $\beta = 1$, and $\eta = 0.58$). To synchronize such sensitive high frequency (spatial dependence) dynamics via boundary signal (1.4) usually is challenging and required subtle analysis of wave traveling.

In Sec. 3, we will construct a slave system that synchronizes (w_x, w_t) and (w, w_x, w_t) of Eq. (1.1), respectively, after a short transition period, regardless of the choice of the initial condition of Eq. (1.1).

We mention here that the approach of this paper can also be viewed as observer construction for Eq. (1.1) in terms of the output measurement (1.4). It is significantly broader than our recent work for the observer design [19]. In Ref. [19], we are not able to construct a dynamical system such that its state $\hat{w}$ converges to the state w (1.1) with (1.4). In this paper, a dynamical (slave) system is constructed (by a delay input) and its state converges to the state of the master system (1.1) globally and exponentially. The introduced delay output reflects a fundamental characteristic from the viewpoint of wave reflections on the boundary: in order to synchronize the wave dynamics of Eq. (1.1), one not only needs to know the outgoing wave at the boundary $x=0$ but also the incoming wave at $x=0$ that is characterized by the delay. Recently, the delay introduced in wave system is also used to stabilize the wave equation when the boundary condition is linear, see Ref. [29], whose approach is not applicable to our case due to the nonlinear boundary condition ($\beta \neq 0$) in our model.

The paper will be organized as follows. In Sec. 2, we will present the main result of our developed synchronizer that uses the boundary observation of Eq. (1.1) at $x=0$ as an input signal. Theoretical justification will be provided in Sec. 3. The main approach is to convert the system (1.1) to a first-order hyperbolic system by making use of the Riemann invariant transformation and to study the wave reflection on the boundary. The coupling gain that determines the exponential transition rate is obtained. In Sec. 4, numerical examples are provided to demonstrate the effectiveness of the proposed approach. In Sec. 5, we present a general approach to synchronize the state (w_x, w_t, w) of Eq. (1.1) via setting delay as a parameter. Further numerical simulations are also provided. The paper ends with concluding remarks in Sec. 6.

2 Main Result for Modeling: Synchronizer

By studying the wave reflections on the boundary, we construct the following synchronized slave system (Eq. (1.1)):

$$\begin{cases} \hat{w}_{tt} - \hat{w}_{xx} = 0, \ x \in (0,1), \ t > 0, \\ \hat{w}_x(0,t) = L(t), \ t > 0, \\ \hat{w}_x(1,t) = R(t), \ t > 0, \\ \hat{w}(x,0) = \hat{w}_0(x), \quad \hat{w}_t(x,0) = \hat{w}_1(x), \quad 0 < x < 1 \end{cases} \tag{2.1}$$

where

$$L(t) = (k-\eta)\hat{w}_t(0,t) + (k-\eta+1)\gamma\hat{w}(0,t) - kw_t(0,t) - (k-\eta+1)\gamma w(0,t), \ t > 0 \tag{2.2}$$

and

$$R(t) = \begin{cases} \alpha\hat{w}_t(1,t) - \beta[\hat{w}_t(1,t) + \gamma(\hat{w}(0,0) - w(0,0))]^3 \\ \quad +(\alpha+1)\gamma(\hat{w}(0,0) - w(0,0)), \ t < 1, \\ \alpha\hat{w}_t(1,t) - \beta[\hat{w}_t(1,t) + \gamma(\hat{w}(0,t-1) - w(0,t-1))]^3 \\ \quad +(\alpha+1)\gamma(\hat{w}(0,t-1) - w(0,t-1)), \ t \geq 1 \end{cases} \tag{2.3}$$

and $\hat{w}_0 \in \mathcal{C}^2([0,1])$ and $\hat{w}_1 \in \mathcal{C}^1([0,1])$. Both constants k and γ are parameters. It is easy to see that when $k = \gamma = 0$, system (2.1) is identical to Eq. (1.1). Let $w_0 \in \mathcal{C}^2([0,1])$ and $w_1 \in \mathcal{C}^1([0,1])$ be the initial states of Eq. (1.1). If we have

(1) the initial data $e_0 = w_0 - \hat{w}_0 \in \mathcal{C}_0^2([0,1])$ and $e_1 = w_1 - \hat{w}_1 \in \mathcal{C}_0^1([0,1])$, and
(2) the parameter k is chosen so that

$$\left| \frac{k-\eta-1}{k-\eta+1} \cdot \frac{1+\alpha}{1-\alpha} \right| \leq 1$$

then, when $\gamma=0$, we have that $(\hat{w}_x, \hat{w}_t)$ of Eq. (2.1) synchronizes (w_x, w_t) of Eq. (1.1), and when $\gamma > 0$, we have that $(\hat{w}, \hat{w}_x, \hat{w}_t)$ of Eq. (2.1) synchronizes (w, w_x, w_t) of Eq. (1.1), after a short transition period of time, respectively. Notice that the synchronizer (2.1) only receives output signal at $x=0$ from the master system (1.1).

Remark 2.1. Here, we use delay= 1 as the normalized wave speed in our system is 1. In Sec. 5, we further discuss the case in which more delay is introduced. ∎

3 Theoretical Approach

In this section, we will provide a detailed justification of our proposed approach. Let us denote the error of states between Eqs. (1.1) and (2.1) to be

$$e := w - \hat{w} \tag{3.1}$$

Then, the error dynamics satisfies the following wave equation:

$$\begin{cases} e_{xx}(x,t) - e_{tt}(x,t) = 0, \quad 0 < x < 1, \ t > 0, \\ e_x(0,t) = (k-\eta)e_t(0,t) + (k-\eta+1)\gamma e(0,t), \ t > 0, \\ e_x(1,t) = \begin{cases} h(t)(e_t(1,t) + \gamma e(0,0)) + \gamma e(0,0), & t < 1, \\ h(t)(e_t(1,t) + \gamma e(0,t-1)) + \gamma e(0,t-1), & t \geq 1, \end{cases} \\ e(\cdot,0) = e_0(\cdot) \in \mathcal{C}_0^2([0,1]), \ e_t(\cdot,0) = e_1(\cdot) \in \mathcal{C}_0^1([0,1]) \end{cases} \tag{3.2}$$

where $e_0 = w_0 - \hat{w}_0$, $e_1 = w_1 - \hat{w}_1$, and $h(t) := \alpha - \beta\zeta(t)$ with

$$\zeta(t) := \begin{cases} w_t^2(1,t) + w_t(1,t)(\hat{w}_t(1,t) - \gamma e(0,0)) \\ +(\hat{w}_t(1,t) - \gamma e(0,0))^2, & t \leq 1 \\ w_t^2(1,t) + w_t(1,t)(\hat{w}_t(1,t) - \gamma e(0,t-1)) \\ +(\hat{w}_t(1,t) - \gamma e(0,t-1))^2, & t > 1 \end{cases} \tag{3.3}$$

It is not difficult to see that $\zeta(t) \geq 0$ and $-\infty < h(t) \leq \alpha$.

THEOREM 3.1. *Assume $\alpha \in (0,1)$, $\beta > 0$, $k - \eta \neq -1$, and $\gamma \geq 0$. For any initial data $e_0 \in \mathcal{C}_0^2([0,1])$ and $e_1 \in \mathcal{C}_0^1([0,1])$, we have*

(1) *When $\gamma=0$, the error dynamics (e_t, e_x) is asymptotically stable in $\mathcal{C}^0$-norm, i.e.,*

$$\lim_{t\to+\infty} (||e_x(\cdot,t)||_{\mathcal{C}^0} + ||e_t(\cdot,t)||_{\mathcal{C}^0}) = 0 \tag{3.4}$$

(2) *When $\gamma > 0$, the error dynamics (e, e_t, e_x) is asymptotically stable in $\mathcal{C}^0$-norm, i.e.,*

$$\lim_{t\to+\infty} (||e(\cdot,t)||_{\mathcal{C}^0} + ||e_x(\cdot,t)||_{\mathcal{C}^0} + ||e_t(\cdot,t)||_{\mathcal{C}^0}) = 0 \tag{3.5}$$

if and only if

$$\left| \frac{k-\eta-1}{k-\eta+1} \cdot \frac{1+\alpha}{1-\alpha} \right| \leq 1 \tag{3.6}$$

Proof. For the simplicity of later discussion, we denote $\xi = k - \eta$. We define two variables (U, V) on $[0,1] \times [0,\infty)$ to be

$$U(x,t) = \frac{e_x(x,t) + e_t(x,t)}{2} \tag{3.7}$$

and

$$V(x,t) = \begin{cases} \dfrac{e_x(x,t) - e_t(x,t)}{2} - \gamma e(0,0), & t < x, \\ \dfrac{e_x(x,t) - e_t(x,t)}{2} - \gamma e(0,t-x), & t \geq x \end{cases} \tag{3.8}$$

Note that U and V are invariant along the characteristics $x+t$ =constant and $x-t$ =constant of Eq. (3.2), respectively, and they are called *Riemann invariants*. The boundary condition of Eq. (3.2) provides the relationship at the left-end $x=0$ and at the right-end $x=1$, respectively, as

$$V(0,t)=\frac{\xi-1}{\xi+1}U(0,t),\quad U(1,t)=\frac{h(t)+1}{h(t)-1}V(1,t) \tag{3.9}$$

Since $\forall t\geq 0$, $h(t)\leq\alpha<1$, one can see that $U(1,t)$ is well-defined. For the convenience of following discussion, we denote

$$F_n(t):=F(h(t-(2n-1)))=\frac{h(t-(2n-1))+1}{h(t-(2n-1))-1},\quad n=0,1,2,\ldots,\ \text{and } t\geq 2n-1 \tag{3.10}$$

By noticing that F is a decreasing function of h, a direct estimation yields

$$|F_n(t)|\leq\left|\frac{\alpha+1}{\alpha-1}\right| \tag{3.11}$$

When $\xi\neq -1$, for $t=2n+\tau$, $n=0,1,2,\ldots$, $0\leq\tau\leq 2$, by the method of characteristics and the induction, the solution (U, V) can be expressed explicitly as follows:

$$U(x,t)=\begin{cases}F_1(x+t)\cdots F_n(x+t)\left(\frac{\xi-1}{\xi+1}\right)^n(U_0(x+\tau)), & 0\leq\tau\leq 1-x,\\ F_1(x+t)\cdots F_{n+1}(x+t)\left(\frac{\xi-1}{\xi+1}\right)^n(V_0(2-x-\tau)), & 1-x<\tau\leq 2-x,\\ F_1(x+t)\cdots F_{n+1}(x+t)\left(\frac{\xi-1}{\xi+1}\right)^{n+1}(U_0(x+\tau-2)), & 2-x<\tau\leq 2\end{cases} \tag{3.12}$$

and

$$V(x,t)=\begin{cases}F_1(t-x)\cdots F_n(t-x)\left(\frac{\xi-1}{\xi+1}\right)^n(V_0(x-\tau)), & 0\leq\tau\leq x,\\ F_1(t-x)\cdots F_n(t-x)\left(\frac{\xi-1}{\xi+1}\right)^{n+1}(U_0(\tau-x)), & x<\tau\leq 1+x,\\ F_1(t-x)\cdots F_{n+1}(t-x)\left(\frac{\xi-1}{\xi+1}\right)^{n+1}(V_0(x-\tau+2)), & 1+x<\tau\leq 2\end{cases} \tag{3.13}$$

where (U_0, V_0) is the initial data. Denote $M=\max_{x\in[0,1]}\{|U_0(x)|,|V_0(x)|\}$. Then, Eqs. (3.12) and (3.13) imply that for $t=2n+\tau$ we have

$$\begin{aligned}\left|U(x,t)\right| &\leq\left|\frac{(\xi-1)}{(\xi+1)}\frac{(\alpha+1)}{(\alpha-1)}\right|^n\cdot\left|\frac{\alpha+1}{\alpha-1}\right|M,\\ \left|V(x,t)\right| &\leq\left|\frac{(\xi-1)}{(\xi+1)}\frac{(\alpha+1)}{(\alpha-1)}\right|^n\cdot\left|\frac{\xi-1}{\xi+1}\right|M\end{aligned}$$

Thus, when

$$\left|\frac{(\xi-1)}{(\xi+1)}\frac{(\alpha+1)}{(\alpha-1)}\right|<1$$

we then have

$$\lim_{t\to\infty}(||U(\cdot,t)||_{C^0}+||V(\cdot,t)||_{C^0})=0$$

Next, we assume that

$$\left|\frac{(\xi-1)}{(\xi+1)}\frac{(\alpha+1)}{(\alpha-1)}\right|=1$$

Let us denote

$$g(t)=|U(1,t)|$$

In the following, we will show that $\lim_{t\to\infty}g(t)=0$.

Without loss of generality, we may assume $\max_{x\in[0,1]}\{|U_0(x)|, |V_0(x)|\}\leq|\alpha-1/\alpha+1|$ for simplicity by scaling both U_0 and V_0. We proceed it by contradiction. Suppose $\lim_{t\to\infty}g(t)\neq 0$, then there exists $\varepsilon_0>0$ such that for any given $T>0$, $\exists t_0>T$ so that we have $g(t_0)\geq\varepsilon_0$. Let $t_0>0$ satisfy $g(t_0)=|U(1,t_0)|\geq\varepsilon_0$.

Let ε be small enough with $\varepsilon_0>\varepsilon>0$ and denote

$$\delta=\frac{2\beta\varepsilon^2}{32(1-\alpha)^2+(1-\alpha)\beta\varepsilon^2},\quad N=\left[\frac{\ln\varepsilon}{\ln\left(1-\delta\left|\frac{\xi-1}{\xi+1}\right|\right)}\right]+1.$$

Let us choose $T=2N+\tau_0$ with $0\leq\tau_0<2$. Choose a $t_0=2n_0+\tau$ with $0\leq\tau<2$ such that $t_0>T+2$ and $g(t_0)=|U(1,t_0)|\geq\varepsilon_0$. Thus, $t_0>T+2$ implies $n_0>N$.

Now, we consider the set

$$S=\{|F_m(t_0+1)|,\text{where } 1\leq m\leq n_0+1\}.$$

If all elements in S satisfy the following:

$$|F_m(t_0+1)|\leq\left|\frac{\alpha+1}{\alpha-1}\right|-\delta$$

then, according to Eq. (3.12) and the choice of N, we would have

$$\begin{aligned}g(t_0)=|U(1,t_0)|&\leq\left(\left(\left|\frac{\alpha+1}{\alpha-1}\right|-\delta\right)\left|\frac{\xi-1}{\xi+1}\right|\right)^{n_0}\\ &\leq\left(1-\delta\left|\frac{\xi-1}{\xi+1}\right|\right)^{n_0}<\left(1-\delta\left|\frac{\xi-1}{\xi+1}\right|\right)^N\leq\varepsilon<\varepsilon_0\end{aligned} \tag{3.14}$$

which leads to a contradiction.

Therefore, there must exist a smallest m with $1 \le m \le n_0 + 1$, such that

$$|F_m(t_0+1)| \in \left(\left|\frac{\alpha+1}{\alpha-1}\right| - \delta, \quad \left|\frac{\alpha+1}{\alpha-1}\right| \right]$$

First, we denote $\Delta = |\alpha + 1/\alpha - 1| - |F_m(t_0+1)|$. When ε is small enough (so is δ), one has $h(t_0 - 2m + 2) \ge 0$. Thus, we have the expression

$$|F_m(t_0+1)| = \frac{1 + h(t_0 - 2m + 2)}{1 - h(t_0 - 2m + 2)}$$

Hence, we arrive at

$$\Delta = \left|\frac{\alpha+1}{\alpha-1}\right| - |F_m(t_0+1)| = \frac{\alpha+1}{1-\alpha} - \frac{\alpha+1-\beta\zeta(t_0-2m+2)}{1-\alpha+\beta\zeta(t_0-2m+2)}$$

where ζ is defined by Eq. (3.3). For simplicity and clarity, we write $\zeta(t_0 - 2m + 2)$ briefly as ζ_0

$$\Delta = \frac{\alpha+1}{1-\alpha} - \frac{\alpha+1-\beta\zeta_0}{1-\alpha+\beta\zeta_0} = \frac{2\beta\zeta_0}{(1-\alpha)(1-\alpha+\beta\zeta_0)} < \delta$$

which implies, by the choice of δ, that

$$\zeta_0 < \frac{\varepsilon^2}{32}$$

Following from the fact $(a^2 + b^2/2) \le a^2 + ab + b^2$ and the definition of ζ_0, we have

$$|w_t(1, t_0 - 2m + 2)| < \frac{\varepsilon}{4},$$
$$|\hat{w}_t(1, t_0 - 2m + 2) - \gamma e(0, t_0 - 2m + 1)| < \frac{\varepsilon}{4}$$

By the boundary conditions of Eqs. (1.1) and (2.1), we obtain

$$|w_x(1, t_0 - 2m + 2)| < \alpha\frac{\varepsilon}{4} + \beta\left(\frac{\varepsilon}{4}\right)^3 < \frac{\varepsilon}{4},$$
$$|\hat{w}_x(1, t - 2m + 2) + \gamma e(0, t_0 - 2m + 1)| < \alpha\frac{\varepsilon}{4} + \beta\left(\frac{\varepsilon}{4}\right)^3 < \frac{\varepsilon}{4}$$

when ε is small enough. Let u and $\hat{u}$ be the Riemann invariants of Eqs. (1.1) and (2.1), respectively, that is,

$$u = \frac{w_x + w_t}{2}, \quad \hat{u} = \frac{\hat{w}_x + \hat{w}_t}{2}$$

Then, it yields

$$|u(1, t_0 - 2m + 2)| = \left|\frac{w_x(1, t_0 - 2m + 2) + w_t(1, t_0 - 2m + 2)}{2}\right| < \frac{\varepsilon}{2}$$

and

$$|\hat{u}(1, t_0 - 2m + 2)| = \left|\frac{\hat{w}_x(1, t_0 - 2m + 2) + \hat{w}_t(1, t_0 - 2m + 2)}{2}\right| < \frac{\varepsilon}{2},$$

respectively. Thus, this leads to the Riemann invariant U of Eq. (3.2) satisfying

$$|U(1, t_0 - 2m + 2)| \le |u(1, t_0 - 2m + 2)| + |\hat{u}(1, t_0 - 2m + 2)| < \varepsilon \tag{3.15}$$

which thus implies that

$$|U(1, t_0)| = \left| \left(\frac{\xi - 1}{\xi + 1}\right)^{m-1} \Pi_{i=1}^{m-1} F_i(1 + t_0) U(1, t_0 - 2m + 2) \right|$$
$$\le |U(x_0, t_0 - 2m + 2)| < \varepsilon \tag{3.16}$$

and gives $g(t_0) = |U(1, t_0)| < \varepsilon$, and this leads to a contradiction.

Therefore, we have

$$\lim_{t\to\infty} |U(1, t)| = 0 \tag{3.17}$$

and obtain

$$\lim_{t\to\infty} \sup_{x\in[0,1]} |U(x, t)| = \lim_{t\to\infty} \sup_{x\in[0,1]} |U(1, t + x - 1)| = 0 \tag{3.18}$$

Following from the reflection at the left end $x = 0$, we have

$$\lim_{t\to\infty} |V(0, t)| = 0 \tag{3.19}$$

and hence

$$\lim_{t\to\infty} \sup_{x\in[0,1]} |V(x, t)| = \lim_{t\to\infty} \sup_{x\in[0,1]} |V(0, t - x)| = 0 \tag{3.20}$$

When $\gamma = 0$, for $t > 1$ and $x \in [0, 1]$, the following expressions

$$e_x(x, t) = U(x, t) + V(x, t),$$
$$e_t(x, t) = U(x, t) - V(x, t)$$

imply

$$\lim_{t\to+\infty} (||e_x(\cdot, t)||_{\mathcal{C}^0} + ||e_t(\cdot, t)||_{\mathcal{C}^0}) = 0$$

If we choose k such that

$$r := \left| \frac{k - \eta - 1}{k - \eta + 1} \cdot \frac{1 + \alpha}{1 - \alpha} \right| < 1$$

then, by the expressions (3.12) and (3.13), one can see the convergence is exponentially decaying with the rate $\ln(1/\sqrt{r})$.

If $\gamma > 0$, $e(0, t)$ can be obtained as

$$e(0, t) = e^{-\gamma t} \int_0^t e^{\gamma s}(U(0, s) - V(0, s))ds, \quad \forall t \ge 0 \tag{3.21}$$

Since $\lim_{t\to+\infty} |U(0, t) - V(0, t)| = 0$, we have

$$\lim_{t\to+\infty} |e(0, t)| = 0 \tag{3.22}$$

It follows from Eqs. (3.22) and expressions (3.7) and (3.8) that

$$\lim_{t\to+\infty} (||e_x(\cdot, t)||_{\mathcal{C}^0} + ||e_t(\cdot, t)||_{\mathcal{C}^0}) = 0$$

Moreover, the solution e of error system (3.2) can be written as

$$\forall t \ge 0, \forall x \in [0, 1], \; e(x, t) = \int_0^x e_x(s, t)ds + e(0, t)$$

Thus, we have obtained

$$\lim_{t\to+\infty} ||e(\cdot, t)||_{\mathcal{C}^0} = 0$$

If equality (3.4) or (3.5) holds, according to Eqs. (3.12) and (3.13), we arrive at

$$\left|\frac{\xi-1}{\xi+1}\cdot\frac{1+\alpha}{1-\alpha}\right|\leq 1$$

Therefore, the proof is complete. ■

Remark 3.1. Here, we make some remarks:

(1) Theorem 3.1 also shows that the slave system (2.1) admits a classical solution provided that the initial condition of Eq. (1.1) satisfies $(w_0, w_1), (\hat{w}_0, \hat{w}_1) \in \mathcal{C}_0^2([0,1]) \times \mathcal{C}_0^1([0,1])$.

(2) In spite of the gradient $(w_t,\ w_x)$ presenting chaotic dynamics, we show not only the convergence of error dynamics but also the gradient $(e_t,\ e_x)$ is convergent in terms of $\mathcal{C}_0^1([0,1])$ norm.

(3) It is easy to see the following equivalence:

$$\left|\frac{k-\eta-1}{k-\eta+1}\cdot\frac{1+\alpha}{1-\alpha}\right|\leq 1 \Longleftrightarrow k\in\left[\eta+\alpha,\eta+\frac{1}{\alpha}\right]$$

(4) To synchronize $(w_x,\ w_t)$, we can set $\gamma=0$, thus the slave system only needs the input signal $w_t(0,t)$ for $t\geq 0$. Moreover, if $k\in(\eta+\alpha,\eta+(1/\alpha))$, the transition time is exponentially fast with the transition rate

$$r=\left|\frac{k-\eta-1}{k-\eta+1}\cdot\frac{1+\alpha}{1-\alpha}\right|<1$$

(5) To synchronize (w, w_x, w_t), we require $\gamma>0$ as well as delay input $w(0,t-1)$. This is because in order to synchronize the state w, one not only needs to know the reflected (outgoing) wave $w(0,\ t)$ but also the incoming wave $w(0,t-1)$ for $t\geq 1$.

(6) Theoretically, we can choose k such that $k-\eta=1$, thus $r=0$. This implies that the error dynamics becomes zero in a finite time; hence, the slave system synchronizes with the master system in a finite time. In numerical implementations, however, due to rounding errors, we may not exactly have $r\equiv 0$; therefore, considering $r<1$ and $r\approx 0$ in such a case is more realistic due to the sensitivity of both master and slave systems.

Next, we provide the details of the parameter setting.

Corollary 3.1. Let $\alpha\in(0,1)$, $\beta>0$ and $\eta\neq 1$. If the parameters k and γ are chosen to satisfy

$$\gamma>0,\ \ 0<\left|\frac{k-\eta-1}{k-\eta+1}\cdot\frac{1+\alpha}{1-\alpha}\right|<1 \tag{3.23}$$

then, for any initial functions $e_0\in\mathcal{C}_0^2([0,1])$ and $e_1\in\mathcal{C}_0^1([0,1])$, the solution of error dynamics (3.2) is exponentially stable. More specifically, there exist constants $M>0$ and $\rho>0$ such that

$$\forall t\geq 0,\ ||e(\cdot,t)||_{\mathcal{C}^0}+||e_x(\cdot,t)||_{\mathcal{C}^0}+||e_t(\cdot,t)||_{\mathcal{C}^0}\leq Me^{-\rho t} \tag{3.24}$$

■

Proof. For simplicity and clarity in the following analysis, we denote

$$r=\left|\frac{k-\eta-1}{k-\eta+1}\cdot\frac{1+\alpha}{1-\alpha}\right| \tag{3.25}$$

For any $e_0\in\mathcal{C}_0^2([0,1])$ and $e_1\in\mathcal{C}_0^1([0,1])$, we then have

$$U_0=\frac{e_0'+e_1}{2}\in\mathcal{C}_0^1([0,1]),\ \ V_0=\frac{e_0'-e_1}{2}-\gamma e_0(0)\in\mathcal{C}_0^1([0,1])$$

Let $M_0>0$ such that

$$\forall x\in[0,1],\ |U_0(x)|\leq\left|\frac{\alpha-1}{\alpha+1}\right|M_0,\ |V_0(x)|\leq M_0 \tag{3.26}$$

By choosing a constant $\rho\in(0,\lambda)$, where $\lambda=\min\{\gamma,\ \ln(1/\sqrt{r})\}$. For $t=2n+\tau$ with $n\in\mathbb{N}$ and $\tau\in[0,2)$, from Eqs. (3.12) and (3.13), we have

$$\begin{aligned}||U(\cdot,t)||_{\mathcal{C}^0}&\leq M_0r^n\leq\frac{M_0}{r}\exp\left(-\left(\ln\frac{1}{\sqrt{r}}\right)t\right)\leq\frac{M_0}{r}\exp(-\rho t),\\ ||V(\cdot,t)||_{\mathcal{C}^0}&\leq M_0r^n\leq\frac{M_0}{r}\exp\left(-\left(\ln\frac{1}{\sqrt{r}}\right)t\right)\leq\frac{M_0}{r}\exp(-\rho t)\end{aligned} \tag{3.27}$$

From the expression (3.21), we arrive at the following estimation:

$$\left|e(0,t)\right|\leq\frac{2M_0}{r}\exp(-\rho t) \tag{3.28}$$

When $\gamma>0$, for $t>1$ and $x\in[0,1]$, the following expressions

$$\begin{aligned}e_x(x,t)&=U(x,t)+V(x,t)+\gamma e(0,t-x),\\ e_t(x,t)&=U(x,t)-V(x,t)-\gamma e(0,t-x),\\ e(x,t)&=\int_0^x e_x(s,t)ds+e(0,t)\end{aligned}$$

as well as both obtained estimations (3.27) and (3.28) yield the following inequality:

$$||e(\cdot,t)||_{\mathcal{C}^0}+||e_x(\cdot,t)||_{\mathcal{C}^0}+||e_t(\cdot,t)||_{\mathcal{C}^0}\leq Me^{-\rho t}$$

where $M\geq(M_0/r)(10+4\gamma)$. Therefore, the proof is completed. ■

Remark 3.2. According to the proof of Corollary 3.1, the condition can be described, equivalently, as

$$r:=\left|\frac{k-\eta-1}{k-\eta+1}\cdot\frac{1+\alpha}{1-\alpha}\right|<1,\quad\forall\rho\in\left(0,\min\left\{\gamma,\ln\frac{1}{\sqrt{r}}\right\}\right)$$

Thus, one can always control the convergent rate ρ by choosing appropriate γ and k. ■

4 Numerical Simulations

In this section, we will provide some numerical simulations to validate the theoretical results of this paper.

For the main system (1.1), we first consider the case $\alpha=0.5$, $\beta=1$ with the following initial data:

$$\forall x\in[0,1],\ w_0(x)=0,\ \ w_1(x)=8\sin^2(2\pi x) \tag{4.1}$$

According to Ref. [14], the gradient $(w_x,\ w_t)$ of Eq. (1.1) undergoes chaotic vibrations when $\eta\in[\eta_0,\eta_0^{-1}]\backslash\{1\}$, where $\eta_0\approx 0.552$, discussed in Sec. 1. Without loss of generality, we use $\hat{w}_0=0$ and $\hat{w}_1=0$ as the initial data of the slave system (2.1). In the simulation of Fig. 1, we set $\gamma=0$. One can see that the gradient $(\hat{w}_x,\hat{w}_t)$ of the slave system (2.1) synchronizes the gradient (w_x, w_t) of Eq. (1.1) in about 4 s even with the very sensitive output $y(t)=w_t(0,t)$. It is interesting to observe that in this case the displacement state $\hat{w}$ of Eq. (2.1) does not synchronize the state w of Eq. (1.1). They are different from a constant. This is due to the fact that the state w of Eq. (1.1) is unobservable by using a single output $y(t)=w_t(0,t)$ (see recent work [18]). Recall here that

$$
\begin{aligned}
||e_x(\cdot,t)||_{C^0} &= \max_{x\in[0,1]} |e_x(x,t)|,\\
||e_t(\cdot,t)||_{C^0} &= \max_{x\in[0,1]} |e_t(x,t)|,\\
||e(\cdot,t)||_{C^0} &= \max_{x\in[0,1]} |e(x,t)|
\end{aligned}
$$

Next, we let $\gamma = 1.5$, which implies that the delay output takes into effect. Figure 2 shows that all states $(\hat{w}, \hat{w}_x, \hat{w}_t)$ of Eq. (2.1) synchronize the states (w, w_x, w_t) of Eq. (1.1) in about 4 s. In the simulated results, one can see that a sharp stepwise decrease in e_x and e_t can be observed but not in the total error term e. This is because (e_x, e_t) at the beginning is quite chaotic before the synchronizing control takes into effect. Also, the total error term is always smaller than the other two since e is the integration of (e_x, e_t). Thus, e's behavior appears to be better than (e_x, e_t) due to the smoothing effect by integration.

Next, we conduct a different simulation. Let $\alpha = 0.4$ and $\beta = 0.1$. Small β implies that the master system has less damping and more self-excited energy is being injected to the system, thus the chaotic behavior will become more severe. For this case, we have $\eta_0 \approx 0.5755$ and choose $\eta = 0.58$. The initial data are set to be

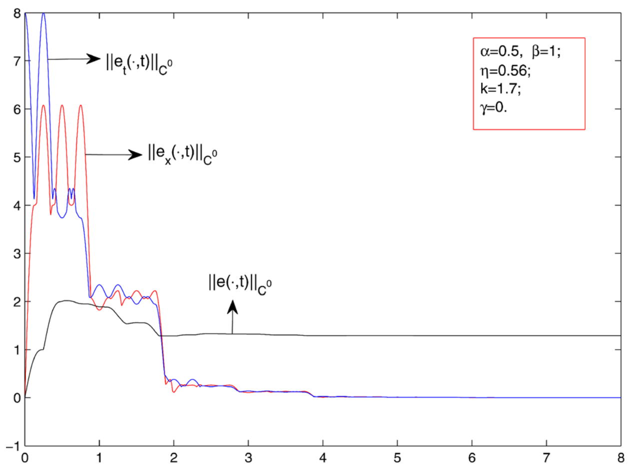

Fig. 1 The profiles of $||e_x(\cdot,t)||_{C^0}$, $||e_t(\cdot,t)||_{C^0}$, and $||e(\cdot,t)||_{C^0}$ for $\gamma = 0$ and $t \in [0, 8]$

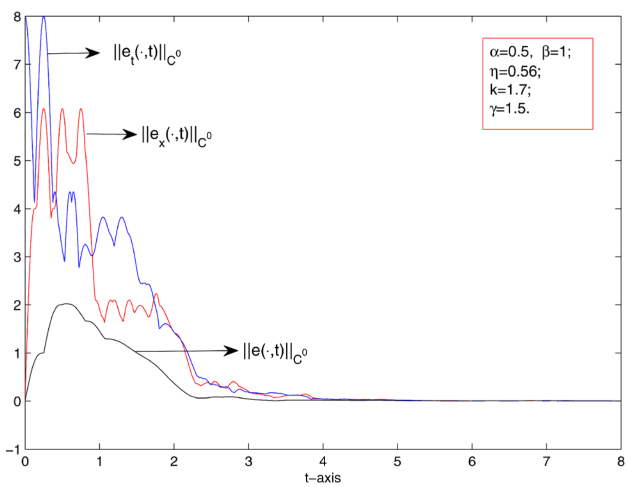

Fig. 2 The profiles of $||e_x(\cdot,t)||_{C^0}$, $||e_t(\cdot,t)||_{C^0}$, and $||e(\cdot,t)||_{C^0}$ for $\gamma = 1.5$ and $t \in [0, 8]$

$$
\forall x \in [0,1], w_0(x) = \frac{1}{10}(2\pi x - \sin(2\pi x)),\ w_1(x) = x^2(1-x)^2 \tag{4.2}
$$

to avoid the similar dynamics as the previous one. The initial states of system (2.1) stay the same, i.e., $\hat{w}_0 = \hat{w}_1 = 0$. The simulations for $\gamma = 0$ and $\gamma = 2$ are given in Figs. 3 and 4, respectively. The results show that the previous arguments remain valid.

5 To Synchronize the States (w_x, w_t, w) by Using Delay as a Parameter

In this section, we further generalize our approach by using delay as a parameter. Throughout this section, let $\ell \geq 0$ be the delay parameter (Eq. (1.1)). The synchronized slave system is still in a form of

$$
\begin{cases}
\hat{w}_{tt} - \hat{w}_{xx} = 0,\ x \in (0,1),\ t > 0,\\
\hat{w}_x(0,t) = L(t),\ t > 0,\\
\hat{w}_x(1,t) = R(t),\ t > 0,\\
\hat{w}(x,0) = \hat{w}_0(x), \quad \hat{w}_t(x,0) = \hat{w}_1(x), \quad 0 < x < 1,
\end{cases} \tag{5.1}
$$

where $L(t)$ and $R(t)$ will be given later. Let us denote the error of states between Eqs. (1.1) and (5.1) to be

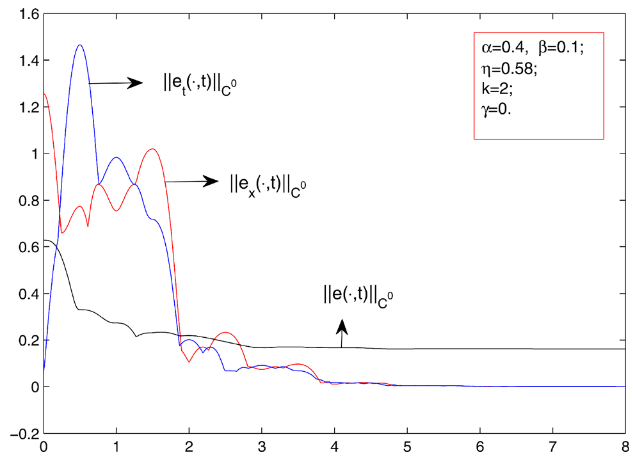

Fig. 3 The profiles of $||e_x(\cdot,t)||_{C^0}$, $||e_t(\cdot,t)||_{C^0}$, and $||e(\cdot,t)||_{C^0}$ for $\gamma = 0$ and $t \in [0, 8]$

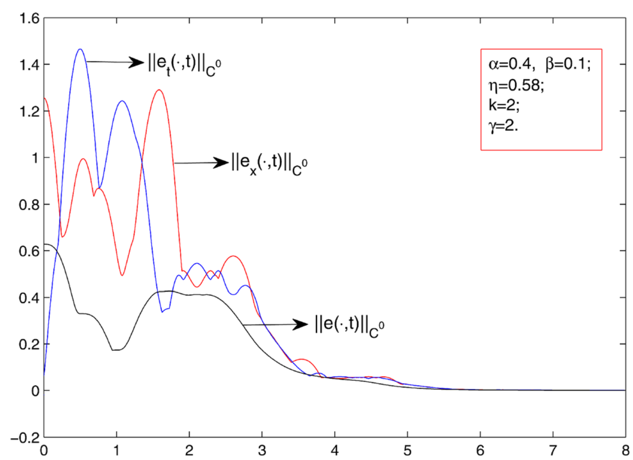

Fig. 4 The profiles of $||e_x(\cdot,t)||_{C^0}$, $||e_t(\cdot,t)||_{C^0}$, and $||e(\cdot,t)||_{C^0}$ for $\gamma = 2$ and $t \in [0, 8]$

$$e := w - \hat{w} \tag{5.2}$$

In order to synchronize the states (w_x, w_t, w) of Eq. (1.1), the challenge results from that the slave system only has the signal $w_t(0,t)$ and $w(0,t)$ from the master system. Our proposed approach, *method of characteristics*, is still applicable when we use delay as a parameter. It will involve a transport equation, which has the same *characteristics* with the Riemann invariant u or v of Eq. (1.1), as shown below. Let us denote

$$v_1(x,t) = \begin{cases} w(0, t-x-\ell), & t \ge x+\ell, \\ w(0,0), & 0 \le t < x+\ell \end{cases} \tag{5.3}$$

$$\hat{v}_1(x,t) = \begin{cases} \hat{w}(0, t-x-\ell), & t \ge x+\ell, \\ \hat{w}(0,0), & 0 \le t < x+\ell \end{cases} \tag{5.4}$$

which are solutions of the transport equation $v_x + v_t = 0$ and are constant along the characteristic direction $\boldsymbol{\xi} = (1,1)$.

We construct the boundary conditions $L(t)$ and $R(t)$ of the slave system, respectively, as follows:

$$L(t) = -\eta \hat{w}_t(0,t) - k e_t(0,t) - \gamma(k-\eta+1)(v_1 - \hat{v}_1)(0,t), \quad t \ge 0 \tag{5.5}$$

$$\begin{aligned} R(t) =& \alpha[\hat{w}_t(1,t) - \gamma(v_1-\hat{v}_1)(1,t)] - \beta[\hat{w}_t(1,t) - \gamma(v_1-\hat{v}_1)(1,t)]^3 \\ & -\gamma(v_1-\hat{v}_1)(1,t), \quad t \ge 0 \end{aligned} \tag{5.6}$$

Then, the error dynamics satisfies the following wave equation:

$$\begin{cases} e_{xx}(x,t) - e_{tt}(x,t) = 0, \quad 0 < x < 1, \ t > 0, \\ e_x(0,t) = (k-\eta)e_t(0,t) + \gamma(k-\eta+1)(v_1-\hat{v}_1)(0,t), \ t > 0, \\ e_x(1,t) = h(t)[e_t(1,t) + (v_1-\hat{v}_1)(1,t)] + \gamma(v_1-\hat{v}_1)(1,t), t > 0, \\ e(\cdot,0) = e_0(\cdot), \quad e_t(\cdot,0) = e_1(\cdot) \end{cases} \tag{5.7}$$

where $e_0 = w_0 - \hat{w}_0$, $e_1 = w_1 - \hat{w}_1$, and $h(t) = \alpha - \beta\zeta(t)$ with

$$\begin{aligned} \zeta(t) =& w_t^2(1,t) + w_t(1,t)[\hat{w}_t(1,t) - \gamma(v_1-\hat{v}_1)(1,t)] \\ & + [\hat{w}_t(1,t) - (v_1-\hat{v}_1)(1,t)]^2, \quad t > 0 \end{aligned} \tag{5.8}$$

One can verify that $\zeta(t) \ge 0$ and $-\infty < h(t) \le \alpha$.

Similar to the discussion given in Sec. 3, the following two Riemann invariants

$$U = \frac{e_x + e_t}{2}, \quad V = \frac{e_x - e_t}{2} - \gamma(v_1 - \hat{v}_1) \tag{5.9}$$

approach to zero exponentially as $t \to +\infty$ if

$$\left| \frac{k-\eta-1}{k-\eta+1} \cdot \frac{1+\alpha}{1-\alpha} \right| < 1 \tag{5.10}$$

i.e.,

$$k \in \left(\eta + \alpha, \eta + \frac{1}{\alpha} \right)$$

The next step is to choose appropriate γ such that $v_1 - \hat{v}_1$ also approaches to zero exponentially as $t \to +\infty$. Since $v_1 - \hat{v}_1$ is invariant along the characteristics $t - x =$constant, one only needs to ensure $(v_1 - \hat{v}_1)(0,t)$ exponentially converges to zeros as $t \to +\infty$. Notice that $(v_1 - \hat{v}_1)(0,t) = e(0, t-\ell)$ when $t \ge \ell$, we need to consider the dynamics of the delayed differential equation

$$\begin{cases} \dot{x}(t) + \gamma x(t-\ell) = f(t), & t > \ell, \\ x(t) = C, \ 0 \le t \le \ell \end{cases} \tag{5.11}$$

where C is a constant and $f(t) := (U-V)(0,t)$. The case $\ell = 0$ has been discussed in Sec. 3. We next assume that $\ell > 0$ and define

$$H(z) = z + \gamma e^{-\ell z}, \quad z \in \mathbb{C} \tag{5.12}$$

$H(z) = 0$ is the so-called *characteristic equation* of Eq. (5.11), see Ref. [30] for more details.

Lemma 5.1. *Consider (5.12). Assume* $0 < \ell\gamma \le 1$, *then*

$$\lambda_0 = \max\{Re(z), H(z) = 0\} \tag{5.13}$$

is well-defined and

$$\lambda_0 < 0$$

Proof . The constant λ_0 is well-defined, followed from Lemma 4.1 in Ref. [30]. Let $z = x + iy$ with $x, y \in \mathbb{R}$, then

$$H(z) = 0 \Rightarrow \begin{cases} x + \gamma e^{-\ell x}\cos(\ell y) = 0, \\ y - \gamma e^{-\ell x}\sin(\ell y) = 0 \end{cases}$$

If $\sin(\ell y) = 0$, then $y = 0$ and $x + \gamma e^{-\ell x} = 0$, which implies $x < 0$. When $\sin(\ell y) \ne 0$, both y and $\sin(\ell y)$ carry the same sign, thus the following equation

$$\ell\gamma e^{-\ell x} = \frac{\ell y}{\sin(\ell y)} > 1, \quad 0 < \ell\gamma \le 1$$

implies $x < 0$. Therefore, the proof is completed. ■

Next, we will provide an explicit estimation of the upper bound of λ_0. We denote $g(x) = x + e^{-x}, x \ge 0$. Then, g is strictly increasing on $[0,\infty)$ and Range $(g) = [1,\infty)$. We define a positive constant

$$\delta_0 = \begin{cases} g^{-1}(-\ln(\ell\gamma)), & 0 < \ell\gamma \le e^{-1}, \\ 0, & e^{-1} < \ell\gamma \end{cases} \tag{5.14}$$

Notice that if $g((-\ln(\ell\gamma)) > -\ln(\ell\gamma)$, then we have $0 \le \delta_0 < -\ln(\ell\gamma))$.

Lemma 5.2. Given ℓ and γ that satisfy $0 < \ell\gamma < 1$ and let δ_0 be defined by Eq. (5.14). Then, we have

$$\lambda_0 \le \frac{\ln(\ell\gamma) + \delta_0}{\ell} < 0$$

where $\lambda_0 = \max\{Re(z), z + \gamma e^{-\ell z} = 0, z \in \mathbb{C}\}$. *Furthermore, for fixed* $\gamma > 0$, *the following holds:*

$$\lim_{\ell \to 0^+} \frac{\ln(\ell\gamma) + \delta_0}{\ell} = -\gamma$$

Proof. Notice that if we denote $z = x + iy$, then

$$z + \gamma e^{-\ell z} = 0 \Rightarrow \begin{cases} x + \gamma e^{-\ell x}\cos(\ell y) = 0, \\ y - \gamma e^{-\ell x}\sin(\ell y) = 0 \end{cases}$$

When ℓ satisfies $e^{-1} < \ell\gamma < 1$, we have $y \ne 0$ since otherwise x would satisfy $x + \gamma e^{-\ell x} = 0$ which does not admit a solution for any $x \in \mathbb{R}$. This implies $\sin(\ell y) \ne 0$, and thus,

$$\ell\gamma e^{-\ell x} = \frac{\ell y}{\sin(\ell y)} > 1$$

which yields $x \le \ln(\ell\gamma)/\ell$. Hence, we have $\lambda_0 \le \ln(\ell\gamma)/\ell$.

When $0 < \ell\gamma \le e^{-1}$, one can verify directly that the real number $z = (\ln(\ell\gamma) + \delta_0/\ell)$ is the unique solution of $z + \gamma e^{-\ell z} = 0$. Hence, we have obtained

$$\lambda_0 = \frac{\ln(\ell\gamma) + \delta_0}{\ell}, \quad \text{for} \quad 0 < \ell\gamma < e^{-1}$$

Furthermore, by the definition of δ_0, we have

$$\delta_0 + e^{-\delta_0} = -\ln(\ell\gamma)$$

and $\lim_{\ell \to 0^+} \delta_0 = +\infty$. Therefore, we have

$$\lambda_0 = \frac{\ln(\ell\gamma) + \delta_0}{\ell} = -\gamma \frac{e^{-\delta_0}}{\ell\gamma} = -\gamma e^{e^{-\delta_0}} \to -\gamma \quad \text{as} \quad \ell \to 0^+$$

and the proof is completed. ■

Lemma 5.3. Consider the delayed differential *equation (5.11)*, and let $x(\cdot)$ be the solution. Suppose f(t) approaches to zero exponentially as $t \to +\infty$ and $0 < \ell\gamma \le 1$, then there exist constants $M > 0$ and $\lambda > 0$ such that

$$\forall t \ge 0, \quad |x(t)| \le Me^{-\lambda t} \tag{5.15}$$

i.e., the solution of Eq. (5.11) approaches to zero exponentially as $t \to +\infty$.

Proof. Choose $\lambda_1 > 0$ with $-\lambda_1 \in (\lambda_0, 0)$, where λ_0 is given by Eq. (5.13). It follows from Theorems 6.1 and 6.2 in Ref. [30] that

$$\forall t \ge \ell, x(t) = CX(t) - C\gamma \int_0^\ell X(t+s-\ell)ds + \int_\ell^t X(t-s+\ell)f(s)ds \tag{5.16}$$

where $X(\cdot)$ satisfies

$$\begin{cases} \dot{X}(t) + \gamma X(t-\ell) = 0, & t > \ell, \\ X(t) = \begin{cases} 1, & t = \ell, \\ 0, & t < \ell \end{cases} \end{cases}$$

and

$$\forall t \ge \ell, \quad |X(t)| \le M_1 e^{-\lambda_1 t} \tag{5.17}$$

for some constant $M_1 = M_1(\lambda_1) > 0$. Choose $M_2 > 0, \lambda_2 > 0$ such that

$$\forall t \ge 0, \quad |f(t)| \le M_2 e^{-\lambda_2 t} \tag{5.18}$$

Without loss of generality, assume that $\lambda_1 \ne \lambda_2$. It follows from Eqs. (5.16)–(5.18) that

$$\forall t \ge \ell, \quad |x(t)| \le Me^{-\lambda t}$$

where $\lambda = \min\{\lambda_1, \lambda_2\}$ and $M = M(\lambda_1, \lambda_2, M_1, M_2) > 0$. Thus, the proof is completed. ■

Theorem 5.1. *Let* $\alpha \in (0, 1)$, $\beta > 0$ *and* $\eta \ne 1$. *If the parameters* ℓ, k, *and* γ *are chosen to satisfy*

$$0 < \ell\gamma \le 1, \; 0 < \left| \frac{k-\eta-1}{k-\eta+1} \cdot \frac{1+\alpha}{1-\alpha} \right| < 1 \tag{5.19}$$

then, for any initial states $e_0 \in \mathcal{C}_0^2([0,1])$ *and* $e_1 \in \mathcal{C}_0^1([0,1])$, *the solution of error dynamics (5.7) is exponentially stable. More specifically, for any* $\lambda \in (0, \lambda_1)$, *where*

$$\lambda_1 = \min\left\{ -\ln\sqrt{r}, -\frac{\ln(\ell\gamma) + \delta_0}{\ell} \right\}$$

there exists a constant $M > 0$ *such that*

$$\forall t \ge 0, \; ||e(\cdot,t)||_{\mathcal{C}^0} + ||e_x(\cdot,t)||_{\mathcal{C}^0} + ||e_t(\cdot,t)||_{\mathcal{C}^0} \le Me^{-\lambda t} \tag{5.20}$$

Proof. According to the discussion in the above theorem, we first have that $||e_x(\cdot,t)||_{\mathcal{C}^0}$, $||e_t(\cdot,t)||_{\mathcal{C}^0}$, and $|e_x(0,t)|$ approach to zero exponentially as $t \to +\infty$. Similar arguments used in the proof of Corollary 3.1 lead to the conclusion. ■

Remark 5.1. Here, we make two remarks:

(i) When $\ell = 0$, Theorem 5.1 is equivalent to Corollary 3.1 (delay= 1) according to Lemma 5.2. In other words, Corollary 3.1 is a special case of Theorem 5.1.

(ii) When $\ell > 0$, due to $\delta_0/\ell \to 0$ as $\ell \to \infty$, the absolute value of λ becomes smaller as ℓ increases, i.e., the convergent rate of error dynamics will become slow, which is reasonable since the longer delay we use, the more responding time is required for the slave system to synchronize with the master system.

To confirm that, we provide the following numerical simulations. Let $\alpha = 0.4$ and $\beta = 0.1$. For this case, we have $\eta_{\approx}0.5755$ and we set $\eta = 0.58$ so that the gradient (w_x, w_t) presents chaotic dynamics. Let parameters $\gamma = 0.5, k = 2$ and the initial condition be

$$w_0(x) = \frac{1}{10}(2\pi x - \sin(2\pi x)), \quad w_1(x) = x^2(1-x)^2, \quad x \in [0,1]$$

The initial state of the slave system is set to be $\hat{w}_0 = \hat{w}_1 = 0$. The simulations for delay $\ell = 1$ and $\ell = 2$ are given in Figs. 5 and 6, respectively. Clearly, the error dynamics for delay $\ell = 2$ appears to be slower than the case for $\ell = 1$. ■

6 Conclusions

In this paper, we model the synchronization of wave equation associated with nonlinear boundary. Wave equation is a standard model for the study of various vibrations in reality in which boundary conditions govern the type of vibrations. The nonlinear

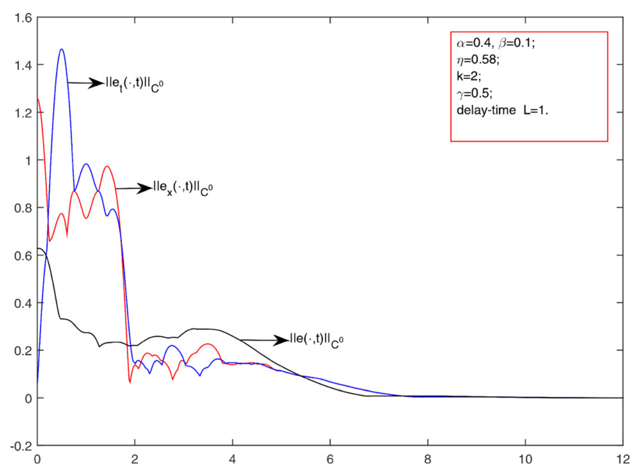

Fig. 5 The profiles of $||e_x(\cdot,t)||_{\mathcal{C}^0}$, $||e_t(\cdot,t)||_{\mathcal{C}^0}$, and $||e(\cdot,t)||_{\mathcal{C}^0}$ for delay $\ell = 1$ and $t \in [0, 12]$

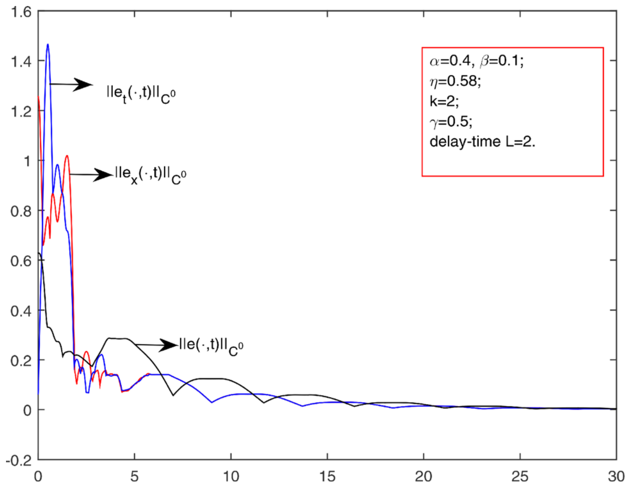

Fig. 6 The profiles of $||e_x(\cdot,t)||_{\mathcal{C}^0}$, $||e_t(\cdot,t)||_{\mathcal{C}^0}$, and $||e(\cdot,t)||_{\mathcal{C}^0}$ for delay $\ell = 2$ and $t \in [0, 30]$

boundary condition adopted in this paper, originated from the well-known Van der Pol oscillator, leads to the chaotic dynamics of the state gradient and causes the irregular, rapid vibration of the state, which induces the rapid changes (high frequencies) of wave propagation that has been received considerable attention in recent years. To the best of our knowledge, there are no existing results for the discussion of possible underlying deterministic structure at this point for this system. This does open a new important question and requires a substantial study for which we will work on it the near future.

Wave equation represents a broad spectrum of mathematical models in real applications and the boundary setting (or control) can significantly affect the wave propagation dynamics. This is particularly important in the development of meta-devices often used to steer wave propagation. The slave system (or called the receiver) is required to be synchronized with the master system in order to recover the true signals locally in a short period of time.

With the only signal being available at the one end of the boundary, we are able to construct a responding system that synchronizes the original system. The transition period can be controlled by selecting appropriate parameters. Even though the original system is very sensitive to its initial condition, the synchronized dynamics responds quite well.

It is worthy of mentioning here that to the best of our knowledge, incorporating the delay output into the responding system seems to be necessary in order to achieve the synchronization of this type of vibrations. As the output signal is only available at the one end of the boundary, one needs to know both outgoing and incoming waves in order to capture the dynamics of the original system. The estimation of error dynamics shown in this paper is new and is not available in the current literature. With the introduction of the delay input, we successfully synchronize all critical state variables: the state as well as the gradient of the state.

The key idea of this paper is to construct two Riemann invariants for the error system that are solvable and allow us to study the wave dynamics via boundary wave reflections. The approach by Riemann invariant essentially is equivalent to *method of characteristics*, an effective method in the study of hyperbolic PDEs. The limitation for this type of approach results from the fact that if the Riemann invariant cannot be solved explicitly (different from the case in this paper), the corresponding wave dynamics is hardly analyzed due to the difficulty in determining the wave reflection on the boundary analytically, in particular, when the system is sensitive to small perturbations. Numerical approach may offer a potential solution provided that a stable algorithm for solving the corresponding Riemann invariant could be developed (for the sensitive case), the synchronization in the discrete level under our framework may become approachable. A further extensive study is required and will be reported elsewhere.

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