N-k Interdiction Modeling for Natural Gas Networks

Mareldi Ahumada-Paras
Department of Electrical Engineering
University of Washington, USA
ahumam@uw.edu

Kaarthik Sundar, Russell Bent, Anatoly Zlotnik
Los Alamos National Laboratory
Los Alamos, NM, USA
{kaarthik,rbent,azlotnik}@lanl.gov

***Abstract*—Although electricity transmission systems are typically very robust, the impacts that arise when they are disrupted motivate methods for analyzing outage risk. For example, N-k interdiction models were developed to characterize disruptions by identifying the sets of k power system components whose failure results in "worst case" outages. While such models have advanced considerably, they generally neglect how failures outside the power system can cause large-scale outages. Specifically, failures in natural gas pipeline networks that provide fuel for gas-fired generators can affect the function of the power grid. In this study, we extend N-k interdiction modeling to gas pipeline networks. We use recently developed convex relaxations for natural gas flow equations to yield tractable formulations for identifying sets of k components whose failure can cause curtailment of natural gas delivery. We then present a novel cutting-plane algorithm to solve these problems. Finally, we use test instances to analyze the performance of the approach in conjunction with simulations of outage effects on electrical power grids.**

***Index Terms*—N-k, interdiction analysis, natural-gas networks, convex relaxations, mixed-integer second-order cone programs**

I. Introduction

In power system operations and planning, N-k contingency analysis is used to assess system reliability and resilience. In these analyses, k components are turned "off" in a computational model of the electrical grid and system-wide effects of this removal are modeled through a computer simulation. These simulations use optimal power flow (OPF)-like optimization models, such as maximal load delivery [1], to estimate outages caused by a contingency.

Contingency analysis is often combined with N-k interdiction modeling to identify sets of k components whose simultaneous failure leads to the worst outcome (typically outages) during a contingency analysis. While solving an interdiction problem itself is challenging, the state-of-the-art has improved considerably over the last several decades and (at least heuristic) solutions are regularly reported on problems with large N and k (see [2]–[8] and references therein).

One of the weaknesses of N-k analysis, in particular for large k, is that it is often implausible for the identified k components to fail simultaneously, i.e., they are geographically separated by a large distances. This has led to the development of new models and methods for identifying sets of k components whose concurrent failure is more likely [7], or that constrain the possibilities of the interdiction plan [9]. In this article, in order to further address this limitation in traditional power system interdiction modeling, we develop an approach for identifying failures in a power system that are caused by exogenous failures, which, in this case, arise in natural gas pipeline networks that deliver fuel to gas-fired generators.

Our investigation is motivated by the increased reliance of many power systems on natural gas-fired generation, which is used to meet increasing production requirements, replace retiring coal and nuclear plants, and provide controllable resources to compensate for the variability from renewable sources like wind and solar [10]. Gas-fired generators now supply a significant fraction of base electric power production in many countries, which creates a fundamental reliance of power grids on gas pipelines for just-in-time fuel delivery. As a result, it has become increasingly likely that unplanned component outages or other contingencies in a natural gas pipeline could cause correlated (large k) electricity generator outages [11], [12]. We develop an approach to support the identification of sets of N-k scenarios in gas pipeline networks that induce large failures in a dependent power system. We also demonstrate the method on models of the Belgian and New England natural gas pipeline networks, as well as the gaslib-582 test instance.

A. Background

In comparison with the power systems literature, there are relatively few studies that apply contingency analysis and interdiction modeling to natural gas pipeline systems. In one

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study, the authors suggest gas pipeline networks as natural candidates for interdiction modeling [13], but do not discuss the complexities associated with modeling natural gas systems in interdiction problems. As a result, many subsequent papers have relied on enumeration methods, used simplified models that neglect the physics of natural gas flows, or are restricted to small problems. For example, one study focused on developing a vulnerability assessment approach in which all combinations of failures up to size k are enumerated, and then performs a max-flow calculation (that does not account for natural gas flow physics) on a 33-node system [14].

The effect that interdicting a gas pipeline network has on a power system that relies heavily on natural gas for generation is another area that has received limited attention. The most relevant study on the behavior of a power system after a natural gas pipeline failure is reference [15]. In this study, the authors develop a model that enumerates all single failures (N-1) in a gas pipeline and then use the results to identify generator outages and security constraint violations in the power system. They do not model the response of the power system, nor does the paper seek to identity the "worst" k-outage for a gas network. References [16]–[18] are the most closely related works to this paper. These studies focus on developing tri-level models for the design or hardening of electric power and natural gas delivery systems such that the lost demand after a worst case k-outage scenario is minimized. To preserve tractability, linear approximations of gas flows are used and empirical results are limited to systems with no more than 40 nodes in the gas pipeline network.

B. Contributions

In this study, we focus on the details of natural gas physical flow modeling in interdiction and, for the first time, *relaxations* of the gas flow are used, which in contrast to approximations are able to provide guarantees on solution quality as well as to scale to a case study with 582 nodes (in a bi-level model).

In summary, the contributions of this article are:

- A comprehensive N-k interdiction model for natural gas systems based on recently developed convex relaxations for gas pipeline networks.
- A tractable computational method for solving natural gas N-k interdiction problems.
- A detailed case study that examines how an N-k interdiction on a gas pipeline network impacts an associated electric power system by estimating the potential loss of generation on gas-fired generators on that system.

Throughout the rest of the paper, we use steady-state equations to model the physics of gas flows. This choice has two primary motivations: (i) to the best of our knowledge there is no work in the literature that considers the N-k interdiction problem in gas networks that takes into account any physics of gas flows, (ii) the steady-state is a good starting point to understand the theoretical and computational limits of the problem before modeling the full transient equations. The rest of the paper is organized as follows. Section II states the N-k interdiction problem for a gas pipeline network and introduces notation. Section III discusses steady-state modeling and Section IV presents the formal problem definition. Section V describes the convex relaxation we use and Section VI describes our methodology. Section VII describes the case study, and we conclude with Section VIII.

II. Problem Statement and Notation

The goal of the N-k interdiction problem for natural gas pipeline networks is to identify k components in the gas network that, when damaged, have the greatest impact on the transportation capacity of the system. For these systems, we measure impact by computing the minimum amount of gas that the system is unable to provide to delivery points, relative to the baseline (unaffected) flow allocation. A subset of these delivery points correspond to power plants that use the natural gas to generate electricity.

Formally, the N-k problem is stated as follows: given a natural gas pipeline network with nodes $\mathcal{N}$, pipelines, $\mathcal{P}$, and compressors $\mathcal{C}$, an N-k interdiction problem identifies k components in $\mathcal{P} \cup \mathcal{C}$ whose loss maximizes the minimum amount of un-served gas loads at delivery points. Gas is injected into or withdrawn from the system from a subset of nodes (receipt and delivery points, $\mathcal{R}$ and $\mathcal{D}$, respectively) in the network. The max-min structure makes the N-k interdiction problem a bi-level optimization problem. These problems are often modeled as Stackelberg games with an attacker and a defender [19], where the attacker's and defender's actions are sequential and the attacker has a perfect model of how the defender will respond to an attack. Such problems are NP-Hard [7] because of the inherent combinatorial nature of the problem. Furthermore, the number of possible N-k contingencies, even for small values of k, makes complete enumeration intractable. This makes such models difficult to scale to large systems, which is a prerequisite to apply the desired interdiction modeling in practice.

The following notation is used for indexing sets, decision variables, and parameters in the optimization formulation:

Sets:

$\mathcal{N}$, $\mathcal{C}$, $\mathcal{P}$ - sets of nodes, compressors, and pipes

$\mathcal{R}$, $\mathcal{D}$ - sets of receipt and delivery points

$\mathcal{R}(i)$, $\mathcal{D}(i)$ - sets of receipt and delivery points at node i

$\mathcal{X}$ - N-k contingency scenario set

$\mathcal{C}(s)$, $\mathcal{P}(s)$ - sets of damaged compressors and pipes in scenario $x \in \mathcal{X}$

$\mathcal{E}(i)$ - subset of pipes and compressors connected to node i and oriented *from* i

$\mathcal{E}^r(i)$ - subset of pipes and compressors connected to node i and oriented *to* i

Decision variables:

π_i - square of pressure at node i (Pa^2)

f_e - mass flow rate across $e \in \mathcal{C} \cup \mathcal{P}$ ($\mathrm{kg\,s^{-1}}$)
s_i - total gas produced at receipt points in $\mathcal{R}(i)$ ($\mathrm{kg\,s^{-1}}$)
λ_i - unserved gas-factor for each node $i \in \mathcal{N}$
γ_e - auxiliary variable for each pipe $e \in \mathcal{P}$
y_e - binary flow direction variable for each $e \in \mathcal{C} \cup \mathcal{P}$
x_e - binary interdiction variable for each $e \in \mathcal{C} \cup \mathcal{P}$
$\boldsymbol{x}$ - vector of interdiction variables x_e
Parameters:
$\boldsymbol{d}_i$ - total gas delivered at delivery points in $\mathcal{D}(i)$ ($\mathrm{kg\,s^{-1}}$)
$\boldsymbol{w}_e$ - resistance of the pipe $e \in \mathcal{P}$
$\boldsymbol{a}$ - speed of sound in the gas ($\mathrm{m\,s^{-1}}$)
$\boldsymbol{\beta}_e$ - friction factor of the pipe $e \in \mathcal{P}$
$\boldsymbol{\ell}_e$, $\boldsymbol{D}_e$ - length, diameter of the pipe $e \in \mathcal{P}$ (m,m)
($\underline{\boldsymbol{\pi}}_i$, $\overline{\boldsymbol{\pi}}_i$) - min and max limits for π_i (Pa^2)
($\underline{\boldsymbol{\alpha}}_e$, $\overline{\boldsymbol{\alpha}}_e$) - min and max compression limits for $e \in \mathcal{C}$
$\boldsymbol{f}_e$ - max flow rate for $e \in \mathcal{C} \cup \mathcal{P}$ ($\mathrm{kg\,s^{-1}}$)

III. Steady State Gas Flow Equations

Before presenting the formulation, we review the physics that govern steady flow of natural gas through pipelines. The physics of flow across a pipeline, $e = (i, j)$, are described by a set of partial differential equations (PDEs) that have dimensions in both time and space [20]. In steady-state, the PDEs reduce to equations of the form

$$\pi_i - \pi_j = \boldsymbol{w}_e f_e |f_e|, \tag{1}$$

where the phenomenological expression on the right hand side quantifies the dissipation of kinetic energy caused by turbulent flow through the pipe. The parameter $\boldsymbol{w}_e$ is called a resistance factor, and is given by

$$\boldsymbol{w}_e = \frac{4\boldsymbol{\beta}_e \boldsymbol{l}_e \boldsymbol{a}^2}{\pi^2 \boldsymbol{D}_e^5}. \tag{2}$$

For a detailed derivation of the parameters in this equation, interested readers are referred to [21]. To compensate for the dissipation of energy along the direction of flow, a gas pipeline utilizes compressors to boost flow and pressure throughout the system. We model these components as short pipes with zero resistance values, which create a jump in pressure while preserving flow in the direction of the compressor's orientation. When the gas flows through the compressor in the opposite direction of its orientation, the compressor is assumed to not offer any pressure boost.

IV. Problem formulation

Given the notations in Sec. II, the N-k interdiction problem is formulated as follows:

$$\max_{\boldsymbol{x} \in \mathcal{X}} \quad \eta(\boldsymbol{x}), \tag{3}$$

where $\mathcal{X} = \{\boldsymbol{x} : \sum_{e \in \mathcal{C} \cup \mathcal{P}} x_e = k\}$ and $\eta(\boldsymbol{x})$ is the total amount of gas unserved at all delivery points in scenario $\boldsymbol{x}$. The elements of $\mathcal{X}$ correspond to N-k contingency scenarios and are implicitly defined by the variables in $\boldsymbol{x}$ that take value 1. The core sub-problem for the N-k problem is the Minimal Gas Shedding (MGS) problem that defines the value of $\eta(\boldsymbol{x})$ as

$$\eta(\boldsymbol{x}) = \min \quad \sum_{i \in \mathcal{N}} \lambda_i \boldsymbol{d}_i, \tag{4a}$$

$$\pi_i - \pi_j = \boldsymbol{w}_e |f_e| f_e \quad \forall (i, j) = e \in \mathcal{P} : x_e = 0, \tag{4b}$$

$$\pi_i - \pi_j = 0, \text{ if } f_e \leqslant 0, \forall (i, j) = e \in \mathcal{C} : x_e = 0, \tag{4c}$$

$$\underline{\boldsymbol{\alpha}}_e^2 \pi_i \leqslant \pi_j \leqslant \overline{\boldsymbol{\alpha}}_e^2 \pi_i, \text{ if } f_e \geqslant 0, \forall (i, j) = e \in \mathcal{C} : x_e = 0 \tag{4d}$$

$$\sum_{e \in \mathcal{E}(i)} f_e - \sum_{e \in \mathcal{E}^r(i)} f_e = s_i - (1 - \lambda_i) \boldsymbol{d}_i \quad \forall i \in \mathcal{N}, \tag{4e}$$

$$\underline{\boldsymbol{\pi}}_i \leqslant \pi_i \leqslant \overline{\boldsymbol{\pi}}_i \quad \forall i \in \mathcal{N}, \tag{4f}$$

$$-\boldsymbol{f}_e \leqslant f_e \leqslant \boldsymbol{f}_e \quad \forall e \in \mathcal{C} \cup \mathcal{P}. \tag{4g}$$

The formulation for MGS, as stated in Eq. (4), is a non-linear disjunctive formulation. Eq. (4b) denotes the steady-state gas flow physics for each pipe that has not been damaged by the N-k scenario, and Eq. (4e) enforces a mass flow balance condition at each node in the system. The Eqs. (4c) and (4d) deactivate pressure boosting and enforce boosting limits of a compressor with flow directed against and along the orientation of the compressor, respectively. Finally, Eqs. (4f) and (4g) enforce pressure and flow rate limits on each node and pipe in the network, respectively. The above formulation is a bi-level optimization problem where the outer maximization problem is given by Eq. (3) and inner minimization problem is given by Eq. (4). In the next section, we present a Mixed-Integer Non-Linear Programming (MINLP) reformulation and a Mixed-Integer Second-Order Cone Programming (MISOCP) relaxation of the MGS using binary flow direction variables y_e for each compressor and pipe in the network.

V. MISOCP Relaxation for the MGS

A. MINLP reformulation

To develop the relaxation for the MGS, we first reformulate the constraints in Eqs. (4b) – (4d) with binary flow direction variables y_e for each $e \in \mathcal{C} \cup \mathcal{P}$ [22]. Given a pipe or a compressor $e = (i, j) \in \mathcal{C} \cup \mathcal{P}$, y_e takes a value 1 if the mass flow is $f_e \geqslant 0$ and 0, otherwise. We remark that if $f_e \leqslant 0$, then gas is flowing from the node j to node i. Given these notations, Eq. (4b), for any $e = (i, j) \in \mathcal{P} : x_e = 0$ equivalently reformulated as

$$\gamma_e = \boldsymbol{w}_e f_e^2 \tag{5a}$$

$$\gamma_e \geqslant \pi_j - \pi_i + 2y_e(\underline{\boldsymbol{\pi}}_i - \overline{\boldsymbol{\pi}}_j) \tag{5b}$$

$$\gamma_e \geqslant \pi_i - \pi_j + 2(y_e - 1)(\overline{\boldsymbol{\pi}}_i - \underline{\boldsymbol{\pi}}_j) \tag{5c}$$

$$\gamma_e \leqslant \pi_j - \pi_i + 2y_e(\overline{\boldsymbol{\pi}}_i - \underline{\boldsymbol{\pi}}_j) \tag{5d}$$

$$\gamma_e \leqslant \pi_i - \pi_j + 2(y_e - 1)(\underline{\boldsymbol{\pi}}_i - \overline{\boldsymbol{\pi}}_j) \tag{5e}$$

$$-\boldsymbol{f}_e(1 - y_e) \leqslant f_e \leqslant \boldsymbol{f}_e y_e \tag{5f}$$

where, γ_e is an auxiliary variable for pipe e. Eqs. (5b) – (5e) are the McCormick envelopes [23] for the equation

$\gamma_e = (2y_e - 1)(\pi_i - \pi_j)$. These envelopes result in an exact reformulation because it is the product of a variable that takes a value of one or negative one, $(2y_e - 1)$, with a continuous variable, $(\pi_i - \pi_j)$. Eq. (5f) bounds the mass flow on the pipe using the flow direction variable y_e. The only nonlinear constraint in the reformulation is Eq. (5a). As for the compressor constraints in Eqs. (4c) and (4d), a linear reformulation of the constraints for every compressor $e = (i,j) \in \mathcal{C} : x_e = 0$ is given by:

$$y_e(\underline{\boldsymbol{\pi}}_i - \overline{\boldsymbol{\pi}}_j) \leqslant \pi_i - \pi_j \leqslant y_e(\overline{\boldsymbol{\pi}}_i - \underline{\boldsymbol{\pi}}_j) \tag{6a}$$
$$\underline{\boldsymbol{\alpha}}_e^2 \pi_i + (1 - y_e)(\underline{\boldsymbol{\pi}}_j - \underline{\boldsymbol{\alpha}}_e^2 \overline{\boldsymbol{\pi}}_i) \leqslant \pi_j \tag{6b}$$
$$\pi_j \leqslant \overline{\boldsymbol{\alpha}}_e^2 \pi_i + (1 - y_e)(\overline{\boldsymbol{\pi}}_j - \overline{\boldsymbol{\alpha}}_e^2 \underline{\boldsymbol{\pi}}_i) \tag{6c}$$
$$-\boldsymbol{f}_e(1 - y_e) \leqslant f_e \leqslant \boldsymbol{f}_e y_e \tag{6d}$$

where, Eqs. (6a) and (6b) – (6c) are disjunctive reformulations of Eqs. (4c) and (4d), respectively. Similar to pipes, Eq. (6d) bounds the mass flow on the compressor using the flow direction variable y_e. Using Eqs. in (5) and (6), the MINLP for the inner problem is then given by

$$\eta(\boldsymbol{x}) = \min \sum_{i \in \mathcal{N}} \lambda_i \boldsymbol{d}_i \text{ subject to: Eqs. (5), (6), (4e) – (4g).}$$

The MINLP reformulation of the MGS is still a difficult problem to solve to global optimality, even for small instances [22] and hence, the remainder of this section is focused on developing a MISOCP relaxation of the MINLP. The MISOCP is based on the formulation introduced in [22] and off-the-shelf commercial and open-source MISOCP solvers effectively solve the inner problem to optimality for a fixed $\boldsymbol{x}$.

Once an MISOCP relaxation is developed, off-the-shelf commercial and open-source MISOCP solvers can be put to effective use to solve the inner problem for a fixed $\boldsymbol{x}$ to optimality.

B. MISOCP relaxation

The only nonlinear constraint in the MINLP reformulation is Eq. (5a). To obtain the MISOCP relaxation, we relax this constraint to

$$\gamma_e \geqslant \boldsymbol{w}_e f_e^2 \tag{7}$$

which is a Second-Order Conic (SOC) constraint. Hence, the MISOCP relaxation of the inner-problem (MGS) is given by

$$\eta(\boldsymbol{x}) = \min \sum_{i \in \mathcal{N}} \lambda_i \boldsymbol{d}_i \text{ subject to:} \tag{8}$$
Eqs. (5b) – (5f), (7), (6), (4e) – (4g).

In the next section, we use the above MISOCP relaxation to develop an iterative algorithm to compute an optimal solution to the bi-level N-k interdiction problem for a natural gas pipeline network with the MISOCP relaxation of the MGS.

VI. Solution Methodology

In this section, we present a generic cutting-plane algorithm that works directly with the bi-level structure of the N-k problem. A number of techniques have been proposed to convert such a bi-level max-min problem into a single mixed-integer program (see [24], [25]). Given the recent success of algorithms that directly exploit the bi-level structure in problems concerning electric transmission systems [7], [9], we adopt them here. The algorithm is generic and is applicable to the MINLP and the MISOCP relaxations as long as they are solved to global optimality. In this article, we restrict our attention to using the algorithm on the MISOCP relaxation of the MGS problem, because it can be solved to global optimality with off-the-shelf commercial or open-source solvers. The algorithm generates cutting planes using solutions of the inner problem, and adds them sequentially to the outer problem.

The algorithm constructs a sequence of piecewise linear functions that bounds from above the total curtailment of scheduled gas delivery given by solutions to the inner problem or its MISOCP relaxation. For any N-k scenario, $\hat{\boldsymbol{x}}$, $\eta(\hat{\boldsymbol{x}})$ denotes the minimum unserved gas for that scenario as given by (4) or its MISOCP relaxation in Eq. (8). Then, the algorithm computes coefficients $\delta_e(\hat{x})$ for each $e \in \mathcal{C} \cup \mathcal{P}$ such that

$$\eta(\boldsymbol{x}) \leqslant \eta(\hat{\boldsymbol{x}}) + \sum_{e \in \mathcal{C} \cup \mathcal{P}} \delta_e(\hat{\boldsymbol{x}}) \cdot x_e \quad \forall \boldsymbol{x} \in \mathcal{X}. \tag{9}$$

The linear cut in (9) is general and there are many choices for the cut coefficients $\delta_e(\hat{\boldsymbol{x}})$. The key challenge is to choose *tight* values for each coefficient that do not remove the optimal N-k scenario. For the N-k problem in gas pipeline networks, the coefficients $\delta_e(\hat{\boldsymbol{x}})$ are computed using a combination of the inner problem solution for the N-k scenario $\hat{\boldsymbol{x}}$ and the physics that governs the steady state flow of gas through the network. Using the inequality in (9), the bi-level problem is equivalently written as

$$(\mathcal{F}) \quad \max \eta(\boldsymbol{x}) \quad \text{subject to:} \tag{10a}$$
$$\eta(\boldsymbol{x}) \leqslant \eta(\hat{\boldsymbol{x}}) + \sum_{e \in \mathcal{C} \cup \mathcal{P}} \delta_e(\hat{\boldsymbol{x}}) \cdot x_e \quad \forall \hat{\boldsymbol{x}} \in \mathcal{X}, \tag{10b}$$

and the algorithm generates a subset of the cuts listed in Eq. (10b). The pseudo-code for the cutting-plane algorithm is shown in Algorithm 1, where the procedure for computing the cut coefficients (line 9-8) is detailed in a forthcoming paragraph.

We now present a technique for computing the coefficients $\delta_e(\hat{\boldsymbol{x}})$ in Eq. (10b) given an N-k scenario $\hat{\boldsymbol{x}}$ and the solution of the MISOCP relaxation of the inner problem (MGS). We first present the mathematical expression of the coefficients and then provide an intuitive justification. The MISOCP relaxation of the inner problem, for a given N-k scenario $\hat{\boldsymbol{x}}$ (let $\hat{s}$ denote the corresponding scenario) gives the value of the mass flow rate, $f_e(\hat{\boldsymbol{x}})$, for every $e \in \mathcal{C} \setminus \mathcal{C}(s)$ and $e \in \mathcal{P} \setminus \mathcal{P}(s)$. For the sake of clarity, the dependence of mass flow rates on

Algorithm 1 Cutting-plane algorithm: pseudo-code

Input: optimality tolerance, $\varepsilon > 0$
Output: $\boldsymbol{x}^* \in \mathcal{X}$, an ε-optimal solution

1: initial problem: $\mathcal{F}$ without constraint (10b)
2: $\eta^* \leftarrow -\infty$ ▷ lower bound on the optimal obj. value
3: $\bar{\eta} \leftarrow +\infty$ ▷ upper bound on the optimal obj. value
4: $\hat{\boldsymbol{x}} \leftarrow$ any initial N-k scenario
5: solve MISOCP relaxation of MGS using $\hat{\boldsymbol{x}}$ and let $\eta(\hat{\boldsymbol{x}})$ be the objective value
6: **if** $\eta(\hat{\boldsymbol{x}}) > \eta^*$ **then** $\eta^* \leftarrow \eta(\hat{\boldsymbol{x}})$ and $\boldsymbol{x}^* \leftarrow \hat{\boldsymbol{x}}$
7: compute $\delta_e(\hat{\boldsymbol{x}})$ for every $e \in \mathcal{C} \cup \mathcal{P}$ satisfying (9)
8: add $\eta(\boldsymbol{x}) \leqslant \eta(\hat{\boldsymbol{x}}) + \sum_{e \in \mathcal{C} \cup \mathcal{P}} \delta_e(\hat{\boldsymbol{x}}) \cdot x_e$ to $\mathcal{F}$ and resolve
9: update $\hat{\boldsymbol{x}}$, and set $\bar{\eta}$ using solution from Step 8
10: **if** $\bar{\eta} - \eta^* \leqslant \varepsilon \eta^*$ **then** $(\boldsymbol{x}^*, \eta^*)$ is the ε-optimal solution, stop
11: return to step: 5

scenario $\hat{\boldsymbol{x}}$ is shown explicitly. The pipes and compressors that constitute the scenario $\hat{s}$ or equivalently, $\hat{\boldsymbol{x}}$, are damaged and hence do not have any gas flowing through them. Given these flow rates, the coefficients are computed by:

$$\delta_e(\hat{\boldsymbol{x}}) = \begin{cases} |f_e(\hat{\boldsymbol{x}})| & \text{if, } e \notin \hat{s} \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Intuitively, setting the coefficients according to Eq. (11) imply that when a pipe or compressor (say e) is removed from a gas network, at most $|f_e|$ will go unserved. This statement is quantitatively true, except in the case of the Braess paradox in natural gas networks [26]. The Braess paradox occurs when adding one or more edges to a transport network can reduce overall throughput under certain conditions. Because a Braess-like condition would be a sub-optimal direction for minimizing the objective function, the paradox does not arise, and thus the coefficient values specified in (11) lead to a valid constraint at each iteration. The algorithm 1 ultimately converges to an ε-optimal solution to the MISOCP relaxation of the N-k problem.

VII. Case Studies

In this section, we present case studies on three networks: (i) the Belgian gas network [27] with a total of 42 pipes and compressors that can be interdicted, (ii) the New-England (NE) natural gas network [28] with 192 pipes and compressors that can be interdicted, and (iii) the gaslib-582 test network [29] with a total of 629 pipes and compressors that can be interdicted. The Belgian and the NE case studies are simplified network models of actual gas pipeline systems in Belgium and the New England region, respectively. The k values for each run of the N-k algorithm is varied from 1 to a value where 100% of the gas load in the system is left unserved by the resulting ε-optimal, N-k contingency. For the gaslib-582 test case, due to the excessive computation time, we restrict the runs to a k value where $> 95\%$ of the gas load is left unserved. The value of ε, the optimality tolerance in Algorithm 1, is set to 0.01% for every run of the algorithm and all the formulations and algorithms were implemented in the Julia programming language using optimization layer JuMP v0.18.6 [30] and GasModels v0.3.5[1]. Finally Gurobi v8.0 was used to solve the MISOCP relaxation of the MGS (the inner problem) for the cutting-plane generation algorithm on a machine with an Intel(R) Core(TM) i7-8700 CPU 3.20GHz.

Furthermore, in order to examine the effects N-k gas contingencies have on power systems, we use models which connect the Belgian and the NE gas networks to the IEEE 14-bus and 36-bus test systems, respectively [28]. In particular, gas-fired generators are attached to nodes in the natural gas networks. These generators withdraw gas from the natural gas pipeline network, and unserved gas load implies that gas-fired power plants receive insufficient gas and operate with reduced capacity. The loss in gas-fired generation capacity is computed using heat rate curves that convert mass flow (kg/s) into available MW capacity. In particular, the burn-rate, i.e., the gas withdrawal d_i from the gas pipeline network at node $i \in \mathcal{N}$, is converted into power production profiles p_g for a generator $g \in \mathcal{G}$ in the power network using a quadratic heat rate curve

$$p_g(\boldsymbol{d}_i) = \beta_0 + \beta_1 \boldsymbol{d}_i + \beta_2 \boldsymbol{d}_i^2. \tag{12}$$

In Eq. (12), the units of p_g is MW, and that of β_0, β_1, and β_2 are MW, $\mathrm{MW\,kg^{-1}}$, and $\mathrm{MW\,s^2\,kg^{-2}}$

A. Performance of the Cutting-Plane Algorithm

First, we present computational results that corroborate the effectiveness of the cutting-plane algorithm in computing ε-optimal ($\varepsilon = 0.01\%$) N-k attacks for the three test systems. Tables I – III show the computation time, the percentage of scheduled gas delivery that was curtailed, and the number of iterations taken by the cutting-plane algorithm to compute the ε-optimal N-k attack for the Belgian, NE, and gaslib-582 cases, respectively for different values of k. Fig. 1 shows the components in the Belgian network that, when interdicted, produce the worst case scenarios for values of k ranging from 1 through 4. Note that the worst k-outage scenarios are a collection of nested sets. It is observed from the tables that for small values of k, $k = 2$, the Belgian, NE, and gaslib-582 cases result in 50%, 40%, and 72%, respectively, of curtailed gas load with respect to the total baseline load levels without outages. This shows the value of developing an algorithm to compute a worst case N-k attack even for small values of k. Though the runs for $k \geqslant 5$ might not seem realistic, i.e., more than 4 components in the gas network failing simultaneously is highly unlikely, these results are shown in order to illustrate the computational limits of our algorithm and can be used as a surrogate to show the fact that our algorithm would

[1]https://github.com/lanl-ansi/GasModels.jl

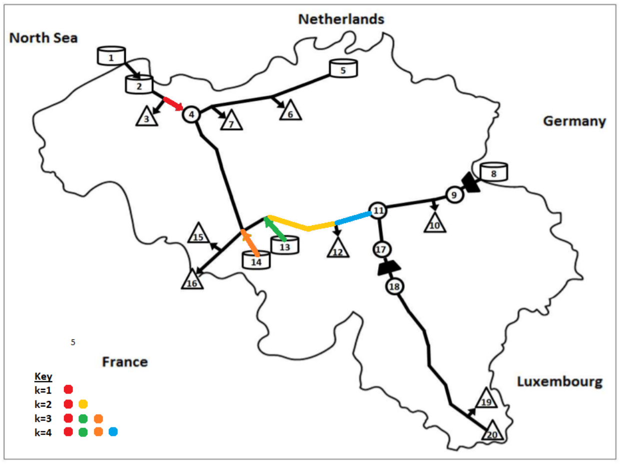

Fig. 1. Belgian Network with interdicted components in worst case scenarios for $k = 1, 2, 3, 4$

TABLE I
BELGIAN GAS NETWORK RESULTS

k	Iterations	Unserved gas (%)	Time (s)
1	4	29.4	0.163
2	4	50.4	0.109
3	4	75.1	0.104
4	7	88.9	0.147
5	6	95.1	0.125
6	8	98.0	0.247
7	9	99.3	0.267
8	13	100.0	0.348

scale to large instances with small values of k. Furthermore, from the iterations column in all the three tables it is clear that the cutting-plane algorithm is effective in computing the ε-optimal solution using only a few iterations. We remark that the computation time of the cutting-plane algorithm is in general proportional to the number of iterations of the algorithm and not related to the value of k. This trend is seen in the results for the NE test case in Table II, though computation time does not always increase with k as the problem is highly nonlinear and solution time depends on initialization. Finally, from Table III, it is clear that despite the low number of iterations of the algorithm even in the larger gaslib-582 case, the computation time per iteration increases because of larger MISOCP problem size for the inner computation in the larger test case.

B. Gas-fired Generation Capacity Loss in the Power Grid

This section presents results that illustrate the impact that N-k gas pipeline contingencies have on electricity transmission networks. We use loss of generation capacity on all the gas-fired generation plants as a measure to quantify this impact. This study is performed only on the Belgian and the NE test cases which were connected to the IEEE 14-bus and 36-bus test systems, respectively. The Belgian-IEEE 14 system is commonly used in the literature for gas-electric system case

TABLE II
NEW ENGLAND GAS NETWORK RESULTS

k	Iterations	Unserved gas (%)	Time (s)
1	2	24.3	3.619
2	4	39.1	5.742
3	12	47.3	15.633
4	19	56.1	27.088
5	15	64.3	23.107
6	17	71.3	23.373
7	16	77.7	21.722
8	12	83.6	14.312
9	9	89.9	10.242
10	7	94.3	9.986
11	7	97.7	8.393
12	9	99.6	9.117
13	14	99.7	15.115
14	24	99.8	34.880
15	28	99.9	73.066
16	32	100.0	716.366

TABLE III
GASLIB-582 RESULTS

k	Iterations	Unserved gas (%)	Time (s)
1	4	43.3	162.917
2	4	72.0	350.705
3	7	84.6	315.970
4	11	91.6	411.532
5	16	95.9	287.466

studies, and the NE-36 bus system is another larger test case. Tables IV and V show lost generation capacity (absolute value (MW) and as a percentage of total power produced by gas-fired generation in the baseline scenario) when the worst case N-k occurs on the gas side. Capacity loss is computed by converting unserved gas to power consumption (MW) using quadratic heat rate curves for gas-fired power plants (Eq. (12)).

VIII. Conclusion and Future Work

This article presents the first systematic algorithm to compute worst-case N-k contingencies on natural gas pipeline networks by modeling relaxations of steady-state gas flow physics. The computational effectiveness of the algorithm, its scalability, and the potential use of such a tool to estimate the impact of a worst-case N-k contingency on the bulk-electric system were shown through extensive computational

TABLE IV
GENERATION CAPACITY LOSS FOR THE BELGIAN-IEEE 14 NETWORK. DURING NORMAL OPERATION THE POWER PRODUCED FROM ALL THE GAS-FIRED POWER PLANTS IS 39.67 MW

k	Loss of capacity (MW)	Loss of capacity (%)
1	13.52	34.09
2	23.22	58.52
3	31.25	78.78
4	37.15	93.63
5	39.34	99.16
6	39.34	99.16

TABLE V

GENERATION CAPACITY LOSS FOR THE NEW ENGLAND-IEEE 36 NETWORK. DURING NORMAL OPERATION THE POWER PRODUCED FROM ALL THE GAS-FIRED POWER PLANTS IS 513.21 MW

k	Loss of capacity (MW)	Loss of capacity (%)
1	129.63	25.26
2	138.39	26.97
3	252.03	49.11
4	319.03	62.16
5	372.38	72.56
6	379.37	73.92
7	405.72	79.06
8	427.22	83.24
9	462.14	90.05
10	485.21	94.54
11	501.92	97.80
12	511.57	99.68

experiments on case studies involving several widely available test networks. Future work will focus on (i) performing a joint N-k interdiction analysis where a total of k components can be interdicted in either the power grid or gas pipeline system, where the modeling involves power flow and steady-state gas flow physics, respectively; and (ii) extension to a transient gas flow model to identify N-k contingencies that occur over time.

REFERENCES

[1] C. Coffrin, R. Bent, B. Tasseff, K. Sundar, and S. Backhaus, "Relaxations of AC Maximal Load Delivery for Severe Contingency Analysis," *IEEE Transactions on Power Systems*, vol. 34, no. 2, pp. 1450–1458, Mar. 2019.

[2] J. Salmeron, K. Wood, and R. Baldick, "Analysis of electric grid security under terrorist threat," *IEEE Transactions on Power Systems*, vol. 19, no. 2, pp. 905–912, 2004.

[3] V. Donde, V. López, B. Lesieutre, A. Pinar, C. Yang, and J. Meza, "Severe multiple contingency screening in electric power systems," *IEEE Transactions on Power Systems*, vol. 23, no. 2, pp. 406–417, 2008.

[4] A. Pinar, J. Meza, V. Donde, and B. Lesieutre, "Optimization strategies for the vulnerability analysis of the electric power grid," *SIAM Journal on Optimization*, vol. 20, no. 4, pp. 1786–1810, 2010.

[5] J. Salmeron, K. Wood, and R. Baldick, "Worst-case interdiction analysis of large-scale electric power grids," *IEEE Transactions on Power Systems*, vol. 24, no. 1, pp. 96–104, 2009.

[6] D. Bienstock and A. Verma, "The n-k problem in power grids: New models, formulations, and numerical experiments," *SIAM Journal on Optimization*, vol. 20, no. 5, pp. 2352–2380, 2010.

[7] K. Sundar, C. Coffrin, H. Nagarajan, and R. Bent, "Probabilistic N-k failure-identification for power systems," *Networks*, vol. 71, no. 3, pp. 302–321, 2018.

[8] T. Kim, S. J. Wright, D. Bienstock, and S. Harnett, "Analyzing vulnerability of power systems with continuous optimization formulations," *IEEE Transactions on Network Science and Engineering*, vol. 3, no. 3, pp. 132–146, 2016.

[9] K. Sundar, S. Misra, R. Bent, and F. Pan, "Spatial and topological interdiction for transmission systems," *arXiv preprint arXiv:1904.08330*, 2019.

[10] C. Lyons and G. Litra, "Gas-power interdependence: Knock-on effects of the das to gas," http://www.scottmadden.com/insight/gaspower-interdependence/, 2013.

[11] S. M. Rinaldi, J. P. Peerenboom, and T. K. Kelly, "Identifying, understanding, and analyzing critical infrastructure interdependencies," *IEEE Control Systems*, vol. 21, no. 6, pp. 11–25, 2001.

[12] T. Li, M. Eremia, and M. Shahidehpour, "Interdependency of natural gas network and power system security," *IEEE Transactions on Power Systems*, vol. 23, no. 4, pp. 1817–1824, 2008.

[13] R. K. Wood, "Bilevel network interdiction models: Formulations and solutions," *Wiley encyclopedia of operations research and management science*, 2010.

[14] H. Su, E. Zio, J. Zhang, and X. Li, "A systematic framework of vulnerability analysis of a natural gas pipeline network," *Reliability Engineering and System Safety*, vol. 175, pp. 79–91, 2018.

[15] Z. Qiao, L. Jia, W. Zhao, Q. Guo, H. Sun, and P. Zhang, "Influence of n-1 contingency in natural gas system on power system," in *2017 IEEE Conference on Energy Internet and Energy System Integration (EI2)*, Nov. 2017, pp. 1–5.

[16] H. Cong, Y. He, X. Wang, and C. Jiang, "Robust optimization for improving resilience of integrated energy systems with electricity and natural gas infrastructures," *Journal of Modern Power Systems and Clean Energy*, vol. 6, no. 5, pp. 1066–1078, Sep. 2018. [Online]. Available: https://doi.org/10.1007/s40565-018-0377-5

[17] S. D. Manshadi and M. E. Khodayar, "Resilient operation of multiple energy carrier microgrids," *IEEE Transactions on Smart Grid*, vol. 6, no. 5, pp. 2283–2292, 2015.

[18] C. Wang, W. Wei, J. Wang, F. Liu, F. Qiu, C. M. Correa-Posada, and S. Mei, "Robust defense strategy for gaselectric systems against malicious attacks," *IEEE Transactions on Power Systems*, vol. 32, no. 4, pp. 2953–2965, Jul. 2017.

[19] G. Brown, M. Carlyle, J. Salmerón, and K. Wood, "Defending critical infrastructure," *Interfaces*, vol. 36, no. 6, pp. 530–544, 2006.

[20] A. Thorley and C. Tiley, "Unsteady and transient flow of compressible fluids in pipelines': a review of theoretical and some experimental studies," *International Journal of Heat and Fluid Flow*, vol. 8, no. 1, pp. 3–15, 1987. [Online]. Available: http://www.sciencedirect.com/science/article/pii/0142727X87900440

[21] K. Sundar and A. Zlotnik, "State and parameter estimation for natural gas pipeline networks using transient state data," *IEEE Transactions on Control Systems Technology*, vol. 27, no. 5, pp. 2110–2124, 2018.

[22] C. Borraz-Sánchez, R. Bent, S. Backhaus, H. Hijazi, and P. V. Hentenryck, "Convex relaxations for gas expansion planning," *INFORMS Journal on Computing*, vol. 28, no. 4, pp. 645–656, 2016.

[23] G. P. McCormick, "Computability of global solutions to factorable nonconvex programs: Part iconvex underestimating problems," *Mathematical programming*, vol. 10, no. 1, pp. 147–175, 1976.

[24] R. E. Alvarez, "Interdicting electrical power grids," Ph.D. dissertation, Naval Postgraduate School, Monterey, California, 2004.

[25] A. L. Motto, J. M. Arroyo, and F. D. Galiana, "A mixed-integer LP procedure for the analysis of electric grid security under disruptive threat," *IEEE Transactions on Power Systems*, vol. 20, no. 3, pp. 1357–1365, 2005.

[26] L. Ayala and S. Blumsack, "The braess paradox and its impact on natural-gas-network performance," *Oil and Gas Facilities*, vol. 2, no. 3, pp. 52–64, Jun. 2013. [Online]. Available: https://www.onepetro.org/journal-paper/SPE-160142-PA

[27] D. de Wolf and Y. Smeers, "Optimal dimensioning of pipe networks with application to gas transmission networks," *Operations Research*, vol. 44, no. 4, pp. 596–608, 1996.

[28] R. Bent, S. Blumsack, P. Van Hentenryck, C. Borraz-Snchez, and M. Shahriari, "Joint electricity and natural gas transmission planning with endogenous market feedbacks," *IEEE Transactions on Power Systems*, vol. 33, no. 6, pp. 6397–6409, Nov. 2018.

[29] M. Schmidt, D. Aßmann, R. Burlacu, J. Humpola, I. Joormann, N. Kanelakis, T. Koch, D. Oucherif, M. E. Pfetsch, L. Schewe *et al.*, "Gaslib – A library of gas network instances," *Data*, vol. 2, no. 4, p. 40, 2017.

[30] I. Dunning, J. Huchette, and M. Lubin, "Jump: A modeling language for mathematical optimization," *SIAM Review*, vol. 59, no. 2, pp. 295–320, 2017.