FRONT PROPAGATION FOR INTEGRO-DIFFERENTIAL KPP REACTION-DIFFUSION EQUATIONS IN PERIODIC MEDIA

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ABSTRACT. We study front propagation phenomena for a large class of KPP-type integro-differential reaction-diffusion equations of order $\alpha \in (0, 2)$ in oscillatory environments, which model various forms of population growth with periodic dependence. We show that, under an exponential rescaling, the solution develops an isotropic advancing front and converges locally uniformly to zero beyond the front and to the periodic stationary state behind the front. The results are the most general available in this general setting.

1. INTRODUCTION

We study the long time/large space asymptotic behavior and front propagation for a class of models governed by integro-differential reaction-diffusion equations of the form

$$u_t + L^\alpha[u] = f(x, u) \text{ in } \mathbb{R}^d \times (0, \infty) \text{ and } u(\cdot, 0) = u_0 \text{ on } \mathbb{R}^d. \tag{1.1}$$

The reaction nonlinearity $f(x, u)$ is C^2 in u and satisfies a periodic KPP-type condition, that is, for each $u \in \mathbb{R}$,

$$x \to f(x, u) \text{ is 1-periodic,} \tag{1.2}$$

and there exists a constant $M > 0$, which is independent of x, such that, for all $x \in \mathbb{R}^d$,

$$s \to f(x, s)/s \text{ is decreasing in } s, \ f(x, 0) = 0, \text{ and } f(x, \cdot) \le 0 \text{ in } [M, \infty); \tag{1.3}$$

here 1-periodic means that the function is periodic in the unit cube. The diffusion L^α belongs to the nonlocal class of singular integral operators given by

$$L^\alpha[u](x) := \int_{\mathbb{R}^d} (u(x) - u(x + y))K(x, y)dy, \tag{1.4}$$

where the "mutation kernel" K satisfies

$$K \text{ is positive, 1-periodic and } C^2 \text{ in } x, \text{ and symmetric in } y, \tag{1.5}$$

and there exists $C_K > 0$ such that, for $\mathcal{K} = K, |D_x K|, \text{ or } |D_x^2 K|$ and all $x, y \in \mathbb{R}^d$,

$$C_K^{-1} \le \mathcal{K}(x, y)|y|^{d+\alpha} \le C_K. \tag{1.6}$$

Such operators generalize the fractional Laplacian $(-\Delta)^{\alpha/2}$. Due to the singularity of the kernel K, the integration in (1.4) should be interpreted in the principal value sense.

Lastly, we assume that $u_0 \in C(\mathbb{R}^d)$ and, for some $c_1 > 0$ and all $x \in \mathbb{R}^d$,

$$0 \le u_0(x) \le \frac{c_1}{1 + |x|^{d+\alpha}}. \tag{1.7}$$

Note that this includes compactly supported initial data. We discuss further the nature and consequences of assumptions (1.2), (1.3), (1.5), (1.6), and (1.7) in Section 2; see in particular Lemma 2.1 and Proposition 2.2.

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The simplest example for a reaction-diffusion equation is the Fisher-KPP equation (with logistic nonlinearity)

$$u_t - \Delta u = u - u^2 \ \text{ in } \ \mathbb{R}^d \times [0, \infty). \tag{1.8}$$

This model, and numerous generalizations, have been studied extensively, going back to the original work of Fisher [18] and Kolmogorov, Petrovskii, and Piskunov [28]. We refer to Aronson and Weinberger [1] and [2], Bramson [11], Freidlin [20], Evans and Souganidis [16], and Majda and Souganidis [31] for the derivations as well as up to date methods to study the long time/large-space asymptotic behavior.

Equations such as (1.1) arise from mesoscopic processes involving Lévy flights seen in models of population dynamics and evolutionary ecology. For example, Jourdain, Méléard, and Woyczynski [27] show a rigorous derivation when modeling Darwinian evolution of phenotypic variation, treating the population as a stochastic point process; see also Fournier and Méléard [19] and Gurney and Nisbet [22]. When the probability distribution of mutations has a heavy tail and belongs to the domain of attraction of a stable law, the corresponding diffusion admits jumps and can include spatial inhomogeneity (see for instance Baeumer, Kovacs, and Meerschaert [3] and Hillen and Othmer [25]) That is, the strength of the nonlocal diffusion realistically depends on the genetic or spatial landscape, which motivates the treatment of models with inhomogeneous reactions and diffusions.

It turns out (see [20], [16], and [31]) that the asymptotic behavior of the solution to (1.8) is recognized by an algebraic scaling $(x, t) \to (x/\epsilon, t/\epsilon^\gamma)$. This is consistent with the fact that the expected advancing level sets (the fronts) move with algebraic (in time) velocity. That is, for every h in the range of u and all t sufficiently large, the set $\{x \ : \ u(x, t) = h\}$ is comparable in a quantified way to the set $\{|x| \approx ct^\gamma\}$ for some c; see [20], [16], and [31] for the results for linear ($\gamma = 1$) growth, and the recent works of Berestycki, Mouhot, and Raoul [7], Bouin, Henderson, and Ryzhik [10], and Henderson, Perthame, Souganidis [24] for the superlinear ($\gamma > 1$) case. See also Hamel and Roques [23] for a treatment of the asymptotic behavior of (1.8) with fat-tailed initial data.

Many physical models, like, for example, neutron emission in nuclear physics, fast migrations in biology, and convective heating in forest fires, use processes that run faster than what would be expected for Brownian motion, either due to a fundamental sparseness of the medium or a long-range dispersion effect in the measured physical quantity; see for instance del-Castillo-Negrete [15] and Mancinelli, Vergni, and Vulpiani [32], and references therein. As a result the nonlocal equations potentially give rise to fronts moving with exponential (in time) velocities. This behavior, that was widely known in the applied literature, was first shown rigorously by Cabré and Roquejoffre [13], who considered the front-propagation and asymptotic dynamics of the nonlocal homogeneous problem

$$u_t + Au = f(u) \ \text{ in } \ \mathbb{R}^d \times [0, \infty),$$

with A the infinitesimal generator of a Feller semigroup and f of KPP-type as in (1.3), but independent of x.

In Cabré, Coulon, and Roquejoffre [12], the authors study a simpler version of (1.1) with fractional Laplacian $L^\alpha = (-\Delta)^{\alpha/2}$ for $\alpha \in (0, 1)$ and $f(x, u) = \mu(x)u - u^2$ with μ periodic, and give an estimate on the exponential (in time) propagation for the low-value level sets of u, that is, for all h sufficiently small and a fixed $\lambda > 0$ which both depend on μ,

$$\{u(x, t) = h\} \approx \{|x|^{d+\alpha} = c_h \exp(\lambda t)\}. \tag{1.9}$$

For more general KPP-type nonlinearities, Roquejoffre and Tarfulea [34] further refine these estimates and show the level sets symmetrize and the solutions become asymptotically flat over time. The results of [34] are not expected to hold in the periodic setting, as solutions in general converge to a steady state which is a nonconstant positive periodic function (and so no flattening can occur). However, it is worth remarking that the propagation of the front implied by (1.9) is radially symmetric even though the reaction nonlinearity is periodic. This is in contrast to the case of local diffusion, where it is known that propagation is anisotropic depending on μ. The lack of such a dependence is due to the strong nonlocal effects of the diffusion, since (1.6) implies K has thick but essentially radial tails.

In this paper, we study the behavior of all level sets for general nonlocal anisotropic L^α with no restriction on the order, that is, for the full range $(0, 2)$. Moreover, the mutation kernel K need not be either translation invariant or homogeneous in y. Also, unlike [12] our methods do not hinge on the construction of explicit sub-and super-solutions

(in the classical sense). Our approach is based on the theory of viscosity solutions and, in particular, a novel iterated improvement scheme that has not appeared before in the literature (see Propositions 3.3 and 3.4). This avoids the much more complicated argument in extending the results of [12] (which the authors acknowledge) to equations like (1.1), and offers an alternative methodology for other models where producing classical sub- and super-solutions is unfeasible.

While completing this paper, we became aware of a recent work of Bouin, Garnier, Henderson, and Patout [9] that considers a homogenization result for the class of equations

$$u_t = J * u - u + u(1 - u) \ \text{ in } \ \mathbb{R} \times (0, \infty),$$

for kernels J that are nonsingular, homogeneous in x, translation-invariant, and monotone. There has been previous work for nonlocal equations with integrable kernels (i.e., when J is a Borel measure); see for instance Garnier [21] and Yagisita [35]. Although the assumptions of [9, 21, 35] on J allow for kernels with more general decay at infinity albeit one dimensional and without singularity, our results are focused on space-dependent kernels with potential singularity at the origin and in any dimension. If J can have a singularity, it generalizes the fractional Laplacian as well as many other integro-differential operators, and creates a different situation than what is explored in [9, 21, 35]. This introduces significant technical difficulties, which we overcome here.

After completing this paper, we also became aware of ongoing work by Leculier [30] that shows a special case of our main result, under the same assumptions as in [12], using techniques similar to [12].

1.1. Rescaling and main results.
The long-time/large-space behavior of the solution to (1.1) is characterized by the limiting properties and behavior as $\epsilon \to 0$ of

$$u^\epsilon(x, t) = u(\widehat{x}|x|^{1/\epsilon}, t/\epsilon), \tag{1.10}$$

which solves

$$\epsilon u_t^\epsilon + \int \left(u(\widehat{x}|x|^{1/\epsilon}, t/\epsilon) - u(\widehat{x}|x|^{1/\epsilon} - y, t/\epsilon) \right) K(\widehat{x}|x|^{1/\epsilon}, y) dy = f(\widehat{x}|x|^{1/\epsilon}, u^\epsilon) \ \text{ in } \mathbb{R}^d \times [0, \infty),$$

with initial datum

$$u^\epsilon(x, 0) = u_0(\widehat{x}|x|^{1/\epsilon}),$$

where, for $x \neq 0$, $\widehat{x} := x/|x|$ is the unit vector in the direction of x.

Given the upper bound of (1.7), we see that $u^\epsilon(\cdot, 0)$ converges to a "patch" function supported on the unit ball, that is,

$$\lim_{\epsilon \to 0} u^\epsilon(x, 0) = \begin{cases} u_0(0) & \text{if } |x| < 1, \\ u_0(x) & \text{if } |x| = 1, \\ 0 & \text{if } |x| > 1. \end{cases}$$

The rescaling (1.10) is motivated by the scaling strategy of Méléard and Mirrahimi [33], who studied the behavior of the solution to

$$u_t + (-\Delta)^{\alpha/2} u = u R \left(u, \int_{\mathbb{R}} u(x, t) dx \right) \ \text{ in } \ \mathbb{R} \times (0, \infty),$$

where $R(u, I)$ is either the local KPP-interaction $1 - u$ or a purely nonlocal reaction $R(I)$ (for example, the mean-field logistic interaction $R(I) = r_0 - I$). The methods of [33] exploit the one-dimensional nature of the problem and the homogeneous in x nature of the nonlocal diffusion and the nonlinearity, in the form of a very particular change of variables, and cannot handle the more complicated estimates necessary for (1.1).

The scaling (1.10) is also motivated by the exponential speed of the invasion front seen for the fractional Fisher-KPP equation (see (1.9)), and also by the general work of Garnier [21] on the approximate front location for reaction-diffusion equations with (radially symmetric) integro-differential diffusion operators.

A key step in our analysis is the classical exponential transformation

$$u^\epsilon = \exp(v^\epsilon/\epsilon),$$

which leads to

$$v_t^\epsilon + \int \left(1 - \frac{\exp(v^\epsilon(\eta^\epsilon(x,y),t)/\epsilon)}{\exp(v^\epsilon(x,t)/\epsilon)}\right) K dy = \frac{f(\widehat{x}|x|^{1/\epsilon}, u^\epsilon)}{u^\epsilon}; \tag{1.11}$$

above and henceforth for convenience of notation

$$\eta^\epsilon(x,y) := (\widehat{x|x|^{1/\epsilon}} - y)|\widehat{x}|x|^{1/\epsilon} - y|^\epsilon \text{ and } K := K(\widehat{x}|x|^{1/\epsilon}, y). \tag{1.12}$$

Before proceeding, we state one final assumption on the parameters of the problem. We assume that the bottom of the spectrum for the linearized stationary operator is negative, that is,

there exists a positive, 1-periodic e^g satisfying $L^\alpha[e^g](x) - \partial_u f(x,0)e^{g(x)} = \lambda_1 e^{g(x)}$ with $\lambda_1 < 0$. (1.13)

For L^α defined as in (1.4), a positive (and unique) first eigenfunction e^g always exists. For completeness, we include a sketch of the proof of this in the Appendix (Proposition B.1).

The requirement on λ_1 ensures that the asymptotic behavior of u is nontrivial. In the case where L^α is the fractional Laplacian $(-\Delta)^{\alpha/2}$, Berestycki, Roquejoffre, and Rossi [8] show that the bottom of the spectrum of the operator $(-\Delta)^{\alpha/2} - \partial_u f(x,0)$ is the indicator to a drastic shift in the asymptotic behavior for (1.1). If $\lambda_1 \geq 0$, the model exhibits extinction, that is, $u(\cdot, t) \to 0$ as $t \to \infty$. On the other hand, if $\lambda_1 < 0$, the model exhibits invasion, that is, u converges, as $t \to \infty$ uniformly on compact sets, to a unique positive steady solution. The sign of λ_1 therefore measures the relative strengths of the depletion and growing zones $\{x : \partial_u f(x,0) \leq 0\}$ and $\{x : \partial_u f(x,0) > 0\}$ with respect to the ambient diffusion. Since we wish to study the nontrivial asymptotic behavior of the solutions to (1.1), we must work with models that exhibit invasion; hence the need for assumption (1.13). Our conclusions extend those of [8] to a much larger class of models.

Our first result concerns the behavior, as $\epsilon \to 0$, of v^ϵ. Note that, in view of the spatially oscillatory behavior of (1.11), the v^ϵ homogenizes in the limit.

Theorem 1.1. *Assume* (1.2), (1.3), (1.5), (1.6), (1.7), *and* (1.13). *Then, as $\epsilon \to 0$ and locally uniformly, the v^ϵ's converge to the unique solution v of the variational inequality*

$$\max(v_t - |\lambda_1|, v) = 0 \text{ in } \mathbb{R}^d \times (0, \infty) \text{ and } v(x,0) = \min(0, -(d+\alpha)\log(|x|)), \tag{1.14}$$

given by

$$v(x,t) = \min(0, |\lambda_1|t - (d+\alpha)\log(|x|)). \tag{1.15}$$

Knowing the asymptotic behavior of the v^ϵ's, we obtain some concrete information about the u^ϵ's.

For this, we recall that (1.1) admits a positive periodic steady state solution $u^+ : \mathbb{R}^d \to \mathbb{R}$ solving

$$L^\alpha[u^+] = f(x, u^+) \text{ in } \mathbb{R}^d.$$

Although the existence of u^+ is classical, for completeness, we sketch a proof summary in Appendix B.

Our second result is:

Theorem 1.2. *Assume* (1.2), (1.3), (1.5), (1.6), (1.7), *and* (1.13). *Then, as $\epsilon \to 0$ and locally uniformly,*

$$\begin{cases} u^\epsilon(x,t) \to 0 \text{ in } \{(x,t) : |x|^{d+\alpha} > e^{|\lambda_1|t}\}, \\[2mm] \frac{u^\epsilon(x,t)}{u^+(\widehat{x}|x|^{1/\epsilon})} \to 1 \text{ in } \{(x,t) : |x|^{d+\alpha} < e^{|\lambda_1|t}\}. \end{cases} \tag{1.16}$$

Let Q be the unit cube in $\mathbb{R}^d$. It follows from (1.16) that, as $\epsilon \to 0$ and for $|x|^{d+\alpha} < e^{|\lambda_1|t}$,

$$u^\epsilon(x,t) \rightharpoonup \overline{U} := \frac{1}{|Q|} \int_Q u^+(y) dy.$$

We remark that, since u^+ is in general non-constant, we cannot expect that u^ϵ converges strongly in the inner region.

The proof of Theorem 1.1 is based on viscosity solution techniques originated in [16], [17], Barles, Evans, and Souganidis [4] and [31]. The main steps are (i) obtaining a priori L^∞-bounds on the v^ϵ's, (ii) using the half-relaxed upper and lower limits in conjunction with the perturbed test function methods to obtain the equation satisfied by the limit (see [16] and Evans and Souganidis [17]), and (iii) deriving the claimed limit properties for the u^ϵ's. With this approach, it is at no point necessary to invent special sub- and super-solutions.

It turns out that steps (i) and (ii) above are connected with each other. Indeed, the obvious bounds implied by (1.7) (see Proposition 2.2 and Lemma 2.3) are not enough to yield the limiting equation. We go around this difficulty by an iterative (inductive) argument along which we improve at each step the upper and lower bounds (see Propositions 3.3 and 3.4), eventually obtaining the claimed result. This is a new argument in the theory of front propagation.

1.2. **Organization of the paper.** The rest of the paper proceeds as follows. In Section 2 we prove a preliminary bound on the solution. In Section 3, we state in Proposition 3.3 and Proposition 3.4 the inductive arguments and give the proof of Theorem 1.1. Proposition 3.3 and Proposition 3.4 are shown, respectively, in Section 4 and Section 5. In Section 6 we give the proof of Theorem 1.2. We also include an Appendix that contains some useful preliminary calculations and context. In Appendix A we recall the definitions of viscosity sub- and super-solutions. In Appendix B we show the existence of a positive periodic first eigenfunction e^g to the linearized operator $L^\alpha - \partial_u f$, as well as a positive steady state u^+ when the corresponding eigenvalue is negative. In Appendix C we prove Lemma 2.1. Finally, in Appendix D we prove a technical lemma used in Section 2.

1.3. **Notation and Terminology.** We write Q and $B_r(x)$ respectively for the unit cube and the open ball of radius r and center x in $\mathbb{R}^d$. Given $A, B \in \mathbb{R}$, $A \lesssim B$ means that there exists some constant $c > 0$, which is independent of the various parameters, such that $A \leq cB$. Throughout the paper, $\|l\|$ denotes the sup-norm of a given bounded function l. We denote by E^c the complement of $E \subset \mathbb{R}^d$. A function which is periodic on the unite cube Q is referred to as 1-periodic. Throughout the paper sub-and super-solutions are understood in the viscosity sense.

2. PRELIMINARY BOUNDS

Observe that, in view of (1.2) and (1.3) (in particular that f is C^2), we may write

$$f(x, u) = \mu(x)u - E(x, u), \tag{2.1}$$

where $\mu(x) := \partial_u f(x, 0)$ and $E(x, u)$ is an error term such that, for some $\overline{M} > 0$,

$$0 < E(x, u) =: \partial_u f(x, 0)u - f(x, u) \leq \overline{M}u^2. \tag{2.2}$$

The algebraic decay of the tail of K, seen in the upper and lower bounds of (1.6) for $|y|$ large, will be very important. Indeed, this is the mechanism which produces exponential (in time) propagation of fronts.

We remark, however, that the proofs of our main results do not rely on K having a singularity as $|y| \to 0$. As a matter of fact the symmetric singularity at $y = 0$ allowed by (1.6) is a technical obstacle which we overcome with careful estimates. Actually all the arguments in the proofs of Theorem 1.1 and Theorem 1.2 remain valid if, instead of (1.6), we assume

$$\frac{C_K^{-1}}{1 + |y|^{d+\alpha}} \leq \mathcal{K} \leq \frac{C_K}{|y|^{d+\alpha}}.$$

The singular lower bound of (1.6) is only needed to prove the existence of a positive principal eigenfunction to the linearized operator, where we need to compare $\langle L^\alpha[u], u \rangle$ with a Sobolev norm; see (1.13) below and Lemma B.2.

The symmetry assumption on K allows us to write, after a change of variables,

$$L^\alpha[u](x) = \frac{1}{2} \int (2u(x) - u(x+y) - u(x-y))K(x, y)dy. \tag{2.3}$$

The following Lemma, which is proved in Appendix C, allows us to assume a nominally stronger condition on the initial data u_0 in the form of an algebraically decaying lower bound.

Lemma 2.1. *Assume* (1.2), (1.3), (1.5), (1.6), *and* (1.7). *Then there exist two positive constants* c_0 *and* c_1 *depending on* u_0 *and the parameters of the problem such that*

$$\frac{c_0}{1 + |x|^{d+\alpha}} \leq u(x, 1) \leq \frac{c_1}{1 + |x|^{d+\alpha}}. \tag{2.4}$$

Henceforth, we shall assume that the initial data u_0 satisfies, for some $c_1 > c_0 > 0$ and all $x \in \mathbb{R}^d$,

$$\frac{c_0}{1 + |x|^{d+\alpha}} \leq u_0(x) \leq \frac{c_1}{1 + |x|^{d+\alpha}}. \tag{2.5}$$

We will prove Theorems 1.1 and 1.2 with (2.5) in place of (1.7). Since they only show locally uniform convergence of the solution away from $t = 0$, in light of Lemma 2.1, we see that the case with more general initial data (1.7) easily follows.

An immediate consequence of (1.2), (1.3), (1.5), (1.6), (2.5), and (1.13) are the bounds given in the next proposition.

Proposition 2.2. *Let* u *be the solution to* (1.1) *and assume* (1.2), (1.3), (1.5), (1.6), (2.5), *and* (1.13). *Then there exist positive constants* $B_0 > |\lambda_1|$, $c_0 < C_0$, *and* A_0 *such that, for all* (x, t),

$$\frac{c_0 e^{-A_0 t}}{1 + e^{-|\lambda_1|t}|x|^{d+\alpha}} \leq u(x, t) \leq \frac{C_0}{1 + e^{-B_0 t}|x|^{d+\alpha}}. \tag{2.6}$$

We remark that the resulting bounds can be improved through more precise analysis (or by specializing to particular nonlinearities such as $\mu(x)u - u^2$), but this is not the purpose of the Proposition. Our main results follow from different techniques, but require an initial rough bound to start the iteration in Section 3.

The estimate follows from the fact that, for some positive constants c, c' and c'' and all x and u,

$$-cu - cu^2 \leq f(x, u) \leq c'u - c''u^2. \tag{2.7}$$

Then Proposition 2.2 is shown by establishing that the lower and upper bounds of (2.6) are respectively sub- and super-solutions to (1.1).

To prove Proposition 2.2 and throughout the paper we will require the following technical lemma which is proved in Appendix D.

Lemma 2.3. *Assume* (1.2), (1.3), (1.5), *and* (1.6), *and let* $h(x, t) = \frac{1}{1 + e^{-\lambda t}|x|^{d+\alpha}}$. *Then there exists a constant* $D > 0$ *depending only on* d *and* K *such that, for all* (x, t),

$$|L^\alpha[h]| \leq \frac{De^{-\alpha\lambda t/(d+\alpha)}}{1 + e^{-\lambda t}|x|^{d+\alpha}}. \tag{2.8}$$

Proof of Proposition 2.2: It follows from (2.5) that (2.6) holds at $t = 0$ for an appropriate choice of c_0 and C_0.

Let $W(x, t) = C_0/(1 + e^{-B_0 t}|x|^{d+\alpha})$ and $w(x, t) = c_0 e^{-A_0 t}/(1 + e^{-|\lambda_1|t}|x|^{d+\alpha})$. Since (1.1) satisfies a comparison principle, we show that W and w are in fact super- and sub-solutions to (1.1) for an appropriate choice of constants.

Let c, c', and c'' be as in (2.7). We remark that $c' > |\lambda_1|$. For $B_0 > D + c' > |\lambda_1|$ and $C_0 > \max(1, B_0/c'')$ large enough that $u_0 < W(\cdot, 0)$, Lemma 2.3 yields

$$W_t + L^\alpha[W] \geq \frac{B_0 C_0 e^{-B_0 t}|x|^{d+\alpha}}{(1 + e^{-B_0 t}|x|^{d+\alpha})^2} - \frac{DC_0 e^{-\alpha B_0 t/(d+\alpha)}}{1 + e^{-B_0 t}|x|^{d+\alpha}} = \left(B_0 - De^{-\alpha B_0 t/(d+\alpha)}\right)W - \frac{B_0}{C_0}W^2 \geq f(x, W),$$

which implies the upper bound of (2.6).

The lower bound follows, after choosing $A_0 > |\lambda_1| + D - c$ and $c_0 < |\lambda_1|/c$ sufficiently small so that $w(\cdot, 0) < u_0$, from the estimate

$$w_t + L^\alpha[w] \leq \frac{|\lambda_1|c_0 e^{-(|\lambda_1|+A_0)t}|x|^{d+\alpha}}{(1 + e^{-|\lambda_1|t}|x|^{d+\alpha})^2} - \frac{A_0 c_0 e^{-A_0 t}}{1 + e^{-|\lambda_1|t}|x|^{d+\alpha}} + \frac{D c_0 e^{-A_0 t} e^{-\alpha|\lambda_1|t/(d+\alpha)}}{1 + e^{-|\lambda_1|t}|x|^{d+\alpha}}$$

$$= \left(|\lambda_1| - A_0 + D e^{-\alpha|\lambda_1|t/(d+\alpha)}\right) w - \frac{|\lambda_1|}{c_0} e^{A_0 t} w^2 \leq f(x, w).$$

$\square$

3. IMPROVEMENT ITERATION AND PROOF OF THEOREM 1.1

We first recall the definition of generalized upper and lower locally uniform limits.

Definition 3.1. *For a bounded family of functions $\{v^\epsilon\}_{\epsilon > 0}$ with each $v^\epsilon : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$ smooth, the half-relaxed upper limit is given by*

$$v^*(x, t) := \limsup_{\epsilon \to 0,\ (y,s) \to (x,t)} v^\epsilon(y, s).$$

Similarly, the half-relaxed lower limit is given by

$$v_*(x, t) := \liminf_{\epsilon \to 0,\ (y,s) \to (x,t)} v^\epsilon(y, s).$$

The half-relaxed upper and lower limits are, respectively, upper- and lower-semicontinuous. As an immediate consequence of the definition, it follows that, on any compact set $K \subset \mathbb{R}^d \times \mathbb{R}_+$,

$$\sup_K (v^\epsilon - v^*) \to 0 \quad \text{and} \quad \inf_K (v^\epsilon - v_*) \to 0 \quad \text{uniformly in } \epsilon. \tag{3.1}$$

We first translate the rough bounds of Proposition 2.2 to the half-relaxed limits.

Lemma 3.2. *For A_0 and B_0 as in (2.6), we have that*

$$\min\left(0, |\lambda_1|t - (d + \alpha)\log(|x|)\right) - A_0 t \leq v_*(x, t) \leq v^*(x, t) \leq \min\left(0, B_0 t - (d + \alpha)\log(|x|)\right), \tag{3.2}$$

Proof. By (2.6) we have that

$$\epsilon \log(c_0) - A_0 t - \epsilon \log(1 + e^{-|\lambda_1|t/\epsilon}|x|^{(d+\alpha)/\epsilon}) \leq v^\epsilon(x, t) \leq \epsilon \log(C_0) - \epsilon \log(1 + e^{-B_0 t/\epsilon}|x|^{(d+\alpha)/\epsilon}). \tag{3.3}$$

If $|x|^{d+\alpha} \leq e^{B_0 t}$, (3.3) yields $v^*(x, t) \leq 0$. However, if $|x|^{d+\alpha} > e^{B_0 t}$, letting $\epsilon \to 0$ in (3.3) shows that $v^*(x, t) \leq B_0 t - (d + \alpha)\log(|x|)$. A similar analysis extends the lower bounds of (3.3) to v_*, yielding (3.2). $\square$

Moreover, letting $t \to 0$, we obtain

$$v_*(x, 0) = v^*(x, 0) = \min(0, -(d + \alpha)\log(|x|)). \tag{3.4}$$

We remark that, in general, the half-relaxed upper and lower limits of solutions satisfy initial and boundary value conditions in a relaxed sense; see Barles and Perthame [6], Ishii [26], and [4]. However, the inequalities in (3.3) hold uniformly, so that we can extract (3.4).

We note for future use that, for any fixed $M > 0$, there exists a $C_M > 0$ such that, for all $(y, s) \in \mathbb{R}^d \times [0, \infty)$,

$$\frac{C_M^{-1}}{1 + e^{-Ms}|y|^{d+\alpha}} \leq \exp\left(\min(0, Ms - (d + \alpha)\log|y|)\right) \leq \frac{C_M}{1 + e^{-Ms}|y|^{d+\alpha}}. \tag{3.5}$$

Having established (3.2) as a base case, we now describe the following inductive procedure.

Let $\{A_k\}_{k \geq 0}$ and $\{B_k\}_{k \geq 0}$ be two monotone decreasing sequences in $(0, \infty)$ with A_0 and B_0 given by the exponents seen in (2.6),

$$A_k \to 0 \quad \text{and} \quad B_k \to |\lambda_1|,$$

and, for each $k \geq 0$,

$$\frac{B_k}{B_{k+1}} < 1 + \frac{\alpha}{d}, \quad B_{k+1} \geq |\lambda_1|, \quad A_k - A_{k+1} < \frac{|\lambda_1|}{4(d+\alpha)} \quad \text{and} \quad A_{k+1} \geq 0. \tag{3.6}$$

We will show that, if A_k and B_k are such that

$$\min\left(0, |\lambda_1|t - (d+\alpha)\log(|x|)\right) - A_k t \le v_*(x,t) \le v^*(x,t) \le \min\left(0, B_k t - (d+\alpha)\log(|x|)\right), \tag{3.7}$$

then the same holds for A_{k+1} and B_{k+1}.

This is the conclusion of the following two propositions that establish appropriate variational inequalities for the half-relaxed upper and lower limits of v^ϵ.

Proposition 3.3. *Assume* (3.6), (3.7), *and let* $V_k(x,t) := \min(0, B_k t - (d+\alpha)\log(|x|))$. *Then*

$$\min(v_t^* - B_{k+1}, v^* - V_{k+1}) \le 0 \ \ in \ \ \mathbb{R}^d \times (0,\infty). \tag{3.8}$$

Proposition 3.4. *Assume* (3.6), (3.7), *and let* $W_k(x,t) := \min(0, |\lambda_1|t - (d+\alpha)\log(|x|)) - A_k t$. *Then*

$$\max(v_{*,t} - |\lambda_1| + A_{k+1}, v_* - W_{k+1}) \ge 0 \ \ in \ \ \mathbb{R}^d \times (0,\infty). \tag{3.9}$$

The proof of Theorem 1.1 is then completed with an argument by induction and the following technical lemma, which is an immediate consequence of the definition of viscosity solution.

Lemma 3.5. *Let* $U \subset \mathbb{R}^d$ *be open and* $F, G : U \times [0,\infty) \to \mathbb{R}$ *be respectively upper- and lower-semicontinuous. Assume that, for a fixed* $C > 0$ *and all* $(x,t) \in U \times [0,\infty)$, $F(x,t) \le Ct$ *and* $G(x,t) \ge -Ct$, *and,*

$$\min(F_t, F) \le 0 \ \ and \ \ \max(G_t, G) \ge 0 \ \ in \ the \ viscosity \ sense.$$

Then

$$F \le 0 \ \ and \ \ G \ge 0 \ \ on \ \ U \times [0,\infty).$$

The proof of Theorem 1.1. It follows from Propositions 3.3 and 3.4 and Lemma 3.5 that, for all $(x,t) \in \mathbb{R}^d \times [0,\infty)$,

$$\min\left(0, |\lambda_1|t - (d+\alpha)\log(|x|)\right) - A_{k+1} t \le v_*(x,t) \le v^*(x,t) \le \min\left(0, B_{k+1} t - (d+\alpha)\log(|x|)\right).$$

Since $A_k \to 0$ and $B_k \to |\lambda_1|$, we conclude that for all $(x,t) \in \mathbb{R}^d \times [0,\infty)$

$$v_*(x,t) = v^*(x,t) = \min(0, |\lambda_1|t - (d+\alpha)\log(|x|)).$$

This in fact shows that v^ϵ has a locally uniform limit v, which is given by (1.15), and clearly satisfies (1.14). □

4. PROOF OF PROPOSITION 3.3

Since the proof is long and technical, we begin with a general outline. First, observe that (3.3) implies $v^* \le 0$ independently of Proposition 3.3, that is, (3.8) trivially holds when $|x| \le 1$. As such, we only consider points (x,t) with $|x| > 1$ and $t > 0$. Let ψ be a smooth test function that touches v^* from above at a point $(x_0, t_0) \in \overline{B}_1^c \times (0,\infty)$ and, arguing by contradiction, we assume that

$$v^*(x_0,t_0) > V_{k+1}(x_0,t_0) = \min(0, B_{k+1}t_0 - (d+\alpha)\log(|x_0|)), \tag{4.1}$$

and

$$\psi_t(x_0,t_0) > B_{k+1}. \tag{4.2}$$

In subsection 4.1, we introduce two important formulas involving L^α. In subsection 4.2, we modify ψ into a new function ψ^ϵ that converges uniformly as $\epsilon \to 0$ to ψ in a neighborhood of (x_0, t_0), and we show that, for all ϵ sufficiently small, each ψ^ϵ is a super-solution to (1.11) in a neighborhood of (x_0, t_0) which is independent of ϵ; see Lemma 4.2 and Lemma 4.3. In subsection 4.3, we essentially use v^ϵ as a test function to show that v^ϵ must remain a fixed distance away from ψ^ϵ in the neighborhood of (x_0, t_0); see Lemma 4.4. Therefore $\psi(x_0, t_0)$ is strictly larger than $v^*(x_0, t_0)$, which contradicts the assumption.

Although the above gives an accurate summary of the scheme of the proof, for the actual argument we need to make a few technical modifications, which, however, do not change the overall strategy of the proof. Firstly, it is necessary to change ψ outside of a neighborhood of (x_0, t_0). Secondly, we will not be able to test ψ^ϵ against v^ϵ directly, since we cannot assume that $v^\epsilon \le v^*$. Instead, we must modify v^ϵ outside of a large but fixed compact set K, and then appeal

to the locally uniform subconvergence of v^ϵ to v^*; see (3.1). Since (1.11) is nonlocal, this will create some error terms that must be bounded.

In proving that ψ^ϵ is a super-solution, we will need a bound on the ratio between ψ^ϵ at an "arbitrary" point and ψ^ϵ near (x_0, t_0). This is used to estimate the size of the nonlocal diffusion term, and employs (4.5) and (4.3). Then (4.1) acts as an impromptu (local) lower bound for ψ at (x_0, t_0). The lower bound will then hold for ψ^ϵ on a sufficiently small neighborhood, depending on ψ but not on ϵ. Combined with the global upper bounds of (3.7), the effect of the nonlocal diffusion is seen to decay to zero as $\epsilon \to 0$, provided (3.6) holds for B_{k+1} and B_k.

4.1. **Some formulas for L^α.** In view of (1.13), if e^g is the first eigenfunction for the linearized operator $L^\alpha - \mu$, then $L^\alpha[e^g]/e^g = -|\lambda_1| + \mu$; here we stress that $g = g(x)$ (unscaled by ϵ) is well-defined as the logarithm of the first eigenfunction since the latter is strictly positive. The estimates for the nonlocal diffusion will require two slightly different approaches depending on the value of α. To that end, we first provide a crucial decomposition in the following lemma; recall that η^ϵ is defined in (1.12).

Lemma 4.1. *Given $a_0 : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$, let $a(x, t) := a_0(x, t) + \epsilon g(\widehat{x}|x|^{1/\epsilon})$. If $\alpha \in (0, 1)$, then*

$$\frac{L^\alpha[e^a]}{e^a} = \frac{L^\alpha[e^g]}{e^g} + \int G_\epsilon(x, y) \left(1 - \frac{\exp(a_0(\eta^\epsilon(x, y), t)/\epsilon)}{\exp(a_0(x, t)/\epsilon)} \right) K \, dy, \tag{4.3}$$

where

$$G_\epsilon(x, y) := \exp \left(g(\widehat{x}|x|^{1/\epsilon} - y) - g(\widehat{x}|x|^{1/\epsilon}) \right) \tag{4.4}$$

is bounded above and below by positive constants uniformly in ϵ.

If $\alpha \in [1, 2)$, then, for any $R > 0$,

$$\frac{L^\alpha[e^a]}{e^a} = \begin{cases} \frac{L^\alpha[e^g]}{e^g} + \frac{1}{2} \int_{|y| < R} \left(H_\epsilon(a_0, x, y) + H_\epsilon(a_0, x, -y) \right) K \, dy \\ + \int_{|y| \geq R} G_\epsilon(x, y) \left(1 - \frac{\exp(a_0(\eta^\epsilon(x, y), t)/\epsilon)}{\exp(a_0(x, t)/\epsilon)} \right) K \, dy, \end{cases} \tag{4.5}$$

where

$$H_\epsilon(a_0, x, y) := G_\epsilon(x, y) \left(1 - \frac{\exp(a_0(\eta^\epsilon(x, y), t)/\epsilon)}{\exp(a_0(x, t)/\epsilon)} \right). \tag{4.6}$$

Proof. When $\alpha \in (0, 1)$, formula (4.3) follows from the identity

$$\frac{L^\alpha[e^a]}{e^a} = \int \left(1 - \frac{\exp(a(\eta^\epsilon(x, y), t)/\epsilon)}{\exp(a(x, t)/\epsilon)} \right) K \, dy$$

$$= \frac{L^\alpha[e^g]}{e^g} + \int \left(\frac{\exp(g(\widehat{x}|x|^{1/\epsilon} - y))}{\exp(g(\widehat{x}|x|^{1/\epsilon}))} - \frac{\exp(a(\eta^\epsilon(x, y), t)/\epsilon)}{\exp(a(x, t)/\epsilon)} \right) K \, dy.$$

When $\alpha \in [1, 2)$, we need a stronger cancellation at $y = 0$ to handle the potential singularity of K. By (2.3), we have, for any $R > 0$,

$$\frac{L^\alpha[e^a]}{e^a} = \int_{|y| < R} \left(1 - \frac{\exp(a(\eta^\epsilon(x, y), t)/\epsilon) + \exp(a(\eta^\epsilon(x, -y), t)/\epsilon)}{2 \exp(a(x, t)/\epsilon)} \right) K \, dy$$

$$+ \int_{|y| \geq R} \left(1 - \frac{\exp(a(\eta^\epsilon(x, y), t)/\epsilon)}{\exp(a(x, t)/\epsilon)} \right) K \, dy.$$

The formula (4.5) then follows. $\qquad \square$

4.2. **Constructing the super-solution from the test function.** We assume that (3.6) and (3.7) hold for v^* and the relevant constants. Let ψ be a smooth function such that $v^* - \psi$ assumes a global maximum value of 0 at (x_0, t_0) and

$$\psi(x_0, t_0) = v^*(x_0, t_0) > \min(0, B_{k+1}t_0 - (d + \alpha)\log(|x_0|)) \tag{4.7}$$

and, for some $\sigma > 0$,

$$\psi_t(x_0, t_0) - B_{k+1} = \sigma > 0.$$

Let $r_0 = \min(t_0, |x_0| - 1)/3$ and consider the cylinder

$$D_{r_0}(x_0, t_0) := B_{r_0}(x_0) \times (t_0 - r_0, t_0 + r_0) \subset \{(x, t) : |x| > 1, \ t > 0\}.$$

We modify ψ outside a fixed neighborhood of (x_0, t_0). Without loss of generality, in light of (3.7), we may assume that

$$\psi(x, t) \leq \min(0, B_k t - (d + \alpha)\log(|x|)). \tag{4.8}$$

Let $\delta > 0$ be such that

$$\frac{B_k d}{d + \alpha} + \delta < B_{k+1}, \ e^{4\delta r_0} < (|x_0| - r_0)^{\alpha/2}, \ \text{and} \ \delta < \frac{\sigma}{3}. \tag{4.9}$$

Notice that, if δ and r_0 satisfy (4.9), then so do any $\delta' < \delta$ and $r_0' < r_0$. We can therefore reduce the values of δ and r_0 without violating the above assumptions.

Taking δ and r_0 smaller, depending on ψ and σ but not on ϵ, we construct a smooth function θ such that, for some $C > 0$ depending on r_0, ψ, B_{k+1}, and B_k but not on ϵ,

$$\begin{cases} (i) & \theta = \psi \ \text{on} \ D_{r_0/2}(x_0, t_0), \\ (ii) & \theta = \psi + \max(0, \delta(t - t_0 + 2r_0)) \ \text{on} \ D_{r_0}(x_0, t_0)^c, \\ (iii) & \theta \leq \psi + \max(0, \delta(t - t_0 + 2r_0)) \ \text{everywhere}, \\ (iv) & \theta \geq \min(0, B_{k+1}t - (d + \alpha)\log(|x|)) - \delta r_0 \ \text{on} \ D_{r_0}(x_0, t_0), \\ (v) & \theta_t - B_{k+1} > 2\sigma/3 \ \text{on} \ D_{r_0}(x_0, t_0), \\ (vi) & \|D\theta\| \ \text{and} \ \|D^2\theta\| < C. \end{cases} \tag{4.10}$$

In light of (4.1) and the regularity of ψ, such a θ always exists; note that to satisfy (4.10)(iv) and (v) it may be necessary to reduce the size of r_0 from its initial value, but this only depends on ψ and not on ϵ.

We now modify θ to obtain a super-solution to the ϵ-problem.

Lemma 4.2. *For (x_0, t_0), r_0, δ, and θ as above, and for all ϵ sufficiently small, the function*

$$\psi^\epsilon(x, t) := \theta(x, t) + \epsilon g(\widehat{x}|x|^{1/\epsilon}) + \delta(t_0 - r_0 - t)$$

is a super-solution (in the viscosity sense) to (1.1) in $D_{r_0}(x_0, t_0)$.

Proof. Let φ be a smooth test function such that $\psi^\epsilon - \varphi$ assumes a minimum value of 0 at $(\bar{x}, \bar{t}) \in D_{r_0}(x_0, t_0)$; recall that $|\bar{x}| > 1$. Since $\varphi = \psi^\epsilon$ at $(\bar{x}, \bar{t})$, for all $y \in \mathbb{R}^d$, we have

$$\frac{\exp(\varphi(\eta^\epsilon(\bar{x}, y), \bar{t})/\epsilon)}{\exp(\varphi(\bar{x}, \bar{t})/\epsilon)} = \frac{\exp(\psi^\epsilon(\eta^\epsilon(\bar{x}, y), \bar{t})/\epsilon)}{\exp(\psi^\epsilon(\bar{x}, \bar{t})/\epsilon)} + \frac{\exp(\varphi(\eta^\epsilon(\bar{x}, y), \bar{t})/\epsilon) - \exp(\psi_\epsilon(\eta^\epsilon(\bar{x}, y), \bar{t})/\epsilon)}{\exp(\varphi(\bar{x}, \bar{t})/\epsilon)},$$

and, hence,

$$1 - \frac{\exp(\varphi(\eta^\epsilon(\bar{x}, y), \bar{t})/\epsilon)}{\exp(\varphi(\bar{x}, \bar{t})/\epsilon)} \geq 1 - \frac{\exp(\psi^\epsilon(\eta^\epsilon(\bar{x}, y), \bar{t})/\epsilon)}{\exp(\psi^\epsilon(\bar{x}, \bar{t})/\epsilon)}. \tag{4.11}$$

Moreover, it follows from (4.10)(v) and (4.9) that

$$\varphi_t(\bar{x}, \bar{t}) \geq \psi_t^\epsilon(\bar{x}, \bar{t}) = \theta_t(\bar{x}, \bar{t}) - \delta > B_{k+1} + \sigma/3. \tag{4.12}$$

Using (4.9), (4.11), and (4.12), for any $R > 0$, we find that, at $(\bar{x}, \bar{t})$,

$$\begin{cases} \varphi_t + \int_{B_R(\bar{x})} \left(1 - \frac{\exp(\varphi(\eta^\epsilon(\bar{x}, y), \bar{t})/\epsilon)}{\exp(\varphi(\bar{x}, \bar{t})/\epsilon)}\right) K \, dy + \int_{B_R(\bar{x})^c} \left(1 - \frac{\exp(\psi^\epsilon(\eta^\epsilon(\bar{x}, y), \bar{t})/\epsilon)}{\exp(\psi^\epsilon(\bar{x}, \bar{t})/\epsilon)}\right) K \, dy \\ \qquad > B_{k+1} + \frac{\sigma}{3} + \int \left(1 - \frac{\exp(\psi^\epsilon(\eta^\epsilon(\bar{x}, y), \bar{t})/\epsilon)}{\exp(\psi^\epsilon(\bar{x}, \bar{t})/\epsilon)}\right) K \, dy. \end{cases} \tag{4.13}$$

At this time we recall once more (1.12), and here $K = K(\widehat{x}|\bar{x}|^{1/\epsilon}, y)$. Next we employ (4.3) and (4.5) with ψ^ϵ in place of a to get, for any $r = r(\epsilon) > 0$,

$$\frac{L^\alpha[e^{\psi^\epsilon}]}{e^{\psi^\epsilon}} = -|\lambda_1| + \mu(\widehat{x}|\bar{x}|^{1/\epsilon}) + \mathcal{I}_{\epsilon,\alpha} + \int_{|y| \geq r} G_\epsilon(\bar{x}, y) \left(1 - \frac{\exp(\theta(\eta^\epsilon(\bar{x}, y), \bar{t})/\epsilon)}{\exp(\theta(\bar{x}, \bar{t})/\epsilon)}\right) K dy \qquad (4.14)$$

where

$$\mathcal{I}_{\epsilon,\alpha} := \begin{cases} \int_{|y| < r} G_\epsilon(\bar{x}, y) \left(1 - \frac{\exp(\theta(\eta^\epsilon(\bar{x}, y), \bar{t})/\epsilon)}{\exp(\theta(\bar{x}, \bar{t})/\epsilon)}\right) K dy & \text{if } \alpha < 1, \\ \frac{1}{2} \int_{|y| < r} (H_\epsilon(\theta, \bar{x}, y) + H_\epsilon(\theta, \bar{x}, -y)) K dy & \text{if } \alpha \geq 1. \end{cases} \qquad (4.15)$$

Note that the quantity $\delta(t_0 - r_0 - t)$ disappears in (4.14) and (4.15) due to the homogeneity of $e^{-\psi^\epsilon} L^\alpha[e^{\psi^\epsilon}]$. Take $r(\epsilon) = \epsilon|\bar{x}|^{1/\epsilon}$ and recall (4.6). It follows from (4.10)(iv) that

$$\theta(\bar{x}, \bar{t}) \geq \min(0, B_{k+1}\bar{t} - (d + \alpha)\log(|\bar{x}|)) - \delta r_0.$$

For the rest of the proof, let

$$h(y) = \frac{\exp(\theta(\eta^\epsilon(\bar{x}, y), \bar{t})/\epsilon)}{\exp(\theta(\bar{x}, \bar{t})/\epsilon)}.$$

Since (4.10)(iii) and (4.8) hold for all (x, t) and $\bar{t} > t_0 - r_0$, we note that (3.5) implies that, for all $y \in \mathbb{R}^d$,

$$\begin{cases} h(y) & \leq \dfrac{\exp(\min(0, B_k\bar{t}/\epsilon - (d+\alpha)\log(|\widehat{x}|\bar{x}|^{1/\epsilon} - y|) + \delta(\bar{t} - t_0 + 2r_0)/\epsilon)}{\exp(\min(0, B_{k+1}\bar{t}/\epsilon - (d+\alpha)\log(|\bar{x}|)/\epsilon) - \delta r_0/\epsilon)} \\ & \lesssim \dfrac{1 + e^{-B_{k+1}\bar{t}/\epsilon}|\bar{x}|^{(d+\alpha)/\epsilon}}{1 + e^{-B_k\bar{t}/\epsilon}|\widehat{x}|\bar{x}|^{1/\epsilon} - y|^{d+\alpha}} e^{\delta(\bar{t} - t_0 + 3r_0)/\epsilon}. \end{cases} \qquad (4.16)$$

and proceed to estimate the last term of (4.14).

Recalling (4.9), and the facts that G_ϵ is bounded uniformly in ϵ and $\bar{t} < t_0 + r_0$ and $|\bar{x}| > |x_0| - r_0$, we observe that, if $e^{B_{k+1}\bar{t}} \geq |\bar{x}|^{d+\alpha}$, then $h(y) < 2e^{4\delta r_0/\epsilon}$ and therefore

$$\int_{|y| \geq r} G_\epsilon(\bar{x}, y)(1 - h(y)) K dy \gtrsim -r^{-\alpha} e^{4\delta r_0/\epsilon} \gtrsim -\epsilon^{-\alpha}|\bar{x}|^{-\alpha/(2\epsilon)}.$$

Otherwise, if $e^{B_{k+1}\bar{t}} < |\bar{x}|^{d+\alpha}$, we use the change of variables $y = |\bar{x}|^{1/\epsilon} z$ to get

$$\int_{|y| \geq r} G_\epsilon(\bar{x}, y)(1 - h(y)) K dy \gtrsim -e^{-B_{k+1}\bar{t}/\epsilon}|\bar{x}|^{(d+\alpha)/\epsilon} \int_{|y| \geq r} \frac{e^{\delta(\bar{t} - t_0 + 3r_0)/\epsilon}}{1 + e^{-B_k\bar{t}/\epsilon}|\widehat{x}|\bar{x}|^{1/\epsilon} - y|^{d+\alpha}} \frac{dy}{|y|^{d+\alpha}}$$

$$= -e^{-B_{k+1}\bar{t}/\epsilon}|\bar{x}|^{d/\epsilon} \int_{|z| \geq \epsilon} \frac{e^{\delta(\bar{t} - t_0 + 3r_0)/\epsilon}}{1 + e^{-B_k\bar{t}/\epsilon}|\bar{x}|^{(d+\alpha)/\epsilon}|\widehat{x} - z|^{d+\alpha}} \frac{dz}{|z|^{d+\alpha}}$$

$$\geq -\epsilon^{-d-\alpha} e^{-B_{k+1}\bar{t}/\epsilon}|\bar{x}|^{d/\epsilon} \int \frac{e^{\delta(\bar{t} - t_0 + 3r_0)/\epsilon} dw}{1 + e^{-B_k\bar{t}/\epsilon}|\bar{x}|^{(d+\alpha)/\epsilon}|w|^{d+\alpha}}$$

$$\gtrsim -\epsilon^{-d-\alpha} \exp\left(\left(\frac{B_k d}{d+\alpha} - B_{k+1}\right)\frac{\bar{t}}{\epsilon} + \delta\frac{\bar{t} - t_0 + 3r_0}{\epsilon}\right).$$

For the above inequalities, the first step used (1.6), the second employed Hölder's inequality, and in the third we made the implicit substitution

$$w = (e^{B_k\bar{t}/\epsilon}|\bar{x}|^{(d+\alpha)/\epsilon})^{-1/(d+\alpha)} \widetilde{w}.$$

Notice that (4.9) and the fact that $3r_0 \leq t_0$ imply the exponent is negative. Therefore, in both cases, the last term of (4.14) becomes nonnegative as $\epsilon \to 0$.

To complete the proof, we need to show that $\lim_{\epsilon \to 0} |\mathcal{I}_{\epsilon,\alpha}| = 0$. Since this is a rather involved calculation, we show this in Lemma 4.3 below.

It follows that, if $B_{k+1} \geq |\lambda_1|$, then for all ϵ sufficiently small (and independent of φ) (4.13) becomes

$$\varphi_t(\bar{x}, \bar{t}) + \int_{B_R(\bar{x})} \left(1 - \frac{\exp(\varphi(\eta^\epsilon(\bar{x}, y), \bar{t})/\epsilon)}{\exp(\varphi(\bar{x}, \bar{t})/\epsilon)}\right) K dy + \int_{B_R(\bar{x})^c} (1 - h(y)) K dy \geq \mu(\hat{x}|\bar{x}|^{1/\epsilon}) + \frac{\sigma}{3},$$

which yields that ψ^ϵ is a super-solution. □

Lemma 4.3. *The term $\mathcal{I}_{\epsilon,\alpha}$ appearing in (4.14) and defined in (4.15) converges to zero as $\epsilon \to 0$.*

Proof. Since (4.15) gives two formulas depending on the value of α, the proof is likewise split into two cases.
Case 1, $\alpha \in [1, 2)$: We must estimate the function h and its derivatives near $y = 0$. To that end, observe the facts that

$$|\partial_i G_\epsilon(\bar{x}, y)| = |G_\epsilon(\bar{x}, y)\partial_i g(\hat{x}|\bar{x}|^{1/\epsilon} - y)| \leq \|G_\epsilon(\bar{x}, \cdot)\| \|Dg\|,$$

and, if $|y| \leq r \leq |\bar{x}|^{1/\epsilon}/2$, then

$$\frac{1}{2}|\bar{x}|^{1/\epsilon} \leq |\hat{x}|\bar{x}|^{1/\epsilon} - y| \leq \frac{3}{2}|\bar{x}|^{1/\epsilon}.$$

Next we need to compute a number of derivatives that enter in the argument below. Recall that, for $1 \leq j \leq d$,

$$\eta_j^\epsilon(x, y) = (\hat{x}_j|x|^{1/\epsilon} - y_j)|\hat{x}|x|^{1/\epsilon} - y|^{\epsilon-1}.$$

Then

$$\partial_i \eta_j^\epsilon(\bar{x}, y) = -\delta_{ij}|\hat{x}|\bar{x}|^{1/\epsilon} - y|^{\epsilon-1} + \eta_j^\epsilon(\bar{x}, y)(1-\epsilon)(\hat{x}_i|\bar{x}|^{1/\epsilon} - y_i)|\hat{x}|\bar{x}|^{1/\epsilon} - y|^{-2},$$

with

$$|\partial_i \eta_j^\epsilon(\bar{x}, y)| \leq 4|\bar{x}|^{1-1/\epsilon},$$

and

$$\partial_{ik}^2 \eta_j^\epsilon(\bar{x}, y) = \delta_{ij}(1-\epsilon)\eta_k^\epsilon(\bar{x}, y)|\hat{x}|\bar{x}|^{1/\epsilon} - y|^{-2} + (1-\epsilon)\partial_k \eta_j^\epsilon(\bar{x}, y)(\hat{x}_i|\bar{x}|^{1/\epsilon} - y_i)|\hat{x}|\bar{x}|^{1/\epsilon} - y|^{-2}$$

$$- \delta_{ik}(1-\epsilon)\eta_j^\epsilon(\bar{x}, y)|\hat{x}|\bar{x}|^{1/\epsilon} - y|^{-2} - 2\eta_j^\epsilon(\bar{x}, y)(1-\epsilon)\frac{(\hat{x}_i|\bar{x}|^{1/\epsilon} - y_i)(\hat{x}_k|\bar{x}|^{1/\epsilon} - y_k)}{|\hat{x}|\bar{x}|^{1/\epsilon} - y|^4},$$

with

$$|\partial_{ik}^2 \eta_j^\epsilon(\bar{x}, y)| \leq 24|\bar{x}|^{1-2/\epsilon}.$$

It follows that

$$\partial_i h(y) = \frac{1}{\epsilon}h(y)D\theta(\eta^\epsilon(\bar{x}, y), \bar{t}) \cdot \partial_i \eta^\epsilon(\bar{x}, y)$$

$$= h(y)\left(-\frac{1}{\epsilon}\frac{\partial_i \theta(\eta^\epsilon(\bar{x}, y), \bar{t})}{|\hat{x}|\bar{x}|^{1/\epsilon} - y|^{1-\epsilon}} + \left(\frac{1}{\epsilon} - 1\right)D\theta(\eta^\epsilon(\bar{x}, y), \bar{t}) \cdot \eta^\epsilon(\bar{x}, y)\frac{\hat{x}_i|\bar{x}|^{1/\epsilon} - y_i}{|\hat{x}|\bar{x}|^{1/\epsilon} - y|^2}\right),$$

and, for $|y| \leq r$,

$$|\partial_{ik}^2 h(y)| \lesssim \frac{\sup_{|z| < r} h(z)}{\epsilon}\left(\frac{1}{\epsilon}\|D\theta\|^2\|D\eta^\epsilon\|^2 + \|D^2\theta\|_{L^\infty}\|D\eta^\epsilon\|^2 + \|D\theta\|\|D^2\eta^\epsilon\|\right)$$

$$\lesssim \left(\sup_{|z| < r} h(z)\right)|\bar{x}|^{-2/\epsilon}\left(\frac{|\bar{x}|^2}{\epsilon^2} + \frac{|\bar{x}|}{\epsilon}\right).$$

Combining all the above and using (4.10)(vi), we get

$$G_\epsilon(\bar{x}, \pm y) = 1 + O(|y|),$$

and

$$h(\pm y) = 1 \pm Dh(0) \cdot y + \left(\sup_{|z| < r} h(z)\right)O\left(|\bar{x}|^{-2/\epsilon}\left(\frac{|\bar{x}|^2}{\epsilon^2} + \frac{|\bar{x}|}{\epsilon}\right)|y|^2\right).$$

Note that

$$Dh(0) = -\frac{1}{\epsilon}D\theta(\bar{x}, \bar{t})|\bar{x}|^{1-1/\epsilon} + \left(\frac{1}{\epsilon} - 1\right)(D\theta(\bar{x}, \bar{t}) \cdot x)\hat{x}|\bar{x}|^{-1/\epsilon}.$$

It follows from (4.16) that, for all $y \in \mathbb{R}^d$,

$$|h(y)| \leq \frac{1 + e^{-B_{k+1}\bar{t}/\epsilon}|\bar{x}|^{(d+\alpha)/\epsilon}}{1 + e^{-B_k\bar{t}/\epsilon}|\hat{x}|\bar{x}|^{1/\epsilon} - y|^{d+\alpha}} e^{\delta(\bar{t}-t_0+3r_0)/\epsilon}$$

$$\leq (1 + \min(e^{-B_{k+1}\bar{t}/\epsilon}|\bar{x}|^{(d+\alpha)/\epsilon}, 2^{d+\alpha}e^{(B_k-B_{k+1})\bar{t}/\epsilon}))e^{\delta(\bar{t}-t_0+3r_0)/\epsilon}, \quad (4.17)$$

where the last inequality used that, for any $(\bar{x}, \bar{t})$ with $|\bar{x}| > 1$ and for all ϵ sufficiently small,

$$\frac{1 + e^{-B_{k+1}\bar{t}/\epsilon}|\bar{x}|^{(d+\alpha)/\epsilon}}{1 + e^{-B_k\bar{t}/\epsilon}|\hat{x}|\bar{x}|^{1/\epsilon} - y|^{d+\alpha}} \leq 1 + e^{-B_{k+1}\bar{t}/\epsilon}|\bar{x}|^{(d+\alpha)/\epsilon},$$

and

$$\frac{1 + e^{-B_{k+1}\bar{t}/\epsilon}|\bar{x}|^{(d+\alpha)/\epsilon}}{1 + e^{-B_k\bar{t}/\epsilon}|\hat{x}|\bar{x}|^{1/\epsilon} - y|^{d+\alpha}} \leq \frac{|\bar{x}|^{-(d+\alpha)/\epsilon} + e^{-B_{k+1}\bar{t}/\epsilon}}{|\bar{x}|^{-(d+\alpha)/\epsilon} + 2^{-(d+\alpha)}e^{-B_k\bar{t}/\epsilon}} \leq 1 + 2^{d+\alpha}e^{(B_k-B_{k+1})\bar{t}/\epsilon}.$$

Since the estimates above hold simultaneously, we can bound the ratio by their minimum. Using (4.5) we obtain

$$\mathcal{I}_{\epsilon,\alpha} = \frac{1}{2}\int_{|y|<r}(H_\epsilon(\theta,\bar{x},y) + H_\epsilon(\theta,\bar{x},-y))K dy = \frac{1}{2}\int_{|y|<r}(G_\epsilon(\bar{x},y)(1-h(y)) + G_\epsilon(\bar{x},-y)(1-h(-y)))K dy$$

$$= \int_{|y|<r} Dh(0) \cdot y(G_\epsilon(\bar{x},-y) - G_\epsilon(\bar{x},y))K dy + \int_{|y|<r}\left(\sup_{|z|<r} h(z)\right)O\left(|\bar{x}|^{-2/\epsilon}\left(\frac{|\bar{x}|^2}{\epsilon^2} + \frac{|\bar{x}|}{\epsilon}\right)|y|^2\right)K dy$$

$$= \int_{|y|<r} O\left(|Dh(0)||y|^2 + \left(\sup_{|z|<r} h(z)\right)|\bar{x}|^{-2/\epsilon}\left(\frac{|\bar{x}|^2}{\epsilon^2} + \frac{|\bar{x}|}{\epsilon}\right)|y|^2\right)K dy$$

so that

$$|\mathcal{I}_{\epsilon,\alpha}| \lesssim \left(|\bar{x}|^{1-1/\epsilon} + \left(\sup_{|z|<r} h(z)\right)|\bar{x}|^{-2/\epsilon}\left(\frac{|\bar{x}|^2}{\epsilon^2} + \frac{|\bar{x}|}{\epsilon}\right)\right)r^{2-\alpha}. \quad (4.18)$$

Observe that, since $|\bar{x}| > 1$ and $r = \epsilon|\bar{x}|^{1/\epsilon}$, $|\bar{x}|^{-1/\epsilon}r^{2-\alpha} \to 0$ for all $\alpha \geq 1$.

It then follows from (4.18) and (4.17) that, for all ϵ sufficiently small and $\alpha \in [1,2)$,

$$|\mathcal{I}_{\epsilon,\alpha}| \lesssim O(\epsilon) + e^{\delta(\bar{t}-t_0+3r_0)/\epsilon}\min(e^{-B_{k+1}\bar{t}}|\bar{x}|^d, e^{(B_k-B_{k+1})\bar{t}}|\bar{x}|^{-\alpha})^{1/\epsilon},$$

while using (3.6) and (4.9) we find uniformly for $(\bar{x}, \bar{t}) \in D_{r_0}(x_0, t_0)$ and $\epsilon > 0$,

$$\min(e^{-B_{k+1}\bar{t}}|\bar{x}|^d, e^{(B_k-B_{k+1})\bar{t}}|\bar{x}|^{-\alpha}) < e^{-\delta\bar{t}} \leq e^{-\delta(\bar{t}-t_0+3r_0)},$$

that is, since

$$\frac{B_{k+1}}{d} > \frac{B_k - B_{k+1}}{\alpha} + \delta\frac{d+\alpha}{d\alpha},$$

we have either

$$|\bar{x}| < e^{B_{k+1}\bar{t}/d}e^{-\delta\bar{t}/d}$$

or

$$|\bar{x}| > e^{(B_k-B_{k+1})\bar{t}/\alpha}e^{\delta\bar{t}/\alpha},$$

and, hence, $\lim_{\epsilon \to 0}|\mathcal{I}_{\epsilon,\alpha}| = 0$.

Case 2, $\alpha \in (0,1)$: This follows almost immediately from the bounds computed above. Indeed, in light of (4.3), we have

$$|\mathcal{I}_{\epsilon,\alpha}| = \left|\int_{|y|<r} G_\epsilon(\bar{x},y)(1-h(y))K dy\right| \lesssim \left(\sup_{|z|<r} h(z)\right)\int_{|y|<r} O(|y|)K dy \lesssim \left(\sup_{|z|<r} h(z)\right)\frac{|\bar{x}|^{1-1/\epsilon}}{\epsilon}r^{1-\alpha} \quad (4.19)$$

and, hence,

$$|\mathcal{I}_{\epsilon,\alpha}| \lesssim e^{\delta(\bar{t}-t_0+3r_0)/\epsilon}\min(e^{-B_{k+1}\bar{t}}|\bar{x}|^d, e^{(B_k-B_{k+1})\bar{t}}|\bar{x}|^{-\alpha})^{1/\epsilon}.$$

Arguing as before and using (4.17), we again conclude that $\lim_{\epsilon\to 0}|\mathcal{I}_{\epsilon,\alpha}| = 0$. □

We note that the decay of $\mathcal{I}_{\epsilon,\alpha}$ and some of the estimates on h are dictated by the decay of $|\bar{x}|^{-1/\epsilon}$ and $e^{-\bar{t}/\epsilon}$, which is not uniform for arbitrary $(\bar{x}, \bar{t})$. However, (x_0, t_0) was fixed with $|x_0| > 1$ and $t_0 > 0$, and $r_0 \leq \min(|x_0| - 1, t_0)/3$. Therefore, the decay of $\mathcal{I}_{\epsilon,\alpha}$ to zero (and any bound on h) is in fact uniform for any $(\bar{x}, \bar{t}) \in D_{r_0}(x_0, t_0)$.

4.3. The proof of Proposition 3.3.

We now wish to compare ψ^ϵ to v^ϵ and deduce a contradiction to (4.7). However, since it does not follow in general that $v^\epsilon \leq \psi^\epsilon$ in $\mathbb{R}^d \times (0, \infty)$, we cannot use v^ϵ in place of φ in definition (A.2). The inductive hypothesis (3.7) only holds for v^* and v_*. These are locally uniform one-sided limits, so the best we can obtain for v^ϵ is (3.1) (one-sided limits that are uniform on compact sets).

Consider the compact set $K = \overline{B}_{R_K}(0) \times [0, t_0 + r_0 + 1]$ with

$$R_K := 3\exp\left(\frac{(B_0 + 4\delta)(t_0 + r_0)}{d + \alpha}\right)(|x_0| + r_0) + 1.$$

Then, recalling (4.8), for all ϵ sufficiently small and all $(x, t) \in K$,

$$v^\epsilon < v^* + \delta r_0 \leq \psi + \delta r_0 \leq \min(0, B_k t - (d + \alpha)\log(|x|)) + \delta r_0.$$

Lemma 4.4. *Given v^ϵ, let $\bar{v}^\epsilon$ be a smooth function such that*

$$\bar{v}^\epsilon = v^\epsilon \text{ in } \overline{B}_{R_K - 1}(0) \times [0, t_0 + r_0], \ \bar{v}^\epsilon \leq \psi \text{ in } K^c, \text{ and } \bar{v}^\epsilon \leq v^\epsilon \text{ in } \mathbb{R}^d \times [0, \infty). \tag{4.20}$$

Then, for all $t < t_0 + r_0$, all $|x| > 1$, and all ϵ sufficiently small, we have that

$$\bar{v}^\epsilon < \psi^\epsilon = \theta + \epsilon g(\widehat{x}|x|^{1/\epsilon}) - \delta(t - t_0 + r_0).$$

This implies that $v^*(x_0, t_0) \leq \psi(x_0, t_0) - \delta r_0$, since $\theta = \psi$ and $\bar{v}^\epsilon = v^\epsilon$ inside of $D_{r_0/2}(x_0, t_0))$, a contradiction to (4.7). This shows that (4.2) cannot hold, thus proving (3.8) and Proposition 3.3.

Proof. For all $(x, t) \in D_{r_0}(x_0, t_0)^c$, (4.10)$(ii)$ implies

$$\psi^\epsilon = \theta + \epsilon g(\widehat{x}|x|^{1/\epsilon}) + \delta(t_0 - r_0 - t) > \psi + \delta r_0 > \bar{v}^\epsilon.$$

It follows that, if there ever comes a first time $\bar{t} \leq t_0 + r_0$ where $\min_{x \in \mathbb{R}^d}(\psi^\epsilon(\cdot, \bar{t}) - \bar{v}^\epsilon(\cdot, \bar{t})) = 0$, it must happen at a point $\bar{x} \in B_{r_0}(x_0)$. Then $\bar{v}^\epsilon$ is an admissible test function (on $B_{r_0}(x_0) \times [t_0 - r_0, \bar{t}]$) and must satisfy

$$\bar{v}_t^\epsilon(\bar{x}, \bar{t}) + \int_{B_R(\bar{x})}\left(1 - \frac{\exp(\bar{v}^\epsilon(\eta^\epsilon(\bar{x}, y), \bar{t})/\epsilon)}{\exp(\bar{v}^\epsilon(\bar{x}, \bar{t})/\epsilon)}\right)K\,dy + \int_{B_R(\bar{x})^c}(1 - h(y))\,K\,dy \geq \mu(\widehat{x}|\bar{x}|^{1/\epsilon}) + \frac{\sigma}{3}. \tag{4.21}$$

Recall from (2.1) that $f(x, u) = \mu(x)u - E(x, u)$. Along with (1.11), we have that

$$v_t^\epsilon(\bar{x}, \bar{t}) + \int\left(1 - \frac{\exp(v^\epsilon(\eta^\epsilon(\bar{x}, y), \bar{t})/\epsilon)}{\exp(v^\epsilon(\bar{x}, \bar{t})/\epsilon)}\right)K\,dy = \mu(\widehat{x}|\bar{x}|^{1/\epsilon}) - Ee^{-v^\epsilon(\bar{x}, \bar{t})/\epsilon},$$

or

$$\bar{v}_t^\epsilon(\bar{x}, \bar{t}) + \int\left(1 - \frac{\exp(\bar{v}^\epsilon(\eta^\epsilon(\bar{x}, y), \bar{t})/\epsilon)}{\exp(\bar{v}^\epsilon(\bar{x}, \bar{t})/\epsilon)}\right)K\,dy + \mathcal{J}_\epsilon = \mu(\widehat{x}|\bar{x}|^{1/\epsilon}) - Ee^{-\bar{v}^\epsilon(\bar{x}, \bar{t})/\epsilon}, \tag{4.22}$$

where $E := E\left(\widehat{x}|\bar{x}|^{1/\epsilon}, e^{v^\epsilon(\bar{x}, \bar{t})/\epsilon}\right)$ and

$$\mathcal{J}_\epsilon := \int \frac{\exp(\bar{v}^\epsilon(\eta^\epsilon(\bar{x}, y), \bar{t})/\epsilon) - \exp(v^\epsilon(\eta^\epsilon(\bar{x}, y), \bar{t})/\epsilon)}{\exp(\bar{v}^\epsilon(\bar{x}, \bar{t})/\epsilon)}K\,dy.$$

Observe that, for all ϵ sufficiently small depending on x_0 and t_0,

$$\{y : |\eta^\epsilon(\bar{x}, y)| > R_K - 1\} = \{y : |\widehat{x}|\bar{x}|^{1/\epsilon} - y| > (R_K - 1)^{1/\epsilon}\} \subseteq S_\epsilon,$$

where

$$S_\epsilon := \{y : |y| > (R_K - 1)^{1/\epsilon} - |\bar{x}|^{1/\epsilon}\} \subset \{y : |y| > 2|\bar{x}|^{1/\epsilon}e^{\frac{(B_0 + 4\delta)\bar{t}}{(d + \alpha)\epsilon}}\}.$$

Since v^ϵ and $\bar{v}^\epsilon$ are equal on S_ϵ^c, we would like to subtract the two equations above. This creates some error terms in the nonlocal piece, but R_K is sufficiently large that the errors vanish as $\epsilon \to 0$. Subtracting (4.22) from (4.21) yields

$$\int_{B_R(\bar{x})^c}\frac{\exp(\bar{v}^\epsilon(\eta^\epsilon(\bar{x}, y), \bar{t})/\epsilon) - \exp(\psi^\epsilon(\eta^\epsilon(\bar{x}, y), \bar{t})/\epsilon)}{\exp(\bar{v}^\epsilon(\bar{x}, \bar{t})/\epsilon)}K\,dy - \mathcal{J}_\epsilon \geq \frac{\sigma}{3} + Ee^{-v^\epsilon(\bar{x}, \bar{t})/\epsilon} > 0. \tag{4.23}$$

The first integral term, however, is nonpositive. Moreover, by (3.3)

$$-\mathcal{J}_\epsilon \leq \int_{S_\epsilon} \frac{\exp(v^\epsilon(\eta^\epsilon(\bar{x},y),\bar{t})/\epsilon)}{\exp(\psi^\epsilon(\bar{x},\bar{t})/\epsilon)} K dy \lesssim \int_{S_\epsilon} \frac{\left(1 + e^{-B_0\bar{t}/\epsilon}|\widehat{x}|\bar{x}|^{1/\epsilon} - y|^{d+\alpha}\right)^{-1}}{\exp(\min(0, B_{k+1}\bar{t} - (d+\alpha)\log(|\bar{x}|))/\epsilon)} e^{\frac{4\delta r_0}{\epsilon}} K dy.$$

Then (3.5) and the choice of R_K yields

$$-\mathcal{J}_\epsilon \lesssim e^{\frac{4\delta r_0}{\epsilon}} \int_{S_\epsilon} \frac{1 + e^{-B_{k+1}\bar{t}/\epsilon}|\bar{x}|^{(d+\alpha)/\epsilon}}{e^{-B_0\bar{t}/\epsilon}|\widehat{x}|x|^{1/\epsilon} - y|^{d+\alpha}} K dy \lesssim \int_{S_\epsilon} \frac{dy}{|y|^{d+\alpha}} \lesssim 2^{-\alpha}|\bar{x}|^{-\alpha/\epsilon} e^{-\frac{\alpha(B_0+4\delta)\bar{t}}{(d+\alpha)\epsilon}}.$$

The middle inequality used the fact that $|\widehat{x}|\widehat{x}|^{1/\epsilon} - y|^{d+\alpha} \geq |\bar{x}|^{(d+\alpha)/\epsilon} e^{(B_0+4\delta)(t_0+r_0)/\epsilon}$ on S_ϵ, which becomes arbitrarily large as $\epsilon \to 0$.

It follows that the left side of (4.23) becomes nonpositive as $\epsilon \to 0$, yet the right side is at least $\sigma/3$. This is a contradiction, implying that $\psi^\epsilon(x,t) > \bar{v}^\epsilon(x,t)$ for all $x \in \mathbb{R}^d$ and $t \leq t_0 + r_0$. $\qquad\square$

5. PROOF OF PROPOSITION 3.4

We follow a strategy similar to the proof of Proposition 3.3. We use the same formulas derived in subsection 4.1. In subsection 5.1, we assume that (3.9) fails for some (x_0, t_0) and use this to build a sub-solution from the associated test function; see Lemma 5.1. In subsection 5.2, we then compare this sub-solution to v^ϵ to derive a contradiction; see Lemma 5.2. There are, however, some key differences, which we explain next.

Recall that, when $|x| \leq 1$, $v^*(x,t) \leq 0$ for all $t > 0$ followed immediately from (3.2). The rest of the proof of Proposition 3.3 assumed that $|x| > 1$, which was crucial for bounding the nonlocal diffusion; see (4.18) and (4.19).

A similar bound will be used in the proof of Proposition 3.4 and will also assume that $|x| > 1$. However, (3.2) by itself does not yield an adequate lower bound on v_* for $|x| \leq 1$, so this case must be treated in a different way.

We first prove (3.9) on $\overline{B}_1^c \times (0, \infty)$ without assuming an improved lower bound on v_* in $\overline{B}_1 \times (0, \infty)$. From this, we extract a lower bound for u, the solution to (1.1) in unscaled coordinates, which then lets us prove (3.9) in $\overline{B}_1 \times (0, \infty)$ via a nonlocal maximum principle; see Lemma 5.3.

5.1. Constructing the sub-solution from the test function.

Assume that ψ is a smooth test function that touches v_* from below at a point (x_0, t_0) with $|x_0| > 1$ and $t_0 > 0$. Arguing by contradiction, we assume that

$$\psi(x_0, t_0) = v_*(x_0, t_0) < -A_{k+1}t_0 + \min(0, |\lambda_1|t_0 - (d+\alpha)\log(|x_0|)) = W_{k+1}(x_0, t_0), \tag{5.1}$$

and, for some $\sigma > 0$,

$$\psi_t(x_0, t_0) - |\lambda_1| + A_{k+1} = -\sigma. \tag{5.2}$$

We need to modify this test function outside of a neighborhood of (x_0, t_0), but the analysis overall is simpler than for Proposition 3.3.

Let $r_0 = \min(t_0, |x_0| - 1)/3$. and define $D_{r_0}(x_0, t_0)$ as before. Without loss of generality, in light of (3.7), we may assume that

$$\min(0, |\lambda_1|t - (d+\alpha)\log(|x|)) - A_{k+1}t \geq \psi(x,t) \geq \min(0, |\lambda_1|t - (d+\alpha)\log(|x|)) - A_k t. \tag{5.3}$$

Let $\delta > 0$ be such that

$$A_k - A_{k+1} + \delta < \frac{|\lambda_1|}{4(d+\alpha)} \quad \text{and} \quad \delta < \min\left(\frac{\sigma}{3}, \frac{A_{k+1}(t_0 - r_0)}{3r_0}\right); \tag{5.4}$$

notice that, if δ and r_0 satisfy (5.4), then so do any $\delta' < \delta$ and $r'_0 < r_0$. We can therefore reduce the values of δ and r_0 without violating (5.4).

Taking r_0 smaller, depending on ψ but not ϵ, we construct a smooth function θ satisfying, for some $C > 0$ independent of ϵ,

$$
\begin{cases}
(i) & \theta = \psi \ \text{on} \ D_{r_0/2}(x_0, t_0), \\
(ii) & \theta = \psi - \max(\delta r_0, \delta(t - t_0 + 2r_0)) \ \text{on} \ D_{r_0}(x_0, t_0)^c, \\
(iii) & \theta \geq \psi - \max(\delta r_0, \delta(t - t_0 + 2r_0)) \ \text{everywhere}, \\
(iv) & \theta \leq \min(0, |\lambda_1| t - (d + \alpha)\log(|x|)) - A_{k+1} t + \delta r_0 < 0 \ \text{on} \ D_{r_0}(x_0, t_0), \\
(v) & \theta_t - |\lambda_1| + A_{k+1} < -2\sigma/3 \ \text{on} \ D_{r_0}(x_0, t_0), \\
(vi) & \|D\theta\|, \ \|D^2\theta\| < C.
\end{cases}
\tag{5.5}
$$

The existence of such a θ follows from (3.7) and the fact that ψ is strictly below v_* away from (x_0, t_0); note that it may be necessary to choose r_0 and δ smaller. We also remark that (5.5)(ii) and (5.3) yield that the upper bound (5.5)(iv) holds for all $(x, t) \in \mathbb{R}^d \times [0, t_0 + r_0]$.

Lemma 5.1. *For (x_0, t_0), r_0, δ, and θ as above, and all ϵ sufficiently small, the function*

$$
\psi^\epsilon(x, t) := \theta(x, t) + \epsilon g(\widehat{x}|x|^{1/\epsilon}) - \delta(t_0 - r_0 - t)
$$

is a sub-solution (in the viscosity sense) to (1.1) in $D_{r_0}(x_0, t_0)$.

Proof. Let φ be a smooth test function such that $\psi^\epsilon - \varphi$ assumes a maximum value of 0 at $(\bar{x}, \bar{t}) \in D_{r_0}(x_0, t_0)$ with $|\bar{x}| > 1$. Since

$$
1 - \frac{\exp(\varphi(\eta^\epsilon(\bar{x}, y), \bar{t})/\epsilon)}{\exp(\varphi(\bar{x}, \bar{t})/\epsilon)} \leq 1 - \frac{\exp(\psi^\epsilon(\eta^\epsilon(\bar{x}, y), \bar{t})/\epsilon)}{\exp(\psi^\epsilon(\bar{x}, \bar{t})/\epsilon)},
\tag{5.6}
$$

in view of (5.5)(v) we have

$$
\varphi_t(\bar{x}, \bar{t}) \leq \psi_t^\epsilon(\bar{x}, \bar{t}) = \theta_t(\bar{x}, \bar{t}) + \delta < |\lambda_1| - A_{k+1} - \sigma/3.
\tag{5.7}
$$

Using (5.6) and (5.7) we find that, for any $R > 0$, at the point $(\bar{x}, \bar{t})$

$$
\varphi_t + \int_{B_R(\bar{x})} \left(1 - \frac{\exp(\varphi(\eta^\epsilon(\bar{x}, y), \bar{t})/\epsilon)}{\exp(\varphi(\bar{x}, \bar{t})/\epsilon)}\right) K dy + \int_{B_R(\bar{x})^c} \left(1 - \frac{\exp(\psi^\epsilon(\eta^\epsilon(\bar{x}, y), \bar{t})/\epsilon)}{\exp(\psi^\epsilon(\bar{x}, \bar{t})/\epsilon)}\right) K dy
$$

$$
\leq \theta_t + \frac{\sigma}{3} + \int \left(1 - \frac{\exp(\psi^\epsilon(\eta^\epsilon(\bar{x}, y), \bar{t})/\epsilon)}{\exp(\psi^\epsilon(\bar{x}, \bar{t})/\epsilon)}\right) K dy
$$

$$
< |\lambda_1| - A_{k+1} - \frac{\sigma}{3} + \int \left(1 - \frac{\exp(\psi^\epsilon(\eta^\epsilon(\bar{x}, y), \bar{t})/\epsilon)}{\exp(\psi^\epsilon(\bar{x}, \bar{t})/\epsilon)}\right) K dy
$$

$$
= -A_{k+1} - \frac{\sigma}{3} + \mu(\widehat{x}|\bar{x}|^{1/\epsilon}) + \mathcal{I}_{\epsilon,\alpha} + \int_{|y| \geq r} G_\epsilon(\bar{x}, y) \left(1 - \frac{\exp(\theta(\eta^\epsilon(\bar{x}, y), \bar{t})/\epsilon)}{\exp(\theta(\bar{x}, \bar{t})/\epsilon)}\right) K dy,
\tag{5.8}
$$

where $r = r(\epsilon) = \epsilon |\bar{x}|^{1/\epsilon}$ and

$$
\mathcal{I}_{\epsilon,\alpha} := \begin{cases}
\int_{|y|<r} G_\epsilon(\bar{x}, y) \left(1 - \frac{\exp(\theta(\eta^\epsilon(\bar{x}, y), \bar{t})/\epsilon)}{\exp(\theta(\bar{x}, \bar{t})/\epsilon)}\right) K dy, & \text{if } \alpha < 1, \\
\frac{1}{2} \int_{|y|<r} (H_\epsilon(\theta, \bar{x}, y) + H_\epsilon(\theta, \bar{x}, -y)) K dy, & \text{if } \alpha \geq 1.
\end{cases}
$$

Again, due to the homogeneity of $e^{-\psi^\epsilon} L^\alpha[e^{\psi^\epsilon}]$, the time-dependent quantity $\delta(t_0 - r_0 - t)$ disappears.

Compared to the proof of Proposition 3.3, the situation here is much simpler. For instance, the direction of the inequality in (A.1) means we only need to bound the last term in (5.8) from above. Since the integrand is bounded above uniformly by $K\|G_\epsilon\|$, we conclude from (1.6) that

$$
\int_{|y| \geq r} G_\epsilon(\bar{x}, y) \left(1 - \frac{\exp(\psi^\epsilon(\eta^\epsilon(\bar{x}, y), \bar{t})/\epsilon)}{\exp(\psi^\epsilon(\bar{x}, \bar{t})/\epsilon)}\right) K dy \lesssim \int_{|y| \geq r} \frac{dy}{|y|^{d+\alpha}} \lesssim r^{-\alpha}.
$$

As for the short range, the formula for $\mathcal{I}_{\epsilon,\alpha}$ above is identical to the one given after (4.14) of the previous section. As such, an identical analysis of $\mathcal{I}_{\epsilon,\alpha}$ as done in the proof of Proposition 3.3 using Taylor's theorem yields that (4.18) and (4.19) still hold, respectively when $\alpha \in [1, 2)$ and $\alpha \in (0, 1)$.

To prove that $\lim_{\epsilon \to 0} |\mathcal{I}_{\epsilon,\alpha}| = 0$, the only remaining issue is to bound the size of $\sup_{|z|<r} h(z)$, where

$$h(y) = \frac{\exp(\theta(\eta^\epsilon(\bar{x}, y), \bar{t})/\epsilon)}{\exp(\theta(\bar{x}, \bar{t})/\epsilon)}.$$

Using (5.3), (5.5)(iii), (5.5)(iv), and (3.6) we have, for all $|y| < r$,

$$h(y) \leq \frac{\exp(-A_{k+1}\bar{t}/\epsilon + \min(0, |\lambda_1|\bar{t}/\epsilon - (d+\alpha)\log(|\widehat{x}||\bar{x}|^{1/\epsilon} - y|)))}{\exp(-A_k\bar{t}/\epsilon + \min(0, |\lambda_1|\bar{t}/\epsilon - (d+\alpha)\log|\bar{x}|/\epsilon))} e^{\delta(\bar{t}-t_0+3r_0)/\epsilon} e^{g(\widehat{x}|\bar{x}|^{1/\epsilon})}$$

$$\lesssim \frac{1 + e^{-|\lambda_1|\bar{t}/\epsilon}|\bar{x}|^{(d+\alpha)/\epsilon}}{1 + e^{-|\lambda_1|\bar{t}/\epsilon}|\widehat{x}||\bar{x}|^{1/\epsilon} - y|^{d+\alpha}} e^{(A_k - A_{k+1} + \delta)\bar{t}/\epsilon} \lesssim \left(e^{(A_k - A_{k+1} + \delta)\bar{t}} \min(1, e^{-|\lambda_1|\bar{t}}|\bar{x}|^{d+\alpha}) \right)^{1/\epsilon}. \qquad (5.9)$$

Since $r = \epsilon|\bar{x}|^{1/\epsilon}$ and $|\bar{x}| > 1$, it then follows from (4.18), (4.19), and (5.9) that, for $\alpha \in (0, 2)$ and all ϵ sufficiently small,

$$|\mathcal{I}_{\epsilon,\alpha}| \lesssim O(\epsilon) + |\bar{x}|^{-\alpha/\epsilon} + \mathcal{H}_\epsilon,$$

with

$$\mathcal{H}_\epsilon := e^{(A_k - A_{k+1} + \delta)\bar{t}/\epsilon}|\bar{x}|^{-1/\epsilon} \min(1, e^{-|\lambda_1|\bar{t}}|\bar{x}|^{d+\alpha})^{1/\epsilon}.$$

If $|\bar{x}|^{d+\alpha} \geq e^{|\lambda_1|\bar{t}/2}$, then

$$\mathcal{H}_\epsilon \leq \left(e^{(A_k - A_{k+1} + \delta)\bar{t}}|\bar{x}|^{-1} \right)^{1/\epsilon} \leq \exp\left(\left(A_k - A_{k+1} + \delta - \frac{|\lambda_1|}{2(d+\alpha)} \right) \frac{\bar{t}}{\epsilon} \right) = c_1^{1/\epsilon},$$

and otherwise

$$\mathcal{H}_\epsilon \leq \left(e^{(A_k - A_{k+1} + \delta)\bar{t}} e^{-|\lambda_1|\bar{t}}|\bar{x}|^{d+\alpha} \right)^{1/\epsilon} \leq \exp\left(\left(A_k - A_{k+1} + \delta - \frac{|\lambda_1|}{2} \right) \frac{\bar{t}}{\epsilon} \right) = c_2^{1/\epsilon},$$

where $0 < c_2 < c_1 < 1$ and both constants are independent of ϵ.

Hence $\lim_{\epsilon \to 0} \mathcal{H}_\epsilon = 0$, and, therefore, for all ϵ sufficiently small and independently of φ, (5.8) becomes

$$\varphi_t(\bar{x}, \bar{t}) + \int_{B_R(\bar{x})} \left(1 - \frac{\exp(\varphi(\eta^\epsilon(\bar{x}, y), \bar{t})/\epsilon)}{\exp(\varphi(\bar{x}, \bar{t})/\epsilon)} \right) K dy + \int_{B_R(\bar{x})^c} (1 - h(y)) K dy \leq \mu(\widehat{x}|\bar{x}|^{1/\epsilon}) - \frac{\sigma}{3} - A_{k+1},$$

that is, ψ^ϵ is a sub-solution. □

5.2. **The proof of Proposition 3.4.** Next we derive a contradiction by comparing ψ^ϵ with v^ϵ. Let $K := \overline{B}_{R_K}(0) \times [0, t_0 + r_0 + 1]$ with

$$R_K := 3 \exp\left(\frac{(A_0 + |\lambda_1| + 4\delta)(t_0 + r_0)}{d+\alpha} \right) (|x_0| + r_0) + 1.$$

It follows from (3.1) and (5.3) that, for all ϵ sufficiently small and all $(x, t) \in K$,

$$v^\epsilon > v_* - \delta r_0/2 \geq \psi - \delta r_0/2 \geq \min(0, |\lambda_1|t - (d+\alpha)\log(|x|)) - A_k t - \delta r_0/2.$$

Lemma 5.2. *Given v^ϵ, let $\bar{v}^\epsilon$ be a smooth function such that*

$$\bar{v}^\epsilon = v^\epsilon \text{ in } \overline{B}_{R_K-1}(0) \times [0, t_0 + r_0], \ \bar{v}^\epsilon \geq \psi \text{ in } K^c, \text{ and } \bar{v}^\epsilon \geq v^\epsilon \text{ in } \mathbb{R}^d \times [0, \infty). \qquad (5.10)$$

Then, for all $t < t_0 + r_0$, all $|x| > 1$, and all ϵ sufficiently small, we have that

$$\psi^\epsilon = \theta + \epsilon g(\widehat{x}|x|^{1/\epsilon}) + \delta r_0/2 < \bar{v}^\epsilon.$$

Since $\theta = \psi$ on this set by (5.5)(i),

$$\psi(x_0, t_0) \leq v_*(x_0, t_0) - \delta r_0/2,$$

contradicting (5.1). This shows that (5.2) cannot hold, thus proving the variational inequality (3.9) for all $(x, t) \in \overline{B}_1^c \times (0, \infty)$.

Proof. In light of (5.5)(ii), we see that, for all $(x,t) \in D_{r_0}(x_0, t_0)^c$ and ϵ sufficiently small that $\epsilon g \leq \delta r_0/2$,

$$\psi^\epsilon = \theta + \epsilon g(\widehat{x}|x|^{1/\epsilon}) - \delta(t_0 - r_0 - t) \leq \psi - \delta r_0/2 < \bar{v}^\epsilon(x,t).$$

If $\bar{t} \leq t_0 + r_0$ is the first time such that $\max_{x \in \mathbb{R}^d}(\psi^\epsilon(\cdot, \bar{t}) - \bar{v}^\epsilon(\cdot, \bar{t})) = 0$, let $\bar{x} \in B_{r_0}(x_0)$ be a point in space where this maximum is achieved.

Then $\bar{v}^\epsilon$ becomes an admissible test function and must satisfy

$$\bar{v}^\epsilon_t(\bar{x}, \bar{t}) + \int_{B_R(\bar{x})} \left(1 - \frac{\exp(\bar{v}^\epsilon(\eta^\epsilon(\bar{x}, y), \bar{t})/\epsilon)}{\exp(\bar{v}^\epsilon(\bar{x}, \bar{t})/\epsilon)}\right) K dy + \int_{B_R(\bar{x})^c} (1 - h(y)) K dy \leq \mu(\widehat{x}|\bar{x}|^{1/\epsilon}) - \frac{\sigma}{3} - A_{k+1}.$$

On the other hand,

$$\bar{v}^\epsilon_t(\bar{x}, \bar{t}) + \int \left(1 - \frac{\exp(\bar{v}^\epsilon(\eta^\epsilon(\bar{x}, y), \bar{t})/\epsilon)}{\exp(\bar{v}^\epsilon(\bar{x}, \bar{t})/\epsilon)}\right) K dy + \mathcal{J}_\epsilon = \mu(\widehat{x}|\bar{x}|^{1/\epsilon}) - Ee^{-\bar{v}^\epsilon(\bar{x}, \bar{t})/\epsilon},$$

where

$$\mathcal{J}_\epsilon := \int \frac{\exp(\bar{v}^\epsilon(\eta^\epsilon(\bar{x}, y), \bar{t})/\epsilon) - \exp(v^\epsilon(\eta^\epsilon(\bar{x}, y), \bar{t})/\epsilon)}{\exp(\bar{v}^\epsilon(\bar{x}, \bar{t})/\epsilon)} K dy.$$

Subtracting the last two equations yields, in light of (2.2),

$$\int_{B_R(\bar{x})^c} \frac{\exp(\bar{v}^\epsilon(\eta^\epsilon(\bar{x}, y), \bar{t})/\epsilon) - \exp(\psi^\epsilon(\eta^\epsilon(\bar{x}, y), \bar{t})/\epsilon)}{\exp(\bar{v}^\epsilon(\bar{x}, \bar{t})/\epsilon)} K dy - \mathcal{J}_\epsilon \leq \overline{M} e^{v^\epsilon(\bar{x}, \bar{t})/\epsilon} - \frac{\sigma}{3} - A_{k+1}. \tag{5.11}$$

Note that the first term in the left hand side of (5.11) is nonnegative, while, as before,

$$\{y: |\eta^\epsilon(\bar{x}, y)| > R_K - 1\} \subset S_\epsilon := \{y: |y| > 2|\bar{x}|^{1/\epsilon} e^{\frac{(A_0 + |\lambda_1| + 4\delta)\bar{t}}{(d+\alpha)\epsilon}}\} \quad \text{and} \quad v^\epsilon = \bar{v}^\epsilon \text{ on } S_\epsilon^c.$$

It follows from (5.3) that

$$\mathcal{J}_\epsilon \leq \int_{S_\epsilon} \frac{\exp(\bar{v}^\epsilon(\eta^\epsilon(\bar{x}, y), \bar{t})/\epsilon)}{\exp(\bar{v}^\epsilon(\bar{x}, \bar{t})/\epsilon)} K dy$$

$$\lesssim \int_{S_\epsilon} \frac{\exp((\min(0, |\lambda_1|\bar{t} - (d+\alpha)\log(|\widehat{x}|\bar{x}|^{1/\epsilon} - y|^\epsilon)) - A_{k+1}\bar{t}/\epsilon)}{\exp((\min(0, |\lambda_1|\bar{t} - (d+\alpha)\log(|\bar{x}|)) - A_k\bar{t})/\epsilon)} \frac{e^{4\delta r_0/\epsilon}}{|y|^{d+\alpha}} dy$$

$$\lesssim e^{(A_k - A_{k+1} + 4\delta r_0)/\epsilon} \int_{S_\epsilon} \frac{1 + e^{-|\lambda_1|\bar{t}/\epsilon}|\bar{x}|^{(d+\alpha)/\epsilon}}{e^{-|\lambda_1|\bar{t}/\epsilon}|\widehat{x}|\bar{x}|^{1/\epsilon} - y|^{d+\alpha}} \frac{dy}{|y|^{d+\alpha}} \lesssim \int_{S_\epsilon} \frac{dy}{|y|^{d+\alpha}} \lesssim 2^{-\alpha}|\bar{x}|^{-\alpha/\epsilon} e^{-\frac{\alpha(A_0 + |\lambda_1| + 4\delta)\bar{t}}{(d+\alpha)\epsilon}}.$$

Hence, as $\epsilon \to 0$, $\mathcal{J}_\epsilon$ becomes nonpositive. Then, for all ϵ sufficiently small,

$$\frac{\sigma}{3} \leq \overline{M} e^{v^\epsilon(\bar{x}, \bar{t})/\epsilon} = \overline{M} e^{\psi^\epsilon(\bar{x}, \bar{t})/\epsilon}. \tag{5.12}$$

The left hand side of (5.12) is fixed and strictly positive. However, in view of (5.5)(iv) and the definition of ψ^ϵ,

$$\psi^\epsilon(\bar{x}, \bar{t}) \leq -A_{k+1}\bar{t} + \delta r_0 - \delta(t_0 - r_0 - \bar{t}) + \epsilon g(\widehat{x}|\bar{x}|^{1/\epsilon}),$$

so that

$$e^{\psi^\epsilon(\bar{x}, \bar{t})/\epsilon} \lesssim e^{-A_{k+1}(t_0 - r_0)/\epsilon} e^{3\delta r_0/\epsilon}. \tag{5.13}$$

In light of (5.4), the right hand side of (5.13) becomes arbitrarily small as $\epsilon \to 0$, a contradiction to (5.12). It follows that there is no such $(\bar{x}, \bar{t})$ in $D_{r_0}(x_0, t_0)$, and we conclude that, for all $(x,t) \in D_{r_0/2}(x_0, t_0)$ and all ϵ sufficiently small, $\psi^\epsilon(x,t) < \bar{v}^\epsilon(x,t)$. $\qquad \square$

5.3. **Lower bound in the short range.** We now establish the improved lower bound in $\overline{B}_1 \times (0, \infty)$. Note that, since we have already proved Proposition 3.3 independently of the rest of this section, it follows that

$$v^* \leq \min(0, |\lambda_1|t - (d + \alpha)\log(|x|)) \text{ for all } (x, t) \in \mathbb{R}^d \times [0, \infty). \tag{5.14}$$

Indeed, since the proof of Proposition 3.3 never used Proposition 3.4 or the lower bound of (3.7) for any k, the inductive argument of Section 3 shows that (5.14) holds independently.

Also, since the previous argument showed that (3.9) holds in $\overline{B}_1^c \times (0, \infty)$, we use Lemma 3.5 (with $U = \overline{B}_1^c$) to conclude that

$$\min(0, |\lambda_1|t - (d + \alpha)\log(|x|)) - A_{k+1}t \leq v_* \text{ for all } (x, t) \in B_1^c \times (0, \infty). \tag{5.15}$$

Assuming the next lemma, we have now concluded the proof of Proposition 3.4.

Lemma 5.3. *Assume (5.14) and (5.15). Then, for all $(x, t) \in \overline{B}_1 \times (0, \infty)$, we have $v_* \geq -A_{k+1}t$.*

Proof. Fix $T > 0$, let $R = e^{2|\lambda_1|T/(d+\alpha)}$ and

$$\overline{S}_{R,T} := \{(x, t) : 1 \leq |x| \leq R, \ T/2 \leq t \leq T\}.$$

Then, for any $\nu > 0$, all $(x, t) \in \overline{S}_{R,T}$, and all ϵ sufficiently small, it follows from (3.1) that

$$v^\epsilon > \min(0, |\lambda_1|t - (d + \alpha)\log(|x|)) - A_{k+1}t - \nu, \tag{5.16}$$

which implies

$$u(\widehat{x}|x|^{1/\epsilon}, t/\epsilon) = u^\epsilon(x, t) = e^{v^\epsilon(x,t)/\epsilon} > \frac{C_1 e^{-A_{k+1}t/\epsilon} e^{-\nu/\epsilon}}{1 + e^{-|\lambda_1|t/\epsilon}|x|^{(d+\alpha)/\epsilon}}.$$

Hence,

$$u(y, s) > \frac{C_1 e^{-A_{k+1}s - \nu/\epsilon}}{1 + e^{-|\lambda_1|s}|y|^{d+\alpha}} \text{ on } \{(y, s) : 1 \leq |y| \leq R^{1/\epsilon}, \ T/(2\epsilon) \leq s \leq T/\epsilon\}. \tag{5.17}$$

Keeping in mind that the solution u is bounded above by a fixed constant for all time, it follows that there exists $C_0 > 0$ such that, uniformly in x,

$$f(x, u) \geq -C_0 u. \tag{5.18}$$

Assume that v^ϵ achieves its minimum on the set $\overline{B}_1(0) \times [T/2, T]$ at (x_0, t_0) and that, for some $\sigma > 0$, $v^\epsilon(x_0, t_0) = -A_{k+1}t_0 - \sigma$. Then u achieves its minimum value on $\overline{B}_1(0) \times [T/(2\epsilon), T/\epsilon]$ at the point $(y_0, s_0) = (\widehat{x_0}|x_0|^{1/\epsilon}, t_0/\epsilon)$, and in view of (1.1) and (5.18),

$$u_t = f(y_0, u) - L^\alpha[u] \geq -C_0 u + \int_{|y_0+z|>1} (u(y_0 + z, s_0) - u(y_0, s_0))K(y_0, z)dz.$$

Note that the inner range $\overline{B}_1(-y_0)$ of integration can be safely omitted from the inequality because, by assumption, $u(w, s_0) - u(y_0, s_0) \geq 0$ for $|w| \leq 1$.

Using (5.16), (5.17), and the size of R, we get that

$$u_t(y_0, s_0) \gtrsim -C_0 e^{-A_{k+1}s_0 - \sigma/\epsilon} + e^{-A_{k+1}s_0 - \nu/\epsilon}I_\epsilon - e^{-A_{k+1}s_0 - \sigma/\epsilon} \int_{|y_0+z|^{d+\alpha} \geq e^{|\lambda_1|s_0}} \frac{dz}{|z|^{d+\alpha}},$$

where

$$I_\epsilon := \int_{1 < |y_0+z|^{d+\alpha} < e^{|\lambda_1|s_0}} \left(\frac{C_1}{1 + e^{-|\lambda_1|s_0}|y_0 + z|^{d+\alpha}} - e^{(\nu-\sigma)/\epsilon} \right) \frac{dz}{|z|^{d+\alpha}}.$$

Choosing $\nu < \sigma/2$ and ϵ sufficiently small depending on σ and C_1, we have

$$e^{(\nu-\sigma)/\epsilon} < e^{-\sigma/(2\epsilon)} < C_1/4.$$

Moreover, since $|y_0| \leq 1$, we have $|y_0 + z| \leq |z| + 1$, and, therefore,

$$\{2 < |z|^{d+\alpha} < e^{|\lambda_1|s_0}/2\} \subseteq \{1 < |y_0 + z|^{d+\alpha} < e^{|\lambda_1|s_0}\} \text{ and } \{|y_0 + z|^{d+\alpha} \geq e^{|\lambda_1|s_0}\} \subseteq \{|z|^{d+\alpha} \geq e^{|\lambda_1|s_0}/2\}.$$

Lastly, because $s_0 > T/(2\epsilon)$, we know that $e^{|\lambda_1|s_0} \to \infty$ as $\epsilon \to 0$. Then, for all ϵ sufficiently small, there exists a fixed $C_1' > 0$ such that

$$I_\epsilon \gtrsim \int_{2 < |z|^{d+\alpha} < e^{|\lambda_1|s_0}} \frac{dz}{(1 + e^{-|\lambda_1|s_0}|z|^{d+\alpha})|z|^{d+\alpha}} \gtrsim C_1'.$$

Therefore, for all ϵ sufficiently small,

$$u_t(y_0, s_0) \gtrsim e^{-A_{k+1}s_0 - \sigma/(2\epsilon)} \left(-C_0 e^{-\sigma/(2\epsilon)} + C_1' - C_2 e^{-\sigma/(2\epsilon)} e^{-\alpha|\lambda_1|s_0} \right),$$

that is, the time derivative of u is positive at its minimum value on $\overline{B}_1(0) \times [T/(2\epsilon), T/\epsilon]$. This can only happen if $s_0 = T/(2\epsilon)$. Since $\sigma > 0$ was arbitrary, we conclude that, if v^ϵ has a minimum on $\overline{B}_1(0) \times [T/2, T]$ that lies below $-A_{k+1}t$, then this minimum must occur at time $T/2$.

It follows that the same must hold true for v_*, which is lower semicontinuous, so it always has a minimum on compact sets. Also recall that $T > 0$ was arbitrary. If there is any time t_0 such that

$$\min_{|x|\leq 1} v_*(x, t_0) < -A_{k+1}t_0,$$

then we apply the previous conclusion for the set $\bar{B}_1(0) \times [t_0/2, t_0]$ and obtain that

$$\min_{|x|\leq 1} v_*(x, t_0/2) < -A_{k+1}t_0 < -A_{k+1}t_0/2.$$

Repeating this argument, we find that for all $m \geq 0$

$$\min_{|x|\leq 1} v_*(x, t_0 2^{-m}) < -A_{k+1}t_0.$$

This is in clear contradiction to (3.7). Therefore, $v_* \geq -A_{k+1}t$ for all $|x| \leq 1$ and $t > 0$.

$\square$

6. PROOF OF THEOREM 1.2

The proof of Theorem 1.2. The first claim of (1.16) follows immediately from the Hopf-Cole transformation and (1.15). That is, if $|x|^{d+\alpha} > e^{|\lambda_1|t}$, then v^ϵ converges locally uniformly to $|\lambda_1|t - (d + \alpha)\log(|x|) < 0$, and so $u^\epsilon = e^{v^\epsilon/\epsilon}$ converges locally uniformly to zero as $\epsilon \to 0$.

Let u be the solution to (1.1) and assume the conclusions of Theorem 1.1. Since both u and u^+ are positive, we write

$$w = u/u^+,$$

and note that, throughout this section, w will always be in unscaled coordinates. Then, by (1.1),

$$u^+ w_t + \int (u^+(x)w(x,t) - u^+(x+y)w(x+y,t))K(x,y)dy = f(x, u^+ w),$$

and

$$u^+ w_t + \int (w(x,t) - w(x+y,t))u^+(x+y)K(x,y)dy = f(x, u^+ w) - f(x, u^+)w,$$

that is,

$$w_t + \int (w(x,t) - w(x+y,t))\frac{u^+(x+y)}{u^+(x)}K(x,y)dy = w\left(\frac{f(x, u^+ w)}{u^+ w} - \frac{f(x, u^+)}{u^+}\right). \tag{6.1}$$

We write

$$\overline{K}(x,y) = \frac{u^+(x+y)}{u^+(x)}K(x,y) \text{ and } N(x,w) = \frac{f(x, u^+ w)}{u^+ w} - \frac{f(x, u^+)}{u^+},$$

and note that $\overline{K}$ also satisfies (1.5), (1.6) with different constants.

It follows from (1.3) that $N(x,w)$ is decreasing in w, negative when $w > 1$, and positive when $0 < w < 1$. Since w decays at infinity like u, it follows that, for a given $t \geq 0$, $w(\cdot, t)$ achieves its maximum value at some $\bar{x} \in \mathbb{R}^d$.

Then, at $(\bar{x}, t)$, (6.1) implies that

$$w_t \leq w\left(\frac{f(\bar{x}, u^+ w)}{u^+ w} - \frac{f(\bar{x}, u^+)}{u^+}\right),$$

and, for any $\sigma > 0$, if

$$\lim_{t_0 \to \infty} \sup_{(x,t)\in\mathbb{R}^d\times[t_0,\infty)} w(x,t) > 1 + \sigma,$$

then there exist $c_\sigma, t_\sigma > 0$ depending only on f, C_0 and the global upper bound for w, such that, for all $(x,t) \in \mathbb{R}^d \times [t_\sigma, \infty)$,

$$w(x,t) < C_0 + (1+\sigma)c_\sigma(t_\sigma - t),$$

that is, $C_0 + (1+\sigma)c_\sigma(t_\sigma - t)$ is a barrier from above for (6.1). This is an obvious contradiction to (2.6). Since $\sigma > 0$ was arbitrary, we conclude that

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^d} w(x,t) \le 1. \tag{6.2}$$

To complete the proof of (1.16), we essentially need the analog of (6.2) for the infimum of w. However, w decays to zero at infinity and lacks a global minimum. Instead, for $\xi > 1$ and $M > 1$, we consider the function

$$h(x,t) := M + e^{-|\lambda_1|t}|x|^{\xi(d+\alpha)}.$$

Writing $W = wh$, it follows from (2.6) that $W(\cdot,t)$ must, for every $t > 0$, achieve a global minimum at some point $\bar{x} \in \mathbb{R}^d$. Moreover, since $\xi > 1$, for all t sufficiently large depending on ξ, $|\bar{x}|^{\xi(d+\alpha)} < e^{|\lambda_1|t}$. It follows from (6.1) that

$$W_t = -|\lambda_1|e^{-|\lambda_1|t}|x|^{\xi(d+\alpha)}w + h\left(wN(x,w) + \int (w(x+y,t) - w(x,t))\overline{K}(x,y)dy\right)$$

$$= W\left(N(x,w) - \frac{|\lambda_1|e^{-|\lambda_1|t}|x|^{\xi(d+\alpha)}}{M + e^{-|\lambda_1|t}|x|^{\xi(d+\alpha)}}\right) + \int (W(x+y,t) - W(x,t))\overline{K}(x,y)dy + \bar{I}, \tag{6.3}$$

with

$$\bar{I} = \int (h(x,t) - h(x+y,t))w(x+y,t)\overline{K}(x,y)dy$$

$$= e^{-|\lambda_1|t}\int (|x|^{\xi(d+\alpha)} - |x+y|^{\xi(d+\alpha)})w(x+y,t)\overline{K}(x,y)dy.$$

For $T > 0$ and for B_0 given by (2.6), let

$$R = \exp\left(\frac{3T(B_0 - |\lambda_1|)}{\alpha - (\xi - 1)(d+\alpha)}\right),$$

and choose ξ sufficiently close to 1 that the exponent above is positive.

Fix $\nu > 0$. Then, for all ϵ sufficiently small, Theorem 1.1 implies that

$$v^\epsilon < \min(0, |\lambda_1|t - (d+\alpha)\log(|x|)) + \nu \quad \text{in} \quad \{(x,t) : |x| \le R, \ T \le t \le 2T\},$$

that is, for some $C_1 > 0$ independent of x, t, or ϵ,

$$u(\widehat{x}|x|^{1/\epsilon}, t/\epsilon) < \frac{C_1 e^{\nu/\epsilon}}{1 + e^{-|\lambda_1|t/\epsilon}|x|^{(d+\alpha)/\epsilon}}.$$

Recall that u^+ is bounded uniformly from above and below. Hence, for a different constant $C_1' > 0$,

$$w(x,t) < \frac{C_1' e^{\nu/\epsilon}}{1 + e^{-|\lambda_1|t}|x|^{d+\alpha}} \quad \text{in} \quad \{(x,t) : |x| \le R^{1/\epsilon}, \ T/\epsilon \le t \le 2T/\epsilon\}. \tag{6.4}$$

Looking at the minimum $(\bar{x}, \bar{t})$ of w on $\mathbb{R}^d \times [T/\epsilon, 2T/\epsilon]$, (6.3) yields

$$W_t \ge (N(\bar{x}, w) - |\lambda_1|/M) W + \bar{I}. \tag{6.5}$$

We now show that, for an appropriate choice of parameters, $\liminf_{\epsilon\to 0} \bar{I} \ge 0$. To do this, we split the domain of integration in the formula for $\bar{I}$ into three parts, that is,

$$\bar{I} \ge \bar{I}_1 + \bar{I}_2 + \bar{I}_3, \tag{6.6}$$

where, for $\rho := \epsilon e^{|\lambda_1|\bar{t}}$,

$$\bar{I}_1 := -\left|\int_{|y|^{\xi(d+\alpha)} \leq \rho} (h(\bar{x},\bar{t}) - h(\bar{x}+y,\bar{t}))w(\bar{x}+y,\bar{t})\bar{K}(\bar{x},y)dy\right|,$$

$$\bar{I}_2 := \int_{|y|^{\xi(d+\alpha)} > \rho, \, |\bar{x}+y| \leq R^{1/\epsilon}} (h(\bar{x},\bar{t}) - h(\bar{x}+y,\bar{t}))w(\bar{x}+y,\bar{t})\bar{K}(\bar{x},y)dy.$$

$$\bar{I}_3 := \int_{|\bar{x}+y| > R^{1/\epsilon}} (h(\bar{x},\bar{t}) - h(\bar{x}+y,\bar{t}))w(\bar{x}+y,\bar{t})\bar{K}(\bar{x},y)dy.$$

We proceed to estimate $\bar{I}_1$. This requires two slightly different treatments depending on the value of α.

When $\alpha \in [1,2)$, a change of variables yields

$$\int_{|y| \leq \rho} (h(\bar{x},\bar{t}) - h(\bar{x}+y,\bar{t}))w(\bar{x}+y,\bar{t})\overline{K}(\bar{x},y)dy$$

$$= \frac{1}{2}\int_{|y| \leq \rho} (2h(\bar{x},\bar{t}) - h(\bar{x}+y,\bar{t}) - h(\bar{x}-y,\bar{t}))w(\bar{x}+y,\bar{t})\overline{K}(\bar{x},y)dy$$

$$+ \frac{1}{2}\int_{|y| \leq \rho} (h(\bar{x},\bar{t}) - h(\bar{x}-y,\bar{t}))(w(\bar{x}+y,\bar{t}) - w(\bar{x}-y,\bar{t}))\overline{K}(\bar{x},y)dy. \tag{6.7}$$

Recall that w and Dw are bounded uniformly, and observe that, if $|x|^{\xi(d+\alpha)} < e^{|\lambda_1|t}$, then

$$|Dh(x,t)| \lesssim e^{-\frac{|\lambda_1|t}{\xi(d+\alpha)}} \quad \text{and} \quad |D^2 h(x,t)| \lesssim e^{-\frac{2|\lambda_1|t}{\xi(d+\alpha)}}. \tag{6.8}$$

It follows from (6.7), (6.8), and Taylor's Theorem that

$$\bar{I}_1 = -\left|\int_{|y|^{\xi(d+\alpha)} \leq \rho} (h(\bar{x},\bar{t}) - h(\bar{x}+y,\bar{t}))w(\bar{x}+y,\bar{t})\overline{K}(\bar{x},y)dy\right|$$

$$\gtrsim -\int_{|y|^{\xi(d+\alpha)} \leq \rho} \left(\sup_{|z|^{\xi(d+\alpha)} < e^{|\lambda_1|\bar{t}}} \left(|D^2 h(z,\bar{t})| + |Dh(z,\bar{t})||Dw(z,\bar{t})|\right)\right) |y|^2 \overline{K}(\bar{x},y)dy$$

$$\gtrsim -e^{-\frac{|\lambda_1|\bar{t}}{\xi(d+\alpha)}} \int_{|y|^{\xi(d+\alpha)} \leq \rho} |y|^{2-d-\alpha}dy \gtrsim -e^{-\frac{|\lambda_1|\bar{t}}{\xi(d+\alpha)}} \left(\epsilon e^{|\lambda_1|\bar{t}}\right)^{\frac{2-\alpha}{\xi(d+\alpha)}}. \tag{6.9}$$

Since $\bar{t} \in [T/\epsilon, 2T/\epsilon]$ and $\alpha \in [1,2)$, it follows that the lower bound converges to zero as $\epsilon \to 0$.

When $\alpha \in (0,1)$, the argument follows immediately from the calculations above. Indeed, using the first bound of (6.8) yields

$$\bar{I}_1 \gtrsim -\int_{|y| \leq \rho} \left(\sup_{|z|^{d+\alpha} \leq e^{|\lambda_1|\bar{t}}} |Dh(z,\bar{t})|\right) |y|\overline{K}(\bar{x},y)dy \gtrsim -e^{-\frac{|\lambda_1|\bar{t}}{d+\alpha}} \left(\epsilon e^{|\lambda_1|\bar{t}}\right)^{\frac{1-\alpha}{\xi(d+\alpha)}}. \tag{6.10}$$

As before, the lower bound converges to zero for any $\alpha \in (0,1)$, hence, in light of (6.9) and (6.10), we find $\liminf_{\epsilon \to 0} \bar{I}_1 \geq 0$.

Next, assume that $|y|^{\xi(d+\alpha)} > \epsilon e^{|\lambda_1|\bar{t}}$. It follows from (1.6) that

$$\bar{K}(\bar{x},y) \lesssim \frac{1}{\epsilon^{1/\xi}e^{|\lambda_1|\bar{t}/\xi} + |y|^{d+\alpha}},$$

while (6.4) yields

$$
\begin{aligned}
\bar{I}_2 &= \int_{\{|y|>\rho,\ |\bar{x}+y|\le R^{1/\epsilon}\}} (h(\bar{x},\bar{t}) - h(\bar{x}+y,\bar{t}))w(\bar{x}+y,\bar{t})\bar{K}(\bar{x},y)dy \\
&\gtrsim -\int_{|x+y|\le R^{1/\epsilon}} \frac{e^{\nu/\epsilon - |\lambda_1|\bar{t}}|\bar{x}+y|^{\xi(d+\alpha)}}{1 + e^{-|\lambda_1|\bar{t}}|\bar{x}+y|^{d+\alpha}} \frac{dy}{\epsilon^{1/\xi}e^{|\lambda_1|\bar{t}/\xi} + |y|^{d+\alpha}} \\
&\ge -e^{\nu/\epsilon}e^{(\xi-1)|\lambda_1|\bar{t}}e^{-\frac{\alpha|\lambda_1|\bar{t}}{d+\alpha}} \int \frac{e^{-\xi|\lambda_1|\bar{t}}|\bar{x}+y|^{\xi(d+\alpha)}}{1 + e^{-|\lambda_1|\bar{t}}|\bar{x}+y|^{d+\alpha}} \frac{e^{-d|\lambda_1|\bar{t}/(d+\alpha)}dy}{\epsilon^{1/\xi}e^{(1-\xi)|\lambda_1|\bar{t}/\xi} + e^{-|\lambda_1|\bar{t}}|y|^{d+\alpha}} \\
&= -\exp\left(\frac{\nu}{\epsilon} + (\xi-1)|\lambda_1|\bar{t} - \frac{\alpha|\lambda_1|\bar{t}}{d+\alpha}\right) \int \frac{|\bar{x}e^{-|\lambda_1|\bar{t}/(d+\alpha)} + z|^{\xi(d+\alpha)}}{1 + |\bar{x}e^{-|\lambda_1|\bar{t}/(d+\alpha)} + z|^{d+\alpha}} \frac{dz}{\epsilon^{1/\xi}e^{(1-\xi)|\lambda_1|\bar{t}/\xi} + |z|^{d+\alpha}}.
\end{aligned}
\tag{6.11}
$$

Recall that $|\bar{x}| < e^{|\lambda_1|\bar{t}/(d+\alpha)}$. The integral term above is finite so long as $\xi(d+\alpha) - (d+\alpha) < \alpha$, in which case it is also bounded by $\epsilon^{-1/\xi}e^{(\xi-1)|\lambda_1|\bar{t}/\xi}$ uniformly in $\bar{x}$. If ν is sufficiently small depending on T but not on ξ, and ξ is sufficiently small depending on ν, we have

$$
\frac{\nu}{\epsilon} + (\xi-1)|\lambda_1|\bar{t} - \frac{\alpha|\lambda_1|\bar{t}}{d+\alpha} + \left(1 - \frac{1}{\xi}\right)|\lambda_1|\bar{t} < 0,
$$

whence $\liminf_{\epsilon\to 0} \bar{I}_2 \ge 0$.

For the remaining term, we use (2.6) to conclude that

$$
\begin{aligned}
\bar{I}_3 &= \int_{|\bar{x}+y|>R^{1/\epsilon}} (h(\bar{x},\bar{t}) - h(\bar{x}+y,\bar{t}))w(\bar{x}+y,\bar{t})\bar{K}(\bar{x},y)dy \\
&\gtrsim -\int_{|\bar{x}+y|>R^{1/\epsilon}} \frac{e^{-|\lambda_1|\bar{t}}|\bar{x}+y|^{\xi(d+\alpha)}}{1 + e^{-B_0\bar{t}}|\bar{x}+y|^{d+\alpha}} \frac{dy}{1 + |y|^{d+\alpha}} \\
&\ge -e^{(B_0-|\lambda_1|)\bar{t}} \int_{|y|>R^{1/\epsilon}/2} \frac{|\bar{x}+y|^{(\xi-1)(d+\alpha)}}{1 + |y|^{d+\alpha}} \gtrsim -e^{(B_0-|\lambda_1|)\bar{t}} R^{(-\alpha+(\xi-1)(d+\alpha))/\epsilon}.
\end{aligned}
\tag{6.12}
$$

In light of the choice of R, the combined exponent above is negative and the lower bound converges to zero as $\epsilon \to 0$. It then follows from (6.6), (6.9), (6.10), (6.11), and (6.12) that

$$
\liminf_{\epsilon\to 0} \bar{I} \ge 0.
\tag{6.13}
$$

Note that in order to estimate $\bar{I}_3$ we needed R to be substantially larger than $e^{|\lambda_1|\bar{t}/(d+\alpha)}$, depending on B_0.

To conclude the proof, fix $\delta > 0$ and choose M large enough such that $N(x, 1-\delta) > 3|\lambda_1|/M$. If $W(x,t) < M(1-\delta)$, then $w(x,t) < 1 - \delta$. Since $N(x,w)$ is decreasing in w, it follows that

$$
N(x,w) > 3|\lambda_1|/M \quad \text{whenever} \quad W < M(1-\delta).
\tag{6.14}
$$

For all $T > 0$, all ϵ sufficiently small, and all $t \in [T/\epsilon, 2T/\epsilon]$, we know that $W(\cdot, t)$ achieves its minimum at some $\bar{x} \in \mathbb{R}^d$. It follows from (6.5), (6.13), and (6.14) that

$$
\max(W(\bar{x},t) - M(1-\delta), W_t(\bar{x},t) - W(\bar{x},t)|\lambda_1|/M) \ge 0,
$$

which implies that

$$
\lim_{t\to\infty} \inf_{|x|^{\xi(d+\alpha)}<e^{|\lambda_1|t}} W(x,t) \ge M(1-\delta).
$$

For any $\xi > 1$ and $M > 0$, $h(\widehat{x}|x|^{1/\epsilon}, t/\epsilon)$ converges locally uniformly to M if $|x|^{\xi(d+\alpha)} < e^{|\lambda_1|t}$. Therefore,

$$
\lim_{t\to\infty} \inf_{|x|^{\xi(d+\alpha)}<e^{|\lambda_1|t}} w(x,t) \ge 1 - \delta.
$$

Since $\delta > 0$ was arbitrary and ξ can be taken arbitrarily close to 1, the second claim of (1.16) follows, and the proof is complete.

We remark that the previous arguments also establish the uniqueness of u^+ from the uniqueness of solutions to (1.1). If $\bar{u}^+$ is another positive periodic steady state, the above argument would not be changed, and we would conclude

that $u^\epsilon(x,t)/u^+(\widehat{x}|x|^{1/\epsilon})$ and $u^\epsilon(x,t)/\bar{u}^+(\widehat{x}|x|^{1/\epsilon})$ both converge locally uniformly to 1 for $|x|^{d+\alpha} < e^{|\lambda_1|t}$, which cannot happen if $u^+ \neq \bar{u}^+$. □

APPENDIX A. VISCOSITY SOLUTIONS

Here we recall the classical definition, as it applies in the context of this paper, for a viscosity solution to a variational inequality.

Definition A.1. *Let $\Psi_{(x_0,t_0)}$ be the collection of functions $\psi : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$ that are smooth in a neighborhood of (x_0, t_0) and Lipschitz continuous everywhere else.*

A function $F : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$ is said to satisfy the variational inequality

$$\min(F_t, F) \leq 0$$

at the point (x_0, t_0) in the viscosity sense *if, for all test functions $\psi \in \Psi_{(x_0,t_0)}$ such that $F - \psi$ achieves a global maximum value of zero at (x_0, t_0), that is, ψ touches F from above at (x_0, t_0), then*

$$\text{either } F(x_0, t_0) \leq 0 \text{ or } \psi_t(x_0, t_0) \leq 0.$$

Similarly, F is said to satisfy the variational inequality

$$\max(F_t, F) \geq 0$$

at the point (x_0, t_0) in the viscosity sense *if, for all test functions $\psi \in \Psi_{(x_0,t_0)}$ such that $F - \psi$ achieves a global minimum value of zero at (x_0, t_0), that is, ψ touches F from below at (x_0, t_0), then*

$$\text{either } F(x_0, t_0) \geq 0 \text{ or } \psi_t(x_0, t_0) \geq 0.$$

We also recall the definitions for viscosity sub- and super-solutions used in Sections 4 and 5; see Barles and Imbert [5].

Definition A.2. *An upper semicontinuous function w_0 is a sub-solution to* (1.11) *if, given a smooth test function φ such that $w_0 - \varphi$ assumes a maximum value of 0 at (x_0, t_0), there exists a family of open balls $B_{\delta_n}(x_0)$ with $\delta_n \to 0$ such that, for all n,*

$$\varphi_t(x_0,t_0) + \int_{B_{\delta_n}(x_0)} \left(1 - \frac{\exp(\varphi(\eta^\epsilon(x_0,y),t_0)/\epsilon)}{\exp(\varphi(x_0,t_0)/\epsilon)} \right) K dy$$
$$+ \int_{B_{\delta_n}(x_0)^c} \left(1 - \frac{\exp(w_0(\eta^\epsilon(x_0,y),t_0)/\epsilon)}{\exp(w_0(x_0,t_0)/\epsilon)} \right) K dy \leq \frac{f(\widehat{x_0}|x_0|^{1/\epsilon}, \exp(w_0(x_0,t_0)/\epsilon))}{\exp(w_0(x_0,t_0)/\epsilon)}. \tag{A.1}$$

A lower semicontinuous function w_1 is a super-solution to (1.11) *if, given a smooth test function φ such that $w_1 - \varphi$ assumes a minimum value of 0 at (x_1, t_1), there exists a family of open balls as above such that*

$$\varphi_t(x_1,t_1) + \int_{B_{\delta_n}(x_1)} \left(1 - \frac{\exp(\varphi(\eta^\epsilon(x_1,y),t_1)/\epsilon)}{\exp(\varphi(x_1,t_1)/\epsilon)} \right) K dy$$
$$+ \int_{B_{\delta_n}(x_1)^c} \left(1 - \frac{\exp(w_1(\eta^\epsilon(x_1,y),t_1)/\epsilon)}{\exp(w_1(x_1,t_1)/\epsilon)} \right) K dy \geq \frac{f(\widehat{x_1}|x_1|^{1/\epsilon}, \exp(w_1(x_1,t_1)/\epsilon))}{\exp(w_1(x_1,t_1)/\epsilon)}. \tag{A.2}$$

APPENDIX B. PRINCIPAL EIGENVALUE FOR THE LINEARIZED OPERATOR AND POSITIVE STEADY STATES

We show that the operator $L^\alpha - \mu$ has a positive periodic first eigenfunction. A similar claim is made in [8] for the case where $L^\alpha = (-\Delta)^{\alpha/2}$, but the authors mainly show a Rayleigh-type formula for the principal eigenfunction. For completeness, we provide here a proof outline for the following claim:

Proposition B.1. *For μ a smooth periodic function and L^α as in* (1.4) *with a kernel K satisfying* (1.5) *and* (1.6)*, the operator $L^\alpha - \mu$ has a unique positive periodic eigenfunction with eigenvalue λ_1, that is*

$$L^\alpha e^g - \mu e^g = \lambda_1 e^g. \tag{B.1}$$

The eigenvalue λ_1 is simple in the algebraic and geometric sense, and is the bottom of the spectrum for $L^\alpha - \mu$.

To handle the issue of periodicity, we construct this eigenfunction on the torus $\mathbb{T}^d = [0,1]^d$. If u is a smooth 1-periodic function on $\mathbb{R}^d$, then it is equivalent to a smooth function defined on $\mathbb{T}^d$. Moreover,

$$L^\alpha[u](x) = \int_{\mathbb{R}^d} K(x,y)(u(x) - u(x+y))dy = \int_{\mathbb{T}^d} \widetilde{K}(x,y)(u(x) - u(x+y))dy, \tag{B.2}$$

where

$$\widetilde{K}(x,y) := \sum_{k \in \mathbb{Z}^d} K(x, y+k);$$

note that, in view of (1.6), the sum above converges for all $x \in \mathbb{T}^d$ and $y \in \mathbb{T}^d \setminus \{0\}$.

It also follows from (1.5) and (1.6) that, for a constant $C > 0$, $\widetilde{\mathcal{K}} = \widetilde{K}$, $|D_x \widetilde{K}|$, or $|D_x^2 \widetilde{K}|$, and all x and y,

$$\widetilde{K} \text{ is bounded from below, symmetric in } y \text{ and } C^{-1} \leq \widetilde{\mathcal{K}}(x,y)|y|^{d+\alpha} \leq C. \tag{B.3}$$

Proposition B.1 then follows from the following lemma:

Lemma B.2. *Let μ a smooth function on $\mathbb{T}^d$ and L^α as in (B.2) satisfying (B.3). Then the operator $L^\alpha - \mu$ has a unique positive eigenfunction e^g satisfying*

$$L^\alpha e^g - \mu e^g = \lambda_1 e^g. \tag{B.4}$$

The eigenvalue λ_1 is simple in the algebraic and geometric sense, and is the bottom of the spectrum for $L^\alpha - \mu$.

Proof. We appeal to the Krein-Rutman Theorem [29]. For X a Banach space and $K : X \to X$ a positive operator, if the spectral radius $r(K)$ is strictly positive, then $r(K)$ is an eigenvalue of K with a positive eigenfunction. We construct K as an inverse to L^α after a suitable shift to gain coercivity.

We first show that L^α satisfies a Gårding-type inequality. For $u \in C^\infty(\mathbb{T}^d)$, we write

$$\langle L^\alpha[u], u \rangle = L_1 + L_2, \tag{B.5}$$

where

$$L_1 := \int \int \widetilde{K}(x,y)(u(x) - u(x+y))^2 dy dx \text{ and } L_2 := \int \int \widetilde{K}(x,y)(u(x) - u(x+y))u(x+y)dydx.$$

It follows from (B.3) that

$$\|u\|_{\dot{H}^{\alpha/2}(\mathbb{T}^d)}^2 \gtrsim L_1 \approx \int \int \frac{(u(x) - u(x+y))^2}{|y|^{d+\alpha}} dx dy \gtrsim \|u\|_{\dot{H}^{\alpha/2}(\mathbb{T}^d)}^2; \tag{B.6}$$

here $\|\cdot\|_{\dot{H}^s(\mathbb{T}^d)}$ and $\|\cdot\|_{H^s(\mathbb{T}^d)}$ denote, respectively, the homogeneous and inhomogeneous Sobolev norm, that is,

$$\|u\|_{\dot{H}^s(\mathbb{T}^d)} := \left(\int_{\mathbb{T}^d} |\Lambda^s u(y)|^2 dy\right)^{1/2} \text{ and } \|u\|_{H^s(\mathbb{T}^d)} := \|u\|_{\dot{H}^s(\mathbb{T}^d)} + \|u\|_{L^2(\mathbb{T}^d)}.$$

We need to show that the remainder L_2 is bounded by u in $L^2(\mathbb{T}^d)$. Changing variables and using (B.3) we find

$$L_2 = \int \int \widetilde{K}(x+y, y)(u(x+y) - u(x))u(x)dydx$$

$$= \frac{1}{2} \int \int (\widetilde{K}(x,y) - \widetilde{K}(x+y, y))(u(x) - u(x+y))u(x)dydx$$

$$= \frac{1}{2} \int \int (2\widetilde{K}(x,y) - \widetilde{K}(x+y, y) - \widetilde{K}(x-y, y))u(x)^2 dydx$$

$$\gtrsim -\int \int |y|^2 \|D_x^2 K\|_{L^\infty} |y|^{-d-\alpha} u(x)^2 dydx.$$

Since $|y|^{2-d-\alpha}$ is integrable on $\mathbb{T}^d$ for all $\alpha \in (0,2)$, we get that, for some fixed positive constants $\widetilde{c}$, $\widetilde{C}$, and $\widetilde{D}$,

$$\widetilde{D}\|u\|_{H^{\alpha/2}} \geq \langle L^\alpha[u], u \rangle \geq \widetilde{c}\|u\|_{\dot{H}^{\alpha/2}}^2 - \widetilde{C}\|u\|_{L^2}^2. \tag{B.7}$$

Then, for a constant μ_0 sufficiently large, the operator $\mathcal{L} := L^\alpha - \mu + \mu_0$ is positive and satisfies the hypotheses of the Lax-Milgram theorem in $H^{\alpha/2}(\mathbb{T}^d)$. Hence it is bijective and, by Sobolev embedding, has a compact everywhere defined inverse

$$K := \mathcal{L}^{-1} = (L^\alpha - \mu + \mu_0)^{-1} : H^{\alpha/2}(\mathbb{T}^d) \to Dom(L^\alpha).$$

By the Krein-Rutman Theorem, there exists a positive eigenfunction e^g of K with eigenvalue $r(K)$. Therefore,

$$L^\alpha[e^g] - \mu e^g = \left(\frac{1}{r(K)} - \mu_0 \right) e^g.$$

This is precisely (B.4) with $\lambda_1 = r(K)^{-1} - \mu_0$. Evidently, λ_1 is the bottom eigenvalue for the spectrum of $L^\alpha - \mu$. $\qquad\square$

When the principal eigenvalue is negative, we also find nontrivial steady states for (1.1). We sketch a proof of this fact in the context of the periodic and nonlocal model (1.1).

Proposition B.3. *Let $\widetilde{K}$ be given by (B.2) and f be as in (1.3) and assume that $\lambda_1 < 0$. Then there exists a positive smooth function $u^+ : \mathbb{T}^d \to \mathbb{R}$ satisfying*

$$L^\alpha[u^+] = f(x, u^+). \tag{B.8}$$

Proof. Let M be the constant given in (1.3). Then $\overline{u} := M$ is a supersolution to (B.8). Since $\lambda_1 < 0$, there exists a sufficiently small $\delta > 0$ so that

$$\mu \delta e^g + \lambda_1 \delta e^g \leq f(x, \delta e^g).$$

Then $\underline{u} := \delta e^g$ is a positive subsolution.

Let $N_0 = \max_{x \in \mathbb{T}^d, \, u \in \mathbb{R}} (-\partial_u f(x, u))$ and note that $N_0 > 0$. We then take $u_0 = \underline{u}$ and obtain the sequence $\{u_k\}_{k \in \mathbb{Z}}$ by iteratively solving

$$L^\alpha[u_{k+1}] + N_0 u_{k+1} = f(x, u_k) + N_0 u_k. \tag{B.9}$$

It follows from the maximum principle that $u_1 \geq u_0$. Arguing by induction and using the maximum principle we get that $u_k \leq \overline{u}$ and $u_{k+1} \geq u_k$ for all k.

There must then exist a pointwise limit u^+ on $\mathbb{T}^d$ which is bounded from below by u_0, and, hence, is positive, smooth and satisfies $L^\alpha[u^+] = f(x, u^+)$. $\qquad\square$

APPENDIX C. PROOF OF LEMMA 2.1

The upper bound of (2.4) follows from the upper bound of (2.6) at time $t = 1$. Although Proposition 2.2 assumed (2.5), the upper bound was obtained by proving that

$$W(x, t) := \frac{C_0}{1 + e^{-B_0 t}|x|^{d+\alpha}}$$

is a super-solution for (1.1) (for suitable C_0 and B_0). Since $W(x, 0)$ can still dominate $u_0(x)$ with assumption (1.7), the upper bound is immediate.

The lower bound of (2.4), however, cannot be obtained by a rough sub-solution. Instead we will use Duhamel's principle with the following estimate on the heat kernel for the homogeneous problem.

Lemma C.1. *The heat kernel $p^\alpha : \mathbb{R}^d \times \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$ which satisfies*

$$\begin{cases} p_t^\alpha(x, z, t) + L^\alpha[p^\alpha(\cdot, z, t)](x) = 0, \\ p^\alpha(x, z, 0) = \delta_z, \end{cases}$$

satisfies, for a fixed $C > 0$ and all $t > 0$,

$$C \min\left(t^{-d/\alpha}, t|x - z|^{-d-\alpha} \right) \leq p^\alpha(x, z, t). \tag{C.1}$$

The proof can be found in Chen and Kumagai [14]; a consequence of their Theorem 1.1.

Using constant functions as sub- and super-solutions, we see from (1.3) that $0 \le u(x,t) \le M$. By (2.1) and (2.2), we know that there exists a $c_0 > 0$ such that any w which solves

$$w_t + L^\alpha[w] = -c_0 w \tag{C.2}$$

with initial data $w(x,0) := w_0 : \mathbb{R}^d \to [0,M]$ is a sub-solution to (1.1).

But the solution to (C.2) is given explicitly by

$$w(x,t) = e^{-c_0 t} \int_{\mathbb{R}^d} p^\alpha(x,z,t) w_0(z) dz.$$

If u_0 satisfies (2.5), then it is possible to choose $w_0 = \delta \chi(B_r(x_0))$ (nontrivial and nonnegative) such that $w_0 < u_0$ on $\mathbb{R}^d$. Therefore,

$$w(x,1) \ge C\delta e^{-c_0} \int_{|z-x_0|<r} \min\left(1, |x-z|^{-d-\alpha}\right) dz \ge \frac{c_1}{1 + |x|^{d+\alpha}}$$

for some $c_1 > 0$ depending on C, c_0, δ, x_0, and r. This concludes the proof of the lower bound.

APPENDIX D. PROOF OF LEMMA 2.3

We prove here a robust bound on the size of the diffusion L^α applied to a particular function, used in the construction of the relaxed upper and lower bounds of Proposition 2.2. Recall that $h(x,t) = (1 + e^{-\lambda t}|x|^{d+\alpha})^{-1}$.

The main observation is that the derivatives of h decay algebraically in $|x|$. That is,

$$\partial_i h(x,t) = -(d+\alpha)e^{-\lambda t} x_i |x|^{d+\alpha-2} h^2$$

and

$$\partial_{ij}^2 h(x,t) = -(d+\alpha)e^{-\lambda t} h^2 |x|^{d+\alpha-2} \left(\delta_{ij} + (d+\alpha-2)\frac{x_i x_j}{|x|^2} - 2(d+\alpha)e^{-\lambda t} x_i x_j |x|^{d+\alpha-2} h\right).$$

It follows from Taylor's Theorem that, for some constants independent of λ, x and t,

$$|h(x,t) - h(x+y,t) + Dh(x,t) \cdot y| \lesssim \sup_{|z|<|y|} \frac{e^{-2\lambda t/(d+\alpha)}}{(1 + e^{-\lambda t}|x+z|^{d+\alpha})^{\frac{d+\alpha+2}{d+\alpha}}} |y|^2. \tag{D.1}$$

Let

$$R = \frac{1}{2}\left(\frac{e^{\lambda t}}{2}\right)^{1/(d+\alpha)} \left(1 + e^{-\lambda t}|x|^{d+\alpha}\right)^{1/(d+\alpha)},$$

and observe that

$$L^\alpha[h] = L_1[h] + L_2[h],$$

where

$$L_1[h] := \int_{|y|\le R} K(x,y)(h(x,t) - h(x+y,t))dy \text{ and } L_2[h] := \int_{|y|>R} K(x,y)(h(x,t) - h(x+y,t))dy.$$

Recall (1.6) and observe that it follows from (D.1) that

$$|L_1[h]| = \left|\int_{|y|\le R} K(x,y)(h(x,t) - h(x+y,t))dy\right| \lesssim \int_{|y|\le R} \|\nabla^2 h(\cdot,t)\|_{B_R(x)} |y|^{2-d-\alpha} dy$$

$$\lesssim R^{2-\alpha} e^{-2\lambda t/(d+\alpha)} \left(1 + e^{-\lambda t}\max(0, |x|-R)^{d+\alpha}\right)^{-\frac{d+\alpha+2}{d+\alpha}}.$$

As long as $e^{-\lambda t}|x|^{d+\alpha} > 1$, we have that $R < |x|/2$. Therefore,

$$|L_1[h]| \lesssim \begin{cases} e^{-\alpha\lambda t/(d+\alpha)}(1 + e^{-\lambda t}|x|^{d+\alpha})^{-1-\alpha/(d+\alpha)}, & \text{if } e^{-\lambda t}|x|^{d+\alpha} > 1 \\ e^{-\alpha\lambda t/(d+\alpha)}, & \text{otherwise.} \end{cases} \tag{D.2}$$

For $L_2[h]$, we have

$$\left|\int_{|y|>R} K(x,y)h(x,t)dy\right| \lesssim h(x,t)R^{-\alpha} \lesssim \frac{e^{-\alpha\lambda t/(d+\alpha)}}{(1+e^{-\lambda t}|x|^{d+\alpha})^{1+\alpha/(d+\alpha)}},$$

and

$$\left|\int_{|y|>R} K(x,y)h(x+y,t)dy\right| \lesssim R^{-d-\alpha}\int \frac{dy}{1+e^{-\lambda t}|x+y|^{d+\alpha}} \lesssim R^{-d-\alpha}e^{-\lambda td/(d+\alpha)} \lesssim \frac{e^{-\alpha\lambda t/(d+\alpha)}}{1+e^{-\lambda t}|x|^{d+\alpha}}.$$

The bounds above, combined with (D.2) establish (2.8), completing the proof.

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