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Bounds on regularity of quadratic monomial ideals

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A B S T R A C T

Castelnuovo-Mumford regularity is a measure of algebraic complexity of an ideal. Regularity of monomial ideals can be investigated combinatorially. We use a simple graph decomposition and results from structural graph theory to prove, improve and generalize many of the known bounds on regularity of quadratic square-free monomial ideals.

1. Introduction

We consider bounds on Castelnuovo-Mumford regularity of a square-free quadratic monomial ideal I over a field of characteristic 0. Many recent papers investigated regularity of such ideals [3,7,12,17,18,26], see also [19] for a survey. One can associate to a quadratic square-free monomial ideal I a graph G, whose vertices are the variables, and edges correspond to quadratic generators of I. Therefore, quadratic square-free monomial ideals are often called edge ideals in the literature.

Another popular approach, which we follow, is to associate to I a graph $G(= G(I))$ where quadratic generators of I are the non-edges of G. We note that the ideal $I(G)$ is

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the Stanley-Reisner ideal of the clique complex of G [24, Chapter 2]. We use $\operatorname{reg}(G)$ to denote Castelnuovo-Mumford regularity of the non-edge ideal $I(G)$ of G.

Our main tool for bounding regularity is the following decomposition theorem, which is based on a straightforward application of Hochster's formula [14].

Theorem 1.1. *Let G be a graph. Let G_1 and G_2 be subgraphs which cover cliques of G (i.e. any clique of G is a clique in either G_1 or G_2). Then,*

$$\operatorname{reg}(G) \leq \max\{\operatorname{reg}(G_1), \operatorname{reg}(G_2), \operatorname{reg}(G_1 \cap G_2) + 1\}$$

We first apply this theorem to the case of a separator of G, i.e. a subset of vertices of G whose deletion disconnects G. For a subgraph H of G we use $G \setminus H$ to denote the induced subgraph on vertices of G that are not in H.

Theorem 1.2 *(Cutset/separator decomposition). Let T be an induced subgraph of G such that $G \setminus T$ is disconnected. Let $C_1, \ldots, C_k$ be the connected components of $G \setminus T$. Then,*

$$\operatorname{reg}(G) \leq \max\{\operatorname{reg}(G_i)_{i=1,\ldots,k}, \operatorname{reg}(T) + 1\}$$

where G_i are the induced subgraphs on vertices of C_i and T, for $i = 1, \ldots, k$.

Theorem 1.2 generalizes a decomposition result used by Dao, Huneke and Schweig in [3, Lemma 3.1]. Recall that an open neighborhood $N_G(v)$ of a vertex v is the induced subgraph on the vertices adjacent to v, and a closed neighborhood $N_G[v]$ of v is the induced subgraph on v and all vertices adjacent to v. Decomposition in [3] arises as a special, but very useful case, where T is the open neighborhood of a vertex v. An additional simplification comes from the fact that regularity of the open and closed neighborhoods of v are the same.

Theorem 1.3 *(Vertex decomposition). Let v be a vertex of G. Then,*

$$\operatorname{reg}(G) \leq \max\{\operatorname{reg}(G \setminus v), \operatorname{reg}(N_G(v)) + 1\}.$$

The above decompositions allow us to leverage existing results in structural graph theory results to derive a number of interesting consequences. A family of graphs is called *hereditary* if it is closed under vertex deletion [16, Chapter 2]. Recall that a graph is chordal if it does not contain a cycle of length at least four as an induced subgraph [4, Section 5.5]. Chordal graphs form a hereditary family, and moreover in every chordal graph there exists a vertex whose neighborhood is the complete graph [4, Proposition 5.5.1]. Therefore, we immediately obtain a result of Fröberg that regularity of chordal graphs is at most 2. With the same idea we obtain the following theorem:

Theorem 1.4 *(Hereditary theorem). Let $\mathscr{G}$ be a hereditary family with the following property: there exists $t \in \mathbb{N}$, such that for any $G \in \mathscr{G}$ there is a separator G' of G with* $\operatorname{reg}(G') \leq t$. *Then regularity of any $G \in \mathscr{G}$ is at most $t + 1$.*

An induced chordless cycle of length at least four is called a *hole*. An interesting connection between notions in algebra and structural graph theory was found by Eisenbud, Green, Hulek, and Popescu. A projective subscheme $X \subseteq \mathbb{P}^r$ satisfies Green-Lazarsfeld condition $N_{2,p}$ for integer $p \geq 1$ if the ideal $I(X)$ of X is generated by quadratics and the first $(p-1)$-steps of the minimal free resolution of the ideal $I(X)$ are linear. It was shown in [6, Theorem 2.1] that a non-edge ideal I satisfies condition $N_{2,p}$ for some integer $p \geq 2$ if and only if G does not contain a hole of length at most $p + 2$. Additionally, the resolution is often simpler if the ideal satisfies property $N_{2,p}$ with $p \geq 2$, (see discussion in [3]). This motivates us to look at graphs with restrictions on holes. We say that a graph satisfies condition $N_{2,p}$ if the corresponding non-edge ideal I does.

A highly studied family of graphs are *perfect* graphs. A graph is perfect if its chromatic number is equal to its clique number, and the same is true for every induced subgraph [4, Section 5.5]. Thus, perfect graphs are simple from the point of view of coloring. One of the most celebrated results in structural graph theory is the Strong Perfect Graph Theorem [1], which states that G is perfect if and only if G and its complement do not contain odd holes. Using Corollary 1.4 we show that perfect graphs satisfying property $N_{2,2}$ have regularity at most 3, since they form a hereditary family, and it is known that 4-hole free perfect graphs have a vertex with a chordal neighborhood in [21]. We note that without property $N_{2,2}$ a perfect graph on $2n$ vertices can have regularity $n + 1$ (see Section 6 for details).

Corollary 1.5. *If a perfect graph G does not contain a hole of length four, then regularity of G is at most* 3.

We observe that graphs without even holes form a hereditary family, and it was shown in [2] that an even-hole free graph contains a vertex with a chordal neighborhood. Therefore we obtain the following corollary:

Corollary 1.6. *If the graph G is even-hole free, then regularity of G is at most* 3.

We also generalize a result of Nevo [20, Theorem 5.1]. Let F be the graph on 5 vertices, consisting of an isolated vertex and two triangles sharing one edge. If G satisfies condition $N_{2,2}$ and does not contain F as an induced subgraph then regularity of G is at most 3.

Corollary 1.7. *Let G be a graph satisfying $N_{2,2}$ which does not contain F as an induced subgraph. Then regularity of G is at most* 3.

Nevo's result assumes that G satisfies $N_{2,2}$ and does not contain the union of an isolated vertex with a triangle as an induced subgraph, which is a stronger condition on G.

So far we only considered decompositions which use induced subgraphs of G, but now we consider an interesting decomposition where subgraphs G_1 and G_2 are not induced. Let M be a subgraph of G (not necessarily induced). We use $G - M$ to denote the subgraph of G obtained by deleting edges of M. Let G_M be the induced subgraph on vertices of M and vertices of G which are adjacent to both vertices of an edge in M. In other words, $G_M = G[V(M) \cup W]$ where W is the subset of vertices of G given by $k \in W$ if $ik, jk \in E(G)$ for some $ij \in E(M)$.

Theorem 1.8. *Let G be a graph and M be a subgraph in G. Then, we have*

$$\operatorname{reg}(G) \leq \max\{\operatorname{reg}(G - M), \operatorname{reg}(G_M), \operatorname{reg}(G_M - M) + 1\}.$$

As a special case, by taking M to be an edge $e = ij$ in G in Theorem 1.8, we get the following edge-neighborhood decomposition theorem.

Theorem 1.9 *(Edge-neighborhood decomposition). Let G be a graph and $e = ij$ be an edge in G with vertices i and j. Then,*

$$\operatorname{reg}(G) \leq \max\{\operatorname{reg}(G - e), \operatorname{reg}(G_e - e) + 1\}.$$

In [8], Fernández-Ramos and Gimenez classified bipartite graphs whose edge ideals have regularity at most 3. Using Corollary 1.9 and structural graph theory results we quickly recover their classification in Theorem 5.2 Another measure of complexity of a graph is its *genus*, which is defined as the genus of the smallest orientable surface on which G can be drawn in such a way that edges of G intersect only at the vertices [27]. The famous case of planar graphs corresponds to genus 0. Woodroofe showed in [28] that planar graphs have regularity at most 4 and this bound is tight. We generalize this bound to arbitrary genus:

Theorem 1.10. *Let S_g be the orientable 2-dimensional manifold of genus g. Suppose that a graph G is embedded into S_g. Then, regularity of G is at most $\lfloor 1 + \sqrt{1 + 3g} \rfloor + 2$.*

Note that this bound is also tight. Indeed, it is known by [23] that the genus of the complete m-partite graph with every part of size two, $K_{m(2)} (= K_{2,2,\ldots,2})$ is $\frac{(m-3)(m-1)}{3}$ for $m \not\equiv 2 \,(\text{mod} 3)$. Then, $\operatorname{reg}(K_{m(2)}) \leq \lfloor 1 + \sqrt{1 + 3g} \rfloor + 2 = m + 1$ which is tight.

Let n be the number of vertices of G. It is well known (see [25, Lemma 2.1]) that $\operatorname{reg}(G) \leq \frac{n}{2} + 1$, and this is tight, by considering $G = K_{\frac{n}{2}(2)}$ with n even. However, as shown in [3], the bounds on regularity become much better if G satisfies condition $N_{2,p}$ for some $p \geq 2$. Specifically, if G satisfies $N_{2,p}$ for $p \geq 2$, then

$$\operatorname{reg}(G) \leq \log_{\frac{p+3}{2}} \frac{n-1}{p} + 3.$$

We prove the following upper bound of regularity of G, which slightly improves on the bound above.

Theorem 1.11. *If G satisfies property $N_{2,p}$ for $p \geq 2$, then,*

$$\operatorname{reg}(G) \leq \log_{\frac{p+4}{2}} \frac{n}{p+1} + 4.$$

2. Graphs and clique complexes

A simple graph G consists of the vertex set $V(G)$ and the edge set $E(G)$. For a subgraph G' of G we use $G \setminus G'$ to denote the induced subgraph on $V(G) \setminus V(G')$, and $G - G'$ to denote the subgraph of G obtained by deleting all edges of G' (but no vertices). An induced subgraph on a subset of vertices W is denoted by $G[W]$.

A simple graph G with vertex set $[n]$ can be identified with a square-free quadratic monomial ideal $I(G)$ over a field k via $I(G) = \langle x_i x_j \mid ij \notin E \rangle$. We call $I(G)$ the non-edge ideal of G.

For a vertex v of G, we define the open neighborhood of v, denoted by $N_G(v)$, to be the induced subgraph of G on vertices which are adjacent to v. We also define the closed neighborhood of v, denoted by $N_G[v]$, to be the induced subgraph on v and the vertices adjacent to v. Now we introduce the clique complex of a graph.

Definition. Given a graph G, the clique complex of G, denoted by ΔG, is a simplicial complex that consists of t-simplices $(x_{i_1}, x_{i_2}, \ldots, x_{i_t})$ whenever $G[\{x_{i_1}, x_{i_2}, \ldots, x_{i_t}\}] = K_t$ where K_t is the complete graph on t vertices.

We observe that the non-edge ideal $I(G)$ is the Stanley-Reisner ideal of the clique complex ΔG [24, Chapter 2]. The graph G associated with the clique complex ΔG is uniquely determined as the closure of the 1-skeleton of ΔG. Thus, there is a one-to-one correspondence between the family of simple graphs G and the family of clique complexes ΔG given by $G \mapsto \Delta G$ and $F \mapsto G(F)$. Homology of clique complexes ΔG gives Betti numbers of non-edge ideal $I(G)$ (or equivalently the Stanley-Reisner rings $k[\Delta] := k[x_1, \ldots, x_n]/I(G)$) via Hochster's formula, if the characteristic of the field k is 0 (see [9, Remark 7.15]).

Theorem 2.1 *(Hochster). Let $I(G)$ be the non-edge ideal of a graph G. Then for $t \geq i+2$,*

$$\beta_{i,t}(I(G)) = \sum_{|W|=t} \dim_k(\tilde{H}_{t-i-2}(\Delta G[W])),$$

where W runs over all subsets of the vertex set of G of size t.

We refer to [24, Corollary 4.9] and [9, Theorem 7.11] for Hochster's formula. We define regularity of a graph G to be the Castelnuovo-Mumford regularity of the corresponding ideal $I(G)$ [22, Definition 18.1].

Definition 2.2. Let $\operatorname{reg}(G)$ be the Castelnuovo-Mumford regularity of the non-edge ideal $I(G)$ of graph G. In detail, $\operatorname{reg}(G) := \max\{r | \beta_{i,i+r}(I(G)) \neq 0 \text{ for some } i\}$. Regularity of the complete graph, which corresponds to the empty ideal, is one.

By Hochster's formula we see that regularity of a graph G is the smallest integer $q \geq 1$ such that $(q-2)$-nd (reduced) homology of the clique complex of any induced subgraph of G is non-zero. Note that regularity of the Stanley-Reisner ring $k[\Delta]$ is one less than regularity of the non-edge ideal $I(G)$.

3. Graph decompositions

Since Betti numbers of a Stanley-Reisner ideal can be obtained by calculating homology of subcomplexes of the corresponding clique complex, two subgraphs that cover all cliques of G give us enough information to bound regularity of G.

Theorem 3.1. *Let G be a graph. Let G_1 and G_2 be subgraphs which cover cliques of G (i.e. any clique of G is a clique in either G_1 or else G_2). Then,*

$$\operatorname{reg}(G) \leq \max\{\operatorname{reg}(G_1), \operatorname{reg}(G_2), \operatorname{reg}(G_1 \cap G_2) + 1\}.$$

Proof. Let W be an induced subgraph of G. Let $W_1 = W \cap G_1$ and $W_2 = W \cap G_2$. We claim that a subcomplex of ΔW is the union of subcomplexes of ΔW_1 and ΔW_2. Let $F = (v_1, \ldots, v_t)$ be a face in ΔW. Then $G(F)$ is a clique in G, and since G_1 and G_2 cover cliques of G, we see that F is a face of either W_1 or W_2, and the claim follows. Additionally, we have $\Delta(W_1 \cap W_2) = \Delta W_1 \cap \Delta W_2$.

Now, we prove the main inequality. Let $m = \max\{\operatorname{reg}(G_1), \operatorname{reg}(G_2), \operatorname{reg}(G_1 \cap G_2) + 1\}$. Given any induced subgraph W, by the Mayer-Vietoris sequence [13, p. 149], we have following exact sequence of complexes

$$\cdots \to \tilde{H}_i(\Delta(W_1 \cap W_2)) \to \tilde{H}_i(\Delta W_1) \oplus \tilde{H}_i(\Delta W_2) \to \tilde{H}_i(\Delta W) \to \tilde{H}_{i-1}(\Delta(W_1 \cap W_2)) \to \cdots$$

Since regularity of $G_1 \cap G_2$ is at most $m-1$, we have $\tilde{H}_i(\Delta(W_1 \cap W_2)) = 0$ for all $i \geq m-2$. Therefore, $\tilde{H}_i(\Delta W) \simeq \tilde{H}_i(\Delta W_1) \oplus \tilde{H}_i(\Delta W_2)$ for all $i \geq m-1$. Since both G_1 and G_2 have regularity at most m, $\tilde{H}_i(\Delta W_1) = \tilde{H}_i(\Delta W_2) = 0$ for all $i \geq m-1$. Thus, $\tilde{H}_i(\Delta W) = 0$ for all $i \geq m-1$ and regularity of G is at most m. □

Our first application deals with the case of defining G_1 and G_2 via a cutset.

Theorem 3.2 *(Cut-set/separator decomposition). Let T be an induced subgraph of G such that the induced graph $G \setminus T$ is disconnected. Let $C_1, \dots, C_k$ be the connected components of $G \setminus T$ and G_i be induced subgraphs on vertices of C_i and T for $i = 1, \dots, k$. Then,* $\operatorname{reg}(G) \leq \max\{\operatorname{reg}(G_i)_{i=1,\dots,k}, \operatorname{reg}(T) + 1\}$.

Proof. Let G_1 be the induced subgraph on vertices of C_1 and T and let G_1' be the induced subgraph on $\cup_{i=2}^k V(C_i) \cup V(T)$. In other words, $G_1' = G \setminus C_1$. Then, we can see that G_1 and G_1' cover all cliques of G. Indeed, if a vertex in C_1 and a vertex in $\cup_{i=2}^k C_i$ are contained in a clique in G, the induced subgraph on the two vertices must be an edge of G. However, it is not possible because C_1 and $\cup_{i=2}^k C_i$ are disjoint. Therefore, two induced subgraphs G_1 and G_1' cover all cliques in G. Then, by Theorem 3.1, we have

$$\operatorname{reg}(G) \leq \max\{\operatorname{reg}(G_1), \operatorname{reg}(G_1'), \operatorname{reg}(T) + 1\}.$$

Now, let G_j' be the induced subgraph on vertices of $C_{j+1}, \dots, C_k$, and T for $j = 2, \dots, k$. Then, by the same process,

$$\operatorname{reg}(G_{j-1}') \leq \max\{\operatorname{reg}(G_j), \operatorname{reg}(G_j'), \operatorname{reg}(T) + 1\}$$

Thus,

$$\begin{aligned}
\operatorname{reg}(G) &\leq \max\{\operatorname{reg}(G_1), \operatorname{reg}(G_1'), \operatorname{reg}(T) + 1\} \\
&\leq \max\{\operatorname{reg}(G_1), \operatorname{reg}(G_2), \operatorname{reg}(G_2'), \operatorname{reg}(T) + 1\} \\
&\vdots \\
&\leq \max\{\operatorname{reg}(G_i)_{i=1,\dots,k}, \operatorname{reg}(T) + 1\}. \quad \square
\end{aligned}$$

We call T in Theorem 3.2 a separator of G. Given any graph G, an open neighborhood of a vertex of G is a separator of G, and so we obtain the following Vertex Neighborhood Decomposition.

Corollary 3.3 *(Vertex neighborhood decomposition). Let v be any vertex of a graph G. Then,*

$$\operatorname{reg}(G) \leq \max\{\operatorname{reg}(G \setminus v), \operatorname{reg}(N_G(v)) + 1\}.$$

Proof. By Theorem 3.2, we have $\operatorname{reg}(G) \leq \max\{\operatorname{reg}(G \setminus v), \operatorname{reg}(N_G[v]), \operatorname{reg}(N_G(v)) + 1\}$, where $N_G(v)$ is the open neighborhood of v in G and $N_G[v]$ is the closed neighborhood of v in G (see Section 2 for definitions). So, it suffices to show that $\operatorname{reg}(N_G[v]) = \operatorname{reg}(N_G(v))$. This follows by a simple application of Hochster's formula, since the clique complex ΔH of an induced subgraph H of $N_G[v]$ with $v \in H$ is contractible. $\square$

So far, we have only considered graph decompositions coming from induced subgraphs, but we now define a useful decomposition where this is not the case. Let M be a subgraph of G. Let G_M be the induced subgraph of G on vertices in M and vertices of G which are adjacent to both vertices of some edge of M. Namely, $G_M = G[V(M) \cup W]$ where W is a subset of vertices in G such that $k \in W$ if $ik \in E(G)$ and $jk \in E(G)$ for some $ij \in E(M)$. Then, we have following decomposition theorem.

Theorem 3.4. *Let M be a subgraph of a graph G. Then,*

$$\operatorname{reg}(G) \leq \max\{\operatorname{reg}(G-M), \operatorname{reg}(G_M), \operatorname{reg}(G_M - M) + 1\}.$$

Proof. We first claim that $G-M$ and G_M cover all cliques of G. Let F be any clique in G. If F does not contain any edges in M, then $G-M$ contains the clique F. Suppose that F contains some edges of M. If all vertices in F are contained in M, then F is contained in G_M since G_M contains M. If v is any vertex in F outside of M, then $uv, wv \in F$ for some $uw \in E(F \cap M)$. This implies that $v \in V(G_M)$ and so $F \subseteq G_M$ since both F and G_M are induced subgraphs of G. Additionally, the intersection of $G-M$ and G_M is $G_M - M$. Indeed, $V(G_M \cap (G-M)) = V(G_M \cap G) = V(G_M)$ and $E(G_M \cap (G-M)) = E(G_M - M)$. Thus, by Theorem 3.1, $\operatorname{reg}(G) \leq \max\{\operatorname{reg}(G-M), \operatorname{reg}(G_M), \operatorname{reg}(G_M - M) + 1\}$. □

Similarly to vertex-neighborhood decomposition in Corollary 3.3, if we take M to be an edge $e = ij$ in Theorem 3.4, then we can bound regularity of G by regularity of two subgraphs.

Corollary 3.5 *(Edge-neighborhood decomposition). Let G be a graph and $e = ij$ be an edge in G. Then,* $\operatorname{reg}(G) \leq \max\{\operatorname{reg}(G-e), \operatorname{reg}(G_e - e) + 1\}$.

Proof. By Theorem 3.4, it suffices to show that $\operatorname{reg}(G_e) \leq \operatorname{reg}(G_e - e) + 1$ for edge e. Indeed, for any graph G, $\operatorname{reg}(G) \leq \operatorname{reg}(G \setminus v) + 1$ for any vertex v by Corollary 3.3, and so we have

$$\begin{aligned} \operatorname{reg}(G_e) &\leq \operatorname{reg}(G_e \setminus i) + 1 \\ &\leq \operatorname{reg}(G_e - e) + 1, \end{aligned}$$

for the edge $e = ij$ because $G_e \setminus i$ is an induced subgraph of $G_e - e$. □

We will use this decomposition to describe complements of bipartite graphs that have regularity 3 in Section 5.

4. Hereditary families

Let $\mathscr{G}$ be a family of graphs. We call $\mathscr{G}$ a hereditary family if it is closed under taking induced subgraphs, or equivalently under deleting vertices.

Theorem 4.1 *(Hereditary theorem). Let $\mathscr{G}$ be a hereditary family with the following property: there exists $t \in \mathbb{N}$, such that for any $G \in \mathscr{G}$ there is a separator G' of G with* $\operatorname{reg}(G') \leq t$. *Then regularity of any $G \in \mathscr{G}$ is at most $t+1$.*

Proof. Let $\mathscr{G}$ be a hereditary family with the above property for some $t \in \mathbb{N}$. We will induct on the number of vertices n in graphs of $\mathscr{G}$. The base case $n = 1$ is trivial, since $t \geq 0$ and $\mathscr{G}$ includes the one vertex graph. Now consider the inductive step. Let $G \in \mathscr{G}$ be a graph on $n+1$ vertices, and let G' be a separator of G. Applying Theorem 3.2 with $T = G'$, we get the desired inequality by the induction assumption. □

Chordal graphs form a hereditary family, and it is known in [5] that any chordal graph contains a vertex v such that neighborhood of v is a complete graph. Therefore we immediately obtain the following result of Fröberg:

Corollary 4.2. *Let G be a chordal graph. Then regularity of G is at most* 2.

Moreover, we can see that regularity of any hole is at least 3 and therefore chordal graphs are the only graphs of regularity at most 2. On the other hand, by combining Fröberg's result with neighborhood decomposition 3.3 we can give a criterion for graphs that have regularity at most 3:

Corollary 4.3. *Let $\mathscr{G}$ be a hereditary family of graphs with the following property: for any $G \in \mathscr{G}$ there is a vertex v of G which has a chordal neighborhood. Then regularity of any $G \in \mathscr{G}$ is at most* 3.

To illustrate the power of the above Corollary 4.3, we give a quick proof of a generalization of a result by Nevo [20, Section 5]. Let F' be a graph on four vertices consisting of an isolated vertex and a triangle. He showed that if G does not contain F' and a four-cycle as induced subgraphs then regularity of G is at most three. We note that not containing a four-cycle as an induced subgraph corresponds to G satisfying condition $N_{2,2}$. Let F be a graph on five vertices consisting of an isolated vertex and two triangles sharing an edge. We show that if G does not contain a four-cycle and F as induced subgraphs, then regularity of G is at most 3, which is a weaker condition on G.

Corollary 4.4. *Let $\mathscr{G}$ be the hereditary family of graphs that do not contain F and the four cycle as induced subgraphs. Then regularity of any $G \in \mathscr{G}$ is at most* 3.

Proof. We will show that any $G \in \mathscr{G}$ contains a vertex with a chordal neighborhood. Suppose not, and let $G \in \mathscr{G}$ be a graph such that no vertex of G has a chordal neighborhood. Let v be the vertex of minimal degree in G. Observe that v is not connected to all vertices of G, otherwise G is the complete graph, which is a contradiction. It follows by our assumption that $N_G(v)$ contains a hole C of length at least 5, and there exists $w \in G$ such that v is not connected to w. Since G is F-free we see that w must be connected to two non-adjacent vertices u_1, u_2 of C. But then the induced subgraph on u_1, v, u_2, w is a 4-cycle, which is a contradiction. □

We also generalize Corollary 4.4 to the case where G does not contain larger cycles as induced subgraphs. Recall that a graph G not containing an ℓ-hole for $\ell = 4, ..., p+2$ with $p \geq 2$ is equivalent to G satisfying condition $N_{2,p}$. Let a fan F_i for $i \geq 1$ be the graph consisting of an isolated vertex and the graph join of a path on $i+1$ vertices and a distinct vertex. With essentially the same proof as Corollary 4.4 we can also show the following:

Corollary 4.5. *If for some $i \geq 2$ a graph G is ℓ-hole free for $\ell = 4, ..., i+2$ and does not contain F_i as an induced subgraph, then regularity of G is at most* 3.

It is known that if G is perfect and does not contain 4-holes or if G is even-hole free, then there is a vertex in G whose neighborhood is chordal (for 4-free perfect graphs see [21] and for even-hole free graphs see [2]). Moreover, both 4-hole free perfect graphs and even-hole free graphs form hereditary families. Thus, we obtain another criterion to make graphs to have regularity 3.

Corollary 4.6. *If G is perfect and does not contain* 4*-holes, or if G is even-hole free then regularity of G is at most* 3.

It follows from the Strong Perfect Graph Theorem [1], that G is perfect and 4-hole free if and only if G is 4-hole free and also odd-hole free. Thus Corollary 4.6 implies that if G is 4-hole free, and regularity of G is at least 4, then G must contain both even and odd holes. This observation is used for improving a bound on regularity in Section 7.

5. Complements of bipartite graphs

Fernández-Ramos and Gimenez gave an explicit description of bipartite graphs associated to edge ideals that have regularity 3 in [8]. We give an independent proof of their result by using Edge Neighborhood Decomposition. Since we consider non-edge ideals, we work with complements of bipartite graphs.

Let G be the complement of a bipartite graph H with bipartition of vertices X and Y. Let B be the subgraph of G with $V(B) = V(G)$ and the edge set consisting of edges of G between vertices in X and vertices in Y. We call B the bipartite part of G. We recall chordal bipartite graphs [10, Section 12.4].

Definition 5.1. A chordal bipartite graph is a bipartite graph which contains no induced cycles of length greater than four.

It is shown in [11] that any chordal bipartite graph G with bipartition of vertices X and Y contains an edge ij for $i \in X$ and $j \in Y$ such that the induced subgraph on vertices of $N_G(i)$ and $N_G(j)$ is a complete bipartite graph. Such an edge ij is called a *bisimplicial* edge. Additionally, it is known in [11] that the subgraph $G - ij$ is again a chordal bipartite graph. This implies that subgraphs obtained by deleting a bisimplicial edge from a chordal bipartite graph are also chordal bipartite graphs.

Combining Corollary 3.5 with property of chordal bipartite graph, we get an exact description of complements of bipartite graphs of regularity 3.

Theorem 5.2. *Let G be the complement of a bipartite graph. Regularity of G is 3 if and only if G contains a hole and the bipartite part B of G is chordal bipartite.*

Proof. Suppose that the complement G of a bipartite graph H has at least one hole and the bipartite part B of G is a chordal bipartite graph. Since G contains at least one hole, regularity of G is at least 3. To show that regularity of G is at most 3 we induct on the number of edges ℓ in B. The base case $\ell = 0$ is simple, since G is then chordal and therefore $\mathrm{reg}(G) \leq 2$. Now we consider the induction step. Let G be the complement of a bipartite graph such that its bipartite part B is a chordal bipartite graph with $\ell + 1$ edges. Then B contains a bisimplicial edge e. By Theorem 3.5,

$$\mathrm{reg}(G) \leq \max\{\mathrm{reg}(G - e), \mathrm{reg}(G_e - e) + 1\}.$$

Since e is a bisimplicial edge in B, $G_e - e$ is a chordal graph, and $\mathrm{reg}(G_e - e) \leq 2$. Additionally, $\mathrm{reg}(G - e) \leq 3$ by the induction assumption, and the desired result follows.

Conversely, suppose that bipartite part B of G contains a hole of length at least 6. We claim that ΔG contains a subcomplex whose 2nd (reduced) homology is not zero. Let G' be the subgraph of G induced by vertices that form the shortest hole in B. Let X' and Y' be the partitions of vertices G' (induced from the partition of vertices of G). Let v be any vertex of X'. Then, the closed neighborhood $N_{G'}[v]$ and the deletion $G' \setminus v$ of v cover cliques of G'. Observe that $\widetilde{H}_1(\Delta N_{G'}[v]) = \widetilde{H}_1(\Delta(G' \setminus v)) = 0$ since $\Delta N_{G'}[v]$ is contractible, and any hole in $G' - v$ is covered by cliques of size 3, but $\widetilde{H}_1(\Delta N_{G'}(v)) \neq 0$ since $N_{G'}(v)$ contains a hole (of length 4). Since $\widetilde{H}_2(\Delta G') \to \widetilde{H}_1(\Delta N_{G'}(v))$ is surjective by the Mayer-Vietoris sequence, $\widetilde{H}_2(\Delta G') \neq 0$, and this implies that regularity of G is at least 4. □

6. Regularity and genus

The following bound on regularity is well-known in [25, Lemma 2.1] (or see [28] for a geometric proof), but we provide a short proof for the sake of completeness.

Lemma 6.1. *If the number of vertices of G is at most $2n - 1$, then regularity of G is at most n.*

Proof. We use induction on n. For $n = 1$, regularity is obviously at most 1 since there are no generators in the non-edge ideal of the graph. Assume that any graph with at most $2\ell - 1$ vertices has regularity at most ℓ. Let G be a graph on $2\ell + 1$ vertices. Note that by Corollary 3.3 we can delete a vertex v without changing regularity if $\mathrm{reg}(G) > \mathrm{reg}(N_G(v)) + 1$. After deleting such vertices, if possible, let v be the vertex of minimal degree in G. If the degree of v is 2ℓ, then G is a complete graph (which has regularity 1). Therefore, we can assume that degree of v is at most $2\ell - 1$. Then, we have

$$\mathrm{reg}(G) \leq \mathrm{reg}(N_G(v)) + 1 \leq \ell + 1,$$

since $N_G(v)$ contains at most $2\ell - 1$ vertices. □

In fact, the bound in Lemma 6.1 is tight. Let $K_{n(2)}$ be the complete n-partite graph, with each part of size two. Since the ideal of $K_{n(2)}$ is a complete intersection of n quadrics, its minimal resolution is given by the Koszul complex. Thus regularity of $K_{n(2)}$ is $n + 1$. We also note that $K_{n(2)}$ is a perfect graph on $2n$ vertices.

Recall that the *genus* of a graph G is the minimal genus of an orientable surface S_g into which G can be embedded (see [27] for reference). Note that any graphs can be embedded into an orientable surface S_g for some genus g and the genus of graphs inscribes a topological complexity of graphs. By using the Lemma 6.1, we can immediately give an alternative proof of a result in [28] that any planar graphs have regularity at most 4 and it is tight. We note that this is the case of genus 0 and we can provide bounds on regularity of graphs in terms of arbitrarily genus.

Theorem 6.2. *Let g be the genus of a graph G. Then, regularity of G is at most $\lfloor 1 + \sqrt{1 + 3g} \rfloor + 2$.*

Proof. Let $|V|$ be the number of vertices, $|E|$ be the number of edges, and $|F|$ be the number of (2-dimensional) faces in the embedding of G. By considering the Euler characteristic of the surface S into which G is embedded, we see that $|V| - |E| + |F| = 2 - 2g$. Recall that $2|E| = \sum_{v \in V} \deg(v) = \sum_{F \in \Delta_2} \ell_F$ where Δ_2 is the set of 2-cells in the embedding and ℓ_F is the number of edges in the face F. In particular, $2|E| = \sum_{F \in \Delta_2} \ell_F \geq 3|F|$ since $\ell_F \geq 3$ for any face F. Let d be the minimal degree of G. Then, $2|E| = \sum_{v \in V} \deg(v) \geq d|V|$. Therefore,

$$\begin{aligned} 2 - 2g &= |V| - |E| + |F| \\ &\leq |V| - |E| + \frac{2}{3}|E| = |V| - \frac{1}{3}|E| \\ &\leq |V| - \frac{d}{6}|V| = \frac{6-d}{6}|V|. \end{aligned}$$

Moreover, we can see that $|V| \geq d + 2$ since $d \leq \deg(v) \leq |V| - 2$. (Note that, if $d = |V| - 1$, the graph is complete graph, which can be excluded.) Thus,

$$6(2g-2) \geq (d-6)|V| \geq (d-6)(d+2) \Rightarrow 0 \geq d^2 - 4d - 12g.$$

This implies that $d \leq 2 + \sqrt{4+12g} = 2 + 2\sqrt{1+3g}$. Let v be the vertex of degree d. Then, $\text{reg}(N_G(v)) \leq \lfloor \frac{1}{2} \lfloor 2 + 2\sqrt{1+3g} \rfloor \rfloor + 1 = \lfloor 1 + \sqrt{1+3g} \rfloor + 1$. By Theorem 4.1, $\text{reg}(G) \leq \lfloor 1 + \sqrt{1+3g} \rfloor + 2$. □

Note that this bound is indeed tight. It is known in [23, Section 4.4] that the genus of 2-regular complete n-bipartite graphs $K_{n(2)} (= K_{2,2,\ldots,2})$ is at least $\frac{(n-3)(n-1)}{3}$. Moreover, the genus of $K_{n(2)}$ is exactly $\frac{(n-3)(n-1)}{3}$ if $n \not\equiv 2 \mod 3$ by [15]. In this case, we have $\text{reg}(K_{n(2)}) = n+1$ and the right hand side of inequality in Theorem 6.2 is $\lfloor 1 + \sqrt{1 + 3\frac{(n-3)(n-1)}{3}} \rfloor + 2 = n+1$.

7. Bounds on regularity of graphs without small holes

Even though regularity of a graph can depend linearly on the number of vertices n, if G does not contain small holes, then regularity of G can be bounded from above by a logarithmic function of n. It was shown in [6] that absence of small holes corresponds to the ideal satisfying property $N_{2,p}$ for some $p \geq 2$.

Theorem 7.1. *Let $p \geq 2$ and $I(G)$ be the non-edge ideal corresponding to a graph G. Then, the followings are equivalent.*

(1) *The minimal graded free resolution of $I(G)$ is $(p-1)$-step linear.*
(2) *The graph G does not contain a hole C_i of length i for $i \leq p+2$.*
(3) *$I(G)$ satisfies $N_{2,i}$ for all $2 \leq i \leq p$.*

It was shown in [3] that if G satisfies $N_{2,p}$ for $p \geq 2$, then

$$\text{reg}(G) \leq \log_{\frac{p+3}{2}} \frac{n-1}{p} + 3.$$

We also provide (a similar and) asymptotically better upper bound on regularity of graphs.

Theorem 7.2. *Suppose that G satisfies property $N_{2,p}$ for $p \geq 2$. Then,*

$$\text{reg}(G) \leq \min\{\log_{\frac{p+3}{2}} \left(\frac{n(p+1)}{p(p+3)}\right) + 3, \log_{\frac{p+4}{2}} \left(\frac{n(p+2)}{(p+1)(p+4)}\right) + 4\}.$$

Proof. Given a graph G, there is an induced subgraph G_0 such that $\text{reg}(G) = \text{reg}(G_0) = \text{reg}(N_{G_0}(v)) + 1$ for any vertex v in G_0. Indeed, we can keep deleting vertices y such that $\text{reg}(G) = \text{reg}(G \setminus y)$ until we arrive at a graph G_0, where $\text{reg}(G_0 \setminus v) = \text{reg}(G_0) - 1$ for any vertex v of G. Then, by Corollary 3.3 we have $\operatorname{reg}(G_0) = \operatorname{reg}(N_{G_0}(v)) + 1$ for any vertex v in G_0. We call such G_0 a trimming of G. Note that a trimming is not unique.

Let x_0 be a vertex of minimal degree in G_0. Let G_1 be a trimming of the open neighborhood $N_{G_0}(x_0)$ of x_0 in G_0. Now we repeat this process: let x_i be a vertex of minimal degree in G_i and let G_{i+1} be a trimming of the open neighborhood of x_i in G_i. We obtain a sequence of induced subgraphs G_i of G such that

$$\operatorname{reg}(G) = \operatorname{reg}(G_0) = \operatorname{reg}(G_1) + 1 = \cdots = \operatorname{reg}(G_t) + t.$$

Let ℓ be the maximal integer such that G_ℓ contains a hole, and let C_m be the hole in G_ℓ of smallest length m, with $m \geq p+3 \geq 5$. Note that C_m is a hole that is present in all graphs G_i, with $0 \leq i \leq \ell$. We use d_i to denote the degree of x_i in G_i.

We claim that for $1 \leq i \leq \ell$ the sum of the degrees of vertices of C_m in $N_{G_{\ell-i}}[x_{\ell-i}]$ is at most

$$md_{\ell-i} - \frac{m^i(m-3)}{2^{i-1}},$$

which we prove by induction on i. The base case is $i = 1$: a vertex of C_m is connected to exactly two vertices of C_m and can be connected to all other vertices in $N_{G_{\ell-1}}[x_{\ell-1}]$. Therefore, the sum of degrees of vertices of C_m is at most $2m + m(d_{\ell-1} + 1 - m) = md_{\ell-1} - m(m-3)$.

For the inductive step, assume that the sum of the degrees of vertices of C_m in $N_{G_{\ell-i+1}}[x_{\ell-i+1}]$ is at most $md_{\ell-i+1} - \frac{m^{i-1}(m-3)}{2^{i-2}}$. Observe that any vertex in $G_{\ell-i+1}$ not connected to $x_{\ell-i+1}$ can be adjacent to at most two vertices of C_m. Otherwise $G_{\ell-i+1}$ is forced to have a 4-hole, which is a contradiction. Since degree of $x_{\ell-i+1}$ in $G_{\ell-i+1}$ is at least the degree of any vertex of C_m is $G_{\ell-i+1}$ we see that there are at least

$$\frac{1}{2}\left(md_{\ell-i+1} - (md_{\ell-i+1} - \frac{m^{i-1}(m-3)}{2^{i-2}})\right) = \frac{m^{i-1}(m-3)}{2^{i-1}} \tag{7.1}$$

vertices in $G_{\ell-i+1} \setminus N_{G_{\ell-i+1}}[x_{\ell-i+1}]$.

Any vertex of $N_{G_{\ell-i}}[x_{\ell-i}]$ belongs to exactly one of $N_{G_{\ell-i}}[x_{\ell-i}] \setminus G_{l-i+1}$, or $G_{l-i+1} \setminus N_{G_{l-i+1}}[x_{l-i+1}]$, or $N_{G_{l-i+1}}[x_{l-i+1}]$. As before, any vertex of $G_{l-i+1} \setminus N_{G_{\ell-i+1}}[x_{\ell-i+1}]$ can be adjacent to at most two vertices in C_m, and a vertex of C_m can be adjacent to all vertices of $N_{G_{\ell-i}}[x_{\ell-i}] \setminus G_{\ell-i+1}$. Therefore,

$$\sum_{v \in C_m} \deg_{N_{G_{\ell-i}}[x_{\ell-i}]}(v) \leq m|N_{G_{\ell-i}}[x_{\ell-i}] \setminus G_{\ell-i+1}| + 2|G_{\ell-i+1} \setminus N_{G_{\ell-i+1}}[x_{\ell-i+1}]| + \sum_{v \in C_m} \deg_{N_{G_{\ell-i+1}}[x_{\ell-i+1}]}(v).$$

Using the induction assumption on $\sum_{v \in C_m} \deg_{N_{G_{\ell-i+1}}[x_{\ell-i+1}]}(v)$ we see that

$$\sum_{v\in C_m} \deg_{N_{G_{\ell-i}}[x_{\ell-i}]}(v) \le md_{\ell-i} - (m-2)\big(|G_{\ell-i+1} \setminus N_{G_{\ell-i+1}}[x_{\ell-i+1}]|\big) - \frac{m^{i-1}(m-3)}{2^{i-2}}.$$

By (7.1) we see that

$$(m-2)\big(|G_{\ell-i+1} \setminus N_{G_{\ell-i+1}}[x_{\ell-i+1}]|\big) + \frac{m^{i-1}(m-3)}{2^{i-2}} \ge \frac{m^i(m-3)}{2^{i-1}},$$

and therefore

$$\sum_{v\in C_m} \deg_{N_{G_{\ell-i}}[x_{\ell-i}]}(v) \le md_{\ell-i} - \frac{m^i(m-3)}{2^{i-1}},$$

as desired. The argument above shows that there are at least $\frac{m^i(m-3)}{2^i}$ vertices in $G_{\ell-i} \setminus N_{G_{\ell-i}}[x_{\ell-i}]$. Since $G_{\ell-i+1}$ is a subgraph of $N_{G_{\ell-i}}[x_{\ell-i}]$, we see that

$$|G_{\ell-i}| - |G_{\ell-i+1}| \ge \frac{m^i(m-3)}{2^i}.$$

Therefore,

$$|G_{\ell-i}| \ge \sum_{t=1}^{i} \frac{m^t(m-3)}{2^t} + m,$$

and by summing the above geometric series we see that

$$|G_{\ell-i}| \ge \frac{m^{i+1}(m-3)}{2^i(m-2)}.$$

Plugging in $i=\ell$, we see that

$$n \ge |G_0| \ge \frac{m^{\ell+1}(m-3)}{2^\ell(m-2)} \ge \frac{p(p+3)^{\ell+1}}{2^\ell(p+1)}.$$

Thus,

$$\operatorname{reg}(G) \le \operatorname{reg}(G_{\ell+1}) + \ell + 1 \le \log_{\frac{p+3}{2}}\left(\frac{n(p+1)}{p(p+3)}\right) + 3,$$

which gives us the first upper bound.

For the second upper bound, we observe that if regularity of G is at least four, then G contains both even and odd holes by Corollary 4.6. With the same setting above, regularity of $N_{G_{\ell-2}}(x_{\ell-2})$ (or equivalently, $G_{\ell-1}$) is four. Let m be the length of the smallest hole in $N_{G_{\ell-2}}(x_{\ell-2})$. Then $N_{G_{\ell-2}}(x_{\ell-2})$ must also contain a hole of size $m+2\alpha+1$ for some positive integer α. We can now apply the same process as above to bound number of vertices of $G_{\ell-i}$ using this larger hole to obtain

$$|G_{\ell-i}| \geq \frac{(m+2\alpha+1)^i(m+2\alpha-2)}{2^{i-1}(m+2\alpha-1)},$$

for $2 \leq i \leq \ell$. By taking $i = \ell$, we see that

$$n \geq |G_0| \geq \frac{(m+2\alpha+1)^\ell(m+2\alpha-2)}{2^{\ell-1}(m+2\alpha-1)} \geq \frac{(p+4)^\ell(p+1)}{2^{\ell-1}(p+2)}.$$

Thus,

$$\operatorname{reg}(G) \leq \operatorname{reg}(G_{\ell+1}) + \ell + 1 \leq \log_{\frac{p+4}{2}} \frac{n(p+2)}{(p+1)(p+4)} + 4. \quad \square$$

Note that the former term in the bound in Theorem 7.2 is slightly better (if $n \geq \frac{p+3}{2}$) than the bound in [3, Theorem 4.9] and the former term will be smaller than the latter term if the size of a graph is relatively small. However, the latter term of the bound is better asymptotically.

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