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Priority evacuation from a disk: The case of $n = 1, 2, 3$ [☆]

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A B S T R A C T

An exit (or target) is at an unknown location on the perimeter of a unit disk. A group of $n+1$ robots (in our case, $n = 1, 2, 3$), initially located at the centre of the disk, are tasked with finding the exit. The robots have unique identities, share the same coordinate system, move at maximum speed 1 and are able to communicate wirelessly the position of the exit once found. Among them there is a distinguished robot called the *queen* and the remainder of the robots are referred to as *servants*. It is known that with two robots searching, the room can be evacuated (i.e., with both robots reaching the exit) in $1 + \frac{2\pi}{3} + \sqrt{3} \approx 4.8264$ time units and this is optimal [11]. Somewhat surprisingly, in this paper we show that if the goal is to have the queen reach the exit, not caring if her servants make it, there is a slightly better strategy for the case of one servant. We prove that this "priority" version of evacuation can be solved in time at most 4.81854. Furthermore, we show that any strategy for saving the queen with one servant requires time at least $3 + \pi/6 + \sqrt{3}/2 \approx 4.3896$ in the worst case. If more servants are available, we show that the time bounds can be improved to 3.8327 for two servants, and 3.3738 for three servants which are better than the known lower bound for the corresponding problems of evacuating three or four robots. Finally, we show lower bounds for these cases of 3.6307 (two servants) and 3.2017 (three servants). The case of more than three servants uses substantially different techniques and is discussed in a separate paper [13].

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1. Introduction

In traditional search, a group of searchers (modelled as mobile autonomous agents or robots) may collaboratively search for an exit (or target) placed within a given search domain [1,2,18]. Although the searchers may have differing capabilities (communication, perception, mobility, memory) search algorithms, previously employed, generally make no distinction between them as they usually play identical roles throughout the execution of the search algorithm and with respect to the termination time (with the exception of faulty robots, which may not contribute to searching or may even try to slow its progress). In this work we are motivated by real-life safeguarding-type situations where a number of agents have the exclusive role of facilitating the execution of the task by a distinguished agent. We introduce and study *Priority Evacuation*, a new form of search in which the search time of the algorithm is measured by the time it takes a special searcher, called the queen, to reach the exit. The remaining searchers in the group, called servants, are participating in the search but are not required to exit.

1.1. Problem definition of Priority Evacuation with n servants (PE_n)

An exit (or target) is hidden at an unknown location on the boundary of a unit disk. The exit can be located by any of the $n+1$ robots (searchers) when they walk over it. Robots have unique identities, share the same coordinate system, start from the centre of the circle, and have maximum speed 1. Among them there is a distinguished robot, called the *queen*, and the remaining n robots are referred to as *servants*. All servants are known to the queen by their identities. Robots may run asymmetric algorithms, and can communicate their findings wirelessly and instantaneously (each message contains the robots identity and location). Feasible solutions to this problem are *evacuation algorithms*, i.e., a set of robot trajectories that guarantee the finding of the hidden exit and the queen reaching it. The cost of an evacuation algorithm is the *evacuation time* of the queen, i.e., the worst case total time until the queen reaches the exit. None of the n servants needs to evacuate.

1.2. Related work

Much of the work related to ours started with the problem of linear search which refers to search on an infinite line. There have been several interesting studies attempting to optimize the search time which were initiated by the influential works of Bellman [7] and Beck [6]. A long list of results followed for numerous variants of the problem, citing all of which is outside the scope of this work. For a comprehensive study of seminal search-type problems see [2,3].

The problem of searching in the plane by one or more searchers, has been considered by [4,5]. The unit disk model considered in our present paper is a form of two-dimensional search that was initiated in the work of [11]. In that paper the authors obtained evacuation algorithms in the wireless and face-to-face communication models both for a small number of robots as well optimal asymptotic results for a large number of robots. Additional evacuation algorithms in the face-to-face communication model were subsequently analyzed for two robots in [14] and later in [8], while, recently, [9] considered worst-case average-case tradeoffs for the same problem. Other variations of the problem include the case of more than one exit, see [10] and [17], triangular and square domains in [15], robots with different moving speeds [16], and evacuation in the presence of crash or byzantine faulty robots [12].

Notably, all relevant previous work in search-type problems considered the objective of minimizing the time it takes either by the first or the last agent to reach the hidden exit or target. In contrast, this paper considers an evacuation (search-type) problem where the completion time is defined with respect to a distinguished mobile agent, the *queen*, while the remaining n servants are not required to reach the exit. Notably, the algorithms we propose significantly improve upon evacuation costs induced by naive trajectories, and in fact the trajectories we propose are surprisingly complex. Our main contribution concerns priority evacuation for each of the cases of $n=1,2,3$ servants, all of which require special treatment. Moreover, all our algorithms are characterized by the fact that the queen contributes effectively to the search for the exit. In sharp contrast, the independent and concurrent work of [13] studies the same problem for $n \geq 4$ servants where in the best known algorithms the queen does not contribute to the search. More importantly, the proposed algorithms of [13] admit a unified description and analysis that does not intersect with the current work.

1.3. Our results & paper organization

Section 2 introduces necessary notation and terminology and discusses preliminaries. Section 3 is devoted to upper bounds for PE_n for $n=1,2,3$ servants (see Subsections 3.1, 3.2, and 3.3, respectively). All our upper bounds are achieved by fixing optimal parameters for families of parameterized algorithms. In Section 4 we derive lower bounds for PE_n, $n=1,2,3$. An interesting corollary of our positive results is that priority evacuation with $n=1,2,3$ servants (i.e., with $n+1$ searchers) can be performed strictly faster than ordinary evacuation with $n+1$ robots where all robots have to evacuate. Indeed, an argument found in [11] can be adjusted to show that the evacuation problem with $n+1$ robots cannot be solved faster than $1+\frac{4\pi}{3(n+1)}+\sqrt{3}$. We show that when one needs to evacuate only one designated robot, the task can provably (due to our upper bounds) be executed faster. All our results, together with the comparison to the lower bounds of [11], are summarized in Table 1. We conclude the paper in Section 5 with a discussion of open problems.

Table 1
Upper bounds (UB) and lower bounds (LB) for priority evacuation.

# of servants	UB for PE_n	LB for PE_n	LB for ordinary evacuation
$n=1$	4.8185 (Theorem 3.1)	4.3896 (Theorem 4.1)	4.826445 (see [11])
$n=2$	3.8327 (Theorem 3.3)	3.6307 (Theorem 4.6)	4.128314 (see [11])
$n=3$	3.3738 (Theorem 3.7)	3.2017 (Theorem 4.10)	3.779248 (see [11])

2. Notation and preliminaries

We use n to denote the number of servants, and we set $[n]=\{1,\dots,n\}$. Queen and servant i will be denoted by $\mathcal{Q}$ and $\mathcal{S}_i$, respectively, where $i\in[n]$. We assume that all robots start from the origin $O=(0,0)$ of a unit circle in $\mathbb{R}^2$. As usual, points in $A\in\mathbb{R}^2$ will be treated, when it is convenient, as vectors from O to A, and $\|A\|$ will denote the euclidean norm of that vector.

2.1. Problem reformulation & solutions' description

Robots' trajectories will be defined by parametric functions $\mathcal{F}(t)=(f(t),g(t))$, where $f,g:\mathbb{R}\mapsto\mathbb{R}$ are continuous and piecewise differentiable. In particular, search algorithms for all robots will be given by trajectories

$$\mathbb{S}_n := \left\{\mathcal{Q}(t), \{\mathcal{S}_i(t)\}_{i\in[n]}\right\},$$

where $\mathcal{Q}(t),\mathcal{S}_i(t)$ will denote the position of $\mathcal{Q}$ and $\mathcal{S}_i$, respectively, at time $t\ge 0$.

Definition 2.1 *(Feasible trajectories).* We say that trajectories $\mathbb{S}_n$ are *feasible* for PE_n if:

(a) $\mathcal{Q}(0)=\mathcal{S}_i(0)=O$, for all $i\in[n]$,
(b) $\mathcal{Q}(t),\{\mathcal{S}_i(t)\}_{i\in[n]}$ induce speed-1 trajectories for $\mathcal{Q},\{\mathcal{S}_i\}_{i\in[n]}$ respectively, and
(c) there is some time $t_0\ge 1$, such that each point of the unit circle is visited (searched) by at least one robot in the time window $[0,t_0]$. We refer to the smallest such t_0 as the *search time* of the circle.

Note that feasible trajectories do indeed correspond to robots' movements for PE_n in which, eventually the entire circle is searched, and hence the search time is bounded. We will describe all our search/evacuation algorithms as feasible trajectories, and we will assume that once the target is reported, $\mathcal{Q}$ will go directly to the location of the exit.

For feasible trajectories $\mathbb{S}_n$ with search time t_0, and for any trajectory $\mathcal{F}(t)$ (either of the queen or of a servant), we denote by $\mathbb{I}(\mathcal{F})$ the subinterval of $[0,t_0]$ that contains all $x\in[0,t_0]$ such that $\|\mathcal{F}(x)\|=1$ (i.e., the robot is on the circle) and no other robot has been to $\mathcal{F}(x)$ before. Since robots start from the origin, it is immediate that $\mathbb{I}(\mathcal{F})\subseteq[1,t_0]$. With this notation in mind, note that the exit can be discovered by some robot $\mathcal{F}$, say at time x, only if $x\in\mathbb{I}(\mathcal{F})$. In this case, the finding is instantaneously reported, so $\mathcal{Q}$ goes directly to the exit, moving along the corresponding line segment between her current position $\mathcal{Q}(x)$ and the reported position of the exit $\mathcal{F}(x)$. Hence, the total time that $\mathcal{Q}$ needs to evacuate equals

$$x+\|\mathcal{Q}(x)-\mathcal{F}(x)\|.$$

Therefore, the *evacuation time* of feasible trajectories $\mathbb{S}_n$ to PE_n is given by expression

$$\max_{\mathcal{F}\in\mathbb{S}_n}\sup_{x\in\mathbb{I}(\mathcal{F})}\{x+\|\mathcal{Q}(x)-\mathcal{F}(x)\|\}.$$

Notice that for "non-degenerate" search algorithms for which the last point on the circle is not searched by $\mathcal{Q}$ alone, the previous maximum can be simply computed over the servants, i.e., the evacuation cost will be

$$\max_{i\in[n]}\sup_{x\in\mathbb{I}(\mathcal{S}_i)}\{x+\|\mathcal{Q}(x)-\mathcal{S}_i(x)\|\}. \tag{1}$$

In other words, we can restate PE_n as the problem of determining feasible trajectories $\mathbb{S}_n$ so as to minimize (1).

2.2. Useful trajectories' components

Feasible trajectories induce, by definition, robots that are moving at (maximum) speed 1. The speed restriction will be ensured by the next condition.

Lemma 2.2. *An object following trajectory $\mathcal{F}(t)=(f(t),g(t))$ has unit speed if and only if*

$$\left(f'(t)\right)^2+\left(g'(t)\right)^2=1,\quad \forall t\ge 0.$$

Proof. For any $t \geq 0$, the velocity of $\mathcal{F}$ is given by $\mathcal{F}'(t) = (df(t)/dt, dg(t)/dt)$, and its speed is calculated as $\left\|\mathcal{F}'(t)\right\|$. $\square$

Robots' trajectories will be composed by piecewise smooth parametric functions. In order to describe them, we introduce some further notation. For any $\theta \in \mathbb{R}$, we introduce abbreviation C_θ for point $\{\cos(\theta), \sin(\theta)\}$. Next we introduce parametric equations for moving along the perimeter of a unit circle (Lemma 2.3), and along a line segment (Lemma 2.4).

Lemma 2.3. *Let $b \in [0, 2\pi)$ and $\sigma \in \{-1, 1\}$. The trajectory of an object moving at speed 1 on the perimeter of a unit circle with initial location C_b is given by the parametric equation*

$$\mathcal{C}(b, \sigma t) := (\cos(\sigma t + b), \sin(\sigma t + b)).$$

If $\sigma = 1$ the movement is counter-clockwise (ccw), and clockwise (cw) otherwise.

Proof. Clearly, $\mathcal{C}(b, 0) = C_b$. Also, it is easy to see that $\|\mathcal{C}(b, t)\| = 1$, i.e., the object is moving on the perimeter of the unit circle. Lastly,

$$\left(\frac{d}{dt}\cos(\sigma t + b)\right)^2 + \left(\frac{d}{dt}\sin(\sigma t + b)\right)^2 = \sigma^2(-\sin(\sigma t + b))^2 + \sigma^2(\cos(\sigma t + b))^2 = 1,$$

so the claim follows by Lemma 2.2. $\square$

Lemma 2.4. *Consider distinct points $A = (a_1, a_2)$, $B = (b_1, b_2)$ in $\mathbb{R}^2$. The trajectory of a speed 1 object moving along the line passing through A, B and with initial position A is given by the parametric equation*

$$\mathcal{L}(A, B, t) := \left(\frac{b_1 - a_1}{\|A - B\|}t + a_1, \frac{b_2 - a_2}{\|A - B\|}t + a_2\right).$$

Proof. It is immediate that the parametric equation corresponds to a line. Also, it is easy to see that $\mathcal{L}(A, B, 0) = A$ and $\mathcal{L}(A, B, \|A - B\|) = B$, i.e., the object starts from A, and eventually visits B. As for the object's speed, we calculate

$$\left(\frac{d}{dt}\left(\frac{b_1 - a_1}{\|A - B\|}t + a_1\right)\right)^2 + \left(\frac{d}{dt}\left(\frac{b_2 - a_2}{\|A - B\|}t + a_2\right)\right)^2 = \left(\frac{b_1 - a_1}{\|A - B\|}\right)^2 + \left(\frac{b_2 - a_2}{\|A - B\|}\right)^2 = 1$$

so, by Lemma 2.2, the speed is indeed 1. $\square$

Robots trajectories will be described in phases. In each phase, robot, say $\mathcal{F}$, will be moving between two explicit points, and the corresponding trajectory $\mathcal{F}(t)$ will be implied by the previous description, using most of the times Lemma 2.3 and Lemma 2.4. We will summarize the details in tables of the following format.

Robot	#	*Description*	*Trajectory*	*Duration*
$\mathcal{F}$	0		$\mathcal{F}(t)$	t_0
	1		$\mathcal{F}(t)$	t_1
	$\vdots$			$\vdots$

Phase 0 will usually correspond to the deployment of $\mathcal{F}$ from the origin to some point of the circle. Also, for each phase we will summarize its duration. With that in mind, trajectory $\mathcal{F}(t)$ during phase i, with duration t_i, will be valid for all $t \geq 0$ with $|t - (t_0 + t_1 + \ldots t_{i-1})| \leq t_i$.

Lastly, the following abbreviation will be useful for the exposition of the trajectories. For any $\rho \in [0, 1]$ and $\theta \in [0, 2\pi)$, we introduce notation

$$K(\theta, \rho) := (1 - \rho)C_{\pi-\theta} + \rho C_{-\theta}.$$

In other words, $K(\theta, \rho)$ is a convex combination of antipodal points $C_{\pi-\theta}, C_{-\theta}$ of the unit circle, i.e., it lies on the diameter of the unit circle passing through these two points. Moreover, it is easy to see that $\|C_{\pi-\theta} - K(\theta, \rho)\| = 2\rho$, and hence

$$\|K(\theta, \rho) - C_{-\theta}\| = 2 - 2\rho.$$

As it will be handy later, we also introduce abbreviation

$$AK(\theta, \rho) := \|C_\pi - K(\theta, \rho)\|.$$

The choice of the abbreviation is clear, if the reader denotes $C_\pi = (-1, 0)$ by A.

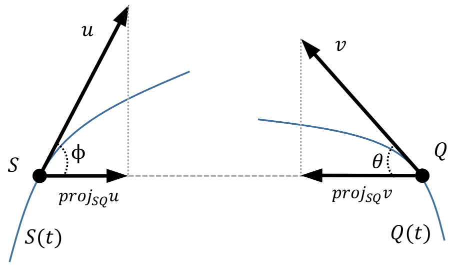

Fig. 1. An illustration of trajectories $\mathcal{S}(t)$, $\mathcal{Q}(t)$, and their critical angles at some fixed time τ, with $\mathcal{S}(\tau) = S$, $\mathcal{Q}(\tau) = Q$, $\mathcal{S}'(\tau) = u$, $\mathcal{Q}'(\tau) = v$.

2.3. Critical angles

The following definition introduces a key concept. In what follows, abstract trajectories will be assumed to be continuous and differentiable, which in particular implies that corresponding velocities are continuous.

Definition 2.5 *(Critical angle).* Let $\mathcal{S}(t) \in \mathbb{R}^2$ denote the trajectory of a speed-1 object, where $t \geq 0$. For some point $Q \in \mathbb{R}^2$, we define the $(\mathcal{S}, Q)$-critical angle at time $t = \tau$ to be the angle between the velocity vector $\mathcal{S}'(\tau)$ and vector $\overrightarrow{\mathcal{S}(\tau)Q}$, i.e. the vector from $\mathcal{S}(\tau)$ to Q.

We make the following critical observation, see also Fig. 1.

Theorem 2.6. *Consider trajectories $\mathcal{S}(t)$, $\mathcal{Q}(t)$ of two speed-1 objects $\mathcal{S}$, $\mathcal{Q}$, where $t \geq 0$. Let also ϕ, θ denote the $(\mathcal{S}, \mathcal{Q}(t))$-critical angle and the $(\mathcal{Q}, \mathcal{S}(t))$-critical angle at time t, respectively. Then $t + \|\mathcal{Q}(t) - \mathcal{S}(t)\|$ is strictly increasing if $\cos(\phi) + \cos(\theta) < 1$, strictly decreasing if $\cos(\phi) + \cos(\theta) > 1$, and constant otherwise.*

Theorem 2.6 is an immediate corollary of the following lemma.

Lemma 2.7. *Consider trajectories $\mathcal{S}(t)$, $\mathcal{Q}(t)$ and their critical angles π, θ, as in the statement of Theorem 2.6. Then*

$$\frac{d}{dt} \|\mathcal{Q}(t) - \mathcal{S}(t)\| = \cos(\phi) + \cos(\theta).$$

Proof. For any fixed t, let d denote $D(t)$, and S, Q denote points $\mathcal{S}(t)$, $\mathcal{Q}(t)$, respectively. Denote also by u, v the velocities of $\mathcal{S}$, $\mathcal{Q}$ at time t, respectively, i.e. $u = \mathcal{S}'(t)$, $v = \mathcal{Q}'(t)$. See also Fig. 1.

With that notation, observe that $\left\|\overrightarrow{SQ}\right\| = d$. Since $\|u\| = \|v\| = 1$, we see that

$$\mathrm{proj}_{SQ}\, u = \frac{\cos(\phi)}{d} \overrightarrow{SQ}$$

and

$$\mathrm{proj}_{SQ}\, v = \frac{\cos(\theta)}{d} \overrightarrow{QS}.$$

Now consider two imaginary objects $\overline{\mathcal{S}}, \overline{\mathcal{Q}}$, with corresponding velocities $\overline{\mathcal{S}}'(t) = \mathrm{proj}_{SQ}\, u$ and $\overline{\mathcal{Q}}'(t) = \mathrm{proj}_{SQ}\, v$. It is immediate that $\|\mathcal{Q}(t) - \mathcal{S}(t)\| = \left\|\overline{\mathcal{Q}}(t) - \overline{\mathcal{S}}(t)\right\|$.

In particular, $\mathrm{proj}_{SQ}\, u - \mathrm{proj}_{SQ}\, v$ is the projection of the relative velocities of $\mathcal{S}, \mathcal{Q}$ on the line segment connecting $\mathcal{S}(t)$, $\mathcal{Q}(t)$. As such, the distance between $\mathcal{S}, \mathcal{Q}$ changes at a rate determined by velocity

$$\mathrm{proj}_{SQ}\, u - \mathrm{proj}_{SQ}\, v = \frac{\cos(\phi) + \cos(\theta)}{d} \overrightarrow{SQ},$$

where $\left\|\mathrm{proj}_{SQ}\, u - \mathrm{proj}_{SQ}\, v\right\| = |\cos(\phi) + \cos(\theta)|$. Moreover, $\mathrm{proj}_{SQ}\, u$, $\mathrm{proj}_{SQ}\, v$ are antiparallel if and only if $\cos(\phi), \cos(\theta) > 0$, in which case the two objects come closer to each other. □

3. Upper bounds

3.1. Evacuation algorithm for PE_1

This subsection is devoted in proving the following.

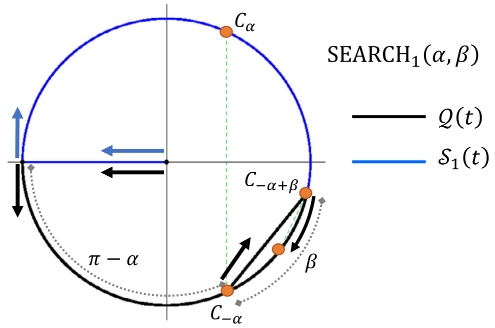

Fig. 2. Algorithm SEARCH$_1(\alpha,\beta)$ depicted for the optimal parameters of the algorithm. In all subsequent figures, as well as here, the orange points on the perimeter of the disc correspond to the worst adversarial placements of the treasure, which due to our optimality conditions induce the same evacuation cost. The orange points in $\mathcal{Q}$'s trajectories correspond to the $\mathcal{Q}$'s positioning when the treasures are reported, in the worst cost induced cases. The green dashed line depicts $\mathcal{Q}$'s trajectory after $\mathcal{Q}$ abandons her trajectory and moves toward the reported exit following a straight line. (For interpretation of the colours in the figures, the reader is referred to the web version of this article.)

Theorem 3.1. *Consider the real function $f(x)=x+\sin(x)$, and denote by $\alpha_0>0$ the solution to equation*

$$f(f(\alpha-\sin(\alpha)))=\sin(\alpha),$$

with $\alpha_0\approx 1.14193$. Then PE$_1$ can be solved in time $1+\pi-\alpha_0+2\sin(\alpha_0)\approx 4.81854$.

The value of α_0 is well defined in the statement of Theorem 3.1. Indeed, by letting $g(x)=f(f(x-\sin(x)))-\sin(x)$, we observe that g is continuous, while $g(1)\approx -0.213934$ and $g(\pi/2)\approx 1.00729$, hence there exists $\alpha_0\in(1,\pi/2)$ with $g(\alpha_0)=0$.

In order to prove Theorem 3.1, and given parameters α,β, we introduce the family of trajectories SEARCH$_1(\alpha,\beta)$, see also Fig. 2.

Algorithm SEARCH$_1(\alpha,\beta)$

Robot	#	*Description*	*Trajectory*	*Duration*
$\mathcal{Q}$	0	Move to point C_π	$\mathcal{L}(O,C_\pi,t)$	1
	1	Search circle ccw till point $C_{-\alpha}$	$\mathcal{C}(\pi,t-1)$	$\pi-\alpha$
	2	Move to point $C_{-\alpha+\beta}$,	$\mathcal{L}(C_{-\alpha},C_{-\alpha+\beta},t-(1+\pi-\alpha))$	$2\sin(\beta/2)$
	3	Search circle cw till point $C_{-\alpha}$	$\mathcal{C}(\beta-\alpha,1+\pi-\alpha+2\sin(\beta/2)-t)$	β
$\mathcal{S}_1$	0	Move to point C_π	$\mathcal{L}(O,C_\pi,t)$	1
	1	Search circle cw till point $C_{\beta-\alpha}$	$\mathcal{C}(\pi,-t+1)$	$\pi+\alpha-\beta$

Partitioning the circle clockwise, we see that the arc with endpoints $C_\pi, C_{\pi+\alpha-\beta}$ is searched by $\mathcal{S}_1$, while the remaining of the circle is searched by $\mathcal{Q}$. Therefore, robots' trajectories in SEARCH$_1(\alpha,\beta)$ are feasible, and it is also easy to see that they are continuous as well. The search time equals $1+\pi+\max\{\alpha-\beta,2\sin(\beta/2)+\beta-\alpha\}$, as well as

$$\mathbb{I}(\mathcal{Q})=[1,1+\pi-\alpha]\cup[1+\pi-\alpha+2\sin(\beta/2),1+\pi-\alpha+2\sin(\beta/2)+\beta],\mathbb{I}(\mathcal{S}_1)=[1,1+\pi+\alpha-\beta].$$

An illustration of the above trajectories for certain values of α,β can be seen in Fig. 2.

First we make some observations pertaining to the monotonicity of the evacuation cost.

Lemma 3.2. *Assuming that $\alpha>\pi/3$ and that $\cos(\alpha)+\cos(\alpha-\beta/2)>1$, the evacuation cost of SEARCH$_1(\alpha,\beta)$ is monotonically increasing if the exit is found by $\mathcal{S}_1$ during $\mathcal{Q}$'s phase 1 and monotonically decreasing if the exit is found by $\mathcal{S}_1$ during $\mathcal{Q}$'s phase 2.*

Proof. Suppose that the exit is found by $\mathcal{S}_1$ during $\mathcal{Q}$'s phase 1, i.e., at time x after robots start searching for the first time, where $0\le x\le\pi-\alpha$. It is easy to see that the critical angles between $\mathcal{Q},\mathcal{S}_1$ are both equal to $\pi-x$. But then $2\cos(\pi-x)\ge 2\cos(\alpha)>2\cos(\pi/3)=1$. Hence, by Theorem 2.6, the evacuation cost is decreasing in this case.

Now suppose that the exit is found by $\mathcal{S}_1$ during $\mathcal{Q}$'s phase 2, i.e., at time x after $\mathcal{Q}$ starts moving along the chord with endpoints $C_{-\alpha},C_{-\alpha+\beta}$, where $0\le x\le 2\sin(\beta/2)$. If ϕ_x,θ_x denote the $\mathcal{S}_1$, $\mathcal{Q}$ critical angles, then it is easy to see that $\phi_0=\cos(\alpha)$ and that $\theta_0=\alpha-\beta/2$. Since $\cos(\phi_0)+\cos(\theta_0)>1$, Theorem 2.6 implies that the evacuation cost is initially decreasing in this phase. For the remaining of $\mathcal{Q}$'s phase 2, it is easy to see that both ϕ_x,θ_x are decreasing in x, hence $\cos(\phi_x)+\cos(\theta_x)$ is increasing in x, hence, the evacuation cost will remain decreasing in this phase. $\square$

Now we can prove Theorem 3.1 by fixing certain values for parameters α, β of $\textsc{Search}_1(\alpha, \beta)$. In particular, we set α_0 as in the statement of Theorem 3.1, and $\beta_0 = 2f(\alpha_0 - \sin(\alpha_0)) \approx 0.925793$. The trajectories of the robots, for the exact same values of the parameters, can be seen in Fig. 2.

Proof. Let f, α_0 be as in the statement of Theorem, and set $\beta_0 = 2f(\alpha_0 - \sin(\alpha_0)) \approx 0.925793$. We argue that the worst evacuation time of $\textsc{Search}_1(\alpha_0, \beta_0)$ is $1 + \pi - \alpha_0 + 2\sin(\alpha_0)$. Note that for the given values of the parameters, we have that $\alpha_0 > \pi/3$, that $\alpha_0 - \sin(\beta_0/2) \le \beta_0$, and that $\cos(\alpha_0) + \cos(\alpha_0 - \beta_0/2) > 1$.

First we observe that if the exit is found by $\mathcal{Q}$, then the worst case evacuation time $E_0(\alpha_0, \beta_0)$ is incurred when the exit is found just before $\mathcal{Q}$ stops searching, that is

$$E_0(\alpha_0, \beta_0) = 1 + \pi - \alpha_0 + 2\sin(\beta_0/2) + \beta_0.$$

Next we examine some cases as to when the exit is found by $\mathcal{S}_1$. If the exit is found by $\mathcal{S}_1$ during the 1st phase of $\mathcal{Q}$, then the evacuation time is, due to Lemma 3.2, given as

$$E_1(\alpha_0, \beta_0) = \sup_{1 \le x \le 1+\pi-\alpha_0} \{x + \|\mathcal{Q}(x) - \mathcal{S}_1(x)\|\} = 1 + \pi - \alpha_0 + 2\sin(\alpha_0).$$

Recall that $\cos(\alpha_0) + \cos(\alpha_0 - \beta_0/2) > 1$, and so, again by Lemma 3.2 we may omit the case that the exit is found by $\mathcal{S}_1$ while $\mathcal{Q}$ is in phase 2. The end of $\mathcal{Q}$'s phase 2 happens at time $\tau := 1 + \pi - \alpha_0 + 2\sin(\beta_0/2)$, when have that $\mathcal{Q}(\tau) = C_{-\alpha+\beta}$, and $\mathcal{S}_1(\tau) = C_{\alpha - 2\sin(\beta_0/2)}$, and both robots are intending to search ccw. Condition $\alpha_0 - \sin(\beta_0/2) \le \beta_0$ says that $\mathcal{S}_1$ will finish searching prior to $\mathcal{Q}$, and this happens when $\mathcal{S}_1$ reaches point $C_{-\alpha+\beta}$. During this phase, the distance between $\mathcal{Q}, \mathcal{S}_1$ stays invariant and equal to $2\alpha_0 - \beta_0 - 2\sin(\beta_0/2)$. We conclude that the cost in this case would be

$$E_2(\alpha_0, \beta_0) = 1 + \pi + \alpha_0 - \beta_0 + 2\sin(\alpha_0 - \beta_0/2 - \sin(\beta_0/2)).$$

Then, we argue that the choice of α_0, β_0 guarantees that $E_0(\alpha_0, \beta_0) = E_1(\alpha_0, \beta_0) = E_2(\alpha_0, \beta_0)$, as wanted.

Indeed, $E_0(\alpha_0, \beta_0) = E_1(\alpha_0, \beta_0)$ implies that $\sin(\beta_0/2) + \beta_0/2 = \sin(\alpha_0)$. But then, we can rewrite $E_2(\alpha_0, \beta_0)$ as

$$E_2(\alpha_0, \beta_0) = 1 + \pi + \alpha_0 - \beta_0 + 2\sin(\alpha_0 - \sin(\alpha_0)).$$

Equating the last expression with $E_1(\alpha_0, \beta_0)$ implies that

$$\beta_0/2 = \alpha_0 - \sin(\alpha_0) + \sin(\alpha_0 - \sin(\alpha_0)) = f(\alpha_0 - \sin(\alpha_0)).$$

Substituting twice $\beta_0/2$ in the already derived condition $\sin(\beta_0/2) + \beta_0/2 = \sin(\alpha_0)$ implies that

$$f(f(\alpha - \sin(\alpha_0))) = \sin(\alpha_0).$$

Fig. 2 depicts the worst placements of the exit, along with the trajectories of the queen (in dashed green lines) after the exit is reported. □

It should be stressed that $\mathcal{Q}$'s Phases 2, 3 are essential for achieving the promised bound. Indeed, had we chosen $\alpha = \beta = 0$, the worst case evacuation time would have been

$$\sup_{1 \le x \le 1+\pi} \{x + \|\mathcal{Q}(x) - \mathcal{S}_1(x)\|\} = \sup_{0 \le x \le \pi} \{1 + x + 2\sin(x)\}.$$

The maximum is attained at $x_0 = 2\pi/3$ (and indeed, both critical angles in this case are $\pi/3$ and in particular $2\cos(\pi/3) = 1$), inducing a cost of $1 + 2\pi/3 + \sqrt{3} \approx 4.82645$. The latter is the cost of the evacuation algorithm for two robots without priority of [11].

3.2. *Evacuation algorithm for* PE_2

In this subsection we prove the following theorem.

Theorem 3.3. PE_2 *can be solved in time 3.8327.*

Given parameters α, ρ, we introduce the family of trajectories $\textsc{Search}_2(\alpha, \rho)$, see also Fig. 3.

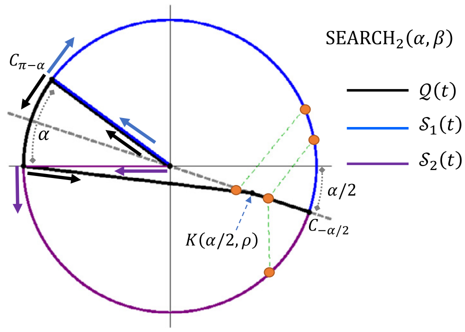

Fig. 3. Algorithm SEARCH$_2(\alpha,\beta)$ depicted for the optimal parameters of the algorithm.

Algorithm SEARCH$_2(\alpha,\rho)$

Robot	#	*Description*	*Trajectory*	*Duration*
$\mathcal{Q}$	0	Move to point $C_{\pi-\alpha}$	$\mathcal{L}(O, C_{\pi-\alpha}, t)$	1
	1	Search the circle ccw till point C_π	$\mathcal{C}(\pi-\alpha, t-1)$	α
	2	Move to point $K(\alpha/2,\rho)$	$\mathcal{L}(C_\pi, K(\alpha/2,\rho), t-(1+\alpha))$	$AK(\alpha/2,\rho)$
	3	Move to point $C_{-\alpha/2}$	$\mathcal{L}(K(\alpha/2,\rho), C_{-\alpha/2})$	$2-2\rho$
$\mathcal{S}_1$	0	Move to point $C_{\pi-\alpha}$	$\mathcal{L}(O, C_{\pi-\alpha})$	1
	1	Search the circle cw till point $C_{-\alpha/2}$	$\mathcal{C}(\pi-\alpha, -t+1)$	$\pi-\alpha/2$
$\mathcal{S}_2$	0	Move to point C_π	$\mathcal{L}(O, C_\pi)$	1
	1	Search the circle ccw till point $C_{-\alpha/2}$	$\mathcal{C}(\pi, t-1)$	$\pi-\alpha/2$

Notice that, by definition of SEARCH$_2(\alpha,\rho)$, robots' trajectories are continuous and feasible, meaning that the entire circle is eventually searched. Indeed, partitioning the circle clockwise, we see that: the arc with endpoints $C_\pi, C_{\pi-\alpha}$ is searched by $\mathcal{Q}$, the arc with endpoints $C_{\pi-\alpha}, C_{-\alpha/2}$ is searched by $\mathcal{S}_1$, and the arc with endpoints $C_{-\alpha/2}, C_\pi$ is searched by $\mathcal{S}_2$.

It is immediate from the description of the trajectories that the search time is $1+\pi-\alpha/2$. Moreover

$$\mathbb{I}(\mathcal{Q}) = [1, 1+\alpha],\ \mathbb{I}(\mathcal{S}_1) = \mathbb{I}(\mathcal{S}_2) = [1, 1+\pi-\alpha/2].$$

An illustration of the above trajectories for certain values of α, ρ can be seen in Fig. 3. Now we make some observations, in order to calculate the worst case evacuation time.

Lemma 3.4. *Suppose that $\pi-\alpha/2 \geq \alpha + AK(\alpha/2,\rho) + 2 - 2\rho$. Then $\|\mathcal{Q}(x) - \mathcal{S}_1(t)\|$ is continuous and differentiable in the time intervals I_1, I_2, I_3 of $\mathcal{Q}$'s phases 1, 2, 3, respectively. Moreover, the worst case evacuation time of* SEARCH$_2(\alpha,\rho)$ *can be computed as*

$$\max\left\{\begin{array}{l} 1+\alpha+2\sin(\alpha), \\ \sup_{t\in I_2}\{t+\|\mathcal{Q}(t)-\mathcal{S}_1(t)\|\} \\ \sup_{t\in I_3}\{t+\|\mathcal{Q}(t)-\mathcal{S}_1(t)\|\} \\ 1+\pi-\alpha/2 \end{array}\right\}$$

where

$$I_2 = [1+\alpha, 1+\alpha+AK(\alpha/2,\rho)],\ I_3 = [1+\alpha+AK(\alpha/2,\rho), 3-2\rho+\alpha+AK(\alpha/2,\rho)].$$

Proof. Note that the line passing through O and $C_{-\alpha/2}$, call it ϵ, has the property that each point of it, including $K(\alpha/2,\rho)$ is equidistant from $\mathcal{S}_1, \mathcal{S}_2$. Moreover, in the time window $[1+\alpha, 1+\alpha+AK(\alpha/2,\rho)]$ that only $\mathcal{S}_1, \mathcal{S}_2$ are searching, $\mathcal{Q}$ stays below line ϵ. At time $1+\alpha+AK(\alpha/2,\rho)$, $\mathcal{Q}$ is, by construction, equidistant from $\mathcal{S}_1, \mathcal{S}_2$, a property that is preserved for the remaining of the execution of the algorithm. As a result, the evacuation time of SEARCH$_2(\alpha,\rho)$ is given by $\sup_{1\leq t\leq 1+\pi-\alpha/2}\{t+\|\mathcal{Q}(t)-\mathcal{S}_1(t)\|\}$.

Now note that condition $\pi-\alpha/2 \geq \alpha + AK(\alpha/2,\rho) + 2 - 2\rho$ guarantees that $\mathcal{Q}$ reaches point $C_{-\alpha/2}$ no later than $\mathcal{S}_1$. Moreover, in each time interval I_1, I_2, I_3, $\mathcal{Q}$'s trajectory is differentiable (and so is $\mathcal{S}_1$'s trajectory). □

Now Theorem 3.3 can be proven by fixing parameters α, ρ for SEARCH$_2(\alpha,\rho)$, in particular, $\alpha = 0.6361, \rho = 0.7944$.

Proof. We choose $\alpha = 0.6361, \rho = 0.7944$. The trajectories of Fig. 3 correspond exactly to those values. The time that $\mathcal{Q}$ needs to reach $C_{-\alpha/2}$ equals $1+\alpha+AK(\alpha/2,\rho)+2-2\rho = 3.6174$, while the time that $\mathcal{S}_1, \mathcal{S}_2$ reach the same point is $1+\pi-\alpha/2 = 3.82354$. Hence, Lemma 3.4 applies.

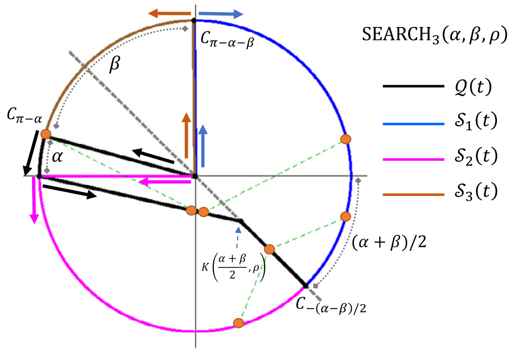

Fig. 4. Algorithm SEARCH3(α,β,ρ) depicted for the optimal parameters of the algorithm.

The worst case evacuation time during phase 1 is $1+\alpha+2\sin(\alpha)=2.82423$. The worst case evacuation time after $\mathcal{Q}$ reaches $C_{-\alpha/2}$, equals $1+\pi-\alpha/2=3.82354$. Hence, it remains to compute the maxima of $t+\|\mathcal{Q}(t)-\mathcal{S}_1(t)\|$ in the two intervals I_2, I_3, which can be done numerically using the trajectories of SEARCH2(α,ρ), since the expression is differentiable in each of the intervals.

To that end, when $t\in I_2=[1.6361, 3.2062]$ we have that

$$\mathcal{Q}(t)=(0.9931t-2.62481, 0.191866-0.11727t)$$

$$\mathcal{S}_1(t)=(\cos(3.50549-t), \sin(3.50549-t)),$$

so that $t+\|\mathcal{Q}(t)-\mathcal{S}_1(t)\|$ becomes

$$t+\sqrt{(-\sin(3.50549-t)-0.11727t+0.191866)^2+(-\cos(3.50549-t)+0.9931t-2.62481)^2}$$

When $t\in I_3=[3.2062, 3.6174]$ we have that

$$\mathcal{Q}(t)=(0.949847t-2.48613, 0.818501-0.312715t)$$

while $\mathcal{S}_1$'s trajectory equation remains unchanged, so that $t+\|\mathcal{Q}(t)-\mathcal{S}_1(t)\|$ becomes

$$t+\sqrt{(-\sin(3.50549-t)-0.312715t+0.818501)^2+(-\cos(3.50549-t)+0.949847t-2.48613)^2}$$

In particular, it follows that

$$\begin{aligned}\sup_{t\in I_2}\{t+\|\mathcal{Q}(t)-\mathcal{S}_1(t)\|\} &\approx \sup_{t\in I_3}\{t+\|\mathcal{Q}(t)-\mathcal{S}_1(t)\|\}\\ &\approx 3.8327\end{aligned}$$

with corresponding maximizers (with approximate values) $\tau_2=3.10066$ and $\tau_3=3.32114$, respectively. Fig. 3 also depicts the locations of the optimizers, i.e., the worst case locations on the circle for the exit to be found, along with the corresponding evacuation trajectory in dashed green colour. □

3.3. *Evacuation algorithm for* PE$_3$

3.3.1. *A simple algorithm*

In this section we prove the following preliminary theorem (to be improved in Section 3.3.2).

Theorem 3.5. PE$_3$ *can be solved in time 3.37882.*

Given parameters α,β,ρ, we introduce the family of trajectories SEARCH3(α,β,ρ), corresponding to robots $\mathcal{Q},\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3$, see also Fig. 4.

Algorithm $\text{Search}_3(\alpha, \beta, \rho)$

Robot	#	*Description*	*Trajectory*	*Duration*
$\mathcal{Q}$	0	Move to point $C_{\pi-\alpha}$	$\mathcal{L}(O, C_{\pi-\alpha}, t)$	1
	1	Search the circle ccw till point C_π	$\mathcal{C}(\pi-\alpha, t-1)$	α
	2	Move to point $K(\frac{\alpha+\beta}{2}, \rho)$	$\mathcal{L}(C_\pi, K(\frac{\alpha+\beta}{2}, \rho), t-(1+\alpha))$	$AK(\frac{\alpha+\beta}{2}, \rho)$
	3	Move to point $C_{-\frac{\alpha+\beta}{2}}$	$\mathcal{L}(K(\frac{\alpha+\beta}{2}, \rho), C_{-\frac{\alpha+\beta}{2}})$	$2-2\rho$
$\mathcal{S}_1$	0	Move to point $C_{\pi-\alpha-\beta}$	$\mathcal{L}(O, C_{\pi-\alpha-\beta})$	1
	1	Search the circle cw till point $C_{-\frac{\alpha+\beta}{2}}$	$\mathcal{C}(\pi-\alpha-\beta, -t+1)$	$\pi-\frac{\alpha+\beta}{2}$
$\mathcal{S}_2$	0	Move to point C_π	$\mathcal{L}(O, C_\pi)$	1
	1	Search the circle ccw till point $C_{-\frac{\alpha+\beta}{2}}$	$\mathcal{C}(\pi, t-1)$	$\pi-\frac{\alpha+\beta}{2}$
$\mathcal{S}_3$	0	Move to point $C_{\pi-\alpha-\beta}$	$\mathcal{L}(O, C_{\pi-\alpha-\beta})$	1
	1	Search the circle ccw till point $C_{-\alpha}$	$\mathcal{C}(\pi-\alpha-\beta, -t+1)$	β

As before, it is immediate that, in $\text{Search}_3(\alpha, \beta, \rho)$, robots' trajectories are continuous and feasible, meaning that the entire circle is eventually searched. In particular, the arc with endpoints $C_\pi, C_{\pi-\alpha}$ is searched by $\mathcal{Q}$, the arc with endpoints $C_{\pi-\alpha-\beta}, C_{-\frac{\alpha+\beta}{2}}$ is searched by $\mathcal{S}_1$, the arc with endpoints $C_{-\pi}, C_{-\frac{\alpha+\beta}{2}}$ is searched by $\mathcal{S}_2$, and the arc with endpoints $C_{\pi-\alpha}, C_{\pi-\alpha-\beta}$ is searched by $\mathcal{S}_3$. Also, the search time is $1+\pi-\frac{\alpha+\beta}{2}$, and

$$\mathbb{I}(\mathcal{Q}) = [1, 1+\alpha],\ \mathbb{I}(\mathcal{S}_1) = \mathbb{I}(\mathcal{S}_2) = [1, 1+\pi-\frac{\alpha+\beta}{2}],\ \mathbb{I}(\mathcal{S}_3) = [1, 1+\beta].$$

An illustration of the above trajectories for certain values of α, β, ρ can be seen in Fig. 4.

Before we prove Theorem 3.5, we need to make some observation, in order to calculate the worst case evacuation time.

Lemma 3.6. *Suppose that $\alpha \le \beta$, $\alpha + AK(\frac{\alpha+\beta}{2}, \rho) \ge \beta$, and $\pi - \frac{\alpha+\beta}{2} \ge \alpha + AK(\frac{\alpha+\beta}{2}, \rho) + 2 - 2\rho$. Then the following functions are continuous and differentiable in each associated time intervals: $\|\mathcal{Q}(x) - \mathcal{S}_3(t)\|$ in $I_1 = \{t \ge 0 : \alpha \le t-1 \le \beta\}$, $\|\mathcal{Q}(x) - \mathcal{S}_1(t)\|$ in $I_2 = \{t \ge 0 : |t-1-\alpha| \le AK(\frac{\alpha+\beta}{2}, \rho)\}$ and in $I_3 = \{t \ge 0 : |t-1-\alpha-AK(\frac{\alpha+\beta}{2}, \rho)| \le 2-2\rho\}$. Moreover, the worst case evacuation time of $\text{Search}_3(\alpha, \beta, \rho)$ can be computed as*

$$\max \left\{ \begin{array}{l} \sup_{t\in I_1} \{t + \|\mathcal{Q}(t) - \mathcal{S}_3(t)\|\} \\ \sup_{t\in I_2} \{t + \|\mathcal{Q}(t) - \mathcal{S}_1(t)\|\} \\ \sup_{t\in I_3} \{t + \|\mathcal{Q}(t) - \mathcal{S}_1(t)\|\} \\ 1+\pi-\frac{\alpha+\beta}{2} \end{array} \right\}.$$

Proof. Conditions $\alpha \le \beta$ and $\alpha + AK(\frac{\alpha+\beta}{2}, \rho) \ge \beta$ mean that $\mathcal{Q}$ stops searching no later than $\mathcal{S}_3$, and that when $\mathcal{S}_3$ stops searching $\mathcal{Q}$ is still in her phase 2, respectively.

The line passing through O and $C_{-(\alpha+\beta)/2}$, call it ϵ, has the property that each point of it, including $K(\frac{\alpha+\beta}{2}, \rho)$ is equidistant from $\mathcal{S}_1, \mathcal{S}_2$. Moreover, while $\mathcal{S}_1, \mathcal{S}_2$ are searching, $\mathcal{Q}$ never goes above line ϵ. At time $1+\alpha+AK(\frac{\alpha+\beta}{2}, \rho)$, $\mathcal{Q}$ is, by construction, equidistant from $\mathcal{S}_1, \mathcal{S}_2$, a property that is preserved for the remaining of the execution of the algorithm. As a result, $\mathcal{S}_2$ can be ignored in the performance analysis, and when it comes to the case that $\mathcal{S}_1$ finds the exit, the evacuation cost is given by the supremum of $t + \|\mathcal{Q}(t) - \mathcal{S}_1(t)\|$ in the time interval I_2 or in the interval I_3. Note that in both intervals, the evacuation cost is continuous and differentiable, by construction.

If the exit is reported by $\mathcal{S}_3$ then the evacuation cost is $t + \|\mathcal{Q}(t) - \mathcal{S}_3(t)\|$ for $t \in [1, 1+\beta]$. However, it is easy to see that the cost is strictly increasing for all $t \in [1, 1+\alpha]$ (in fact it is linear). Since the evacuation cost is also continuous, we may restrict the analysis in interval I_1.

Lastly, observe that $\pi - \frac{\alpha+\beta}{2} \ge \alpha + AK(\frac{\alpha+\beta}{2}, \rho) + 2 - 2\rho$ implies that $\mathcal{S}_1, \mathcal{S}_2$ reach point $C_{-(\alpha+\beta)/2}$ no earlier than $\mathcal{Q}$. Hence $\mathcal{Q}$ waits at $C_{-(\alpha+\beta)/2}$ until the search of the circle is over, which can be easily seen to induce the worse evacuation time after $\mathcal{Q}$ reaches $C_{-(\alpha+\beta)/2}$. □

Next, we prove Theorem 3.5 by fixing parameters α, β, ρ for $\text{Search}_3(\alpha, \beta, \rho)$.

Proof. We choose $\alpha = 0.26738$, $\beta = 1.2949$, $\rho = 0.70685$. The trajectories of Fig. 4 correspond exactly to those values. The time that $\mathcal{Q}$ needs to reach $C_{-\frac{\alpha+\beta}{2}}$ equals $1+\alpha+AK(\frac{\alpha+\beta}{2}, \rho)+2-2\rho = 3.17984$, while the time that $\mathcal{S}_1, \mathcal{S}_2$ reach the same point is $1+\pi-\frac{\alpha+\beta}{2} = 3.36045$. Hence, Lemma 3.6 applies.

From the above, it is immediate that the worst evacuation time after $\mathcal{Q}$ reaches $C_{-(\alpha+\beta)/2}$ equals $1+\pi-\frac{\alpha+\beta}{2} = 3.36045$. Hence, it remains to compute the maxima of $t + \|\mathcal{Q}(t) - \mathcal{S}_3(t)\|$ in interval I_1, and of $t + \|\mathcal{Q}(t) - \mathcal{S}_1(t)\|$ in intervals I_2, I_3.

To that end, when $t \in I_1 = [1.26738, 2.2949]$ we have that

$$\mathcal{Q}(t) = (-2.23643 + 0.97558t, 0.278372 - 0.219643t)$$
$$\mathcal{S}_3(t) = (\cos(t + 0.579313), \sin(t + 0.579313)),$$

so that $t + \|\mathcal{Q}(t) - \mathcal{S}_3(t)\|$ becomes

$$t + \sqrt{(-0.219643t - \sin(t + 0.579313) + 0.278372)^2 + (0.97558t - \cos(t + 0.579313) - 2.23643)^2}$$

in which case

$$\sup_{t \in I_1} \{t + \|\mathcal{Q}(t) - \mathcal{S}_3(t)\|\} = 1 + \beta + \|\mathcal{Q}(1 + \beta) - \mathcal{S}_3(1 + \beta)\| \approx 3.37882$$

When $t \in I_2 = [1.26738, 2.59354]$, $\mathcal{Q}$'s trajectory is the same as in I_1 and

$$\mathcal{S}_1(t) = (\cos(2.57931 - t), \sin(2.57931 - t)),$$

so that $t + \|\mathcal{Q}(t) - \mathcal{S}_1(t)\|$ becomes

$$t + \sqrt{(-\sin(2.57931 - t) - 0.219643t + 0.278372)^2 + (-\cos(2.57931 - t) + 0.97558t - 2.23643)^2}.$$

When $t \in I_3 = [2.59354, 3.17984]$, $\mathcal{S}_1$'s trajectory is the same as in I_2 and

$$\mathcal{Q}(t) = (-1.54793 + 0.710111t, 1.5348 - 0.704089t),$$

so that $t + \|\mathcal{Q}(t) - \mathcal{S}_1(t)\|$ becomes

$$t + \sqrt{(\sin(2.57931 - t) + 0.704089t - 1.5348)^2 + (\cos(2.57931 - t) - 0.710111t + 1.54793)^2}.$$

Numerically

$$\sup_{t \in I_2} \{t + \|\mathcal{Q}(t) - \mathcal{S}_1(t)\|\} = \tau_2 + \|\mathcal{Q}(\tau_2) - \mathcal{S}_1(\tau_2)\| \approx 3.37882$$
$$\sup_{t \in I_3} \{t + \|\mathcal{Q}(t) - \mathcal{S}_1(t)\|\} = \tau_3 + \|\mathcal{Q}(\tau_3) - \mathcal{S}_1(\tau_3)\| \approx 3.37882$$

where $\tau_2 \approx 2.34029$ and $\tau_3 \approx 2.84758$. □

3.3.2. Improved search algorithm

In this section we improve the upper bound of Theorem 3.5 by 0.00495 additive term.

Theorem 3.7. PE$_3$ *can be solved in time 3.37387.*

The main idea can be described, at a high level, as a cost preservation technique. By the analysis of Algorithm Search3(α, β, ρ) for the value of parameters of α, β, ρ as in the proof of Theorem 3.5, we know that there is a critical time window $[\tau_2, \tau_3]$ so that the total evacuation time is the same if the exit is found by $\mathcal{S}_1$ either at time τ_2 or τ_3, and strictly less for time moments strictly in-between. In fact, during time $[\tau_2, 1 + \alpha + AK(\frac{\alpha+\beta}{2}, \rho)]$ $\mathcal{Q}$ is executing phase 2, and in the time window $[1 + \alpha + AK(\frac{\alpha+\beta}{2}, \rho), \tau_3]$ $\mathcal{Q}$ is executing phase 3 of Search3(α, β, ρ).

From the above, it is immediate that we can lower $\mathcal{Q}$'s speed in the time window $[\tau_2, \tau_3]$ so that the evacuation time remains *unchanged* no matter when $\mathcal{S}_1$ finds the exit in the same time interval (notably, $\mathcal{S}_3$ has finished searching prior to τ_2 and $\|\mathcal{Q}(t) - \mathcal{S}_1\| \geq \|\mathcal{Q}(t) - \mathcal{S}_2\|$). But this also implies that we must be able to maintain the evacuation time even if we preserve speed 1 for $\mathcal{Q}$, that will in turn allow us to twist parameters α, β, ρ, hopefully improving the worst case evacuation time. We show this improvement is possible by using the following technical observation:

Theorem 3.8. *Consider point $Q = (q_1, q_2) \in \mathbb{R}^2$. Let $\mathcal{S}(t)$ be the trajectory of an object $\mathcal{S}$ moving at speed 1, where $t \geq 0$, and denote by ϕ the $(\mathcal{S}, Q)$-critical angle at time $t = 0$. Assuming that $\cos(\phi) \geq 0$, then there is some $\tau > 0$, and a trajectory $\mathcal{Q}(t) = (f(t), g(t))$ of a speed-1 object, where $t \geq 0$, so that $t + \|\mathcal{Q}(t) - \mathcal{S}(t)\|$ remains constant, for all $t \in [0, \tau]$. Moreover, $\mathcal{Q}(t)$ can be determined by solving the system of differential equations*

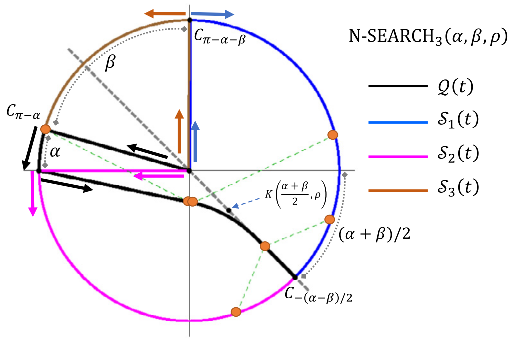

Fig. 5. Algorithm SEARCH$_3(\alpha,\beta,\rho)$ depicted for the optimal parameters of the algorithm.

$$\left(f'(t)\right)^2 + \left(g'(t)\right)^2 = 1 \tag{2}$$

$$t + \|\mathcal{Q}(t) - \mathcal{S}(t)\| = \|\mathcal{S}(0) - Q\| \tag{3}$$

$$(f(0), g(0)) = (q_1, q_2). \tag{4}$$

Proof. An object with trajectory $(f(t), g(t))$ satisfying (2) and (4) has speed 1 (by Lemma 2.2), and starts from point $Q = (q_1, q_2)$. We need to examine whether we can choose f, g so as to satisfy (3).

By Lemma 2.7, such a trajectory $\mathcal{Q}(t)$ exists exactly when we can guarantee that $\cos(\phi) + \cos(\theta) = 1$ over time t. When $t = 0$ we are given that $\cos(\phi) > 0$, hence there exists θ satisfying $\cos(\phi) + \cos(\theta) = 1$. This uniquely determines the velocity of $\mathcal{Q}$ at $t = 0$.

By continuity of the velocities, there must exist a $\tau > 0$ such that $\cos(\phi) + \cos(\theta) = 1$ admits a solution for θ also as ϕ changes over time $t \in [0, \tau]$, in which time window the cosine of the $(\mathcal{S}, \mathcal{Q}(t))$-critical angle at time t remains non-negative. $\square$

Note that condition $\cos(\phi) \geq 0$ of Theorem 3.8 translates to $\|\mathcal{S}(t) - Q\|$ is not increasing at $t = \tau$, i.e., that $\mathcal{S}$ does not move away from point Q.

Now fix parameters α, β, ρ together with the trajectories of $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$ as in the description of Algorithm SEARCH$_3(\alpha,\beta,\rho)$. The description of our *new algorithm* N-SEARCH$_3(\alpha,\beta,\rho)$ will be complete once we fix a new trajectory for $\mathcal{Q}$. Naming specific values for parameters α, β, ρ will eventually prove Theorem 3.7. In order to do so, we introduce some *further notation and conditions*, denoted below by *(Conditions i-iv)*, that we later make sure are satisfied.

Consider $\mathcal{Q}$'s trajectory as in SEARCH$_3(\alpha,\beta,\rho)$. Let τ_0 denote a local maximum of

$$t + \|\mathcal{Q}(t) - \mathcal{S}_1(t)\|$$

as it reads for $t \geq 0$ with $|t - 1 - \alpha| \leq AK(\frac{\alpha+\beta}{2}, \rho)$ (recall that in this time window, the expression is differentiable by Lemma 3.6), i.e.,

$$|\tau_0 - 1 - \alpha| \leq AK(\frac{\alpha+\beta}{2}, \rho). \tag{Condition i}$$

Set $Q = \mathcal{Q}(\tau_0)$, and assume that

"The cosine of the $(\mathcal{S}, Q)$-critical angle at time τ_0 is non-negative." (Condition ii)

Then obtain from Theorem 3.8 trajectory $(f(t), g(t))$ that has the property that it preserves $\tau_0 + \|\mathcal{Q}(\tau_0) - \mathcal{S}_1(\tau_0)\|$ in the time window $[\tau_0, \tau']$. Assume also that

"There is time $\tau_1 \leq \tau'$ such that point $K_1 := (f(\tau_1), g(\tau_1))$ is equidistant from $\mathcal{S}_1(\tau_1), \mathcal{S}_2(\tau_1)$," (Condition iii)

for the first time after time τ_0, such that

$$\tau_1 \leq 1 + \pi - \frac{\alpha+\beta}{2}. \tag{Condition iv}$$

Then consider the following modification of SEARCH$_3(\alpha,\beta,\rho)$, where the trajectories of $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ remain unchanged, see also Fig. 5.

Algorithm N-Search$_3(\alpha, \beta, \rho)$

Robot	#	Description	Trajectory	Duration
$\mathcal{Q}$	0	Move to point $C_{\pi-\alpha}$	$\mathcal{L}(O, C_{\pi-\alpha}, t)$	1
	1	Search the circle ccw till point C_π	$\mathcal{C}(\pi-\alpha, t-1)$	α
	2	Move toward point $K(\frac{\alpha+\beta}{2}, \rho)$	$\mathcal{L}(C_\pi, K(\frac{\alpha+\beta}{2}, \rho), t-(1+\alpha))$	$\tau_0 - 1 - \alpha$
	3	Preserve $\tau_0 + \|\mathcal{Q}(\tau_0) - \mathcal{S}_1(\tau_0)\|$	$(f(t), g(t))$	$\tau_1 - \tau_0$
	4	Move to point $C_{-\frac{\alpha+\beta}{2}}$	$\mathcal{L}(K_1, C_{-\frac{\alpha+\beta}{2}})$	$\left\|K_1 - C_{-\frac{\alpha+\beta}{2}}\right\|$

Note that in phase 2, $\mathcal{Q}$ is not reaching (necessarily) point K rather it moves toward it for a certain duration. The search time is still $1+\pi-\frac{\alpha+\beta}{2}$. Trajectories of $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ are continuous as before, and

$$\mathbb{I}(\mathcal{S}_1) = \mathbb{I}(\mathcal{S}_2) = [1, 1+\pi-\frac{\alpha+\beta}{2}],\ \mathbb{I}(\mathcal{S}_3) = [1, 1+\beta],$$

as well as $\mathbb{I}(\mathcal{Q}) = [1, 1+\alpha]$.

Condition i makes sure that while $\mathcal{Q}$ is in phase 2, and before it reaches $K(\frac{\alpha+\beta}{2}, \rho)$, there is a time moment τ_0 when the rate of change of $t + \|\mathcal{Q}(t) - \mathcal{S}_1(t)\|$ is 0. Together with condition ii, this implies that Theorem 3.8 applies. In fact, for the corresponding critical angles ϕ, θ between $\mathcal{S}_1, \mathcal{Q}$ at time τ_0, we have that $\cos(\phi) + \cos(\theta) = 1$ by construction. Hence the trajectory $(f(t), g(t))$ of phase 3 is well defined, and indeed, $\mathcal{Q}$ jumps from phase 2 to phase 3 while $\mathcal{Q}$ is still moving toward point K. Notably, $\mathcal{Q}$'s trajectory is even differentiable at $t = \tau_0$ (but not necessarily at $t = \tau_1$). Then, Condition iii says that $\mathcal{Q}$ eventually will enter phase 4, and that this will happen before $\mathcal{S}_1, \mathcal{S}_2$ finish the exploration of the circle. Overall, we conclude that in N-Search$_3(\alpha, \rho)$, robots' trajectories are continuous and feasible. An illustration of the above trajectories for certain values of α, β, ρ can be seen in Fig. 5.

Now we make some observations, in order to calculate the worst case evacuation time.

Lemma 3.9. *Suppose that $\alpha \le \beta$, $1+\beta \le \tau_0$, and $1+\pi-\frac{\alpha+\beta}{2} \ge \tau_1 + \left\|K_1 - C_{-\frac{\alpha+\beta}{2}}\right\|$ as well as Conditions i-iv are satisfied. Then the following functions are continuous and differentiable in each associated time intervals: $\|\mathcal{Q}(x) - \mathcal{S}_3(t)\|$ in $I_1 = \{t \ge 0 :\ \alpha \le t-1 \le \beta\}$, $\|\mathcal{Q}(x) - \mathcal{S}_1(t)\|$ in $I_2 = \{t \ge 0 :\ 1+\alpha \le t \le \tau_0\}$ and in $I_3 = \left\{t \ge 0 :\ |t-\tau_1| \le \left\|K_1 - C_{-\frac{\alpha+\beta}{2}}\right\|\right\}$. Moreover, the worst case evacuation time of* N-Search$_3(\alpha, \beta, \rho)$ *can be computed as*

$$\max \left\{ \begin{array}{l} \sup_{t \in I_1} \{t + \|\mathcal{Q}(t) - \mathcal{S}_3(t)\|\} \\ \sup_{t \in I_2} \{t + \|\mathcal{Q}(t) - \mathcal{S}_1(t)\|\} \\ \sup_{t \in I_3} \{t + \|\mathcal{Q}(t) - \mathcal{S}_1(t)\|\} \\ 1+\pi-\frac{\alpha+\beta}{2} \end{array} \right\}.$$

Proof. Conditions $\alpha \le \beta$ and $1+\beta \le \tau_0$ mean that $\mathcal{Q}$ stops searching no later than $\mathcal{S}_3$, and that when $\mathcal{Q}$ enters phase 3 after $\mathcal{S}_3$ is done searching, respectively.

The line passing through O and $C_{-(\alpha+\beta)/2}$, call it ϵ, has the property that each point of it, including $K(\frac{\alpha+\beta}{2}, \rho)$ is equidistant from $\mathcal{S}_1, \mathcal{S}_2$. Moreover, while $\mathcal{S}_1, \mathcal{S}_2$ are searching, $\mathcal{Q}$ never goes above line ϵ. Also, while $\mathcal{Q}$ is executing phase 3, $\mathcal{Q}$ remains equidistant from $\mathcal{S}_1, \mathcal{S}_2$ and this is preserved for the remainder of the execution of the algorithm. As a result, $\mathcal{S}_2$ can be ignored in the performance analysis, and when it comes to the case that $\mathcal{S}_1$ finds the exit, the evacuation cost is given by the supremum of $t + \|\mathcal{Q}(t) - \mathcal{S}_1(t)\|$ in the time interval I_2 or in the interval I_3. Note that in both intervals, the evacuation cost is continuous and differentiable, by construction.

If the exit is reported by $\mathcal{S}_3$ then the evacuation cost is $t + \|\mathcal{Q}(t) - \mathcal{S}_3(t)\|$ for $t \in [1, 1+\beta]$. However, it is easy to see that the cost is strictly increasing for all $t \in [1, 1+\alpha]$ (in fact it is linear). Since the evacuation cost is also continuous, we may restrict the analysis in interval I_1.

Lastly, observe that $1+\pi-\frac{\alpha+\beta}{2} \ge \tau_1 + \left\|K_1 - C_{-\frac{\alpha+\beta}{2}}\right\|$ implies that $\mathcal{S}_1, \mathcal{S}_2$ reach point $C_{-(\alpha+\beta)/2}$ no earlier than $\mathcal{Q}$. Hence $\mathcal{Q}$ waits at $C_{-(\alpha+\beta)/2}$ till the search of the circle is over, which can be easily seen to induce the worse evacuation time after $\mathcal{Q}$ reaches $C_{-(\alpha+\beta)/2}$. □

Next we prove Theorem 3.7 by fixing parameters α, β, ρ for N-Search$_3(\alpha, \beta, \rho)$.

Proof. We choose $\alpha = 0.27764$, $\beta = 1.29839$, $\rho = 0.68648$. The trajectories of Fig. 4 correspond exactly to those values. For these values we see that $AK(\frac{\alpha+\beta}{2}, \rho) = 1.29041$, while $\tau_0 - \alpha - 1 = 1.04877$. Hence the transition between phase 1 and phase 2 of $\mathcal{Q}$ is well defined.

The time that $\mathcal{Q}$ needs to reach $C_{-\frac{\alpha+\beta}{2}}$ equals $1 + \tau_1 + \left\|K_1 - C_{-\frac{\alpha+\beta}{2}}\right\| = 3.18073$, while the time that $\mathcal{S}_1, \mathcal{S}_2$ reach the same point is $1+\pi-\frac{\alpha+\beta}{2} = 3.35358$. Therefore we may attempt to solve numerically the differential equation of

Theorem 3.8. It turns out that for the resulting trajectory $(f(t), g(t)$, and for $\tau_1 = 2.89288$, point $(f(\tau_1), g(\tau_1)$ is equidistant from $\mathcal{S}_1, \mathcal{S}_2$. Moreover, $\mathcal{Q}$ enters phase 4 at time $\tau_1 = 2.89288$, prior to $1 + \pi - \frac{\alpha+\beta}{2}$. Hence, Conditions i-iv are all met, as well as Lemma 3.9 applies.

From the above, it is immediate that the worst evacuation time after $\mathcal{Q}$ reaches $C_{-(\alpha+\beta)/2}$ equals $1 + \pi - \frac{\alpha+\beta}{2} = 3.35358$. Hence, it remains to compute the maxima of $t + \|\mathcal{Q}(t) - \mathcal{S}_3(t)\|$ in interval I_1, and of $t + \|\mathcal{Q}(t) - \mathcal{S}_1(t)\|$ in intervals I_2, I_3.

To that end, when $t \in I_1 = [1.27764, 2.29839]$ we have that

$$\mathcal{Q}(t) = (0.978782t - 2.25053, 0.261795 - 0.204905t)$$
$$\mathcal{S}_3(t) = (\cos(t + 0.565563), \sin(t + 0.565563)),$$

so that $t + \|\mathcal{Q}(t) - \mathcal{S}_3(t)\|$ becomes

$$t + \sqrt{(-0.204905t - \sin(t + 0.565563) + 0.261795)^2 + (0.978782t - \cos(t + 0.565563) - 2.25053)^2}$$

in which case

$$\sup_{t\in I_1} \{t + \|\mathcal{Q}(t) - \mathcal{S}_3(t)\|\} = 1 + \beta + \|\mathcal{Q}(1 + \beta) - \mathcal{S}_3(1 + \beta)\| \approx 3.37387$$

When $t \in I_2 = [1.27764, 2.32641]$, $\mathcal{Q}$'s trajectory is the same as in I_1 and

$$\mathcal{S}_1(t) = (\cos(2.56556 - t), \sin(2.56556 - t)),$$

so that $t + \|\mathcal{Q}(t) - \mathcal{S}_1(t)\|$ becomes

$$t + \sqrt{(-\sin(2.56556 - t) - 0.204905t + 0.261795)^2 + (-\cos(2.56556 - t) + 0.978782t - 2.25053)^2}.$$

When $t \in I_3 = [2.89288, 3.18073]$, $\mathcal{S}_1$'s trajectory is the same as in I_2 and

$$\mathcal{Q}(t) = (0.705254t - 1.53797, 1.54604 - 0.708955t0.706399t - 1.53762, 1.5407 - 0.707814t),$$

so that $t + \|\mathcal{Q}(t) - \mathcal{S}_1(t)\|$ becomes

$$t + \sqrt{(-\sin(2.56556 - t) - 0.708955t + 1.54604)^2 + (-\cos(2.56556 - t) + 0.705254t - 1.53797)^2}.$$

Numerically,

$$\begin{aligned}\sup_{t\in I_2} \{t + \|\mathcal{Q}(t) - \mathcal{S}_1(t)\|\} &= \tau_0 + \|\mathcal{Q}(\tau_0) - \mathcal{S}_1(\tau_0)\| = \tau_1 + \|\mathcal{Q}(\tau_1) - \mathcal{S}_1(\tau_1)\| \\ &= \sup_{t\in I_3} \{t + \|\mathcal{Q}(t) - \mathcal{S}_1(t)\|\} \approx 3.37387.\end{aligned}$$

The reader may also consult Fig. 5. □

4. Lower bounds

In this section we derive lower bounds for evacuation. In Section 4.1 we treat the case of $n = 1$ (see Theorem 4.1) and in Section 4.2 we treat the case of $n = 2$ and 3 (see Theorem 4.3).

4.1. Lower bound for PE$_1$

We will derive the lower bound using an adversarial argument placing the exit at an unknown vertex of a regular hexagon.

Theorem 4.1. *The worst-case evacuation time for* PE$_1$ *is at least* $3 + \pi/6 + \sqrt{3}/2 \approx 4.3896$.

Proof. At time $1 + \pi/6$, at most $\pi/3$ of the perimeter of the circle can have been explored by the queen and servant. Thus, there is a regular hexagon, none of whose vertices have been explored. If the exit is at one of these vertices, by Theorem 4.2, it takes $2 + \sqrt{3}/2$ for the queen to evacuate. The total time is $1 + \pi/6 + 2 + \sqrt{3}/2$. □

Next we proceed to provide a lower bound on a unit-side hexagon. Label the vertices of the hexagon V as $A, \ldots, F$ as shown in Fig. 6. Fix an evacuation algorithm $\mathcal{A}$. For any vertex v of the hexagon, we call $f(v)$ the time of *first visit* of the vertex v by either the servant or the queen, according to algorithm $\mathcal{A}$. We call $q(v)$ the time that the queen gets to the vertex v. Clearly, $q(v) \geq f(v)$, and if the queen arrives at the vertex no later than the servant, $q(v) = f(v)$.

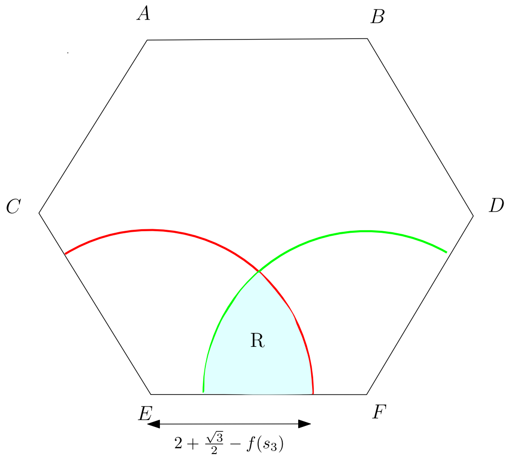

Fig. 6. The queen must be in region R at time $f(s_3)$. Here $s_3 = E$ and $q_3 = F$.

Theorem 4.2. *For any algorithm $\mathcal{A}$, the evacuation time for the queen when the exit is at one of the vertices of the hexagon is* $max_{v\in V}\{q(v)\} \geq 2+\sqrt{3}/2$.

Proof. Suppose there is an algorithm in which the queen can always evacuate in time $< 2+\sqrt{3}/2$. Consider the trajectories of the servant and the queen. If either the queen or the servant are the first to visit 4 vertices, then for the fourth such vertex v, we have $f(v) \geq 3$, a contradiction. Therefore, the queen is the first to visit three vertices, and the servant is the first to visit three vertices. We denote the three vertices visited first by the servant as s_1, s_2, s_3 (in the order they are visited) and the three vertices visited first by the queen as q_1, q_2, q_3, and note that they are all distinct.

Notice that neither s_3 nor q_3 can be visited before time 2, that is, $f(s_3), f(q_3) \geq 2$. If $f(q_3) \leq f(s_3)$, then we place the exit at s_3, and the queen needs time at least 1 to get to s_3, which implies that $T \geq q(s_3) \geq f(q_3)+1 \geq 3$, a contradiction. We conclude that at time $f(s_3)$, the queen is yet to visit q_3. Since the exit can be at either s_3 or q_3, at time $f(s_3)$, the queen must be at distance $< 2+\sqrt{3}/2 - f(s_3) \leq \sqrt{3}/2$ from *both* s_3 and q_3.

Assume without loss of generality that $s_3 = E$ (see Fig. 6). Since A, B, D are all at distance at least $\sqrt{3}$ from E, we conclude that q_3 is either C or F. Assume without loss of generality that $q_3 = F$. Let R denote the lens-shaped region that is at distance $< 2+\sqrt{3}/2 - f(s_3)$ from both E and F. Recall that at time $f(s_3)$, the queen must be inside the region R. Notice that if $f(s_3) \geq 1.5+\sqrt{3}/2$, the region R is empty, yielding a contradiction. So it must be that $2 \leq f(s_3) < 1.5+\sqrt{3}/2$.

We now work backwards to deduce the trajectories of the servant and the queen. Clearly $s_2 \neq F$ since $q_3 = F$. If $s_2 \neq C$, then $f(s_3) \geq \sqrt{3}+1 > 1.5+\sqrt{3}/2$, a contradiction. Therefore, $s_2 = C$. By the same reasoning, $s_1 = A$. Therefore, the queen is the first to visit D and B. If $q_1 = D$ and $q_2 = B$, we place the exit at E; since $f(q_2) \geq 1$ and $dist(B, E) = 2$, we have $T \geq q(E) \geq 3$, a contradiction. Thus, $q_2 = D$ and $q_1 = B$.

Consider the location of the queen at time 1. If she is at distance $\geq 1+\sqrt{3}/2$ from C at time 1, then if the exit is at C, $q(C) \geq 2+\sqrt{3}/2$. So at time 1, the queen must be at distance $< 1+\sqrt{3}/2$ from C and consequently she is at distance $\geq 1-\sqrt{3}/2$ from vertex D. Therefore $f(q_2) = f(D) \geq 2-\sqrt{3}/2$. Also, $f(D) < 1.5$ since if the queen reaches D at or after time 1.5, she cannot reach the region R before time $1.5+\sqrt{3}/2 > f(s_3)$. So $f(D) \leq f(s_3)$. If the exit is at $E = s_3$, the queen cannot reach the exit before time $f(D) + dist(D, E) \geq 2-\sqrt{3}/2+\sqrt{3} = 2+\sqrt{3}$, concluding the proof by contradiction. □

We remark that the above bound is optimal, and is achieved by the algorithm depicted in Fig. 7.

4.2. Lower bounds for PE_2 *and* PE_3 *- proof outline*

In the case of $n = 2$ and $n = 3$ the proof is rather technical. Next we present a high level outline as to why the lower bounds hold.

Theorem 4.3. *The worst-case evacuation time for* PE_2 *is at least* 3.6307 *and for* PE_3 *at least* 3.2017.

Throughout this section we will use $\mathcal{T}$ to refer to the evacuation time of an arbitrary algorithm and use $\mathcal{U}$ to refer to the unit circle which must be evacuated.

The main thrust of the proof relies on a simple idea – the queen should aid in the exploration of $\mathcal{U}$. This is immediately evident for the particular case of $n = 2$ since, if the queen does not explore, it will take time at least $1+\pi$ for the servants to search all of $\mathcal{U}$ and we already have an upper bound smaller than this (Theorem 3.3). Thus, a general overview of the proof is as follows: we show that in order to evacuate in time $\mathcal{T}$ the queen must explore some minimum length of the

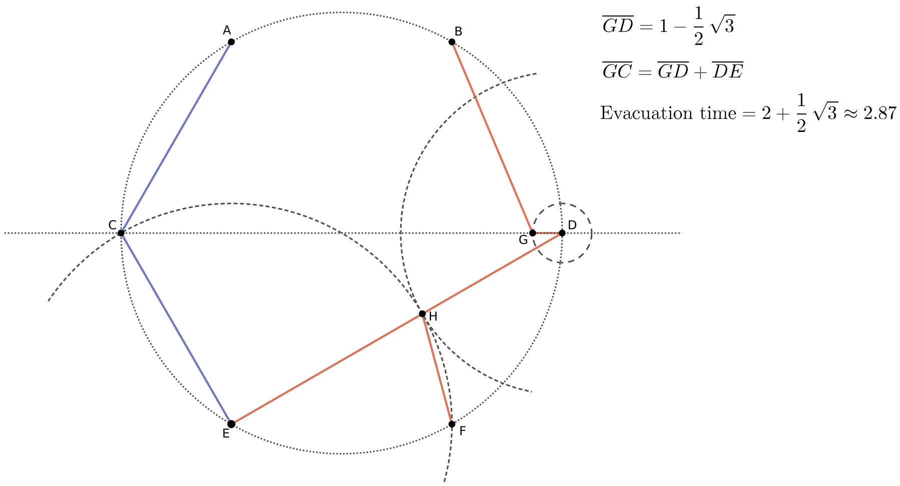

Fig. 7. Blue trajectory: servant and red trajectory: queen. At point H, if the queen hears of an exit at E, she goes there, otherwise she goes to F.

perimeter of $\mathcal{U}$. We will then demonstrate that the queen is not able to explore this minimum amount in any algorithm with evacuation time smaller than what is given in Theorem 4.3.

To be concrete, consider the case of $n = 2$ and assume that we have an algorithm with evacuation time $\mathcal{T} < 1 + \pi$. Then, in order for the robots to have explored all of $\mathcal{U}$ in time $\mathcal{T}$, the queen must explore a subset of the perimeter of total length at least $2(1 + \pi - \mathcal{T})$. Intuitively, this minimum length of perimeter will increase in size as $\mathcal{T}$ decreases.

Now consider that it is not possible for the queen to always remain on the perimeter (indeed, in each of the algorithms presented, the queen leaves the perimeter). To see why this is consider that, in any algorithm with evacuation time $\mathcal{T}$, it must be the case that all unexplored points of $\mathcal{U}$ are located a distance no more than $\mathcal{T} - t$ from the queen at all times $t \le \mathcal{T}$. If the queen is on the perimeter at any time t satisfying $\mathcal{T} - t \le 2$, then, there will be some arc $\theta(t, \mathcal{T}) \subset \mathcal{U}$ (see Lemma 4.4) such that all points of $\theta(t, \mathcal{T})$ are at a distance at least $\mathcal{T} - t$ from the queen. Thus, if the queen is to be on the perimeter at the time t we can conclude that all of the arc $\theta(t, \mathcal{T})$ must have already been discovered. However, we will find (see Lemma 4.5) that $\theta(t, \mathcal{T})$ will often grow at a rate much larger than the robots can collectively explore and at some point the queen will have to leave the perimeter. In fact, there will be an interval of time during which it is not possible for the queen to be exploring and this in turn implies that there is a maximum amount of perimeter that can be explored by the queen. Intuitively, the maximum length of perimeter that can be explored by the queen will decrease as $\mathcal{T}$ decreases. The lower bound will result by balancing the minimum amount of perimeter the queen needs to search and the maximum amount of perimeter that the queen is able to search.

The above argument will need a slight modification in the case of $n = 3$. In this case we will show that there is some critical time t_* before which the queen must have explored some minimum amount of perimeter. Again, the lower bound follows by balancing the maximum amount of perimeter the queen can explore by the time t_* and the minimum amount of perimeter the queen needs to explore before the time t_*.

4.3. Lower bounds for PE_2 and PE_3 - proof details

In this section we present the complete details of the proofs for the lower bounds in the cases $n = 2$ and $n = 3$. Throughout this section we will use $\mathcal{T}$ to refer to the evacuation time of an arbitrary algorithm and use $\mathcal{U}$ to refer to the unit circle which must be evacuated.

The idea of the proofs is to bound the amount of perimeter the queen can search for a given evacuation time $\mathcal{T}$ and then show that the queen must search a minimum amount of the perimeter in order to achieve the evacuation time $\mathcal{T}$. The lower bounds result by balancing the minimum amount of perimeter the queen must search with the maximum amount of perimeter the queen can search.

We begin with two lemmas which will be used for both the $n = 2$ and $n = 3$ bounds. Their necessity will become apparent shortly.

Lemma 4.4. *Consider any $r < 2$ and a point $P \in \mathcal{U}$. Define the circle $\mathcal{D}_P$ as the disk centred on P with radius r. Then the subset of the perimeter of $\mathcal{U}$ which is not contained in $\mathcal{D}_P$ has length $\theta = 4\cos^{-1}\left(\frac{r}{2}\right)$.*

Proof. Without loss of generality assume that the point P is located at $(-1,\ 0)$. Since $r < 2$ the disks $\mathcal{U}$ and $\mathcal{D}_P$ will intersect at two boundary points A and B between which the distance along the perimeter of $\mathcal{U}$ is θ. This situation is depicted in Fig. 8. Referring to this figure, one can easily observe that $r = 2\sin\left(\frac{\pi}{2} - \frac{\theta}{4}\right) = 2\cos\left(\frac{\theta}{4}\right)$. Rearranging for θ we find that $\theta = 4\cos^{-1}\left(\frac{r}{2}\right)$. $\square$

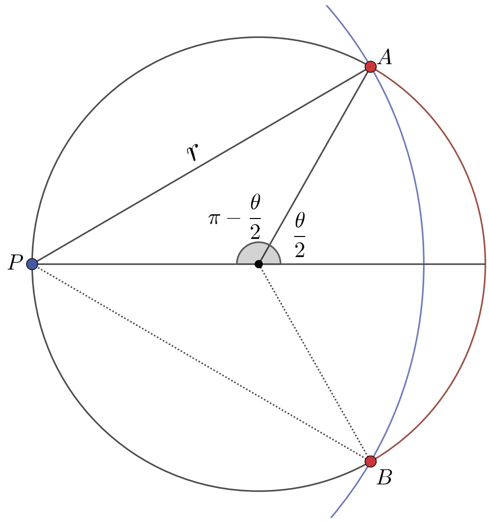

Fig. 8. Setup for the proof of Lemma 4.4. The boundary of the disk $\mathcal{D}_P$ is indicated in blue. The arc of $\mathcal{U}$ which is excluded from $\mathcal{D}_P$ is highlighted in red and has length θ.

Lemma 4.5. *Consider the function $\theta(t, \mathcal{T}) = 4\cos^{-1}\left(\frac{\mathcal{T}-t}{2}\right)$ with $\mathcal{T} > 0$. Then $\frac{d\theta}{dt} > 2$ for all t satisfying $\mathcal{T} - 2 < t < \mathcal{T}$ and $\frac{d\theta}{dt} > 3$ for t satisfying $\mathcal{T} - 2 < t < \mathcal{T} - \frac{2}{3}\sqrt{5}$. Furthermore, $\frac{d\theta}{d\mathcal{T}} < -2$ for all $\mathcal{T} - 2 < t < \mathcal{T}$.*

Proof. The rate of change of $\theta(t, \mathcal{T})$ with t is given by

$$\frac{d\theta}{dt} = \frac{4}{\sqrt{4 - (\mathcal{T} - t)^2}}.$$

From this relation it is simple to confirm that $\frac{d\theta}{dt} > 2$ for $\mathcal{T} - 2 < t < \mathcal{T}$ and that $\frac{d\theta(r)}{dr} > 3$ for $\mathcal{T} - 2 < t < \mathcal{T} - \frac{2}{3}\sqrt{5}$. It should also be obvious by the symmetry of $\mathcal{T}$ and t in the function $\theta(t, \mathcal{T})$ that $\frac{d\theta}{d\mathcal{T}} < -2$ for all $\mathcal{T} - 2 < t < \mathcal{T}$. $\square$

4.3.1. Lower bound for $n = 2$

We begin with the main result of the section.

Theorem 4.6. *For $n = 2$ and any algorithm the queen cannot be evacuated in time less than $\mathcal{T}_2$ which is the solution to the equations*

$$\tau = \mathcal{T}_2 - 2\cos\left(\frac{\tau - 1}{2}\right)$$

$$t_* = \frac{1}{2}(\mathcal{T}_2 + 1)$$

$$\mathcal{T}_2 = t_* + 2\cos\left(\frac{2t_* + \tau}{4} - \frac{3}{4}\right).$$

Solving these equations numerically gives $\tau \approx 1.7815$, $t_* \approx 2.3154$, and $\mathcal{T}_2 \approx 3.6307$.

We will see that the queen cannot be located on the perimeter of the circle during the interval of time $(\tau,\ t_*)$ and thus $\tau - 1$ represents the maximum amount of perimeter that can be explored by the queen before the time t_*. The time t_* is chosen such that for all $\mathcal{T} < \mathcal{T}_2$ a solution to the equations in Theorem 4.6 does not exist, and, as such, $\tau - 1$ will represent the maximum length of the perimeter that can be explored by the queen. In the following lemma we show that the queen must explore a length of the perimeter greater than $\tau - 1$ in order to evacuate in time less than $\mathcal{T}_2$.

Lemma 4.7. *For $n = 2$ and any evacuation algorithm with $\mathcal{T} < 1 + \pi$, the queen must explore a subset of the perimeter of length $y \geq 2(1 + \pi - \mathcal{T})$. In particular, if $\mathcal{T} < \mathcal{T}_2$, we need $y > 2(1 + \pi - \mathcal{T}_2) \approx 1.0217$.*

Proof. If the queen explores a subset of the perimeter of length y then the robots will take time $1 + \frac{2\pi - y}{2}$ to explore the circle. The robots need to at least explore the entire circle in time $\mathcal{T}$ and therefore $1 + \frac{2\pi - y}{2} \leq \mathcal{T}$, or, equivalently, $y \geq 2(1 + \pi - \mathcal{T}_2)$. For $\mathcal{T} < \mathcal{T}_2 \approx 3.6307$ we need $y > 1.0217$. $\square$

We will now show that the maximum length of perimeter the queen can explore is less than $\tau - 1$ if $\mathcal{T} < \mathcal{T}_2$. This will be the goal of the next two lemmas.

Lemma 4.8. *Consider the equation $\mathcal{T} = t + 2\cos\left(\frac{1}{2}(t-1) + \frac{1}{2}\alpha\right)$ with $\mathcal{T} > 0$, α satisfying $0 < \alpha \le t$ and t satisfying $1 < t \le \mathcal{T}$. Then $\frac{dt}{d\mathcal{T}} \ge \frac{1}{2}$, and, if $0 < t < 1 + 2\pi - \frac{\alpha}{2}$ then $\frac{dt}{d\alpha} > 0$.*

Proof. Implicitly differentiating the equation $\mathcal{T} = t + 2\cos\left(\frac{1}{2}(t-1) + \frac{1}{4}\alpha\right)$ with respect to $\mathcal{T}$ gives us

$$\frac{dt}{d\mathcal{T}} = \frac{1}{1 - \sin\left(\frac{1}{2}(t-1) + \frac{1}{4}\alpha\right)}.$$

Since the sine function ranges from -1 to 1 we can easily see that $\frac{dt}{d\mathcal{T}} \ge \frac{1}{2}$.

Implicitly differentiating the equation $\mathcal{T} = t + 2\cos\left(\frac{1}{2}(t-1) + \frac{1}{4}\alpha\right)$ with respect to α gives us

$$\frac{dt}{d\alpha} = \frac{1}{2} \cdot \frac{\sin\left(\frac{1}{2}(t-1) + \frac{1}{4}\alpha\right)}{1 - \sin\left(\frac{1}{2}(t-1) + \frac{1}{4}\alpha\right)}.$$

We can easily see that the denominator of $\frac{dt}{d\alpha}$ will never be negative and thus $\frac{dt}{d\alpha} > 0$ provided that the numerator is positive. This clearly occurs for $\frac{1}{2}(t-1) + \frac{1}{4}\alpha < \pi$ or $t < 1 + 2\pi - \frac{\alpha}{2}$. $\square$

Lemma 4.9. *Define τ as in Theorem 4.6. Then, for $n = 2$ and any evacuation algorithm with $\mathcal{T} < \mathcal{T}_2$, the queen cannot explore a subset of the perimeter with length $y > \tau - 1$.*

Proof. We start with an observation: if the queen is to evacuate in time $\mathcal{T}$, then, at any time $t < \mathcal{T}$, all points of $\mathcal{U}$ that are a distance greater than $\mathcal{T} - t$ from the queen must be explored by a robot. If the queen is located on the perimeter at the time $t > \mathcal{T} - 2$ then by Lemma 4.4 there is an arc of length

$$\theta(t, \mathcal{T}) = 4\cos^{-1}\left(\frac{\mathcal{T} - t}{2}\right)$$

all points of which lie a distance greater than $\mathcal{T} - t$ from the queen (as an abuse of notation we will refer to the arc with length $\theta(t, \mathcal{T})$ as $\theta(t, \mathcal{T})$). Thus, in order for the queen to be on the perimeter at the time t, the arc $\theta(t, \mathcal{T})$ must be explored. As we have 3 robots in total the maximum length of $\theta(t, \mathcal{T})$ that can be explored at any time t is $3(t-1)$. However, we claim that the queen cannot have explored any of $\theta(t, \mathcal{T})$ if the time t satisfies $t < \frac{1}{2}(\mathcal{T} + 1)$. Indeed, observe that the endpoints of $\theta(t, \mathcal{T})$ lie a distance $\mathcal{T} - t$ away from the queen (by definition) and the queen – who took a unit of time to reach the perimeter – could have explored a point on the perimeter at most a distance $t - 1$ from her current position. Thus, if $t - 1 < \mathcal{T} - t$, or, alternatively, $t < \frac{1}{2}(\mathcal{T} + 1)$, the queen cannot have explored any of the arc $\theta(t, \mathcal{T})$. We must therefore have $\theta(t, \mathcal{T}) \le 2(t-1)$ for times t that satisfy $t < \frac{1}{2}(\mathcal{T} + 1)$.

We note that there is a trivial lower bound of $1 + \frac{2\pi}{3} > 3$ and thus we can assume that $\mathcal{T} > 3$. We make the following claim: if $\mathcal{T} < \mathcal{T}_2$ then the smallest time $t_0 > 0$ solving $\theta(t_0, \mathcal{T}) = 2(t_0 - 1)$ satisfies $\left.\frac{d\theta}{dt}\right|_{t=t_0} > 2$ and $t_0 < \frac{1}{2}(\mathcal{T} + 1)$. We note that, if this is the case, the queen will have to leave the perimeter at the time t_0 (since she has not explored any of the arc $\theta(t, \mathcal{T})$ and, immediately after the time t_0, $\theta(t, \mathcal{T})$ will be too large to have been explored by the servants alone).

We first show that $t_0 < \frac{1}{2}(\mathcal{T} + 1)$. To this end we rearrange the equation $\theta(t_0) = 2(t_0 - 1)$ to get

$$t_0 = \mathcal{T} - 2\cos\left(\frac{t_0 - 1}{2}\right)$$

which is the definition of τ in Theorem 4.6 (in the case that $\mathcal{T} = \mathcal{T}_2$). One can easily confirm that in the case of $\mathcal{T} = \mathcal{T}_2$ we have $\left.\frac{d\theta}{dt}\right|_{t=\tau} \approx 5.2511 > 2$ and $\tau < \frac{1}{2}(\mathcal{T} + 1)$. Now observe that $\theta(t, \mathcal{T})$ is a decreasing function of $\mathcal{T}$ and this implies that for $\mathcal{T} < \mathcal{T}_2$ we have $\theta(\tau, \mathcal{T}) > \theta(\tau, \mathcal{T}_2)$. We can therefore conclude that the time t_0 must occur earlier than the time τ. We note that $\tau < 2$ and, since $\mathcal{T} \ge 3$, we have $\tau < \frac{1}{2}(\mathcal{T} + 1)$. Since $t_0 < \tau$ we can conclude that $t_0 < \frac{1}{2}(\mathcal{T} + 1)$.

The second part of the claim follows directly from Lemma 4.5 where we show that $\frac{d\theta}{dt} > 2$ for all t satisfying $\mathcal{T} - 2 < t < \mathcal{T}$.

As the queen must leave the perimeter at the time $t_0 < \tau$, by Lemma 4.7, we can say that the queen must be able to return to the perimeter and explore before the algorithm terminates. Thus, consider the smallest time $t_1 > t_0$ at which the queen may return to the perimeter. In order for the queen to be on the perimeter we will still need the arc $\theta(t, \mathcal{T})$ to be completely explored. However, in this case it may be possible that $t_1 \ge \frac{1}{2}(\mathcal{T} + 1)$ and as such the queen could have explored at most a length $t_0 - 1$ of $\theta(t, \mathcal{T})$ at the time t_1. We can therefore conclude that t_1 will satisfy $\theta(t_1) = 2(t_1 - 1) + y$ with

$y = 0$ if $t_1 < \frac{1}{2}(\mathcal{T}+1)$, and $y \le t_0 - 1$ if $t_1 \ge \frac{1}{2}(\mathcal{T}+1)$. Writing the equation $\theta(t_1) = 2(t_1-1) + y$ in full and rearranging we find that

$$t_1 = \mathcal{T} - 2\cos\left(\frac{1}{2}(t_1-1) + \frac{1}{4}y\right).$$

We will now consider the cases $t_1 < \frac{1}{2}(\mathcal{T}+1)$ and $t_1 \ge \frac{1}{2}(\mathcal{T}+1)$ separately.

Case 1: $t_1 < \frac{1}{2}(\mathcal{T}+1)$
In this case t_1 can be observed to satisfy the same equation as t_0. We claim that this is not possible if $t_1 > t_0$. Indeed, by Lemma 4.5 we have $\frac{d\theta}{dt} > 2$ and the arc $\theta(t, \mathcal{T})$ will always grow at a rate larger than the servants alone can explore. Thus, a solution to the equation $\theta(t_1) = 2(t_1-1)$ with $t_1 > t_0$ does not exist. This implies that the queen can explore a maximum subset of the perimeter of total length $t_0 - 1 < \tau - 1$ if $t_1 < \frac{1}{2}(\mathcal{T}+1)$.

Case 2: $t_1 \ge \frac{1}{2}(\mathcal{T}+1)$
In this case t_1 satisfies

$$t_1 = \mathcal{T} - 2\cos\left(\frac{1}{2}(t_1-1) + \frac{1}{4}y\right).$$

Although it can be confirmed that $\frac{dt_1}{dy} > 0$ (see Lemma 4.8) we will show that, even when t_1 is as large as possible (i.e. $y = t_0 - 1$), we cannot have $t_1 \ge \frac{1}{2}(\mathcal{T}+1)$. Thus we assume that t_1 satisfies

$$t_1 = \mathcal{T} - 2\cos\left(\frac{1}{2}(t_1-1) + \frac{1}{4}(t_0-1)\right).$$

Now write $t_1 = t_1(\mathcal{T})$ as a function of $\mathcal{T}$ and note that, by Lemma 4.8, we have $\frac{dt_1}{d\mathcal{T}} > \frac{1}{2}$. Using this we can say that $t_1(\mathcal{T}_2) - t_1(\mathcal{T}) > \frac{1}{2}(\mathcal{T}_2 - \mathcal{T})$. By definition of $\mathcal{T}_2$ we have $t_1(\mathcal{T}_2) = \frac{1}{2}(\mathcal{T}_2+1)$ and we can therefore write $\frac{1}{2}(\mathcal{T}_2+1) - t_1(\mathcal{T}) > \frac{1}{2}(\mathcal{T}_2 - \mathcal{T})$. Rearranging this inequality gives us $t_1(\mathcal{T}) < \frac{1}{2}(\mathcal{T}+1)$ which contradicts with our assumption that $t_1 \ge \frac{1}{2}(\mathcal{T}+1)$ and we must conclude that $t_1 < \frac{1}{2}(\mathcal{T}+1)$. This concludes the proof. □

At this point the proof of Theorem 4.6 is rather trivial.

Proof. Assume that we have an algorithm with evacuation time $\mathcal{T} < \mathcal{T}_2$. Then, by Lemma 4.7, the queen must explore a subset of the perimeter of length at least $y > 1.0217$. However, by Lemma 4.9, the queen can only explore a subset of the perimeter of length $y < \tau - 1 \approx 0.7815$ if $\mathcal{T} < \mathcal{T}_2$. It is therefore not possible for the queen to evacuate in time less than $\mathcal{T}_2$. □

4.3.2. Lower bound for $n = 3$

The main result of this section is given below:

Theorem 4.10. *For $n = 3$ and any algorithm the queen cannot be evacuated in time less than $\mathcal{T}_3$ which is the solution to the equations*

$$\tau = \mathcal{T}_3 - 2\cos\left(\frac{3}{4}(\tau-1)\right)$$
$$t_* = 1 + \frac{2}{3}\cos^{-1}\left(\frac{-2}{3}\right) - \frac{(\tau-1)}{3}$$
$$\mathcal{T}_3 = t_* + \sin\left(\frac{3(t_*-1) + (\tau-1)}{2}\right).$$

Solving these equations numerically gives $\tau \approx 1.2319$, $t_* \approx 2.4564$, and $\mathcal{T}_3 \approx 3.2017$.

As before, τ represents the beginning of an interval of time during which the queen cannot be located on the perimeter. In this case, however, t_* is not the first time at which it is possible for the queen to return to the perimeter. Instead it represents a particularly critical time of any algorithm with $n = 3$ at which the evacuation time is maximized (although it will happen that t_* occurs before the queen can return to the perimeter). We will show that the queen must explore a subset of the perimeter with total length more than $\tau - 1$ before the time t_* in order to evacuate in time less than $\mathcal{T}_2$.

We begin with a lemma that was first introduced in [11]:

Lemma 4.11. *Consider a perimeter of a disk whose subset of total length $u + \epsilon > 0$ has not been explored for some $\epsilon > 0$ and $\pi \ge u > 0$. Then there exist two unexplored boundary points between which the distance along the perimeter is at least u.*

This next lemma is used to determine the critical time t_*.

Lemma 4.12. *Consider an evacuation algorithm with n servants and assume that at the time t the queen has explored a total subset of the perimeter of length y. Then, for x and y satisfying* $1+\frac{\pi-y}{n} \le t \le 1+\frac{2\pi-y}{n}$, *it takes time at least* $\mathcal{T}=t+\sin\left(\frac{n(t-1)+y}{2}\right)$ *to evacuate the queen.*

Proof. Consider an algorithm with evacuation time $\mathcal{T}$ and with n servants. Then, at the time t, the total length of perimeter that the robots have explored is at most $n(t-1)+y \ge \pi$ (since each robot may search at a maximum speed of one, the queen has explored a subset of length y, and the robots need at least a unit of time to reach the perimeter). Thus, by Lemma 4.11, there exist two unexplored boundary points between which the distance along the perimeter is at least $2\pi - n(t-1) - y - \epsilon$ for any $\epsilon > 0$. The chord connecting these points has length at least $2\sin\left(\pi - \frac{n(t-1)+y}{2} - \frac{\epsilon}{2}\right)$ and an adversary may place the exit at either endpoint of this chord. The queen will therefore take at least $\sin\left(\pi - \frac{n(t-1)+y}{2} - \frac{\epsilon}{2}\right)$ more time to evacuate and the total evacuation time will be at least $t+\sin\left(\pi - \frac{n(t-1)+y}{2} - \frac{\epsilon}{2}\right)$. As this is true for any $\epsilon > 0$ taking the limit $\epsilon \to 0$ we obtain

$$\mathcal{T} \ge t+\sin\left(\pi - \frac{n(t-1)+y}{2}\right) = t+\sin\left(\frac{n(t-1)+y}{2}\right). \qquad \square$$

In the next two lemmas we show that in order to evacuate in time $\mathcal{T} < \mathcal{T}_2$ the queen must explore a length of the perimeter greater than $\tau - 1$ and then demonstrate that this is not possible.

Lemma 4.13. *Define τ and t_* as in Theorem 4.10. Then, for $n = 3$ and any evacuation algorithm with $\mathcal{T} < \mathcal{T}_3$, the queen must explore a subset of $\mathcal{U}$ with total length $y > \tau - 1$ before the time t_*.*

Proof. Consider an algorithm with evacuation time $\mathcal{T} < \mathcal{T}_3$. We make the assumption that the queen has only explored a subset of total length $y < \tau - 1$ at the time t_* and show that this leads to a contradiction.

Observe that t_* satisfies $1+\frac{\pi-y}{3} \le t_* \le 1+\frac{2\pi-y}{3}$ for all y satisfying $0 \le y \le \tau-1$ and thus, by Lemma 4.12, we can write

$$\mathcal{T} \ge t_* + \sin\left(\frac{3(t_*-1)+y}{2}\right).$$

Since $\mathcal{T} < \mathcal{T}_3$ we also have

$$\mathcal{T}_3 > t_* + \sin\left(\frac{3(t_*-1)+y}{2}\right).$$

Since $\mathcal{T}_3 = t_* + \sin\left(\frac{3(t_*-1)+(\tau-1)}{2}\right)$ we further have

$$\sin\left(\frac{3(t_*-1)+(\tau-1)}{2}\right) > \sin\left(\frac{3(t_*-1)+y}{2}\right).$$

Finally, since $t_* \ge 1+\frac{\pi-y}{3}$ we know that $\sin\left(\frac{3(t_*-1)+y}{2}\right)$ is a decreasing function of its argument and thus we get

$$\frac{3(t_*-1)+(\tau-1)}{2} < \frac{3(t_*-1)+y}{2}$$

which implies that $y > \tau - 1$ which contradicts with our assumption that $y < \tau - 1$. $\square$

Lemma 4.14. *Define τ and t_* as in Theorem 4.10. Then, for $n = 3$ and any evacuation algorithm with $\mathcal{T} < \mathcal{T}_3$, the queen cannot explore a subset of the perimeter with length $y > \tau - 1$ before the time t_*.*

Proof. As was the case for $n = 2$, if the queen is to be on the perimeter at the time t then all of the arc $\theta(t, \mathcal{T}) = 4\cos^{-1}\left(\frac{\mathcal{T}-t}{2}\right)$ must be explored. Since we have 4 robots in total, the maximum length of arc that can be explored at any time t is $4(t-1)$. However, we can again say that the queen cannot search any of the arc $\theta(t)$ if $t \le \frac{1}{2}(\mathcal{T}+1)$. We must therefore have $\theta(t, \mathcal{T}) \le 3(t-1)$ for times t that satisfy $t < \frac{1}{2}(\mathcal{T}+1)$.

Assume first that $\mathcal{T} \ge 3$. We make the following claim: if $3 \le \mathcal{T} < \mathcal{T}_3$ then the smallest time $t_0 > 0$ solving $\theta(t_0, \mathcal{T}) = 3(t_0-1)$ satisfies $\left.\frac{d\theta}{dt}\right|_{t=t_0} > 3$ and $t_0 < \frac{1}{2}(\mathcal{T}+1)$. If this is the case the queen will have to leave the perimeter at the time t_0.

We first demonstrate that $t_0 < \frac{1}{2}(\mathcal{T}+1)$. Let us rearrange the equation $\theta(t_0,\mathcal{T}) = 3(t_0-1)$ to get

$$t_0 = \mathcal{T} - 2\cos\left(\frac{3}{4}(t_0-1)\right)$$

which is the definition of τ in Theorem 4.10 (in the case that $\mathcal{T} = \mathcal{T}_3$). One can easily confirm that in the case of $\mathcal{T} = \mathcal{T}_3$, both $\frac{d\theta}{dt}\Big|_{t=\tau} > 3$ and $\tau < \frac{1}{2}(\mathcal{T}+1)$. Now observe that $\theta(t,\mathcal{T})$ is a decreasing function of $\mathcal{T}$ and this implies that for $\mathcal{T} < \mathcal{T}_3$ we have $\theta(\tau,\mathcal{T}) > \theta(\tau,\mathcal{T}_3)$. The time t_0 must therefore occur earlier than the time τ. We note that $\tau < 2$ and, since we are assuming that $\mathcal{T} \geq 3$, we have $\tau < \frac{1}{2}(\mathcal{T}+1)$. Since $t_0 < \tau$ we can finally conclude that $t_0 < \frac{1}{2}(\mathcal{T}+1)$.

The second part of the claim follows from Lemma 4.5 if we can show that $t_0 < \mathcal{T} - \frac{2}{3}\sqrt{5}$. We note that $\mathcal{T} \geq 3$ and thus $\mathcal{T} - \frac{2}{3}\sqrt{5} \geq 1.5093$. Since $\tau \approx 1.2319$ and $t_0 < \tau$ we can clearly see that $t_0 < \mathcal{T} - \frac{2}{3}\sqrt{5}$.

If $\mathcal{T} < 3$ then it should be obvious that the queen cannot even be at the perimeter at the time $t = 1$. Thus, in this case, we take $t_0 = 1$.

Since the queen must leave the perimeter at the time $t_0 < \tau$, by Lemma 4.13, we know that the queen must be able to return to the perimeter and explore before the time t_*. We claim that this is not possible. Indeed, observe that the queen cannot return to the perimeter until the earliest time $t > t_0$ at which $\theta(t) = 3(t-1) + y$ (where we have set $y < \tau - 1$ as the length of the arc $\theta(t)$ explored by the queen). Thus, in order for the queen to have returned to the perimeter before the time t_* we must have $\theta(t_*) \leq 3(t-1) + y$. However, since $\mathcal{T} < \mathcal{T}_3$ we have

$$\theta(t_*) = 4\cos^{-1}\left(\frac{\mathcal{T} - t_*}{2}\right) > 4\cos^{-1}\left(\frac{\mathcal{T}_3 - t_*}{2}\right).$$

We note that

$$\mathcal{T}_3 - t_* = \sin\left(\frac{3(t_*-1) + (\tau-1)}{2}\right) = \sin\left(\cos^{-1}\left(\frac{-2}{3}\right)\right) = \sqrt{\frac{5}{9}}$$

and thus

$$\theta(t_*) > 4\cos^{-1}\left(\frac{\sqrt{5}}{6}\right) \approx 4.7556.$$

Since $\tau \approx 1.2319$, and $t_* \approx 2.4564$ we have

$$3(t_*-1) + y \leq 3(t_*-1) + (\tau-1) \approx 4.6010.$$

We can therefore see that it is not the case that $\theta(t_*) \leq 3(t-1) + y$ and thus the queen cannot have returned to the perimeter before the time t_*. We can finally conclude that the queen can only explore a subset of the perimeter of length $t_0 - 1 < \tau - 1$ before the time t_*. □

At this point the proof of Theorem 4.10 is trivial.

Proof. Assume we have an algorithm with evacuation time $\mathcal{T} < \mathcal{T}_3$. Then, by Lemma 4.13, the queen must explore a subset of the perimeter of length at least $\tau - 1$ by the time t_*. However, by Lemma 4.14, the queen can only explore a subset of the perimeter of length $y < \tau - 1$ if $\mathcal{T} < \mathcal{T}_3$. We must therefore conclude that it is not possible for the queen to evacuate in time less than $\mathcal{T}_3$. □

5. Conclusion

We considered an evacuation problem concerning priority searching on the perimeter of a unit disk where only one robot (the queen) needs to reach the exit. In addition to the queen, there are $n \leq 3$ other robots (servants) aiding the queen by contributing to the exploration of the disk but which do not need to evacuate. We proposed evacuation algorithms and studied non-trivial tradeoffs on the queen's evacuation time depending on the number n of servants. In addition to analyzing tradeoffs and improving the bounds obtained for the wireless communication model, an interesting open problem would be to investigate other communication models, e.g., the face-to-face model studied in [11] and elsewhere.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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