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J. Differential Equations 269 (2020) 10239–10276

Journal of Differential Equations

www.elsevier.com/locate/jde

Two-dimensional solutions of a mean field equation on flat tori

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Received 9 February 2020; revised 29 June 2020; accepted 4 July 2020

Abstract

We study the mean field equation on the flat torus $T_\sigma := \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\sigma)$

$$\Delta u + \rho\left(\frac{e^u}{\int_{T_\sigma} e^u} - \frac{1}{|T_\sigma|}\right) = 0,$$

where ρ is a real parameter. For a general flat torus, we obtain the existence of two-dimensional solutions bifurcating from the trivial solution at each eigenvalue (up to a multiplicative constant $|T_\sigma|$) of Laplace operator on the torus in the space of even symmetric functions. We further characterize the subset of all eigenvalues through which only one bifurcating curve passes. Finally local convexity near bifurcating points of the solution curves are obtained.

MSC: 35B32; 35A01; 35J61; 58J55

Keywords: Mean field equation; Torus; Bifurcation; Symmetry

1. Introduction

In this paper, we consider the mean field equation on the flat torus $T_\sigma := \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\sigma)$

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https://doi.org/10.1016/j.jde.2020.07.012
0022-0396/

$$\Delta u + \rho \left(\frac{e^u}{\int_{T_\sigma} e^u} - \frac{1}{|T_\sigma|} \right) = 0, \qquad (x, y) \in T_\sigma, \tag{1}$$

where $\sigma := \alpha + i\beta$ with $\alpha \geq 0$, $\beta = |T_\sigma| > 0$. Since the above equation is invariant by adding a constant to a solution, we introduce

$$\mathcal{H}_\sigma = \left\{ u \in H^1(T_\sigma) \mid \int_{T_\sigma} u = 0 \right\}.$$

Note that $\mathcal{H}_\sigma$, equipped with the $H^1(T_\sigma)$ norm, is a Hilbert space.

The corresponding energy functional of (1) is

$$J_{\rho,\sigma}(u) = \frac{1}{2} \int_{T_\sigma} |\nabla u|^2 - \rho \log \left(\frac{1}{|T_\sigma|} \int_{T_\sigma} e^u \right). \tag{2}$$

This kind equation is related to the prescribed Gauss curvature problem from the geometric point of view (see [3], [4], [11], [12], [13], [17], [26], [27], [31], [32], [35]). It also arises from vortex theory of two dimensional turbulence (see [2], [7], [8], [9], [10], [19], [23], [29]) and Chern-Simons-Higgs gauge field theory (see [6], [18], [20], [30], [39], [40]).

Mean field equations on tori have been studied by many researchers and there are also many known results. In [36], the authors proved that non-zero one-dimensional solutions to (1) on a square torus exist if and only if $\rho > 4\pi^2$ and the solutions are evenly symmetric. Concerning the nontrivial two-dimensional solutions, the authors in [38] used the Min-Max scheme to establish the existence for $\rho \in (8\pi, 4\pi^2)$. Both their results could be extended to a rectangular torus by a simple scaling argument while the bound $4\pi^2$ is replaced by $\lambda_1(T_\sigma)|T_\sigma|$ where $\lambda_1(T_\sigma)$ denotes the first eigenvalue of Laplace operator on T_σ. Recently it is found in [15] that two dimensional solutions exist for $\rho > 8\pi$ but close to 8π on a rectangular torus. The authors in [1] generalized the result to any flat torus.

Based on the above stated results, a natural task is to understand the structures of solutions of (1) for various ranges of ρ. Now the structure of solutions has been completely understood for $\rho \leq 8\pi$. In [5] the authors proved a one-dimensional symmetry result for ρ up to an upper bound which is smaller than 8π and can be written explicitly in terms of the maximum conformal radius of a rectangular torus. Later, it is proven in [33] that zero is the unique solution of (1) provided that $\rho \leq \min\{8\pi, 32\frac{l^2}{|T_\sigma|}\}$ where l denotes the length of the shortest geodesic of an arbitrary torus. Note that their result is sharp if $\frac{l^2}{|T_\sigma|} \geq \frac{\pi}{4}$. In [34] the one-dimensional symmetry of any global minimizer of $J_{\rho,\sigma}$ is proven for $\rho \leq 8\pi$. It is also conjectured that $u \equiv 0$ is the unique solution of (1) on an arbitrary flat torus whenever $\rho \leq \min\{8\pi, \lambda_1(T_\sigma)|T_\sigma|\}$. The conjecture is validated by the second author and Moradifam in [24] for the case of rectangular tori. Their proof relies on a "sphere covering inequality" developed in [25]. Precisely, they first show that the solutions are evenly symmetric about both axes if the origin is a critical point of the solution by applying the "sphere covering inequality", then prove that symmetric solutions about two axes must be steiner-symmetric on some "sub-torus" and then must be one-dimensional. Recently, the conjecture is proved by in [22] for the case of a general flat torus.

In this paper, we will try to find multiple two-dimensional solutions of (1) for large ρ. Dealing separately for the rectangular torus case and the generic flat torus case, we will use the bifurcation method to obtain two-dimensional solutions, which are bifurcating from the trivial solution at each eigenvalue (up to a multiplicative constant $\beta = |T_\sigma|$) of Laplace operator on the torus in the spaces of functions with certain symmetries. We will further find out the subset of all eigenvalues, from which only one bifurcating curve emanates. Local convexity near bifurcating points of the solution curves will also be obtained. Multiple non-axially symmetric solutions of the mean field equation on unit sphere bifurcating from trivial solution can be found in [21].

First we consider the rectangular torus case, namely $\alpha = 0$. We denote T_σ as T_β. Let Γ_β be the lattice generated by 1 and $\beta\sqrt{-1}$ in $\mathbb{C}$, namely

$$\Gamma_\beta = \mathrm{span}\{(1,0), (0,\beta)\}.$$

We may suppose $\beta \geq 1$. The dual lattice Γ_β^* denotes the set

$$\Gamma_\beta^* := \{\xi \in \mathbb{R}^2 : \langle \xi, (x,y)\rangle \in \mathbb{Z}, \quad \forall (x,y) \in \Gamma_\beta\},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product in $\mathbb{R}^2$. So

$$\Gamma_\beta^* = \mathrm{span}\{(1,0), (0,\beta^{-1})\}.$$

It is well known that the function

$$f_\xi(x,y) = e^{2\pi\sqrt{-1}\langle \xi,(x,y)\rangle}, \quad \xi = (\xi_1,\xi_2) = (i, \beta^{-1} j) \in \Gamma_\beta^*, \ \forall\, i, j \in \mathbb{Z}$$

is an eigenfunction of the Laplace operator on T_β with eigenvalue $4\pi^2|\xi|^2$, namely

$$-\Delta f_\xi = 4\pi^2 |\xi|^2 f_\xi.$$

Moreover, the family $\{f_\xi\}_{\xi \in \Gamma_\beta^*}$ is a complete system of eigenfunctions. For given $i, j \in \mathbb{Z}$, $\xi = (\xi_1, \xi_2) = (i, \beta^{-1} j)$ associates with the following two independent eigenfunctions corresponding to the eigenvalue $4\pi^2|\xi|^2 = 4\pi^2(i^2 + \beta^{-2} j^2)$

$$\cos(2\pi(\xi_1 x + \xi_2 y)), \quad \sin(2\pi(\xi_1 x + \xi_2 y)).$$

Note that, if $ij \neq 0$, the three symmetric elements $(-\xi_1, -\xi_2)$, $(-\xi_1, \xi_2)$, $(\xi_1, -\xi_2) \in \Gamma_\beta^*$ give the following other two independent eigenfunctions corresponding to the eigenvalue $4\pi^2(i^2 + \beta^{-2} j^2)$

$$\cos(2\pi(\xi_1 x - \xi_2 y)), \quad \sin(2\pi(\xi_1 x - \xi_2 y)).$$

Hence there are four independent eigenfunctions to the eigenvalue $4\pi^2|\xi|^2 = 4\pi^2(i^2 + \beta^{-2} j^2)$, or equivalently the following four independent eigenfunctions

$$\cos(2\pi\xi_1 x)\cos(2\pi\xi_2 y), \ \sin(2\pi\xi_1 x)\sin(2\pi\xi_2 y),$$

$$\cos(2\pi\xi_1 x)\sin(2\pi\xi_2 y), \ \sin(2\pi\xi_1 x)\cos(2\pi\xi_2 y).$$

If $ij = 0$, namely $\xi_1\xi_2 = 0$, say $\xi_2 = 0$, there are only two independent eigenfunctions to the eigenvalue $4\pi^2|\xi|^2$

$$\cos(2\pi\xi_1 x), \quad \sin(2\pi\xi_1 x).$$

If we impose axial symmetry on functions

$$f_\xi(-x,-y) = f_\xi(x,-y) = f_\xi(-x,y),$$

then the eigenfunction space is 1-dimensional in either case, and its basis is

$$\cos(2\pi\xi_1 x)\cos(2\pi\xi_2 y).$$

Hence, in what follows, we may assume that the components i, j in $\xi = (\xi_1, \xi_2) = (i, j\beta^{-1}) \in \Gamma_\beta^*$ satisfy $i, j \geq 0 \in \mathbb{Z}$.

Denote

$$\lambda_{i,j} := 4\pi^2(i^2 + \beta^{-2}j^2), \;\; \forall\, i, j \geq 0 \in \mathbb{Z},$$

then the set of all eigenvalues is

$$\{4\pi^2|\xi|^2 : \; \xi \in \Gamma_\beta^*\} = \{\lambda_{i,j}\}_{i,j\geq 0, i^2+j^2\neq 0} =: S_\beta.$$

The reason for $i^2 + j^2 \neq 0$ is that $\int_{T_\beta} f_\xi = 0$.

To make $\lambda_{m,n} = \lambda_{i,j}$, we require that $\beta^2(m^2 - i^2) = j^2 - n^2$. Denote

$$\Lambda_\beta := \left\{\lambda_{i,j} : \exists\, m, n \geq 0 \in \mathbb{Z}, \; (m,n) \neq (i,j) \text{ s.t. } \beta^2 = \frac{j^2 - n^2}{m^2 - i^2}\right\},$$

$$K_\beta := S_\beta \backslash \Lambda_\beta.$$

Further we define

$$\beta K_\beta := \{\beta\lambda_{i,j} : \; \lambda_{i,j} \in K_\beta\}, \quad \beta S_\beta := \{\beta\lambda_{i,j} : \; \lambda_{i,j} \in S_\beta\}.$$

We denote the sets of all rational numbers and irrational numbers by Q and $\bar{Q}$.

In this paper, we always denote a positive rational number as $\frac{p}{q}$, where $p, q \in \mathbb{N}$ and their largest common factor $(p, q) = 1$.

Our results in the rectangular torus case are as follows.

Theorem 1.1. *All elements in βS_β are bifurcation points for the curve of trivial solutions $(\rho, 0)$.*

In particular, for any $\rho_{i,j} = \beta\lambda_{i,j} \in \beta K_\beta$, there exists $\varepsilon_0 > 0$, and for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, (1) admits a family of solutions $(\rho_{i,j}(\varepsilon), u_\varepsilon)$, where $\rho_{i,j}(\varepsilon)$ is an analytic curve and

$$\begin{cases} \rho_{i,j}(0) = \rho_{i,j}, \\ u_\varepsilon(x,y) = \varepsilon\cos(2i\pi x)\cos(2j\pi\beta^{-1}y) + \varepsilon Z_\varepsilon(x,y). \end{cases} \tag{3}$$

Here $Z_\varepsilon(x, y)$ satisfies that $Z_\varepsilon(-x, -y) = Z_\varepsilon(x, -y) = Z_\varepsilon(-x, y)$ and $Z_0 = 0$. Moreover, all the bifurcation curves can be extended globally, and are unbounded with either ρ or $|u|_{H^1(T_\sigma)}$ tending to infinity.

Theorem 1.2. *(i) For $\beta^2 \in \bar{Q}$, we have $K_\beta = S_\beta$;*

(ii) For $\beta = 1$, we have

$$K_\beta = \{\lambda_{i,i}\}_{i=1}^{+\infty} \backslash \{\lambda_{ki,ki} : \forall k \in \mathbb{N}, \exists a, b \in \mathbb{N}, a > b, (a, b) = 1 \text{ s.t. } i = a^2 + b^2\};$$

(iii) For $\beta = \frac{p}{q} > 1$, we have

$$\begin{aligned} K_\beta = S_\beta \backslash [&\{\lambda_{qi,pj}\}_{i \neq j} \\ &\cup \{\lambda_{kqi,kpi} : \forall k \in \mathbb{N}, \exists a, b \in \mathbb{N}, a > b, (a, b) = 1 \text{ s.t. } i = a^2 + b^2\} \\ &\cup \{\lambda_{mq^2-n,sp^2+t} : m, n, s, t \in \mathbb{N}, mq^2 - n, sp^2 - t \geq 0, mn = st, \frac{n}{q} \text{ or } \frac{t}{p} \notin \mathbb{N}\} \\ &\cup \{\lambda_{\frac{mq^2-n}{j}, \frac{sp^2+t}{j}} : m, n, s, t \in \mathbb{N}, mn = st, 1 < j \leq 2p^2q^2, \\ &\frac{mq^2-n}{j}, \frac{mq^2+n}{j}, \frac{sp^2-t}{j}, \frac{sp^2+t}{j} \in \mathbb{N} \cup \{0\}\}]; \end{aligned}$$

(iv) For $\beta \in \bar{Q}$ and $\beta^2 = \frac{p}{q}$, we have

$$\begin{aligned} K_\beta = S_\beta \backslash [&\{\lambda_{mq-n,sp+t} : m, n, s, t \in \mathbb{N}, mq - n, sp - t \geq 0, mn = st\} \\ &\cup \{\lambda_{\frac{mq-n}{j}, \frac{sp+t}{j}} : m, n, s, t \in \mathbb{N}, mn = st, 1 < j \leq 2pq, \\ &\frac{mq-n}{j}, \frac{mq+n}{j}, \frac{sp-t}{j}, \frac{sp+t}{j} \in \mathbb{N} \cup \{0\}\}]. \end{aligned}$$

For the generic flat torus case, namely the case $\alpha > 0$, and we denote T_σ as $T_{\alpha,\beta}$. Hence the corresponding lattice is

$$\Gamma_{\alpha,\beta} = \text{span}\{(1, 0), (\alpha, \beta)\}.$$

We may suppose $\alpha^2 + \beta^2 \geq 1$. The dual lattice $\Gamma^*_{\alpha,\beta}$ is

$$\Gamma^*_{\alpha,\beta} = \text{span}\left\{\left(1, -\frac{\alpha}{\beta}\right), (0, \beta^{-1})\right\}.$$

Similarly, the family functions

$$f_\xi(x, y) = e^{2\pi\sqrt{-1}\langle \xi, (x,y)\rangle}, \ \xi = (\xi_1, \xi_2) = (i, \beta^{-1}(j - \alpha i)) \in \Gamma^*_{\alpha,\beta}, \ \forall\, i, j \in \mathbb{Z}$$

is a complete system of eigenfunctions of the Laplace operator on $T_{\alpha,\beta}$. For given $i, j \in \mathbb{Z}$, $\xi = (\xi_1, \xi_2) = (i, \beta^{-1}(j - \alpha i))$ gives the following two independent eigenfunctions corresponding to the eigenvalue $4\pi^2|\xi|^2 = 4\pi^2(i^2 + \beta^{-2}(\alpha i - j)^2)$

$$\cos 2\pi[ix+\beta^{-1}(j-\alpha i)y], \quad \sin 2\pi[ix+\beta^{-1}(j-\alpha i)y].$$

If we impose even symmetry on functions

$$f_\xi(-x,-y)=f_\xi(x,y),$$

then the eigenfunction space is 1-dimensional, and its basis is

$$\cos 2\pi[ix+\beta^{-1}(j-\alpha i)y].$$

All eigenvalues of Laplace operator on $T_{\alpha,\beta}$ are

$$4\pi^2[i^2+\beta^{-2}(\alpha i-j)^2]:=\mu_{i,j}, \ \ \forall\, i,j\in\mathbb{Z}.$$

Moreover $i^2+j^2\neq 0$. For simplicity, during the computation in the whole section 4 we will omit the coefficient $4\pi^2\beta^{-2}$ of $\mu_{i,j}$, and its following two forms will be used

$$\mu_{i,j}=\beta^2 i^2+(\alpha i-j)^2=(\alpha^2+\beta^2)i^2+j^2-2\alpha ij.$$

Note that both (i,j) and $(-i,-j)$ deduce the same eigenvalue and eigenfunction, so we denote

$$\Lambda_{\alpha,\beta}:=\left\{\mu_{i,j}:\exists\, m,n\in\mathbb{Z},(m,n)\neq(i,j),\ (-i,-j)\text{ s.t. }\mu_{m,n}=\mu_{i,j}\right\}.$$

Similarly we introduce the sets

$$S_{\alpha,\beta}:=\{\mu_{i,j}\}_{i^2+j^2\neq 0}, \qquad K_{\alpha,\beta}:=S_{\alpha,\beta}\backslash\Lambda_{\alpha,\beta}.$$

Our main results in the generic flat torus case are as follows.

Theorem 1.3. *All elements in $\beta S_{\alpha,\beta}$ are bifurcation points for the curve of trivial solutions $(\rho,0)$.*
In particular for any $\rho_{i,j}=\beta\mu_{i,j}\in\beta K_{\alpha,\beta}$, there exists $\varepsilon_0>0$, and for $\varepsilon\in(-\varepsilon_0,\varepsilon_0)$, (1) admits a family of solutions $(\rho_{i,j}(\varepsilon),u_\varepsilon)$, where $\rho_{i,j}(\varepsilon)$ is an analytic curve and

$$\begin{cases}\rho_{i,j}(0)=\rho_{i,j},\\ u_\varepsilon(x,y)=\varepsilon\cos 2\pi[ix+\beta^{-1}(j-\alpha i)y]+\varepsilon Z_\varepsilon(x,y).\end{cases} \tag{4}$$

Here $Z_\varepsilon(x,y)$ satisfies that

$$Z_\varepsilon(-x,-y)=Z_\varepsilon(x,y)\text{ and }Z_0=0. \tag{5}$$

Moreover the bifurcation is global and the Rabinowitz alternative holds true.

In the following, Theorems 1.4-1.11 give a detailed and complete classification of the bifurcations points according to the simplicity of the associated eigenvalues.

Theorem 1.4. *Assume $\alpha \in \bar{Q}$.*

(i) If $\sqrt{\alpha^2+\beta^2} \in \bar{Q}$ and $\alpha^2+\beta^2 = \frac{p}{q}$, then $K_{\alpha,\beta} = S_{\alpha,\beta}$;

(ii) If $\sqrt{\alpha^2+\beta^2} = \frac{p}{q}$, then

$$K_{\alpha,\beta} = S_{\alpha,\beta} \backslash \left\{ \mu_{i,j} : \exists m, n \in \mathbb{Z},\ (m,n) \neq (i,j), (-i,-j) \text{ s.t. } \left\{ \begin{array}{l} pi = qn, \\ pm = qj \end{array} \right\} \right\}.$$

In particular, if $\alpha^2+\beta^2 = 1$, then $K_{\alpha,\beta} = \{\mu_{i,i}, \mu_{i,-i}\}_{i\neq 0}$.

Theorem 1.5. *Suppose $\alpha \in \bar{Q}$ and $\alpha^2+\beta^2 \in \bar{Q}$.*

(i) If $\frac{\alpha^2+\beta^2}{\alpha} \in Q$ then

$$K_{\alpha,\beta} = S_{\alpha,\beta} \backslash \left\{ \mu_{i,j} : \exists m \in \mathbb{Z},\ m \neq i \text{ s.t. } \frac{j}{i+m} = \frac{\alpha^2+\beta^2}{\alpha} \right\};$$

(ii) If $\frac{\alpha^2+\beta^2}{\alpha} \in \bar{Q}$, and further, either the following homogeneous linear algebraic equation does not admit non-zero integer solutions (e, f, g)

$$\frac{e(\alpha^2+\beta^2)}{\alpha} = \frac{f}{\alpha} + g, \tag{6}$$

or it admits non-zero integer solutions and each such solution (e, f, g) satisfies $\sqrt{g^2+4ef} \in \bar{Q}$, then $K_{\alpha,\beta} = S_{\alpha,\beta}$;

(iii) If $\frac{\alpha^2+\beta^2}{\alpha} \in \bar{Q}$, and further, if (6) admits non-zero integer solutions (e, f, g) satisfying $\sqrt{g^2+4ef} \in Q$, then

$$K_{\alpha,\beta} = S_{\alpha,\beta} \backslash \{ \mu_{i,j} : \exists m, n \in \mathbb{Z},\ (m,n) \neq (i,j), (-i,-j), (0,0) \text{ s.t. } \left\{ \begin{array}{l} i = \frac{2d_1 n + (d_1 d_2 - 1)m}{1+d_1 d_2}, \\ j = \frac{2d_2 m + (1-d_1 d_2)n}{1+d_1 d_2} \end{array} \right\},$$

where

$$d_1 = \frac{2e}{g+\sqrt{g^2+4ef}}, \quad d_2 = \frac{2f}{g+\sqrt{g^2+4ef}}. \tag{7}$$

Theorem 1.6. *Suppose $\alpha = \frac{p}{q}$ and $\alpha^2+\beta^2 = 1$. Then*

$$K_{\alpha,\beta} = \{\mu_{i,i}, \mu_{i,-i}\}_{i\neq 0} \backslash [\{ \mu_{i,i} : \exists m, n \in \mathbb{Z}, mn \neq i^2,\ \text{s.t. } \frac{m^2+n^2-2i^2}{mn-i^2} = \frac{2p}{q} \} \\ \cup \{ \mu_{i,-i} : \exists m, n \in \mathbb{Z}, mn \neq -i^2,\ \text{s.t. } \frac{m^2+n^2-2i^2}{mn+i^2} = \frac{2p}{q} \}].$$

In particular, if $\alpha = \frac{1}{2}$, then $K_{\alpha,\beta} = \emptyset$.

Theorem 1.7. *Suppose $\alpha = \frac{p}{q}$, $\alpha^2 + \beta^2 > 1$ and $\beta^2 \in \bar{Q}$. Then*

$$K_{\alpha,\beta} = \begin{cases} S_{\alpha,\beta} \backslash \left\{ \mu_{qk,j} \right\}_{k \neq 0 \in \mathbb{Z},\, j \neq pk}, & \text{if } q \text{ is odd}, \\ S_{\alpha,\beta} \backslash \left\{ \mu_{\frac{qk}{2},j} \right\}_{k \neq 0 \in \mathbb{Z},\, j \neq \frac{pk}{2}}, & \text{if } q \text{ is even}. \end{cases}$$

We denote

$$b_\alpha := \min\{\alpha - [\alpha], [\alpha] + 1 - \alpha\},$$

where $[\alpha]$ stands for the largest integer not exceeding α. Note that $0 \leq b_\alpha \leq \frac{1}{2}$, and $b_\alpha = 0$ if $\alpha \in \mathbb{N}$.

Theorem 1.8. *Suppose $\alpha = \frac{p}{q}$, $\alpha^2 + \beta^2 > 1$ and $\beta^2 \in Q$.*
(i) If $\beta > 1$, then $\emptyset \neq K_{\alpha,\beta} \subsetneq S_{\alpha,\beta}$;
(ii) If $\alpha \in \mathbb{N}$ and $\beta = 1$, then $K_{\alpha,\beta} = \emptyset$;
(iii) If $\alpha \in \mathbb{N}$ and $\beta < 1$, then $\emptyset \neq K_{\alpha,\beta} \subsetneq S_{\alpha,\beta}$;
(iv) If $\alpha \notin \mathbb{N}$ and $\sqrt{1 - b_\alpha^2} \leq \beta \leq 1$, then $\emptyset \neq K_{\alpha,\beta} \subsetneq S_{\alpha,\beta}$;
(v) If $\alpha \notin \mathbb{N}$ and $\beta = \sqrt{1 - b_\alpha^2}$, then $\emptyset \neq K_{\alpha,\beta} \subsetneq S_{\alpha,\beta}$ for $q > 2$, $K_{\alpha,\beta} = \emptyset$ for $q = 2$.

Theorem 1.9. *Suppose $\alpha = \frac{p}{q} \notin \mathbb{N}$, $\beta^2 \in Q$, $\alpha^2 + \beta^2 > 1$ and $\beta < \sqrt{1 - b_\alpha^2}$. Furthermore, if $q = 2$, then*
(i) $K_{\alpha,\beta} = \emptyset$ for $\beta = \frac{\sqrt{3}}{6}$ or $\frac{1}{2}$;
(ii) $\emptyset \neq K_{\alpha,\beta} \subsetneq S_{\alpha,\beta}$ for $\beta \in (0, \frac{\sqrt{3}}{6}) \cup (\frac{\sqrt{3}}{6}, \frac{1}{2}) \cup (\frac{1}{2}, \frac{\sqrt{3}}{2})$.

Theorem 1.10. *Suppose $\alpha = \frac{p}{q} \notin \mathbb{N}$, $\beta^2 \in Q$, $\alpha^2 + \beta^2 > 1$, $\beta < \sqrt{1 - b_\alpha^2}$ and $q > 2$. Furthermore, if $b_\alpha = \frac{1}{q}$, then $\emptyset \neq K_{\alpha,\beta} \subsetneq S_{\alpha,\beta}$.*

Remark 1.1. For $\alpha = \frac{p}{q} \notin \mathbb{N}$ and $q = 2, 3, 4, 6$, the unique value of b_α is $\frac{1}{q}$.

Theorem 1.11. *Suppose $\alpha = \frac{p}{q} \notin \mathbb{N}$, $\beta^2 \in Q$, $\alpha^2 + \beta^2 > 1$, $\beta < \sqrt{1 - b_\alpha^2}$ and $b_\alpha \geq \frac{2}{q}$ (so $q = 5$ or $q \geq 7$). Then*
(i) If $0 < \beta \leq \frac{1}{q\sqrt{q^2-4}}$, then $\emptyset \neq K_{\alpha,\beta} \subsetneq S_{\alpha,\beta}$;
(ii) If $\frac{1}{q\sqrt{q^2-4}} < \beta < \sqrt{\frac{1}{3}(b_\alpha^2 - \frac{1}{q^2})}$, and q is odd and $b_\alpha = \frac{q-1}{2q}$, then $\emptyset \neq K_{\alpha,\beta} \subsetneq S_{\alpha,\beta}$;
(iii) If $\beta = \sqrt{\frac{1}{3}(b_\alpha^2 - \frac{1}{q^2})}$, then $\emptyset \neq K_{\alpha,\beta} \subsetneq S_{\alpha,\beta}$ for $q \geq 7$; $K_{\alpha,\beta} = \emptyset$ for $q = 5$;
(iv) If $\sqrt{\frac{1}{3}(b_\alpha^2 - \frac{1}{q^2})} < \beta < \sqrt{1 - b_\alpha^2}$, then $\emptyset \neq K_{\alpha,\beta} \subsetneq S_{\alpha,\beta}$.

Finally we obtain local convexity near bifurcating points of bifurcating curves.

Theorem 1.12. *The parameter function $\rho_{i,j}(\varepsilon)$ in Theorem 1.1 satisfies $\rho'_{i,j}(0) = 0$ and*

$$\rho''_{i,j}(0) = -\frac{1}{2}\lambda_{i,j}\left(\frac{\lambda_{i,j}}{\lambda_{2i,0} - \lambda_{i,j}} + \frac{\lambda_{i,j}}{\lambda_{0,2j} - \lambda_{i,j}} - \frac{1}{3}\right).$$

Corollary 1.1. *(i) If* $\frac{\sqrt{3}}{3}\beta^{-1}j < i < \sqrt{3}\beta^{-1}j$*, then* $\rho''_{i,j}(0) < 0$*;*

(ii) If $0 \le i < \frac{\sqrt{3}}{3}\beta^{-1}j$ *or* $i > \sqrt{3}\beta^{-1}j$*, then* $\rho''_{i,j}(0) > 0$*.*

Remark 1.2. For $\beta = 1$, from Theorem 1.2 we know that any element of βK_β admits the form $\rho_{i,i}$, namely $i = j$. Then Corollary 1.1 shows that for $i \in \mathbb{N}\backslash\{\lambda_{ki,ki} : \forall k \in \mathbb{N}, \exists a, b \in \mathbb{N}, a > b, (a,b) = 1 \text{ s.t. } i = a^2 + b^2\}$ the following inequality holds $\rho''_{i,i}(0) < 0$.

Theorem 1.13. *The parameter function* $\rho_{i,j}(\varepsilon)$ *in Theorem 1.3 satisfies* $\rho'_{i,j}(0) = 0$ *and* $\rho''_{i,j}(0) = \frac{\mu_{i,j}}{3} > 0$*.*

We would like to point out that the analyses for the rectangular torus case and the generic flat torus case are different in the sense that the latter is not a straightforward generalization of the former. Indeed, we have to consider the bifurcation solutions in functional spaces with different symmetry. In the rectangular flat torus case, we have to pose the axial symmetry, i.e., the even symmetry about both x-axis and y-axis; while in the generic flat torus case, such symmetry is too strong and only allows the trivial solution. So we only pose a weaker even symmetry, i.e., even symmetry about the origin. In both cases, equipped with the above suitable symmetric functional spaces, there are bifurcations points on the trivial solution curve $(\rho, 0)$ where the kernels of the linearized operators are one dimensional, but are of different type, i.e., the eigenfunction in (3) can not be obtained by letting $\alpha = 0$ in the eigenfunction in (4), while the former can not be generalized to obtain the latter. This is also the reason why the local convexity results are different for both cases as shown in Theorem 1.12, Corollary 1.1 and Theorem 1.13.

We also note that in this paper we only consider in some details the local bifurcation curves while the global bifurcation pictures are merely explained as unbounded. Indeed, according to the global theory of bifurcation (the Rabinowitz alternative), each bifurcation curve (ρ, u) from $(\rho_{i,j}, 0)$ either meets the trivial solution curve at another bifurcation point $(\rho^*, 0)$ or extends to infinity. The first scenario can be excluded by using the techniques developed in [22,24,25]. In the latter scenario, it can be shown, according to [31,24,22], that either ρ tends to infinity, or ρ goes $8\pi N$ for some positive integer N when the solution blows up at exactly N points. In the special case when we consider only one dimensional solution, such bifurcation curves do not consist of blow-up solutions and hence ρ goes to infinity. In this one dimensional setting, there indeed exists a nontrivial one dimensional solution for $\rho > \max\{8\pi, \lambda_1(T_\sigma)|T_\sigma|\}$ which may be regarded as bifurcating from the trivial solution (see [36,22]). However, it remains open whether or not ρ must go to infinity for other bifurcation curves. The full global bifurcation picture is being studied in an on-going project.

The paper is organized as follows. In section 2 we prove Theorems 1.1 and 1.3 by verifying the hypotheses of Crandall-Rabinowitz's bifurcation theory. We find out the single bifurcating curve set for rectangular torus case in section 3, and in section 4 we characterize the single bifurcating curve set for the generic flat torus case. The local convexity of bifurcating curves will be obtained in section 5.

2. Verification of the hypotheses of Crandall-Rabinowitz's theory

In this section we first introduce two properties which will be used to prove Theorems 1.1, 1.3, 1.12 and 1.13, and subsequently we finish the proof of Theorems 1.1, 1.3.

Let $F(t, x)$ be an operator mapping from $\mathbb{R} \times X$ to Y. Denote $\partial_x F$ and $\partial_t F$ as the Fréchet partial derivatives of F with respect to x and t respectively.

Proposition 2.1. *([14]) Let X, Y be Bananch spaces, $V \subset X$ a neighborhood of 0 and $F : \mathbb{R} \times V \to Y$ a map with the following properties*

(1) $F(t, 0) = 0$ for any $t \in \mathbb{R}$,

(2) $\partial_t F$, $\partial_x F$ and $\partial^2_{t,x} F$ exist and are continuous,

(3) $\ker(\partial_x F(t^*, 0)) = span\{w_0\}$ and $\mathcal{R}(\partial_x F(t^*, 0))^\perp$ has dimension 1,

(4) $\partial^2_{t,x} F(t^*, 0)w_0 \notin \mathcal{R}(\partial_x F(t^*, 0))$.

If Z is any complement of $\ker(\partial_x F(t^*, 0))$ in X, then there exists $\varepsilon_0 > 0$, a neighborhood $U \subset \mathbb{R} \times X$ of $(t^*, 0)$, and continuously differentiable maps $\eta : (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ and $z : (-\varepsilon_0, \varepsilon_0) \to Z$ such that

$$\begin{cases} \eta(0) = t^*, \\ z(0) = 0, \\ F^{-1}(0) \cap U \setminus (\mathbb{R} \times \{0\}) = \{(\eta(\varepsilon), \varepsilon w_0 + \varepsilon z(\varepsilon)) \mid \varepsilon \in (-\varepsilon_0, \varepsilon_0)\backslash\{0\}\}. \end{cases}$$

Proposition 2.2. *([37] or [28] chapter I6) Assume all the hypotheses of Proposition 2.1 are satisfied. Select $\psi \neq 0 \in Y^*$, where Y^* is the dual space of Y, such that $\mathcal{R}(\partial_x F(t^*, 0)) = \{y \in Y \mid \langle \psi, y\rangle = 0\}$, then the derivative $\eta'(0)$ of $\eta(\varepsilon)$ at $\varepsilon = 0$ is given by*

$$\eta'(0) = -\frac{\langle \partial^2_{x,x} F(t^*, 0)[w_0, w_0], \psi\rangle}{2\|w_0\|\langle \partial^2_{t,x} F(t^*, 0)w_0, \psi\rangle}.$$

Moreover, if $\eta'(0) = 0$ and F is of class C^3, then we have

$$\eta''(0) = -\frac{\langle \partial^3_{x,x,x} F(t^*, 0)[w_0]^3 - 3\partial^2_{x,x} F(t^*, 0)\left[w_0, (\partial_x F(t^*, 0))^{-1}(I - Q)\partial^2_{x,x} F(t^*, 0)[w_0]^2\right], \psi\rangle}{3\|w_0\|^2\langle \partial^2_{t,x} F(t^*, 0)w_0, \psi\rangle},$$

where $Q : y \to \frac{\langle y,\psi\rangle}{\|\psi\|^2}\psi$ is the projection from Y to $\mathcal{R}(\partial_x F(t^*, 0))^\perp$ and $(\partial_x F(t^*, 0))^{-1} : \mathcal{R}(\partial_x F(t^*, 0)) \to \ker(\partial_x F(t^*, 0))^\perp$ is the inverse of $\partial_x F(t^*, 0)$ restricted to the complementary of its kernel.

According to equation (1), we define an operator $F : \mathbb{R} \times \mathcal{H}_\sigma \to L^2(T_\sigma)$ as

$$F : (\rho, u) \to \Delta u + \rho \left(\frac{e^u}{\int_{T_\sigma} e^u} - \frac{1}{|T_\sigma|} \right).$$

A direct computation shows that

$$\partial_u F(\rho, 0)\phi = \Delta\phi + \frac{\rho}{|T_\sigma|}\phi = \Delta\phi + \frac{\rho}{\beta}\phi.$$

We set

$$\mathcal{L}_\sigma = \left\{ u \in L^2(T_\sigma) \mid \int_{T_\sigma} u = 0 \right\}.$$

Clearly F maps $\mathbb{R} \times \mathcal{H}_\sigma$ into $\mathcal{L}_\sigma$.

Given any element $\lambda_{i,j} \in \Lambda_\beta$, for any $(m,n) \neq (i,j)$ such that $\lambda_{m,n} = \lambda_{i,j}$, there are two possible cases. The first case is that all of i, j, m, n are not zero. From the formula of $\lambda_{i,j}$, we know that $\frac{m}{i}, \frac{n}{j} \in \mathbb{N} \cup \{0\}$ and $\frac{i}{m}, \frac{j}{n} \in \mathbb{N} \cup \{0\}$ cannot take place simultaneously. Without loss generality, we assume that $\frac{m}{i} \notin \mathbb{N} \cup \{0\}$ or $\frac{n}{j} \notin \mathbb{N} \cup \{0\}$. Then we consider the smaller torus T_β^{ij} with the corresponding lattice

$$\Gamma_\beta^{ij} = \text{span}\{(i^{-1}, 0), (0, j^{-1}\beta)\}.$$

The dual lattice of Γ_β^{ij} is

$$(\Gamma_\beta^{ij})^* = \text{span}\{(i, 0), (0, j\beta^{-1})\}.$$

The complete system of axial symmetric eigenfunctions of the Laplace operator on T_β^{ij} is

$$\{\cos(2\pi\xi_1 x)\cos(2\pi\xi_2 y)\}_{(\xi_1,\xi_2)\in(\Gamma_\beta^{ij})^*} = \{\cos(2\pi r i x)\cos(2\pi\beta^{-1} s j y)\}_{r,s\geq 0, r^2+s^2\neq 0}.$$

Clearly $\cos(2m\pi x)\cos(2n\beta^{-1}\pi y)$ does not belong to this set, since $\frac{m}{i}$ or $\frac{n}{j} \notin \mathbb{N} \cup \{0\}$. Moreover, it is easy to verify that for any other $\lambda_{l,k}$ satisfying that $\lambda_{l,k} = \lambda_{i,j} = \lambda_{m,n}$ and $(l,k) \neq (i,j), (m,n)$, its eigenfunction $\cos(2l\pi x)\cos(2k\beta^{-1}\pi y)$ does not belong to the above set. All in all, the kernel corresponding to the eigenvalue $\lambda_{i,j}$ in T_β^{ij} is 1-dimensional, and its basis is

$$\cos(2i\pi x)\cos(2j\pi\beta^{-1} y).$$

The second case is that at least one of i, j, m, n is zero. Without loss of generality, we assume $i = 0$, then $m \neq 0$. Then we consider the smaller torus T_β^{ij} with the corresponding lattice

$$\Gamma_\beta^{ij} = \text{span}\{((m+1)^{-1}, 0), (0, j^{-1}\beta)\}.$$

The dual lattice of Γ_β^{ij} is

$$(\Gamma_\beta^{ij})^* = \text{span}\{(m+1, 0), (0, j\beta^{-1})\}.$$

The complete system of axially symmetric eigenfunctions of the Laplace operator on T_β^{ij} is

$$\{\cos(2\pi\xi_1 x)\cos(2\pi\xi_2 y)\}_{(\xi_1,\xi_2)\in(\Gamma_\beta^{ij})^*} = \{\cos(2\pi r(m+1)x)\cos(2\pi\beta^{-1} s j y)\}_{r,s\geq 0, r^2+s^2\neq 0}.$$

Clearly $\cos(2m\pi x)\cos(2n\beta^{-1}\pi y)$ does not belong to this set, since $\frac{m}{m+1} \notin \mathbb{N} \cup \{0\}$. Now the kernel corresponding to the eigenvalue $\lambda_{i,j}$ in T_β^{ij} is 1-dimensional, and its basis is

$$\cos(2j\pi\beta^{-1}y).$$

For $\lambda_{i,j} \in K_\beta$, we define $T_\beta^{ij} = T_\beta$.

We introduce the following spaces

$$\mathcal{X}_\beta^{ij} := \{\phi \in \mathcal{H}_\beta^{ij} \mid \phi(-x,-y) = \phi(x,-y) = \phi(-x,y)\},$$
$$\mathcal{Y}_\beta^{ij} := \{\psi \in \mathcal{L}_\beta^{ij} \mid \psi(-x,-y) = \psi(x,-y) = \psi(-x,y)\},$$

where

$$\mathcal{H}_\beta^{ij} := \{u \in H^1(T_\beta^{ij}) \mid \int\limits_{T_\beta^{ij}} u = 0\},$$

$$\mathcal{L}_\beta^{ij} := \{u \in L^2(T_\beta^{ij}) \mid \int\limits_{T_\beta^{ij}} u = 0\}.$$

Similarly, for a generic flat torus and any given element $\mu_{i,j} \in \Lambda_{\alpha,\beta}$, if there is some $(m,n) \neq (i,j), (-i,-j)$ such that $\mu_{m,n} = \mu_{i,j}$, then we have also two cases. The first case is that all of i, j, m, n are not zero. From the formula of $\mu_{i,j}$, we know that $\frac{m}{i}, \frac{n}{j} \in \mathbb{Z}$ and $\frac{i}{m}, \frac{j}{n} \in \mathbb{Z}$ also cannot take place simultaneously. If $\frac{m}{i} \notin \mathbb{Z}$ or $\frac{n}{j} \notin \mathbb{Z}$. We consider the smaller torus $T_{\alpha,\beta}^{ij}$ with the corresponding lattice

$$\Gamma_{\alpha,\beta}^{ij} = \mathrm{span}\{(i^{-1}, 0), (j^{-1}\alpha, j^{-1}\beta)\}.$$

The dual lattice of $\Gamma_{\alpha,\beta}^{ij}$ is

$$(\Gamma_{\alpha,\beta}^{ij})^* = \mathrm{span}\{(i, -i\beta^{-1}\alpha), (0, j\beta^{-1})\}.$$

The complete system of evenly symmetric eigenfunctions of the Laplace operator on $T_{\alpha,\beta}^{ij}$ is

$$\{\cos 2\pi(\xi_1 x + \xi_2 y)\}_{(\xi_1,\xi_2)\in(\Gamma_{\alpha,\beta}^{ij})^*} = \{\cos 2\pi[rix + \beta^{-1}(sj - \alpha ri)y]\}_{r^2+s^2\neq 0}.$$

Clearly $\cos 2\pi[mx + \beta^{-1}(n - \alpha m)y]$ does not belong to this set, so the kernel corresponding to the eigenvalue $\mu_{i,j}$ in $T_{\alpha,\beta}^{ij}$ is 1-dimensional, and its basis is

$$\cos 2\pi[ix + \beta^{-1}(j - \alpha i)y].$$

The second case is that at least one of i, j, m, n is zero. We assume $i = 0$, then $m \neq 0$. We consider the smaller torus $T_{\alpha,\beta}^{ij}$ with the corresponding lattice

$$\Gamma^{ij}_{\alpha,\beta} = \text{span}\{((|m|+1)^{-1}, 0), (j^{-1}\alpha, j^{-1}\beta)\}.$$

The dual lattice of $\Gamma^{ij}_{\alpha,\beta}$ is

$$(\Gamma^{ij}_{\alpha,\beta})^* = \text{span}\{(|m|+1, -(|m|+1)\beta^{-1}\alpha), (0, j\beta^{-1})\}.$$

The complete system of evenly symmetric eigenfunctions of the Laplace operator on $T^{ij}_{\alpha,\beta}$ is

$$\{\cos 2\pi(\xi_1 x + \xi_2 y)\}_{(\xi_1,\xi_2)\in(\Gamma^{ij}_{\alpha,\beta})^*} = \{\cos 2\pi[r(|m|+1)x + \beta^{-1}(sj - \alpha r(|m|+1))y]\}_{r^2+s^2\neq 0}.$$

Clearly $\cos 2\pi[mx + \beta^{-1}(n-\alpha m)y]$ does not belong to this set, since $\frac{m}{|m|+1} \notin \mathbb{Z}$. So the kernel corresponding to the eigenvalue $\mu_{i,j}$ in $T^{ij}_{\alpha,\beta}$ is 1-dimensional, and its basis is

$$\cos(2\pi\beta^{-1}jy).$$

For $\mu_{i,j} \in K_{\alpha,\beta}$, we define $T^{ij}_{\alpha,\beta} = T_{\alpha,\beta}$.

We introduce the following spaces

$$\mathcal{X}^{ij}_{\alpha,\beta} := \{\phi \in \mathcal{H}^{ij}_{\alpha,\beta} \mid \phi(-x,-y) = \phi(x,y)\},$$
$$\mathcal{Y}^{ij}_{\alpha,\beta} := \{\psi \in \mathcal{L}^{ij}_{\alpha,\beta} \mid \psi(-x,-y) = \psi(x,y)\}.$$

We have the following lemmas.

Lemma 2.1. *The restriction $\mathcal{F} := F\mid_{\mathbb{R}\times\mathcal{X}^{ij}_{\beta}}$ (or $\mathcal{F} := F\mid_{\mathbb{R}\times\mathcal{X}^{ij}_{\alpha,\beta}}$) maps its domain into $\mathcal{Y}^{ij}_{\beta}$ (or $\mathcal{Y}^{ij}_{\alpha,\beta}$). Moreover, for $\rho = \beta\lambda_{i,j}$ (or $\rho = \beta\mu_{i,j}$),*

$$\dim\{\ker(\partial_u\mathcal{F}(\rho,0))\} = 1,$$

and the basis is

$$\cos(2\pi\xi_1 x)\cos(2\pi\xi_2 y)(or\ \cos 2\pi(\xi_1 x + \xi_2 y)),$$

where $\xi := (\xi_1, \xi_2) := (i, \beta^{-1}j)$ (or $\xi := (i, -\beta^{-1}(\alpha i - j))$).

Lemma 2.2. *For $\rho = \beta\lambda_{i,j}$ (or $\rho = \beta\mu_{i,j}$), the range of the operator $\partial_u\mathcal{F}(\rho,0)$ has co-dimension one and is given by*

$$\mathcal{R}(\partial_u\mathcal{F}(\rho,0)) = \left\{\phi \in L^2(T^{ij}_{\beta}) \mid \int_{T^{ij}_{\beta}} \phi(x,y)\cos(2\pi ix)\cos(2\pi\beta^{-1}jy) = 0\right\}$$

(or $\mathcal{R}(\partial_u\mathcal{F}(\rho,0)) = \left\{\phi \in L^2(T^{ij}_{\alpha,\beta}) \mid \int_{T^{ij}_{\alpha,\beta}} \phi(x,y)\cos 2\pi[ix + \beta^{-1}(j-\alpha i)y] = 0\right\}$).

Proof. By the definition of the operator F and the well-known spectral properties of Δ on tori, the range of $\partial_u \mathcal{F}(\rho, 0)$ coincides with the orthogonal of its kernel. This and the result of Lemma 2.1 yield the desired results of this lemma. □

Lemma 2.3. *For $\rho = \beta\lambda_{i,j}$ (or $\rho = \beta\mu_{i,j}$), we have*

$$\partial^2_{\rho,u}\mathcal{F}(\rho,0)\{\cos(2\pi i x)\cos(2\pi\beta^{-1} j y)\} \notin \mathcal{R}(\partial_u \mathcal{F}(\rho,0))$$

(or $\partial^2_{\rho,u}\mathcal{F}(\rho,0)\{\cos 2\pi[ix + \beta^{-1}(j - \alpha i)y]\} \notin \mathcal{R}(\partial_u \mathcal{F}(\rho,0))$).

Proof. Differentiating $\partial_u \mathcal{F}$ with respect to ρ, we get

$$\partial^2_{\rho,u}\mathcal{F}(\rho,0)\phi = \frac{\phi}{\beta}.$$

Therefore, owing to

$$\int_{T^{ij}_\beta} \Big(\cos(2\pi i x)\cos(2\pi\beta^{-1} j y)\Big)^2 dxdy \neq 0$$

$$(\text{or} \int_{T^{ij}_{\alpha,\beta}} \Big(\cos 2\pi[ix + \beta^{-1}(j - \alpha i)y]\Big)^2 dxdy \neq 0),$$

the desired results follow, where the result in Lemma 2.2 is used. □

Proof of Theorem 1.1. We apply Proposition 2.1 with $\mathcal{F} : \mathbb{R} \times \mathcal{X}^{ij}_\beta \to \mathcal{Y}^{ij}_\beta$, Lemmas 2.1-2.3 show the existence of a continuously differentiable local branch. Namely there exists a branch of non-trivial solutions $(\rho_{i,j}(\varepsilon), u_\varepsilon)$ nn the torus T^{ij}_β, where $\rho_{i,j}(0) = \rho_{i,j}$ and u_ε satisfies

$$u_\varepsilon(x, y) = \varepsilon\cos(2\pi i x)\cos(2\pi\beta^{-1} j y) + \varepsilon Z_\varepsilon(x, y)$$

with Z_ε satisfying

$$Z_\varepsilon(-x, -y) = Z_\varepsilon(x, -y) = Z_\varepsilon(-x, y)$$

and $Z_0 = 0$. In fact, since $\mathcal{F}$ is real analytic and $\mathcal{F}_u(\rho, 0)$ is a Fredholm operator, from [16] we know that $\rho_{i,j}(\varepsilon)$ is an analytic curve.

In order to show that the bifurcation is global we use a degree argument. We introduce operators G and $\mathcal{G}$ as follows

$$G : (\rho, u) \to \frac{u}{\rho} - (-\Delta)^{-1}\left(\frac{e^u}{\int_{T^{ij}_\beta} e^u} - \frac{1}{|T^{ij}_\beta|}\right) =: \frac{u}{\rho} - \mathcal{G}(u).$$

Note that $G = -(-\Delta)^{-1}F$. Clearly 0 is the simple eigenvalue of the operator $\partial_u G|_{\rho=\rho_{i,j},u=0}$. Note that the operator $\mathcal{G}$ is a compact operator from $\mathcal{H}^{ij}_\beta$ to itself. Hence classical results in the bifurcation theory ensure the existence of a global continuum of solutions to (1) satisfying the Rabinowitz alternative. Furthermore, we can exclude the possibility that the bifurcation curve meet the trivial solution curve at another bifurcation point $(\rho^*, 0)$ by using using the techniques developed in [22,24,25]. The global bifurcation curve then has to be unbounded. In this case, it can be shown, according to [31,24,22], that either ρ tends to infinity, or ρ goes $8\pi N$ for some positive integer N when the solution blows up at exactly N points.

We extend u_ε periodically in axes-x and y respectively from torus T_β^{ij} to torus T_β, and we still denote it as u_ε. Plainly u_ε is a solution in T_β. □

Proof of Theorem 1.3. We apply Proposition 2.1 with $\mathcal{F} : \mathbb{R} \times \mathcal{X}_{\alpha,\beta}^{ij} \to \mathcal{Y}_{\alpha,\beta}^{ij}$, Lemmas 2.1-2.3 show the existence of a local branch. Namely there exists a branch of non-trivial solutions $(\rho_{i,j}(\varepsilon), u_\varepsilon)$, where $\rho_{i,j}(0) = \rho_{i,j}$ and u_ε satisfies

$$u_\varepsilon(x, y) = \varepsilon \cos 2\pi[ix + \beta^{-1}(j - \alpha i)y] + \varepsilon Z_\varepsilon(x, y)$$

with Z_ε satisfying $Z_\varepsilon(-x, -y) = Z_\varepsilon(x, y)$ and $Z_0 = 0$.

The argument of the global bifurcation result is similar as that of Theorem 1.1, we omit it. □

3. Rectangular torus case

Lemma 3.1. *If $\beta^2 \in \bar{Q}$, then $\Lambda_\beta = \emptyset$.*

Proof. Combining the formulas $\beta^2 = \frac{j^2-n^2}{m^2-i^2}$ in the set Λ_β and the assumption that $\beta^2 \in \bar{Q}$, we know that there does not exist $(m, n) \neq (i, j)$ such that $\lambda_{m,n} = \lambda_{i,j}$, so $\Lambda_\beta = \emptyset$. □

Lemma 3.2. *If $\beta = 1$, then*

$$\Lambda_\beta = \{\lambda_{i,j}\}_{i\neq j} \cup \{\lambda_{ki,ki} : \forall k \in \mathbb{N}, \exists a, b \in \mathbb{N}, a > b, (a, b) = 1 \ s.t. \ i = a^2 + b^2\}.$$

Proof. Clearly $\lambda_{i,j} = \lambda_{j,i}, \forall i, j$. For $i \neq j$, one has $(i, j) \neq (j, i)$, so $\{\lambda_{i,j}\}_{i\neq j} \subseteq \Lambda_\beta$. For $i = j$, to the end that $\lambda_{i,i} = \lambda_{m,n}$, it requires that

$$2i^2 = m^2 + n^2.$$

All coprime positive integer solutions (i, m, n) of this equation are

$$i = a^2 + b^2, \ m = |a^2 - b^2 - 2ab|, \ n = a^2 - b^2 + 2ab,$$

where $a, b \in \mathbb{N}, a > b, (a, b) = 1$.

Note that if $\lambda_{i,j} \in \Lambda_\beta$, then $\lambda_{ki,kj} \in \Lambda_\beta, \forall k \in \mathbb{N}$. This is because that $\lambda_{i,j} = \lambda_{m,n}$ gives $\lambda_{ki,kj} = \lambda_{km,kn}$. So the subset of the points with the form $\lambda_{i,i}$ in Λ_β is

$$\{\lambda_{ki,ki} : \forall k \in \mathbb{N}, \exists a, b \in \mathbb{N}, a > b, (a, b) = 1 \text{ s.t. } i = a^2 + b^2\}. \quad \square$$

Lemma 3.3. *If $\beta = \frac{p}{q} > 1$, then*

$$\begin{aligned}
&\Lambda_\beta = \{\lambda_{qi,pj}\}_{i\neq j} \\
&\cup \{\lambda_{kqi,kpi} : \forall k \in \mathbb{N}, \exists a, b \in \mathbb{N}, a > b, (a,b) = 1 \ \ s.t.\ i = a^2 + b^2\} \\
&\cup \{\lambda_{mq^2-n,sp^2+t} : m,n,s,t \in \mathbb{N}, mq^2 - n, sp^2 - t \geq 0, mn = st, \frac{n}{q} or \frac{t}{p} \notin \mathbb{N}\} \\
&\cup \{\lambda_{\frac{mq^2-n}{j},\frac{sp^2+t}{j}} : m,n,s,t \in \mathbb{N}, mn = st, 1 < j \leq 2p^2q^2, \\
&\frac{mq^2-n}{j}, \frac{mq^2+n}{j}, \frac{sp^2-t}{j}, \frac{sp^2+t}{j} \in \mathbb{N}\cup\{0\}\}.
\end{aligned}$$

Proof. Clearly $\lambda_{qi,pj} = \lambda_{qj,pi}, \forall i, j$. For $i \neq j$, one has $(qi, pj) \neq (qj, pi)$, so $\{\lambda_{qi,pj}\}_{i\neq j} \subseteq \Lambda_\beta$. For $i = j$, to the end that $\lambda_{qi,pi} = \lambda_{qm,pn}$, it also requires that $2i^2 = m^2 + n^2$. So

$$\{\lambda_{kqi,kpi} : \forall k \in \mathbb{N}, \exists a, b \in \mathbb{N}, a > b, (a,b) = 1 \text{ s.t. } i = a^2 + b^2\} \subseteq \Lambda_\beta.$$

Other than the points $\{\lambda_{qi,pj}\}$, the set S_β also includes some points which cannot be written in the form $\lambda_{qi,pj}$. It is easy to verify that for any $\lambda_{qj,pi}$ there could not exist $\lambda_{l,k}$ satisfying $\lambda_{l,k} = \lambda_{qj,pi}$ for $\frac{l}{q}$ or $\frac{k}{p} \notin \mathbb{Z}$.

For any $m, n, s, t \in \mathbb{Z}$ satisfying $mn = st$, we have that

$$\lambda_{mq^2-n,sp^2+t} = \lambda_{mq^2+n,sp^2-t}. \tag{8}$$

We may suppose that $mn = st \geq 0$, otherwise we replace n, t by $-n, -t$ respectively. We claim that $mn = st \neq 0$ if $\lambda_{mq^2-n,sp^2+t} = \lambda_{mq^2+n,sp^2-t} \in \Lambda_\beta$. Indeed, if $mn = st = 0$, from (8) and the fact that $mq^2 - n, sp^2 + t, mq^2 + n, sp^2 - t$ are all nonnegative integers, we deduce that $(mq^2 - n, sp^2 + t) = (mq^2 + n, sp^2 - t)$. Hence we may assume that $mn = st > 0$. From this and the fact that $sp^2 + t, mq^2 + n$ are both nonnegative, so actually $m, n, s, t \in \mathbb{N}$. Therefore (8) holds true for $m, n, s, t \in \mathbb{N}, mq^2 - n \geq 0, sp^2 - t \geq 0, mn = st$.

If at least one of $\frac{n}{q}$ and $\frac{t}{p}$ is not an integer, then the elements $\lambda_{mq^2-n,sp^2+t} (= \lambda_{mq^2+n,sp^2-t})$ in (8) actually belong to the subset of the points in Λ_β that cannot be written as the form $\lambda_{qi,pj}$.

Note that it is possible that there exists $\lambda_{c,d}, \lambda_{e,f}$, satisfying $\lambda_{c,d} = \lambda_{e,f}$, belong to the subset of the points in Λ_β that cannot be written as the form $\lambda_{qi,pj}$, and cannot be written as the form (8). Namely at least one of the following two linear systems of unknown quantities (m, n) and (s, t)

$$\begin{cases} mq^2 - n = c, \\ mq^2 + n = e, \end{cases} \qquad \begin{cases} sp^2 + t = d, \\ sp^2 - t = f, \end{cases}$$

does not admit positive integer solutions. However, observe that there exists some positive integer j not exceeding $2p^2q^2$ such that both the following two systems

$$\begin{cases} mq^2 - n = cj, \\ mq^2 + n = ej, \end{cases} \qquad \begin{cases} sp^2 + t = dj, \\ sp^2 - t = fj, \end{cases}$$

necessarily admit positive integer solutions (m, n), (s, t).

Hence the subset of the points that cannot be written as the form $\lambda_{qi,pj}$ in Λ_β is

$$\{\lambda_{mq^2-n,sp^2+t} : m,n,s,t \in \mathbb{N}, mq^2-n, sp^2-t \geq 0, mn = st, \frac{n}{q} \text{ or } \frac{t}{p} \notin \mathbb{N}\}$$
$$\cup \{\lambda_{\frac{mq^2-n}{j},\frac{sp^2+t}{j}} : m,n,s,t \in \mathbb{N}, mn = st, 1 < j \leq 2p^2q^2,$$
$$\frac{mq^2-n}{j}, \frac{mq^2+n}{j}, \frac{sp^2-t}{j}, \frac{sp^2+t}{j} \in \mathbb{N} \cup \{0\}\}. \quad \square$$

Remark 3.1. For $\beta = 1$ or $\beta = \frac{p}{q} > 1$, any elements $\lambda_{i,j}, \lambda_{k,l} \in \Lambda_\beta$ satisfying $\lambda_{i,j} = \lambda_{k,l}$ and $(i, j) \neq (k, l)$, neither $\frac{k}{i}, \frac{l}{j} \in \mathbb{N} \cup \{0\}$ nor $\frac{i}{k}, \frac{j}{l} \in \mathbb{N} \cup \{0\}$ takes place.

Lemma 3.4. *If $\beta \in \bar{Q}$ and $\beta^2 = \frac{p}{q}$, then*

$$\Lambda_\beta = \{\lambda_{mq-n,sp+t} : m,n,s,t \in \mathbb{N}, mq-n, sp-t \geq 0, mn = st\}$$
$$\cup \{\lambda_{\frac{mq-n}{j},\frac{sp+t}{j}} : m,n,s,t \in \mathbb{N}, mn = st, 1 < j \leq 2pq,$$
$$\frac{mq-n}{j}, \frac{mq+n}{j}, \frac{sp-t}{j}, \frac{sp+t}{j} \in \mathbb{N} \cup \{0\}\}.$$

Proof. For any $m, n, s, t \in \mathbb{Z}$ satisfying $mn = st$, we have that

$$\lambda_{mq-n,sp+t} = \lambda_{mq+n,sp-t}. \tag{9}$$

We also assume that $mn = st \geq 0$. We claim that $mn = st \neq 0$ if $\lambda_{mq-n,sp+t} = \lambda_{mq+n,sp-t} \in \Lambda_\beta$. Indeed, if $mn = st = 0$, combining (9) and the fact that $mq - n, sp + t, mq + n, sp - t$ are all nonnegative integers, we deduce that $(mq - n, sp + t) = (mq + n, sp - t)$. Hence we may assume that $mn = st > 0$. From this and the fact that $sp + t, mq + n$ are both nonnegative, we can actually derive that $m, n, s, t \in \mathbb{N}$. So (9) holds true for $m, n, s, t \in \mathbb{N}, mn = st, mq - n \geq 0, sp - t \geq 0$.

Note that it is possible that there exists $\lambda_{c,d}, \lambda_{e,f} \in \Lambda_\beta$ satisfying $\lambda_{c,d} = \lambda_{e,f}$ that cannot be written in the form (9). Namely at least one of the following two linear systems

$$\begin{cases} mq - n = c, \\ mq + n = e, \end{cases} \quad \begin{cases} sp + t = d, \\ sp - t = f, \end{cases}$$

does not admit positive integer solutions. However, there exists some positive integer j not exceeding $2pq$ such that both the following two systems

$$\begin{cases} mq - n = cj, \\ mq + n = ej, \end{cases} \quad \begin{cases} sp + t = dj, \\ sp - t = fj, \end{cases}$$

necessarily admit positive integer solutions (m, n), (s, t). $\square$

Remark 3.2. For $\beta \in \bar{Q}$ and $\beta^2 = \frac{p}{q}$, it is possible there exist elements $\lambda_{i,j}, \lambda_{k,l} \in \Lambda_\beta$ satisfying $\lambda_{i,j} = \lambda_{k,l}$ and $(i,j) \neq (k,l)$ such that $\frac{k}{i}, \frac{l}{j} \in \mathbb{N} \cup \{0\}$ or $\frac{i}{k}, \frac{j}{l} \in \mathbb{N} \cup \{0\}$. An example is that $p = 3, q = 2, \lambda_{5,0} = 25 = \lambda_{1,6}$.

Proof of Theorem 1.2. Theorem 1.2 follows from Lemmas 3.1-3.4. □

4. Generic flat torus case

4.1. Case $\alpha \in \bar{Q}$

Proof of Theorem 1.4. (i) From $\mu_{i,j} = \mu_{m,n}$, due to $\alpha \in \bar{Q}$ and $\alpha^2 + \beta^2 \in Q$, one has

$$ij = mn, \quad \frac{p}{q} i^2 + j^2 = \frac{p}{q} m^2 + n^2,$$

which lead to $\sqrt{\frac{p}{q}} i + j = \pm(\sqrt{\frac{p}{q}} m + n)$. Then $(m,n) = (i,j)$ or $(-i,-j)$, because $\sqrt{\frac{p}{q}}$ is an irrational number. So $\Lambda_{\alpha,\beta} = \emptyset$, which gives $K_{\alpha,\beta} = S_{\alpha,\beta}$.

(ii) From $\mu_{i,j} = \mu_{m,n}$, we deduce that $ij = mn$ and

$$\frac{p}{q} i + j = \pm\left(\frac{p}{q} m + n\right), \tag{10}$$

$$\frac{p}{q} i - j = \pm\left(\frac{p}{q} m - n\right). \tag{11}$$

For (10) we first consider the case $\frac{p}{q} i + j = \frac{p}{q} m + n$, which holds true if and only if $m - i = kq$, $j - n = kp$, $k \neq 0 \in \mathbb{Z}$. Substituting $m = i + kq$, $j = n + kp$ into $ij = mn$, we obtain $pi = qn$. Multiplying j on two sides of this equality, we have $qj = pm$. In fact $(m,n) \neq (i,j)$ corresponds to $k \neq 0$.

Similarly for (11) we consider the case $\frac{p}{q} i - j = -\left(\frac{p}{q} m - n\right)$, which holds true if and only if $m + i = kq$, $j + n = kp$. Substituting $m = kq - i$, $j = kp - n$ into $ij = mn$, we obtain $pi = qn$. Multiplying j on both sides of this equality, we have $qj = pm$. Note that $(m,n) \neq (-i,-j)$ corresponds to $k \neq 0$.

For the other case in (10) and (11), similar argument yields $pi = -qn, qj = -pm$. However this is equivalent to the result in the above case, since (i,j) and $(-i,-j)$ yield the same eigenvalue and eigenfunction.

Hence

$$\Lambda_{\alpha,\beta} = \left\{ \mu_{i,j} : \exists m, n \in \mathbb{Z},\ (m,n) \neq (i,j), (-i,-j) \text{ s.t. } \left\{ \begin{array}{l} pi = qn, \\ pm = qj \end{array} \right. \right\}.$$

In particular, if $\alpha^2 + \beta^2 = \frac{p}{q} = 1$, namely $p = q = 1$, then the equalities in the above set become $m = j, n = i$, namely $(m,n) = (j,i)$. Therefore

$$\left\{ \mu_{i,j} : \exists (m,n) \neq (i,j), (-i,-j) \text{ s.t. } (m,n) = (j,i) \right\} = \{\mu_{i,j}\}_{i \neq j, i \neq -j},$$

which yields

$$K_{\alpha,\beta} = S_{\alpha,\beta} \backslash \Lambda_{\alpha,\beta} = \{\mu_{i,j}\}_{i^2+j^2\neq 0} \backslash \{\mu_{i,j}\}_{i\neq j, i\neq -j} = \{\mu_{i,i}, \mu_{i,-i}\}_{i\neq 0}. \quad \square$$

Proof of Theorem 1.5. (i). Since $\frac{\alpha^2+\beta^2}{\alpha} \in Q$, from the relation $\frac{\mu_{i,j}}{\alpha} = \frac{\mu_{m,n}}{\alpha}$, we deduce that $j^2 = n^2$. If $j = n$, the equality $\mu_{i,j} = \mu_{m,n}$ yields $\frac{\alpha^2+\beta^2}{2\alpha} = \frac{j}{i+m}(i \neq m)$. If $j = -n$, the equality $\mu_{i,j} = \mu_{m,n}$ yields $\frac{\alpha^2+\beta^2}{2\alpha} = \frac{j}{i-m}(i \neq -m)$, which is equivalent to the result in the case $j = n$, since (i, j) and $(-i, -j)$ yield the same eigenvalue and eigenfunction.

(ii) An elementary computation shows that

$$\frac{\mu_{m,n} - \mu_{i,j}}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha}(m^2 - i^2) - \frac{1}{\alpha}(j^2 - n^2) - 2(mn - ij).$$

If (6) does not admit non-zero integer solutions, then from $\mu_{m,n} = \mu_{i,j}$ we deduce that $(m, n) = (i, j)$ or $(-i, -j)$, so $\Lambda_{\alpha,\beta} = \emptyset$.

It is easy to compute that

$$\sqrt{4(mn - ij)^2 + 4(m^2 - i^2)(j^2 - n^2)} = 2|mj - in|, \tag{12}$$

which contradicts with the assumption that any non-zero solution (e, f, g) of (6) satisfying $\sqrt{g^2 + 4ef} \in \bar{Q}$. Therefore $\mu_{m,n} - \mu_{i,j} \neq 0$ for any $(m, n) \neq (i, j), (-i, -j)$, which shows $\Lambda_{\alpha,\beta} = \emptyset$.

(iii) Given non-zero integer solution (e, f, g) of (5) satisfying $\sqrt{g^2 + 4ef} \in Q$. If the following system admits integer solutions $(i, j, m, n, k)(k \neq 0)$

$$\begin{cases} m^2 - i^2 = ek, \\ j^2 - n^2 = fk, \\ 2(mn - ij) = gk, \end{cases} \tag{13}$$

then $\mu_{i,j} - \mu_{m,n} = 0$. From (12) we deduce

$$2(mj - in) = k\sqrt{g^2 + 4ef},$$

where we assume $mj - in, k \geq 0$. This equation and the third equation in (13) lead to

$$2(m - i)(j + n) = k(g + \sqrt{g^2 + 4ef}).$$

Combining this with the first and second equation in (13) respectively, we obtain

$$\begin{cases} m + i = d_1(j + n), \ i \neq m, \\ j - n = d_2(m - i), \ j \neq -n, \end{cases}$$

where d_1, d_2 are defined in (7). From this system we obtain $(m, n) \neq (i, j), (-i, -j)$ and

$$\begin{cases} i = \frac{2d_1 n + (d_1 d_2 - 1)m}{1 + d_1 d_2}, \\ j = \frac{2d_2 m + (1 - d_1 d_2)n}{1 + d_1 d_2}. \end{cases} \quad \square$$

Remark 4.1. Under the assumptions that $\alpha \in \bar{Q}$, $\alpha^2+\beta^2 \in \bar{Q}$ and $\frac{\alpha^2+\beta^2}{\alpha} \in \bar{Q}$, any integer solution (e, f, g) of (6) satisfies either $e=f=g=0$ or $e \neq 0$, $f \neq 0$, $g \neq 0$. Moreover, for non-zero solution (e, f, g), the components e and f necessarily possess the same sign, so $d_1 d_2 > 0$.

4.2. *Case* $\alpha = \frac{p}{q}$

4.2.1. *Subcase* $\alpha^2+\beta^2 = 1$

Proof of Theorem 1.6. Clearly $\mu_{i,j} = \mu_{j,i}$, $\forall i, j$. From $\mu_{i,i} = \mu_{m,n}$, we have

$$\frac{2(q-p)}{q} i^2 = m^2 + n^2 - \frac{2p}{q} mn.$$

If $i^2 = mn$, then $m^2+n^2 = 2i^2$, and we deduce that $m=n=i$ or $m=n=-i$. If $i^2 \neq mn$, then

$$\frac{m^2+n^2-2i^2}{mn-i^2} = \frac{2p}{q}. \tag{14}$$

Similarly, from $\mu_{i,-i} = \mu_{m,n}$, we have $\frac{2(q+p)}{q} i^2 = m^2+n^2-\frac{2p}{q}mn$. If $i^2 = -mn$, then $m^2+n^2 = 2i^2$, and we deduce that $m=-n=i$ or $m=-n=-i$. If $i^2 \neq -mn$, then

$$\frac{m^2+n^2-2i^2}{mn+i^2} = \frac{2p}{q}. \tag{15}$$

So

$$\Lambda_{\alpha,\beta} = \{\mu_{i,j}\}_{i\neq j} \cup \left\{\mu_{i,i} : \exists m, n \in \mathbb{Z}, mn \neq i^2, \text{ s.t. } \frac{m^2+n^2-2i^2}{mn-i^2} = \frac{2p}{q}\right\}$$
$$\cup \left\{\mu_{i,-i} : \exists m, n \in \mathbb{Z}, mn \neq -i^2, \text{ s.t. } \frac{m^2+n^2-2i^2}{mn+i^2} = \frac{2p}{q}\right\}.$$

In (14), if we take $m = i$, then (14) become $\frac{n}{i} = \frac{2p-q}{q}$. Furthermore, if q is odd then we take $i = q$, and if q is even then we take $i = \frac{q}{2}$. Then we obtain $n = 2p - q$ and $n = p - \frac{q}{2}$ respectively. Namely we obtain

$$\begin{cases} \mu_{q,q} = \mu_{q,2p-q}, & \text{if } q \text{ is odd,} \\ \mu_{\frac{q}{2},\frac{q}{2}} = \mu_{\frac{q}{2},p-\frac{q}{2}}, & \text{if } q \text{ is even.} \end{cases} \tag{16}$$

Similarly, in (15), if we take $m = i$, then (15) become $\frac{n}{i} = \frac{2p+q}{q}$. If q is odd then we take $i = q$, and if q is even then we take $i = \frac{q}{2}$, and we obtain $n = 2p+q$ and $n = p + \frac{q}{2}$ respectively. Namely we have

$$\begin{cases} \mu_{q,-q} = \mu_{q,2p+q}, & \text{if } q \text{ is odd,} \\ \mu_{\frac{q}{2},-\frac{q}{2}} = \mu_{\frac{q}{2},p+\frac{q}{2}}, & \text{if } q \text{ is even.} \end{cases} \tag{17}$$

If $\alpha = \frac{1}{2}$, namely $q = 2$, then (16) and (17) give that

$$\mu_{1,1} = \mu_{1,0} = \mu_{0,1}, \quad \mu_{1,-1} = \mu_{1,2} = \mu_{2,1},$$

so

$$\mu_{i,i} = \mu_{i,0} = \mu_{0,i}, \quad \mu_{i,-i} = \mu_{i,2i} = \mu_{2i,i}, \ \forall\, i \in \mathbb{Z}.$$

Hence $\Lambda_{\alpha,\beta} = S_{\alpha,\beta}$. □

4.2.2. Subcase $\alpha^2 + \beta^2 > 1$

Recall that $\alpha = \frac{p}{q}$. To make $\mu_{i,j} = \mu_{i,n}$ and $n \neq j, n \neq -j$, we deduce $j + n = 2\alpha i = \frac{2p}{q} i$. From $n \neq -j$ we have $i \neq 0$. If q is odd, then $i = qk$, $j + n = 2pk$, and if q is even, then $i = \frac{q}{2}k$, $j + n = pk$ for some $k \in \mathbb{Z}$, since $i \neq 0$, so $k \neq 0$. Note that from $\mu_{i,j} = \mu_{-i,n}$ and $(-i,n) \neq (i,j), (-i,-j)$, we deduce $j - n = 2\alpha i = \frac{2p}{q} i (j \neq -n, i \neq 0)$, which is equivalent to the above case $\mu_{i,j} = \mu_{i,n}$, since (i,j) and $(-i,-j)$ associate with the same eigenvalue and eigenfunction. Hence

$$\Lambda_{\alpha,\beta} \supseteq \begin{cases} \left\{\mu_{qk,j}\right\}_{k\neq 0, j \neq pk}, & \text{if } q \text{ is odd}, \\ \left\{\mu_{\frac{q}{2}k,j}\right\}_{k\neq 0, j\neq \frac{p}{2}k}, & \text{if } q \text{ is even}. \end{cases} \tag{18}$$

Proof of Theorem 1.7. Since $\alpha \in Q$, $\beta^2 \in \bar{Q}$, then $\alpha^2 + \beta^2 \in \bar{Q}$. So the equation $\mu_{i,j} = \mu_{m,n}$ yields $i^2 = m^2$. The above analysis gives the desired result of this theorem. □

If $\beta^2 \in Q$, to have $\mu_{i,j} = \mu_{m,j}$ and $m \neq i, m \neq -i$, we require $\frac{j}{i+m} = \frac{\alpha^2+\beta^2}{2\alpha} (i \neq m, j \neq 0)$. This means that other than the elements in (18), $\Lambda_{\alpha,\beta}$ includes also the points satisfying that $\mu_{i,j} = \mu_{m,j}$, $\frac{j}{i+m} = \frac{\alpha^2+\beta^2}{2\alpha} (i \neq m, j \neq 0)$. Hence $\Lambda_{\alpha,\beta} \neq \emptyset$, namely $K_{\alpha,\beta} \subsetneq S_{\alpha,\beta}$.

Proof of Theorem 1.8. (i) We claim that $\mu_{0,1} \notin \Lambda_{\alpha,\beta}$. Indeed it is clear that for any $|j| \geq 2$, one has $\mu_{0,j} > \mu_{0,1}$. For $i \neq 0$, we have

$$\mu_{i,j} = (\alpha i - j)^2 + \beta^2 i^2 \geq \beta^2 i^2 > i^2 \geq 1 = \mu_{0,1}, \ \forall\, j.$$

Hence the claim is true. So $\Lambda_{\alpha,\beta} \subsetneq S_{\alpha,\beta}$, namely $K_{\alpha,\beta} \neq \emptyset$.

(ii) Note that for any $i \in \mathbb{Z}$, we have

$$\begin{cases} \mu_{i,j} = \mu_{i,2\alpha i - j}, & \forall j \neq \alpha i, \\ \mu_{i,\alpha i} = \mu_{0,i}, \end{cases}$$

hence $\Lambda_{\alpha,\beta} = S_{\alpha,\beta}$ and $K_{\alpha,\beta} = \emptyset$.

(iii) We claim that $\mu_{1,\alpha} = \beta^2 \notin \Lambda_{\alpha,\beta}$. Indeed, since $\beta^2 < 1$, it is clear that $\mu_{0,j} > \mu_{1,\alpha}, \forall j \neq 0$. For any $j \neq \alpha$, we have

$$\mu_{-1,-j} = \mu_{1,j} = (\alpha - j)^2 + \beta^2 > \beta^2 = \mu_{1,\alpha}.$$

For any $|i| \geq 2$ and j, we have

$$\mu_{i,j} = (\alpha i - j)^2 + \beta^2 i^2 \geq 4\beta^2 > \beta^2 = \mu_{1,\alpha}.$$

Hence the claim is true.

(iv) We claim that $\mu_{0,1} \notin \Lambda_{\alpha,\beta}$. It is clear that $\mu_{0,j} > \mu_{0,1}, \forall |j| > 1$. Since $\alpha \notin \mathbb{N}$, if $\beta = 1$, then for any $i \neq 0$, we have

$$\mu_{i,j} = (\alpha i - j)^2 + i^2 > i^2 \geq 1 = \mu_{0,1}, \ \forall j.$$

Hence $\mu_{0,1} \notin \Lambda_{\alpha,\beta}$.

If $\sqrt{1 - b_\alpha^2} < \beta < 1$, namely $b_\alpha^2 + \beta^2 > 1$, since $b_\alpha^2 \leq \frac{1}{4}$, we have $\beta^2 > \frac{3}{4}$. Note that $\min_j \mu_{1,j} = b_\alpha^2 + \beta^2$. So the assumption $b_\alpha^2 + \beta^2 > 1$ gives $\mu_{1,j} = \mu_{-1,-j} > \mu_{0,1}, \forall j$. For any $|i| \geq 2$ and j, we have $\mu_{i,j} \geq 4\beta^2 > 3 > \mu_{0,1}$. So $\mu_{0,1} \notin \Lambda_{\alpha,\beta}$.

(v) We first consider the case $\beta = \sqrt{1 - b_\alpha^2}$ and $q > 2$. Since $q > 2$, then $\alpha - [\alpha] \neq [\alpha] + 1 - \alpha$. We claim that if $b_\alpha = \alpha - [\alpha]$ then $\mu_{1,[\alpha]+1} \notin \Lambda_{\alpha,\beta}$, and if $b_\alpha = [\alpha] + 1 - \alpha$ then $\mu_{1,[\alpha]} \notin \Lambda_{\alpha,\beta}$.

Let $b_\alpha = \alpha - [\alpha]$. It is easy to see that

$$\mu_{-1,-[\alpha]} = \mu_{1,[\alpha]} = b_\alpha^2 + \beta^2 = 1 < \mu_{1,[\alpha]+1} = (1 - b_\alpha)^2 + \beta^2 < 2.$$

For any $j \neq [\alpha],\ [\alpha] + 1$, we have $\mu_{-1,-j} = \mu_{1,j} > \mu_{1,[\alpha]+1}$. For $|i| \geq 2$, we have

$$\mu_{i,j} \geq i^2\beta^2 \geq 3 > \mu_{1,[\alpha]+1}, \ \ \forall\, j,$$

where we used the fact that $\beta^2 \geq \frac{3}{4}$, since $b_\alpha^2 + \beta^2 = 1$. Clearly $\mu_{0,j} \neq \mu_{1,[\alpha]+1}, \ \forall\, j$.

To sum up, $\mu_{1,[\alpha]+1} \notin \Lambda_{\alpha,\beta}$. The argument of this claim for $b_\alpha = [\alpha] + 1 - \alpha$ is similar, we omit it.

Now we consider the case $q = 2$. One has $b_\alpha = \frac{1}{2}, \beta^2 = \frac{3}{4}$. From the derivation of (18), we have $\mu_{i,j} = \mu_{i,pi-j}, \forall i, j$. Note that p is odd. So for odd i, we have $j \neq pi - j$. For non-zero even i, we have $j \neq pi - j$ as $j \neq \frac{pi}{2}$. Observe that $\mu_{2,p} = 3 = \mu_{1,\frac{p+3}{2}}$. Hence for any non-zero even i, we have $\mu_{i,\frac{pi}{2}} = \frac{3i^2}{4} = \mu_{\frac{i}{2},\frac{(p+3)i}{4}}$. On the other hand, the relation $\mu_{0,1} = 1 = \mu_{1,\frac{p+1}{2}}$ gives $\mu_{0,j} = j^2 = \mu_{j,\frac{(p+1)j}{2}}, \forall j \neq 0$.

To sum up, $\Lambda_{\alpha,\beta} = S_{\alpha,\beta}$, so $K_{\alpha,\beta} = \emptyset$. □

Next we consider the case that $\beta^2 \in Q$, $\alpha = \frac{p}{q} \notin \mathbb{N}$ (so $q \geq 2$) and $\beta < \sqrt{1 - b_\alpha^2}$. We discuss it by dividing into $q = 2$ and $q \geq 3$. For $q = 2$ we have $b_\alpha = \frac{1}{2}$ and $\beta < \frac{\sqrt{3}}{2}$.

Proof of Theorem 1.9. (i) We claim that $\mu_{2,p} \notin \Lambda_{\alpha,\beta}$.

We first consider the case $\beta < \frac{\sqrt{3}}{6}$, namely $\frac{1}{4} + \beta^2 > 4\beta^2$. Note that $4\beta^2 = \mu_{2,p}$. Clearly $\mu_{-2,-j} = \mu_{2,j} > \mu_{2,p}, \forall j \neq p$. Note that $\min_j \mu_{1,j} = \frac{1}{4} + \beta^2$, hence $\mu_{1,j} = \mu_{-1,-j} > \mu_{2,p}, \forall j$. We have

$$\mu_{i,j} \geq 4k^2\beta^2 > 4\beta^2 = \mu_{2,p}, \forall j, |i| = 2k, k \geq 2 \tag{19}$$

and

$$\mu_{i,j} \geq \frac{1}{4} + i^2\beta^2 > 9\beta^2 > 4\beta^2 = \mu_{2,p}, \forall j, |i| = 2k-1, k \geq 2. \tag{20}$$

We also have

$$\mu_{2,p} = 4\beta^2 < \frac{1}{4} + \beta^2 < 1 \leq \mu_{0,j}, \forall j \neq 0.$$

Hence $\mu_{2,p} \notin \Lambda_{\alpha,\beta}$.

We consider the other case $\beta \in (\frac{\sqrt{3}}{6}, \frac{1}{2}) \cup (\frac{1}{2}, \frac{\sqrt{3}}{2})$, namely $\frac{1}{4} + \beta^2 < 4\beta^2$ and $\beta^2 \neq \frac{1}{4}$. Note that $\mu_{2,p} = 4\beta^2 < 3$ and $\mu_{2,p} \neq 1$. Hence $\mu_{2,p} \neq \mu_{0,j}, \forall j \neq 0$. It is easy to see that $\mu_{2,p} = 4\beta^2 < \frac{9}{4} + \beta^2$. Combining this and the fact that $\min_j \mu_{1,j} = \frac{1}{4} + \beta^2 < 4\beta^2 = \mu_{2,p}$, we obtain that $\mu_{2,p} \neq \mu_{1,j} = \mu_{-1,-j}, \forall j$. Clearly $\mu_{2,j} > \mu_{2,p}, \forall j \neq p$. Combining this analysis and (19)-(20), we deduce that $\mu_{2,p} \notin \Lambda_{\alpha,\beta}$.

(ii) We first consider the case $\beta = \frac{1}{2}$. From (18), for any i and $j \neq \frac{pi}{2}$, $\mu_{i,j} \in \Lambda_{\alpha,\beta}$. If i is odd, owing to p is odd, then $\frac{pi}{2} \notin \mathbb{Z}$. If i is even, note that

$$\mu_{i,\frac{pi}{2}} = \frac{i^2}{4} = \mu_{0,\frac{i}{2}}, \quad \forall\, i \neq 0.$$

Hence $\Lambda_{\alpha,\beta} = S_{\alpha,\beta}$.

Now we consider the case $\beta = \frac{\sqrt{3}}{6}$, namely the case that $\frac{1}{4} + \beta^2 = 4\beta^2$. For any given (i, j), to find $(m, n) \neq (i, j), (-i, -j)$ such that $\mu_{m,n} = \mu_{i,j}$, we need to solve

$$[p(i+m) - 2(j+n)][p(i-m) - 2(j-n)] = \frac{1}{3}(m+i)(m-i).$$

Set $m - i = 3k$, then

$$[p(2i+3k) - 2(j+n)][-3kp - 2(j-n)] = (2i+3k)k.$$

Let

$$\begin{cases} n + j = \frac{(p+1)(2i+3k)}{2}, \\ n - j = \frac{(3p-1)k}{2}, \end{cases}$$

we obtain $j = \frac{(p+1)i}{2} + k$ and

$$n = \frac{1}{2}[(p+1)i + (3p+1)k]. \tag{21}$$

Substituting $k = j - \frac{(p+1)i}{2}$ into $m = i + 3k$ and (21), we obtain

$$\begin{cases} m = 3j - \frac{(3p+1)i}{2}, \\ n = \frac{(3p+1)j}{2} - \frac{(p+1)(3p-1)i}{4}. \end{cases} \tag{22}$$

Note that p is odd, so both the values of m, n obtained in (22) are necessarily integers. To the end that $(m, n) \neq (i, j)$, we deduce $k \neq 0$, so $j \neq \frac{(p+1)i}{2}$. For $j = \frac{(p+1)i}{2}$, we have

$$\mu_{i, \frac{(p+1)i}{2}} = \frac{i^2}{3} = \mu_{2i, pi}, \quad \forall i \neq 0.$$

An elementary computation shows that $m + i = 2i + 3k = 3\left(j - \frac{(3p-1)i}{6}\right)$ and

$$n + j = \frac{(p+1)(2i+3k)}{2} = \frac{3(p+1)}{2}\left(j - \frac{(3p-1)i}{6}\right).$$

To have that $(m, n) \neq (-i, -j)$, one requires that $j \neq \frac{(3p-1)i}{6}$. Note that $\frac{(3p-1)i}{6} \in \mathbb{Z}$ if and only if $i = 3s$ for some $s \in \mathbb{Z}$. Observe that when $i = 3s$,

$$\mu_{i, \frac{(3p-1)i}{6}} = \frac{i^2}{9} = s^2 = \mu_{0,s}, \quad \forall i \neq 0.$$

To sum up, $\Lambda_{\alpha,\beta} = S_{\alpha,\beta}$. □

Proof of Theorem 1.10. Note that $\beta < \sqrt{1 - b_\alpha^2} = \frac{\sqrt{q^2-1}}{q}$, since $b_\alpha = \frac{1}{q}$.

Claim 1: For the case $\beta < \frac{1}{q\sqrt{q^2-1}}$, namely $b_\alpha^2 + \beta^2 > \beta^2 q^2$, we have $\mu_{q,p} \notin \Lambda_{\alpha,\beta}$.

Clearly $\mu_{-q,-j} = \mu_{q,j} > \mu_{q,p}, \forall j \neq p$. We have

$$\mu_{qk,j} \geq q^2 k^2 \beta^2 > q^2 \beta^2 = \mu_{q,p}, \forall j, i = qk, |k| \geq 2. \tag{23}$$

Since $b_\alpha = \frac{1}{q}$, we have

$$\mu_{i,j} \geq b_\alpha^2 + i^2 \beta^2 \geq b_\alpha^2 + \beta^2 > q^2 \beta^2 = \mu_{q,p}, \forall j, 1 \leq |i| \neq q|k|. \tag{24}$$

We also have

$$\mu_{q,p} = q^2 \beta^2 < b_\alpha^2 + q^2 < 1 \leq \mu_{0,j}, \forall j \neq 0.$$

Hence $\mu_{q,p} \notin \Lambda_{\alpha,\beta}$.

Claim 2: For the case $\frac{1}{q\sqrt{q^2-1}} < \beta < \frac{\sqrt{q^2-1}}{q}$, namely $b_\alpha^2 + \beta^2 < \beta^2 q^2$, if $b_\alpha = \alpha - [\alpha]$ then $\mu_{1,[\alpha]} \notin \Lambda_{\alpha,\beta}$, and if $b_\alpha = [\alpha] + 1 - \alpha$ then $\mu_{1,[\alpha]+1} \notin \Lambda_{\alpha,\beta}$.

Let $b_\alpha = \alpha - [\alpha]$. It is easy to see that

$$\mu_{1,[\alpha]} = b_\alpha^2 + \beta^2 = \min_j \mu_{1,j} < \mu_{1,j} = \mu_{-1,-j}, \; j \neq [\alpha].$$

For $i = kq, k \neq 0 \in \mathbb{Z}$, we have

$$\mu_{kq,j} \geq k^2 q^2 \beta^2 \geq q^2 \beta^2 > b_\alpha^2 + \beta^2 = \mu_{1,[\alpha]}, \forall j.$$

For $2 \le |i| \ne |k|q$, due to $b_\alpha = \frac{1}{q}$, we have

$$\mu_{i,j} \ge b_\alpha^2 + i^2\beta^2 > b_\alpha^2 + \beta^2 = \mu_{1,[\alpha]}, \quad \forall\, j.$$

The assumption $b_\alpha^2 + \beta^2 < 1$ yields $\mu_{0,j} > \mu_{1,[\alpha]}, \ \forall\, j \ne 0$. Hence $\mu_{1,[\alpha]} \notin \Lambda_{\alpha,\beta}$. The argument of this claim for $b_\alpha = [\alpha] + 1 - \alpha$ is similar, we omit it.

Claim 3: For the case $\beta = \frac{1}{q\sqrt{q^2-1}}$, namely $b_\alpha^2 + \beta^2 = \beta^2 q^2$, if $b_\alpha = \alpha - [\alpha]$ then $\mu_{q-1,(q-1)[\alpha]+1} \notin \Lambda_{\alpha,\beta}$, and if $b_\alpha = [\alpha] + 1 - \alpha$ then $\mu_{q-1,(q-1)([\alpha]+1)-1} \notin \Lambda_{\alpha,\beta}$. We only prove the case of $b_\alpha = \alpha - [\alpha]$.

Let $b_\alpha = \alpha - [\alpha]$. Note that $b_\alpha^2 + \beta^2 = \beta^2 q^2 = \frac{1}{q^2-1}$. Plainly we have

$$\mu_{q-1,(q-1)[\alpha]+1} < \mu_{q-1,j} = \mu_{-(q-1),-j}, \ \forall j \ne (q-1)[\alpha] + 1.$$

We have

$$\mu_{q-1,(q-1)[\alpha]+1} = b_\alpha^2 + (q-1)^2\beta^2 = \frac{2}{q(q+1)} > \frac{1}{q^2-1} = b_\alpha^2 + \beta^2 = \mu_{1,[\alpha]} = \mu_{-1,-[\alpha]}.$$

From this and the relation $\mu_{q,p} = \beta^2 q^2 = b_\alpha^2 + \beta^2 = \mu_{1,[\alpha]}$, we have

$$\mu_{q-1,(q-1)[\alpha]+1} > \mu_{q,p} = \mu_{-q,-p}.$$

For any $j \ne [\alpha]$, we have

$$\begin{aligned} \mu_{q-1,(q-1)[\alpha]+1} - \mu_{1,j} &\le \mu_{q-1,(q-1)[\alpha]+1} - \mu_{1,[\alpha]+1} \\ &= \frac{2}{q(q+1)} - (\beta^2 + (1-b_\alpha)^2) \\ &= \frac{2}{q(q+1)} - \frac{1}{q^2(q^2-1)} - \frac{(q-1)^2}{q^2} \\ &= -\frac{q^3 - 2q^2 - 2q + 4}{q(q+1)(q-1)} < 0. \end{aligned} \tag{25}$$

So $\mu_{q-1,(q-1)[\alpha]+1} < \mu_{1,j} = \mu_{-1,-j}, \forall j \ne [\alpha]$. From

$$\mu_{q-1,(q-1)[\alpha]+1} = \frac{2}{q(q+1)} < \frac{4}{q^2-1} = 4(b_\alpha^2 + \beta^2),$$

we deduce that

$$\mu_{q-1,(q-1)[\alpha]+1} < \mu_{i,j}, \quad \forall\, j, \ \forall\, 2 \le |i| \le q-2.$$

We also have

$$\mu_{q-1,(q-1)[\alpha]+1} = b_\alpha^2 + (q-1)^2\beta^2 < b_\alpha^2 + q^2\beta^2 < 1 + q^2\beta^2 \le \mu_{q,j} = \mu_{-q,-j}, \quad \forall\, j \ne p.$$

For $q < |i| \ne |k|q$, an elementary computation shows that

$$\mu_{q-1,(q-1)[\alpha]+1} = b_\alpha^2 + (q-1)^2\beta^2 < b_\alpha^2 + i^2\beta^2 \le \mu_{i,j}, \ \forall\ j.$$

For $i = kq\,(|k| \ge 2)$, we have

$$\mu_{kq,kp} = k^2q^2\beta^2 = k^2(b_\alpha^2 + \beta^2) = \frac{k^2}{q^2-1} \ge \frac{4}{q^2-1} > \mu_{q-1,(q-1)[\alpha]+1},$$

which yields that

$$\mu_{q-1,(q-1)[\alpha]+1} < \mu_{kq,j}, \ \ \forall\ j.$$

Finally it is clear that

$$\mu_{q-1,(q-1)[\alpha]+1} < \mu_{0,j}, \ \ \forall\ j \ne 0.$$

To sum up, we have $\mu_{q-1,(q-1)[\alpha]+1} \notin \Lambda_{\alpha,\beta}$.

Claims 1-3 give the desired result of this theorem. □

Finally we need to deal with the case that $b_\alpha \ge \frac{2}{q}$ (so $q = 5$ or $q \ge 7$). An elementary computation shows that

$$\frac{1}{q^2(q^2-4)} < \frac{b_\alpha^2}{q^2-1} < \frac{1}{3}(b_\alpha^2 - \frac{1}{q^2}). \tag{26}$$

Moreover it is easy to verify that the following relation does not hold true

$$q^2\beta^2 = b_\alpha^2 + \beta^2 \le \frac{1}{q^2} + 4\beta^2,$$

since $b_\alpha \ge \frac{2}{q}$ and $q \ge 5$.

Proof of Theorem 1.11. (i) We first consider the case $\beta < \frac{1}{q\sqrt{q^2-4}}$, namely the case

$$q^2\beta^2 < \min\left\{b_\alpha^2 + \beta^2, \frac{1}{q^2} + 4\beta^2\right\}. \tag{27}$$

We claim that $\mu_{q,p} = q^2\beta^2 \notin \Lambda_{\alpha,\beta}$. Indeed, from (27) we know that for any $1 \le |i| \ne q|k|$ and any j, $\mu_{q,p} < \mu_{i,j}$. Clearly $\mu_{q,p} < \mu_{q,j} = \mu_{-q,-j}, j \ne p$, $\mu_{q,p} < \mu_{kq,j}, \forall |k| > 1, \forall j$ and $\mu_{q,p} < \mu_{0,j}, \forall j \ne 0$. Hence $\mu_{q,p} \notin \Lambda_{\alpha,\beta}$.

Now we consider the case $\beta = \frac{1}{q\sqrt{q^2-4}}$, namely the case

$$q^2\beta^2 = \frac{1}{q^2} + 4\beta^2 < b_\alpha^2 + \beta^2. \tag{28}$$

Claim 1: If either q is even or q is odd and $b_\alpha \ne \frac{q-1}{2q}$, then $\mu_{q,p} \notin \Lambda_{\alpha,\beta}$.

If q is even, then $\mu_{2,j} \geq \frac{4}{q^2} + 4\beta^2, \forall j$. Combining this and (28), we obtain $\mu_{q,p} < \mu_{2,j} = \mu_{-2,-j}, \forall j$. From (28) we also know that $\mu_{q,p} < \mu_{1,j}, \forall j$, and for any $3 \leq |i| \neq qk$ and j, $\mu_{q,p} < \mu_{i,j}$. Clearly $\mu_{q,p} < \mu_{q,j} = \mu_{-q,-j}, j \neq p$, $\mu_{q,p} < \mu_{0,j}$ and $\mu_{q,p} < \mu_{kp,j}, \forall |k| > 1, \forall j$. Hence $\mu_{q,p} \notin \Lambda_{\alpha,\beta}$.

If q is odd and $b_\alpha \neq \frac{q-1}{2q}$, then $\mu_{2,j} \geq \frac{4}{q^2} + 4\beta^2, \forall j$. The rest argument is the same as that of the case q is even, we omit it.

Claim 2: If q is odd and $b_\alpha = \frac{q-1}{2q} = \alpha - [\alpha]$, then $\mu_{q-2,(q-2)[\alpha]+\frac{q-3}{2}} \notin \Lambda_{\alpha,\beta}$. Note that

$$\mu_{q-2,(q-2)[\alpha]+\frac{q-3}{2}} = \frac{1}{q^2} + (q-2)^2\beta^2. \tag{29}$$

So $\mu_{q-2,(q-2)[\alpha]+\frac{q-3}{2}} < \mu_{q-2,j} = \mu_{-(q-2),-j}, j \neq (q-2)[\alpha] + \frac{q-3}{2}$. By $\beta^2 = \frac{1}{q^2(q^2-4)}$, we obtain

$$\frac{1}{q^2} + (q-2)^2\beta^2 < \frac{4}{q^2} + 16\beta^2 = 4q^2\beta^2,$$

which means that

$$\mu_{q-2,(q-2)[\alpha]+\frac{q-3}{2}} < \mu_{kq,j}, \ \forall\, |k| \geq 2, \ \forall\, j,$$

and

$$\mu_{q-2,(q-2)[\alpha]+\frac{q-3}{2}} < \mu_{i,j}, \ \forall\, 4 \leq |i| \leq q-1, |i| \neq q-2\,, \forall\, j.$$

It is easy to verify that

$$\frac{1}{q^2} + (q-2)^2\beta^2 < b_\alpha^2 + \beta^2,$$

$$\frac{1}{q^2} + (q-2)^2\beta^2 < \frac{(q-3)^2}{4q^2} + 9\beta^2,$$

$$\frac{1}{q^2} + (q-2)^2\beta^2 < \frac{(q-1)^2}{q^2} + 4\beta^2,$$

and

$$\frac{1}{q^2} + (q-2)^2\beta^2 < 1 + q^2\beta^2.$$

The above four inequalities respectively leads to

$$\mu_{q-2,(q-2)[\alpha]+\frac{q-3}{2}} < \mu_{1,j} = \mu_{-1,-j}, \ \forall\, j,$$

$$\mu_{q-2,(q-2)[\alpha]+\frac{q-3}{2}} < \mu_{3,j} = \mu_{-3,-j}, \ \forall\, j,$$

$$\mu_{q-2,(q-2)[\alpha]+\frac{q-3}{2}} < \mu_{2,j} = \mu_{-2,-j}, \ \forall\, j \neq 2[\alpha]+1,$$

and

$$\mu_{q-2,(q-2)[\alpha]+\frac{q-3}{2}} < \mu_{q,j} = \mu_{-q,-j}, \ \forall\ j \neq p.$$

From (28) and (29) we know that

$$\mu_{q-2,(q-2)[\alpha]+\frac{q-3}{2}} > \mu_{2,2[\alpha]+1} = \mu_{-2,-(2[\alpha]+1)},$$
$$\mu_{q-2,(q-2)[\alpha]+\frac{q-3}{2}} > \mu_{q,p} = \mu_{-q,-p}.$$

Finally, it is clear that

$$\mu_{q-2,(q-2)[\alpha]+\frac{q-3}{2}} < \mu_{i,j}, \ \forall\ q < |i| \neq |k|q, \ \forall\ k, \ \ j,$$
$$\mu_{q-2,(q-2)[\alpha]+\frac{q-3}{2}} < \mu_{0,j}, \ \forall\ j.$$

To sum up, we have $\mu_{q-2,(q-2)[\alpha]+\frac{q-3}{2}} \notin \Lambda_{\alpha,\beta}$.

Claim 3: If q is odd and $b_\alpha = \frac{q-1}{2q} = [\alpha] + 1 - \alpha$, then $\mu_{q-2,(q-2)[\alpha]+\frac{q-1}{2}} \notin \Lambda_{\alpha,\beta}$. The argument is similar as that of Claim 2, we omit it.

(ii) For $\frac{1}{q\sqrt{q^2-4}} < \beta < \sqrt{\frac{1}{3}(b_\alpha^2 - \frac{1}{q^2})}$, namely

$$\frac{1}{q^2} + 4\beta^2 < \min\left\{b_\alpha^2 + \beta^2, q^2\beta^2\right\}, \tag{30}$$

we claim that $\mu_{2,2[\alpha]+1} \notin \Lambda_{\alpha,\beta}$. Note that $\mu_{2,2[\alpha]+1} = \frac{1}{q^2} + 4\beta^2$, since q is odd and $b_\alpha = \frac{q-1}{2q}$. Combining this and (30), we have

$$\mu_{2,2[\alpha]+1} < \mu_{1,j} = \mu_{-1,-j}, \ \forall\ j,$$
$$\mu_{2,2[\alpha]+1} < \mu_{i,j}, \ \forall\ i = kq, \ \forall\ k \neq 0, \ \ j.$$

Clearly

$$\mu_{2,2[\alpha]+1} < \mu_{i,j}, \ \forall\ 3 \leq |i| \neq kq, \ \forall\ k, \ \ j,$$
$$\mu_{2,2[\alpha]+1} < \mu_{2,j} = \mu_{-2,-j}, \ \forall\ j \neq 2[\alpha]+1,$$
$$\mu_{2,2[\alpha]+1} < \mu_{0,j}, \ \forall\ j \neq 0.$$

In another word, $\mu_{2,2[\alpha]+1} \notin \Lambda_{\alpha,\beta}$.

(iii) The case $\beta = \sqrt{\frac{1}{3}(b_\alpha^2 - \frac{1}{q^2})}$ corresponds to the case

$$\frac{1}{q^2} + 4\beta^2 = b_\alpha^2 + \beta^2 < q^2\beta^2. \tag{31}$$

Claim A: if either q is even or q is odd and $b_\alpha \neq \frac{q-1}{2q}$, then

$$\begin{cases} \mu_{1,[\alpha]} \notin \Lambda_{\alpha,\beta}, & \text{if } b_\alpha = \alpha - [\alpha], \\ \mu_{1,[\alpha]+1} \notin \Lambda_{\alpha,\beta}, & \text{if } b_\alpha = [\alpha] + 1 - \alpha. \end{cases}$$

Assume q is even. Note that $\mu_{2,j} \geq \frac{4}{q^2} + 4\beta^2, \forall j$. If $b_\alpha = \alpha - [\alpha]$, then $\mu_{1,[\alpha]} = b_\alpha^2 + \beta^2$. Combining these and (31), we obtain $\mu_{1,[\alpha]} < \mu_{2,j} = \mu_{-2,-j}, \forall j$. From (31) we also know that $\mu_{1,[\alpha]} < \mu_{kp,j}, \forall k \neq 0, \forall j$, and for any $3 \leq |i| \neq q|k|$ and j, $\mu_{1,[\alpha]} < \mu_{i,j}$. Clearly $\mu_{1,[\alpha]} < \mu_{1,j} = \mu_{-1,-j}, \forall j \neq [\alpha]$ and $\mu_{1,[\alpha]} < \mu_{0,j}, \forall j$. Hence $\mu_{1,[\alpha]} \notin \Lambda_{\alpha,\beta}$. For the case $b_\alpha = [\alpha] + 1 - \alpha$, note that $\mu_{1,[\alpha]+1} = b_\alpha^2 + \beta^2$. The rest argument of this claim in this case is similar as that of $b_\alpha = \alpha - [\alpha]$, we omit it.

Assume q is odd and $b_\alpha \neq \frac{q-1}{2q}$. Then $\mu_{2,j} \geq \frac{4}{q^2} + 4\beta^2, \forall j$. The rest argument is the same as that of the case q is even, we omit it.

Claim B: If $q \geq 7$ is odd and $b_\alpha = \frac{q-1}{2q} = \alpha - [\alpha]$, then $\mu_{1,[\alpha]+1} \notin \Lambda_{\alpha,\beta}$. Note that

$$\mu_{1,[\alpha]+1} = \frac{(q+1)^2}{4q^2} + \beta^2 > \frac{(q-1)^2}{4q^2} + \beta^2 = b_\alpha^2 + \beta^2 = \frac{1}{q^2} + 4\beta^2, \tag{32}$$

so

$$\mu_{-1,-[\alpha]} = \mu_{1,[\alpha]} < \mu_{1,[\alpha]+1} < \mu_{1,j} = \mu_{-1,-j}, \ j \neq [\alpha], \ [\alpha]+1,$$

and

$$\mu_{1,[\alpha]+1} > \mu_{2,2[\alpha]+1} = \mu_{-2,-(2[\alpha]+1)}.$$

By $\beta^2 = \frac{1}{3}(b_\alpha^2 - \frac{1}{q^2}) = \frac{(q+1)(q-3)}{12q^2}$, we obtain

$$\frac{(q+1)^2}{4q^2} + \beta^2 < \frac{(q-1)^2}{q^2} + 4\beta^2,$$

so

$$\mu_{1,[\alpha]+1} < \mu_{2,j} = \mu_{-2,-j}, \ \forall \ j \neq 2[\alpha]+1.$$

It is easy to verify that

$$\frac{(q+1)^2}{4q^2} + \beta^2 < q^2\beta^2, \quad \frac{(q+1)^2}{4q^2} + \beta^2 < 1,$$

which mean that

$$\mu_{1,[\alpha]+1} < \mu_{kq,j}, \ \forall \ k \neq 0, \ \forall \ j, \ \text{and} \ \mu_{1,[\alpha]+1} < \mu_{0,j}, \ \forall \ j \neq 0.$$

We have

$$\frac{(q+1)^2}{4q^2}+\beta^2-\left(\frac{1}{q^2}+9\beta^2\right)=-\frac{(q-5)(5q+3)}{12q^2}<0,$$

which means that

$$\mu_{1,[\alpha]+1}<\mu_{i,j},\ \forall\, 3\le |i|\ne q|k|,\ \forall\, k,\ j.$$

All in all, $\mu_{q-2,(q-2)[\alpha]+\frac{q-3}{2}}\notin \Lambda_{\alpha,\beta}$.

Claim C: If $q\ge 7$ is odd and $b_\alpha=\frac{q-1}{2q}=[\alpha]+1-\alpha$, then $\mu_{1,[\alpha]}\notin\Lambda_{\alpha,\beta}$. The argument is similar as that of Claim B, we omit it.

Claim D: If $q=5$ and $b_\alpha=\frac{q-1}{2q}=\frac{2}{5}$, then $\Lambda_{\alpha,\beta}=S_{\alpha,\beta}$. Indeed, $\beta^2=\frac{1}{3}(b_\alpha^2-\frac{1}{q^2})=\frac{1}{25}$. Recall that

$$\mu_{i,j}=\frac{1}{25}i^2+\left(\frac{pi}{5}-j\right)^2.$$

For any $i\ne 0$, we have

$$\mu_{5i,j}=\mu_{5i,2pi-j},\ \forall\, j\ne pi,\quad \mu_{5i,pi}=i^2=\mu_{0,i}.$$

On the other hand, for any i satisfying $\frac{i}{5}\notin\mathbb{Z}$, we have

$$\mu_{i,j}=\frac{i^2}{25}+\frac{(pi-5j)^2}{25}=\mu_{pi-5j,\frac{(p^2+1)i}{5}-pj},\ \forall\, j.$$

Note that $\frac{p^2+1}{5}\in\mathbb{Z}$, since $b_\alpha=\frac{2}{5}$. Moreover, it is easy to verify that $(pi-5j,\frac{(p^2+1)i}{5}-pj)\ne(i,j),(-i,-j)$. Observe that

$$\mu_{0,j}=j^2=\mu_{5j,pj},\ \forall\, j\ne 0.$$

To sum up, $\Lambda_{\alpha,\beta}=S_{\alpha,\beta}$.

(iv) For $\sqrt{\frac{1}{3}(b_\alpha^2-\frac{1}{q^2})}<\beta<\sqrt{1-b_\alpha^2}$, namely

$$b_\alpha^2+\beta^2<\min\left\{\frac{1}{q^2}+4\beta^2,q^2\beta^2\right\},\tag{33}$$

we claim that if $b_\alpha=\alpha-[\alpha]$ then $\mu_{1,[\alpha]}\notin\Lambda_{\alpha,\beta}$, and if $b_\alpha=[\alpha]+1-\alpha$ then $\mu_{1,[\alpha]+1}\notin\Lambda_{\alpha,\beta}$. Assume $b_\alpha=\alpha-[\alpha]$. Note that $\mu_{1,[\alpha]}=b_\alpha^2+\beta^2$. Combining this and (33), we have

$$\mu_{1,[\alpha]}<\mu_{i,j},\ \forall\, 2\le|i|\ne|k|q,\ \forall\, k,\ j,$$
$$\mu_{1,[\alpha]}<\mu_{i,j},\ \forall\, i=kq,\ \forall\, k\ne 0,\ j.$$

Moreover

$$\mu_{1,[\alpha]}<\mu_{1,j}=\mu_{-1,-j},\ \forall\, j\ne[\alpha],$$

$$\mu_{1,[\alpha]} < \mu_{0,j}, \ \forall j \neq 0.$$

Hence $\mu_{1,[\alpha]} \notin \Lambda_{\alpha,\beta}$. The argument of this claim for $b_\alpha = [\alpha] + 1 - \alpha$ is similar, we omit it. □

5. Local convexity of bifurcating curves

We first consider local convexity of bifurcating curves for rectangular torus case. To this end we need to establish the following lemma.

Lemma 5.1. *Assume $\rho_{i,j} = \beta\lambda_{i,j} \in \beta K_\beta$. For any $\zeta \in \mathcal{R}(\partial_u F(\rho_{i,j}, 0))$, the only solution $\phi \in \ker(\partial_u F(\rho_{i,j}, 0))^\perp$ of*

$$\partial_u F(\rho_{i,j}, 0)\phi = \Delta\phi + \lambda_{i,j}\phi = \zeta$$

is given by

$$\phi(x, y) = C_0 + \sum_{r,s \in \mathbb{N}\cup\{0\}, r^2+s^2 \neq 0} A_{r,s} \cos(2\pi r x) \cos(2\pi\beta^{-1} s y),$$

where

$$C_0 = \frac{\tilde{C}_0}{\lambda_{i,j}}, \quad A_{i,j} = 0, \quad A_{r,s} = \frac{\tilde{A}_{r,s}}{\lambda_{i,j} - \lambda_{r,s}}, \ (r, s) \neq (i, j).$$

Here $\tilde{A}_{r,s}, \tilde{C}_0$ are the Fourier coefficients of ζ

$$\zeta(x, y) = \tilde{C}_0 + \sum_{r,s \in \mathbb{N}\cup\{0\}, r^2+s^2 \neq 0} \tilde{A}_{r,s} \cos(2\pi r x) \cos(2\pi\beta^{-1} s y).$$

Proof. Substituting the Fourier expansions of ϕ and ζ into the equation $\Delta\phi + \lambda_{i,j}\phi = \zeta$, we obtain

$$\begin{aligned} &C_0\lambda_{i,j} + \sum_{r^2+s^2 \neq 0} A_{r,s}[\lambda_{i,j} - 4\pi^2(r^2 + \beta^{-2}s^2)] \cos(2\pi r x) \cos(2\pi\beta^{-1} s y) \\ &= \tilde{C}_0 + \sum_{r^2+s^2 \neq 0} \tilde{A}_{r,s} \cos(2\pi r x) \cos(2\pi\beta^{-1} s y), \end{aligned}$$

which gives

$$C_0\lambda_{i,j} = \tilde{C}_0, \quad A_{r,s}(\lambda_{i,j} - \lambda_{r,s}) = \tilde{A}_{r,s}, \ (r, s) \neq (i, j).$$

In fact $A_{i,j} = 0$, since $\phi \in \ker(\partial_u F(\rho_{i,j}, 0))^\perp$. □

Proof of Theorem 1.12. A simple computation shows that

$$\partial^2_{u,u} F(\rho,0)[\varphi,\varsigma] = \rho\left(\frac{\varphi\varsigma}{\beta} - \frac{\int_0^1\int_0^\beta \varphi\varsigma\, dxdy}{\beta^2}\right).$$

Take $\psi = w_0 = \cos(2\pi ix)\cos(2\pi\beta^{-1}jy)$. Due to $\int_0^1\int_0^\beta w_0 dxdy = \int_0^1\int_0^\beta w_0^3 dxdy = 0$, we have

$$\begin{aligned}\langle \partial^2_{u,u} F(\rho_{i,j},0)[w_0,w_0],\psi\rangle &= \int_0^1\int_0^\beta w_0\rho_{i,j}\left(\frac{w_0^2}{\beta} - \frac{\int_0^1\int_0^\beta w_0^2}{\beta^2}\right)\\ &= \frac{\rho_{i,j}}{\beta}\int_0^1\int_0^\beta w_0^3 - \frac{\rho_{i,j}}{\beta^2}\int_0^1\int_0^\beta w_0\int_0^1\int_0^\beta w_0^2\\ &= 0.\end{aligned}$$

So $\rho'_{i,j}(0) = 0$ and

$$Q\partial^2_{u,u} T(\rho_{i,j},0)[w_0,w_0] = 0.$$

Then

$$(\partial_u F(\rho_{i,j},0))^{-1}(I-Q)\partial^2_{u,u}F(\rho_{i,j},0)[w_0,w_0] = (\partial_u F(\rho_{i,j},0))^{-1}\partial^2_{u,u}F(\rho_{i,j},0)[w_0,w_0],$$

and we denote this term as ϕ. Correspondingly we denote

$$\zeta := \partial^2_{u,u}F(\rho_{i,j},0)[w_0,w_0] = \rho_{i,j}\left(\frac{w_0^2}{\beta} - \frac{\int_0^1\int_0^\beta w_0^2}{\beta^2}\right),$$

then

$$\zeta = \frac{\lambda_{i,j}}{4}(\cos(4\pi ix) + \cos(4\pi\beta^{-1}jy) + \cos(4\pi ix)\cos(4\pi\beta^{-1}jy)).$$

From Lemma 5.1 we have

$$\phi = A_{2i,0}\cos(4\pi ix) + A_{0,2j}\cos(4\pi\beta^{-1}jy) + A_{2i,2j}\cos(4\pi ix)\cos(4\pi\beta^{-1}jy),$$

where

$$A_{2i,0} = \frac{\lambda_{i,j}}{4(\lambda_{i,j}-\lambda_{2i,0})},\quad A_{0,2j} = \frac{\lambda_{i,j}}{4(\lambda_{i,j}-\lambda_{0,2j})},\quad A_{2i,2j} = -\frac{1}{12}.$$

So

$$\langle \partial^2_{u,u} F(\rho_{i,j},0)\Big[w_0, (\partial_u F(\rho_{i,j},0))^{-1}(I-Q)\partial^2_{u,u}F(\rho_{i,j},0)[w_0,w_0]\Big], \psi\rangle$$
$$= \langle \partial^2_{u,u} F(\rho_{i,j},0)[w_0,\phi],\psi\rangle = \int_0^1\int_0^\beta w_0\rho_{i,j}\left(\frac{w_0\phi}{\beta} - \int_0^1\int_0^\beta \frac{w_0\phi}{\beta^2}\right)$$
$$= \lambda_{i,j}\int_0^1\int_0^\beta w_0^2\phi$$
$$= \lambda_{i,j}\int_0^1\int_0^\beta w_0^2[A_{2i,0}\cos(4\pi i x) + A_{0,2j}\cos(4\pi\beta^{-1}jy) + A_{2i,2j}\cos(4\pi i x)\cos(4\pi\beta^{-1}jy)]$$
$$= \frac{\beta\lambda_{i,j}}{8}\left(A_{2i,0} + A_{0,2j} - \frac{1}{24}\right). \tag{34}$$

It is not difficult to compute that

$$\langle \partial^3_{u,u,u}F(\rho_{i,j},0)[w_0,w_0,w_0],\psi\rangle$$
$$= \int_0^1\int_0^\beta w_0\rho_{i,j}\left(\frac{w_0^3}{\beta} - 3\frac{w_0\int_0^1\int_0^\beta w_0^2}{\beta^2} - \frac{\int_0^1\int_0^\beta w_0^3}{\beta^2}\right)$$
$$= \frac{\rho_{i,j}}{\beta}\int_0^1\int_0^\beta w_0^4 - \frac{3\rho_{i,j}}{\beta^2}\left(\int_0^1\int_0^\beta w_0^2\right)^2$$
$$= \frac{-3\rho_{i,j}}{64}. \tag{35}$$

Finally, we have

$$\|w_0\|^2\langle \partial^2_{\rho,u}F(\rho_{i,j},0)w_0,\psi\rangle = \|w_0\|^2\frac{1}{\beta}\int_0^1\int_0^\beta w_0^2 = \frac{\beta}{16}. \tag{36}$$

Combining the formula of second-order derivative in Proposition 2.2 and (34)-(36), we obtain the desired result of Theorem 1.12. □

Proof of Corollary 1.1. We denote

$$\rho''_{i,j}(0) = -\frac{1}{2}\lambda_{i,j}\left(\frac{\lambda_{i,j}}{\lambda_{2i,0}-\lambda_{i,j}} + \frac{\lambda_{i,j}}{\lambda_{0,2j}-\lambda_{i,j}} - \frac{1}{3}\right) =: -\frac{1}{2}\lambda_{i,j}\left(\frac{1}{\delta}+\frac{1}{\sigma}-\frac{1}{3}\right).$$

Note that $\lambda_{2i,0}+\lambda_{0,2j} = 4\lambda_{i,j}$, so $\delta+\sigma = 2$.

(i) If $\frac{\sqrt{3}}{3}\beta^{-1}j < i < \sqrt{3}\beta^{-1}j$, namely

$$(\lambda_{2i,0} - \lambda_{i,j})(\lambda_{0,2j} - \lambda_{i,j}) = -(3i^2 - \beta^{-2}j^2)(i^2 - 3\beta^{-2}j^2) > 0, \tag{37}$$

then both δ and σ are positive. Hence the mean value inequality $\frac{2}{\frac{1}{\delta}+\frac{1}{\sigma}} \le \frac{\delta+\sigma}{2}$ and the fact $\delta+\sigma = 2$ yield $\frac{1}{\delta} + \frac{1}{\sigma} \ge 2$, and the desired result $\rho''_{i,j}(0) < 0$ follows.

(ii) If $i < \frac{\sqrt{3}}{3}\beta^{-1}j$ or $i > \sqrt{3}\beta^{-1}j$, then the sign in (37) is in opposite direction. So the desired result $\rho''_{i,j}(0) > 0$ follows. □

Next we consider the local convexity for the generic flat torus case. Similarly, we establish the following lemma.

Lemma 5.2. *Assume $\rho_{i,j} = \beta\mu_{i,j} \in \beta K_{\alpha,\beta}$. For any $\zeta \in \mathcal{R}(\partial_u F(\rho_{i,j}, 0))$, the only solution $\phi \in \ker(\partial_u F(\rho_{i,j}, 0))^\perp$ of*

$$\partial_u F(\rho_{i,j}, 0)\phi = \Delta\phi + \mu_{i,j}\phi = \zeta$$

is given by

$$\phi(x, y) = C_0 + \sum_{r,s\in\mathbb{Z}, r^2+s^2\neq 0} A_{r,s} \cos 2\pi[rx + \beta^{-1}(s - \alpha r)y],$$

where

$$C_0 = \frac{\tilde{C}_0}{\mu_{i,j}}, \quad A_{i,j} = 0, \quad A_{r,s} = \frac{\tilde{A}_{r,s}}{\mu_{i,j} - \mu_{r,s}}, \ (r, s) \neq (i, j).$$

Here $\tilde{A}_{r,s}, \tilde{C}_0$ are the Fourier coefficients of ζ

$$\zeta(x, y) = \tilde{C}_0 + \sum_{r,s\in\mathbb{Z}, r^2+s^2\neq 0} \tilde{A}_{r,s} \cos 2\pi[rx + \beta^{-1}(s - \alpha r)y].$$

Proof. Substituting the Fourier expansions of ϕ and ζ into the equation $\Delta\phi + \mu_{i,j}\phi = \zeta$, we obtain

$$\begin{aligned} &C_0\mu_{i,j} + \sum_{r^2+s^2\neq 0} A_{r,s}(\mu_{i,j} - \mu_{r,s}) \cos 2\pi[rx + \beta^{-1}(s - \alpha r)y] \\ &= \tilde{C}_0 + \sum_{r^2+s^2\neq 0} \tilde{A}_{r,s} \cos 2\pi[rx + \beta^{-1}(s - \alpha r)y], \end{aligned}$$

which gives

$$C_0\mu_{i,j} = \tilde{C}_0, \quad A_{r,s}(\mu_{i,j} - \mu_{r,s}) = \tilde{A}_{r,s}, \ (r, s) \neq (i, j).$$

In fact $A_{i,j} = 0$, since $\phi \in \ker(\partial_u F(\rho_{i,j}, 0))^\perp$. □

We set

$$\begin{cases} \tilde{x} = x - \dfrac{\alpha}{\beta} y, \\ \tilde{y} = \dfrac{\sqrt{\alpha^2+\beta^2}}{\beta} y. \end{cases} \tag{38}$$

A simple computation shows that

$$\frac{\partial(x,y)}{\partial(\tilde{x},\tilde{y})} = \frac{\beta}{\sqrt{\alpha^2+\beta^2}}.$$

Set $w_0 = \cos 2\pi[ix + \beta^{-1}(j-\alpha i)y]$. We claim that

$$\int_{T_{\alpha,\beta}} w_0 dxdy = \int_{T_{\alpha,\beta}} w_0^3 dxdy = 0$$

and

$$\int_{T_{\alpha,\beta}} w_0^2 dxdy = \frac{\beta}{2}, \quad \int_{T_{\alpha,\beta}} w_0^4 dxdy = \frac{3\beta}{8}.$$

Indeed, by (38) we have

$$\begin{aligned} \int_{T_{\alpha,\beta}} w_0 dxdy &= \int_0^1 \int_0^{\sqrt{\alpha^2+\beta^2}} \cos 2\pi \left(i\tilde{x} + \frac{j}{\sqrt{\alpha^2+\beta^2}} \tilde{y} \right) \frac{\partial(x,y)}{\partial(\tilde{x},\tilde{y})} d\tilde{x} d\tilde{y} \\ &= \frac{\beta}{\sqrt{\alpha^2+\beta^2}} \int_0^1 \int_0^{\sqrt{\alpha^2+\beta^2}} \cos(2\pi i\tilde{x}) \cos\left(\frac{2\pi j\tilde{y}}{\sqrt{\alpha^2+\beta^2}} \right) d\tilde{x} d\tilde{y} \\ &- \frac{\beta}{\sqrt{\alpha^2+\beta^2}} \int_0^1 \int_0^{\sqrt{\alpha^2+\beta^2}} \sin(2\pi i\tilde{x}) \sin\left(\frac{2\pi j\tilde{y}}{\sqrt{\alpha^2+\beta^2}} \right) d\tilde{x} d\tilde{y} \\ &= 0. \end{aligned}$$

Similar computation can give the other three equalities in this claim.

Proof of Theorem 1.13. For

$$\partial_{u,u}^2 F(\rho,0)[\varphi,\varsigma] = \rho \left(\frac{\varphi\varsigma}{\beta} - \frac{\int_{T_{\alpha,\beta}} \varphi\varsigma dxdy}{\beta^2} \right),$$

taking $\psi = w_0$, we have

$$\begin{aligned}\langle \partial^2_{u,u} F(\rho_{i,j},0)[w_0,w_0],\psi\rangle &= \int\limits_{T_{\alpha,\beta}} w_0\rho_{i,j}\left(\frac{w_0^2}{\beta}-\frac{\int_{T_{\alpha,\beta}} w_0^2}{\beta^2}\right)\\ &= \frac{\rho_{i,j}}{\beta}\int\limits_{T_{\alpha,\beta}} w_0^3 - \frac{\rho_{i,j}}{\beta^2}\int\limits_{T_{\alpha,\beta}} w_0 \int\limits_{T_{\alpha,\beta}} w_0^2\\ &= 0.\end{aligned}$$

So $\rho'_{i,j}(0)=0$ and $Q\partial^2_{u,u}T(\rho_{i,j},0)[w_0,w_0]=0$. Then

$$(\partial_u F(\rho_{i,j},0))^{-1}(I-Q)\partial^2_{u,u}F(\rho_{i,j},0)[w_0,w_0]=(\partial_u F(\rho_{i,j},0))^{-1}\partial^2_{u,u}F(\rho_{i,j},0)[w_0,w_0],$$

and we denote this term as ϕ. Correspondingly we denote

$$\zeta := \partial^2_{u,u}F(\rho_{i,j},0)[w_0,w_0]=\rho_{i,j}\left(\frac{w_0^2}{\beta}-\frac{\int_{T_{\alpha,\beta}} w_0^2}{\beta^2}\right),$$

then

$$\zeta = \frac{\mu_{i,j}}{2}\cos 2\pi[2ix+\beta^{-1}(2j-2\alpha i)y].$$

From Lemma 5.2 we have

$$\phi = -\frac{1}{6}\cos 2\pi[2ix+\beta^{-1}(2j-2\alpha i)y],$$

where we used the relation $\mu_{2i,2j}=4\mu_{i,j}$. So

$$\begin{aligned}&\langle \partial^2_{u,u}F(\rho_{i,j},0)\Big[w_0,(\partial_u F(\rho_{i,j},0))^{-1}(I-Q)\partial^2_{u,u}F(\rho_{i,j},0)[w_0,w_0]\Big],\psi\rangle\\ &= \langle \partial^2_{u,u}F(\rho_{i,j},0)[w_0,\phi],\psi\rangle = \int\limits_{T_{\alpha,\beta}} w_0\rho_{i,j}\left(\frac{w_0\phi}{\beta}-\int\limits_{T_{\alpha,\beta}}\frac{w_0\phi}{\beta^2}\right)\\ &= \mu_{i,j}\int\limits_{T_{\alpha,\beta}} w_0^2\phi\\ &= -\frac{\mu_{i,j}}{6}\int\limits_{T_{\alpha,\beta}}\frac{1+\cos 2\pi[2ix+\beta^{-1}(2j-2\alpha i)y]}{2}\cos 2\pi[2ix+\beta^{-1}(2j-2\alpha i)y]dxdy\\ &= -\frac{\rho_{i,j}}{24}. \qquad (39)\end{aligned}$$

We also have

$$\begin{aligned}
&\langle \partial^3_{u,u,u} F(\rho_{i,j}, 0)[w_0, w_0, w_0], \psi \rangle \\
&= \int\limits_{T_{\alpha,\beta}} w_0 \rho_{i,j} \left(\frac{w_0^3}{\beta} - 3 \frac{w_0 \int_{T_{\alpha,\beta}} w_0^2}{\beta^2} - \frac{\int_{T_{\alpha,\beta}} w_0^3}{\beta^2} \right) \\
&= \frac{\rho_{i,j}}{\beta} \int\limits_{T_{\alpha,\beta}} w_0^4 - \frac{3\rho_{i,j}}{\beta^2} \left(\int\limits_{T_{\alpha,\beta}} w_0^2 \right)^2 \\
&= \frac{-3\rho_{i,j}}{8},
\end{aligned} \tag{40}$$

and

$$\|w_0\|^2 \langle \partial^2_{\rho,u} F(\rho_{i,j}, 0) w_0, \psi \rangle = \|w_0\|^2 \frac{1}{\beta} \int\limits_{T_{\alpha,\beta}} w_0^2 = \frac{\beta}{4}. \tag{41}$$

Combining the formula of second-order derivative in Proposition 2.2 and (39)-(41), we obtain the desired result in Theorem 1.13. □

Acknowledgments

The first author is supported by the Natural Science Foundation of Hunan Province, China (Grant No. 2016JJ2018). The second author is partially supported by NSF grants DMS-1601885 and DMS-1901914 and a Simons Foundation grant Award 617072.

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