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Periodic solutions of Allen–Cahn system with the fractional Laplacian

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ABSTRACT

We consider periodic solutions of the following nonlinear system associated with the fractional Laplacian

$$(-\partial_{xx})^s \mathbf{u}(x) + \nabla F(\mathbf{u}(x)) = 0 \quad \text{in } \mathbb{R},$$

where $\mathbf{u}(x) = (u(x), v(x))$. The function $F : \mathbb{R}^2 \to \mathbb{R}$ is a smooth double-well potential. We prove the existence of periodic solutions with large period T by using variational methods. Moreover, we draw a conclusion that the second component of periodic solution is identical to zero if the origin is a saddle point of F, whereas the second component is not identical to zero if the origin is a local maximum point of F. A Hamiltonian identity for periodic solutions is also established.

1. Introduction

We consider the following Allen–Cahn system involving the fractional Laplacian

$$(-\partial_{xx})^s \mathbf{u}(x) + \nabla F(\mathbf{u}(x)) = 0, \quad \mathbf{u}(x) = \mathbf{u}(x+T) \quad \text{in } \mathbb{R}, \tag{1}$$

where $(-\partial_{xx})^s$, $s \in (0,1)$, denotes the usual fractional Laplace operator, a Fourier multiplier of symbol $|\xi|^{2s}$. The function F is a smooth double-well potential with wells at $\mathbf{b}_1$ and $\mathbf{b}_2$. Without loss of generality we may assume that $\mathbf{b}_1 = -\mathbf{b}_2 = (1,0)$. More precisely, we assume that $F(\mathbf{w})$ grows rapidly to infinity as $|\mathbf{w}| \to \infty$ and satisfies

$$\begin{cases} F(\mathbf{b}_i) = 0 < F(\mathbf{w}), & \forall \mathbf{w} \neq \mathbf{b}_i, i = 1,2, \\ \nabla F(\mathbf{w}) \cdot \mathbf{w} \geq 0 & \text{for } |\mathbf{w}| \geq 1. \end{cases} \tag{2}$$

For the simplicity of the exposition of the paper, we also assume throughout the paper that

$$F_u(u,0) > 0, \text{ for all } u \in (-1,0); \quad F_u(u,0) < 0, \text{ for all } u \in (0,1). \tag{3}$$

Note that conditions (2)–(3) mean that $F(0,0) = \max_{-1 \leq u \leq 1} F(u,0) > 0$.

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The fractional Laplace operator $(-\Delta)^s$ can be defined as a Dirichlet-to-Neumann map for a so-called s-harmonic extension problem (see [10]). Given a function ϕ, the solution Φ of the following problem

$$\begin{cases} \mathrm{div}(y^a \nabla \Phi) = 0 & \text{in } \mathbb{R}^{n+1}_+ = \{(x, y) : x \in \mathbb{R}^n, y > 0\}, \\ \Phi(x, 0) = \phi(x) & \text{on } \mathbb{R}^n \end{cases}$$

is called the s-harmonic extension of ϕ. It is well-known that Φ has finite energy $\int_{\mathbb{R}^{n+1}_+} |\nabla \Phi|^2 y^a dxdy < +\infty$. The parameter a is related to the power s of the fractional Laplacian $(-\Delta)^s$ by the formula $a = 1 - 2s \in (-1, 1)$. The authors in [10] proved that

$$(-\Delta)^s \phi(x) = d_s \frac{\partial \Phi}{\partial \nu^a} \quad \text{in} \quad \mathbb{R}^n = \partial \mathbb{R}^{n+1}_+,$$

where

$$\frac{\partial \Phi}{\partial \nu^a} := -\lim_{y \downarrow 0} y^a \frac{\partial \Phi}{\partial y}, \quad d_s = 2^{2s-1} \frac{\Gamma(s)}{\Gamma(1-s)}.$$

For the corresponding scalar problem of (1)

$$(-\partial_{xx})^s u + G'(u) = 0, \quad u(x + T) = u(x) \quad \text{in } \mathbb{R}, \tag{4}$$

where G is a smooth double-well potential, the authors and Zhang in [17] obtain the existence of periodic solution u_T for a large period T. In [13] more general existence result is obtained, and an upper bound of the least positive period is given. Moreover, a Hamiltonian identity, Modica-type inequalities and an estimate of the energy functional for periodic solutions are also established. In [14], the authors generalize these results to the corresponding non-autonomous Allen–Cahn Equations. Existence and multiplicity of periodic solutions to the so-called pesudo-relativistic Schrödinger equations are also established in [1–3]. In [18], the authors establish interior and boundary Harnack's inequalities for nonnegative solutions to $(-\Delta)^s u = 0$ with periodic boundary conditions, and they also obtain regularity properties of the fractional Laplacian with periodic boundary conditions and the pointwise integro-differential formula for the operator.

In this paper we shall prove the existence of periodic solutions to system (1), according to the cases whether $F(u, v)$ is even with respect to u and v respectively.

System (1) can be realized in a local manner through the nonlinear boundary value problem

$$\begin{cases} \mathrm{div}(y^a \nabla \mathbf{U}) = 0 & \text{in } \mathbb{R}^2_+ = \{(x, y) : x \in \mathbb{R}, y > 0\}, \\ \frac{\partial \mathbf{U}}{\partial \nu^a} = -\nabla F(\mathbf{U}) & \text{on } \mathbb{R}. \end{cases} \tag{5}$$

Problem (5) is related to (1) in the sense that, if $\mathbf{U}$ is a solution of (5), then a positive constant multiple of $\mathbf{u}(x) := \mathbf{U}(x, 0)$ satisfies (1).

In the rest of this article we always assume that F satisfies conditions (2)–(3).

We first obtain the existence result of solutions of (1) by finding a minimizer of the corresponding energy functional.

Theorem 1.1. *Let $s \in (0, 1)$ and assume that $F(u, v)$ is even with respect to u and v respectively. Then there exists $T_1 > 0$ such that for any $T > T_1$, (1) admits a periodic solution $\mathbf{u}_T$ with period T and $|\mathbf{u}_T| \le 1$, and its two components u_T and v_T are odd function and even function respectively. Moreover, $u_T(x) \in (0, 1)$ for all $x \in (0, \frac{T}{2})$.*

Under the conditions in Theorem 1.1, it is easy to see that (1) possesses a periodic solution $(\hat{u}_T, 0)$, where $\hat{u}_T$ is a periodic solution to scalar problem (4) with $G(u) := F(u, 0)$.

A natural question arises: whether must the second component v_T of $\mathbf{u}_T$ be identical to zero?

We will draw different conclusions for two cases of F. The first case is that the origin is a saddle point of F, and the second case is that the origin is a local maximum point of F.

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Theorem 1.2. *(i). Assume*

$$F(u,0) \le F(u,v) \ \textit{for any} \quad |u| \le 1 \quad \textit{and} \quad v. \tag{6}$$

Then $v_T \equiv 0$;

(ii). Let $s \in (1/2, 1)$. Assume

$$\begin{aligned} &\exists \ \ \lambda > 0 \quad s.t. \quad D^2F(0) \le -\lambda I, \ \textit{and for any} \ \ |u| < 1: \\ &\exists \ \ \delta(u) > 0 \quad s.t. \quad F_{vv}(u,v) < 0 \ \textit{for} \ \ |v| < \delta(u). \end{aligned} \tag{7}$$

Then $v_T \not\equiv 0$ for large T.

Remark. (1). From conditions (2) and (7) we know that $\lim_{|u|\to 1} \delta(u) = 0$ and $F_{vv}(\mathbf{b}_i) = 0, i = 1, 2$.

(2) Condition (7) and the fact $\nabla F(0,0) = 0$ imply that the origin is a local maximum point of F, and for each $|u| < 1$, $F(u,0) > F(u,v)$ for any $|v| \le \delta(u)$.

(3) A typical example of F satisfying condition (6) is $F(\mathbf{u}) = \frac{|1-\mathbf{u}^2|^2}{4}$, where we identify $\mathbf{u}$ with a complex number $u + iv$. Hence any minimizer solution (u_T, v_T) of the following problem

$$(-\partial_{xx})^s \mathbf{u} + \nabla \left(\frac{|1-\mathbf{u}^2|^2}{4} \right) = 0, \qquad \mathbf{u}(x) = \mathbf{u}(x+T), \qquad \text{in } \mathbb{R}$$

must have $v_T \equiv 0$, and u_T is a periodic solution of the scalar problem

$$(-\partial_{xx})^s u + u(1-u^2) = 0.$$

(4) For the scalar Allen–Cahn equation with a fractional Laplacian, there exists a large literature, in particular regarding the one dimensional symmetry and layered solutions. See for example, [4–9,11,12,15,16,19,20,22], etc.

For the general case that $F(u,v)$ is not necessary even in u or v, we have the following existence result by finding mountain-pass solutions.

Theorem 1.3. *Let $s \in (0,1)$. Then there exists $T_2 > 0$ such that for any $T > T_2$, (1) admits a periodic solution $\mathbf{u}_T$ with period T.*

We also establish the so-called Hamiltonian identity.

Theorem 1.4 (*Hamiltonian Identity*)**.** *Assume $\mathbf{U}$ is the s-harmonic extension of a periodic solution $\mathbf{u}$ of (1). Then for all $x \in \mathbb{R}$ we have*

$$\frac{1}{2}\int_0^\infty [|\mathbf{U}_x(x,y)|^2 - |\mathbf{U}_y(x,y)|^2] y^a dy - F(\mathbf{U}(x,0)) \equiv C_T.$$

2. Proof of Theorem 1.1

For $\mathbf{u} = (u,v)$, we denote

$$|\mathbf{u}| = \sqrt{u^2+v^2},$$

and for a matrix $A = (a_{ij})$, we denote

$$|A|^2 = A : A = \sum_{i,j} a_{ij}^2.$$

Proof of Theorem 1.1. Denote

$$\Omega_T := [-\frac{T}{2}, \frac{T}{2}] \times [0, +\infty).$$

We consider the corresponding energy functional

$$J(\mathbf{U}, \Omega_T) := \frac{1}{2}\int_{\Omega_T} y^a |\nabla \mathbf{U}(x,y)|^2 dxdy + \int_{-\frac{T}{2}}^{\frac{T}{2}} F(\mathbf{U}(x,0))dx. \tag{8}$$

We denote the admissible set of the energy J as

$$\begin{aligned} \Lambda_T := \{\mathbf{U} = (U,V) : \mathbf{U} \in \left(H^1(\Omega_T, y^a)\right)^2, \quad \mathbf{U}(x+T,y) = \mathbf{U}(x,y), \\ U(-x,y) = -U(x,y),\ U \geq 0 \text{ in } [0,\frac{T}{2}] \times [0,+\infty), \\ V(-x,y) = V(x,y), \quad V(0,y) \leq 0 \leq V(\frac{T}{2}, y)\}. \end{aligned}$$

Here

$$H^1(\Omega_T, y^a) := \{W(x,y) : y^a(W^2 + |\nabla W|^2) \in L^1(\Omega_T)\}.$$

Note that $J(\mathbf{U}, \Omega_T) \geq 0$. On the other hand, we have that $(0,0) \in \Lambda_T$ and $J(0, \Omega_T) = F(0,0)T < +\infty$. Hence there exists a minimizing sequence $\{\mathbf{U}_k\} \subseteq \Lambda_T$ of J, namely

$$\lim_{k\to\infty} J(\mathbf{U}_k, \Omega_T) = m_T := \inf_{\mathbf{U}\in\Lambda_T} J(\mathbf{U}, \Omega_T).$$

By condition (2) we may assume that $|\mathbf{U}_k| \leq 1$. Since $F \geq 0$, from the definition of J, we have

$$\int_{\Omega_T} y^a |\nabla \mathbf{U}_k(x,y)|^2 dxdy \leq 2m_T + 1. \tag{9}$$

From this, weighted Poincaré inequality and the fact that $\mathbf{U}_k$ is bounded, we obtain

$$\int_{\Omega_T} y^a |\mathbf{U}_k(x,y)|^2 dxdy \leq C < +\infty, \quad \forall\ k. \tag{10}$$

From (9)–(10) we deduce that there exists a subsequence of $\{\mathbf{U}_k\}$, still denoted as $\{\mathbf{U}_k\}$, converging weakly in $\left(H^1(\Omega_T, y^a)\right)^2$ to a function $\mathbf{U}_T \in \left(H^1(\Omega_T, y^a)\right)^2$. By weak lower-semi continuity of the norm, we obtain that

$$\int_{\Omega_T} y^a |\nabla \mathbf{U}_T(x,y)|^2 dxdy \leq \liminf_{k\to\infty} \int_{\Omega_T} y^a |\nabla \mathbf{U}_k(x,y)|^2 dxdy.$$

By Fatou's Lemma, we also have

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} F(\mathbf{U}_T(x,0))dx \leq \liminf_{k\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} F(\mathbf{U}_k(x,0))dx.$$

Hence $J(\mathbf{U}_T, \Omega_T) \leq m_T$. Note that the set Λ_T is weakly closed, so $\mathbf{U}_T \in \Lambda_T$. Then $J(\mathbf{U}_T, \Omega_T) = m_T$, namely $\mathbf{U}_T$ is a minimizer of $J(\mathbf{U}, \Omega_T)$ in Λ_T.

Fix any $\eta = (\eta_1, \eta_2) \in \Lambda_T$. It is clear that for all small $\sigma > 0$, $\mathbf{U}_T + \sigma\eta \in \Lambda_T$. Thus if we set a real-valued function

$$\varphi(\sigma) := J(\mathbf{U}_T + \sigma\eta, \Omega_T),$$

then

$$\begin{aligned} 0 &\leq \frac{d}{d\sigma}\varphi(\sigma)|_{\sigma=0} \\ &= \int_{\Omega_T} y^a \nabla \mathbf{U}_T(x,y) : \nabla\eta(x,y)dxdy + \int_{-\frac{T}{2}}^{\frac{T}{2}} \nabla F(\mathbf{U}_T(x,0)) \cdot \eta(x,0)dx \end{aligned}$$

$$= -\int_{\Omega_T} \eta_1 \mathrm{div}(y^a \nabla U_T) dxdy + \int_{-\frac{T}{2}}^{\frac{T}{2}} \left[\frac{\partial U_T}{\partial \nu^a} + F_u(\mathbf{U}_T(x,0))\right] \eta_1(x,0)dx$$
$$- \int_{\Omega_T} \eta_2 \mathrm{div}(y^a \nabla V_T) dxdy + \int_{-\frac{T}{2}}^{\frac{T}{2}} \left[\frac{\partial V_T}{\partial \nu^a} + F_v(\mathbf{U}_T(x,0))\right] \eta_2(x,0)dx$$
$$= -2\int_0^{\frac{T}{2}}\int_0^{\infty} \eta_1 \mathrm{div}(y^a \nabla U_T) dxdy + 2\int_0^{\frac{T}{2}} \left[\frac{\partial U_T}{\partial \nu^a} + F_u(\mathbf{U}_T(x,0))\right] \eta_1(x,0)dx$$
$$-2\int_0^{\frac{T}{2}}\int_0^{\infty} \eta_2 \mathrm{div}(y^a \nabla V_T) dxdy + 2\int_0^{\frac{T}{2}} \left[\frac{\partial V_T}{\partial \nu^a} + F_v(\mathbf{U}_T(x,0))\right] \eta_2(x,0)dx.$$

Hence, by the arbitrariness of η, we obtain

$$\begin{cases} \mathrm{div}(y^a \nabla V_T) = 0 & \text{in } [0, \frac{T}{2}] \times [0, +\infty), \\ \frac{\partial V_T}{\partial \nu^a} = -F_v(\mathbf{U}_T) & \text{on } [0, \frac{T}{2}] \end{cases} \tag{11}$$

and

$$\begin{cases} \mathrm{div}(y^a \nabla U_T) \leq 0 & \text{in } [0, \frac{T}{2}] \times [0, +\infty), \\ \frac{\partial U_T}{\partial \nu^a} \geq -F_u(\mathbf{U}_T) & \text{on } [0, \frac{T}{2}]. \end{cases} \tag{12}$$

We now prove that $U_T \not\equiv 0$. Set $\mu := \min_{v\in[-1,1]} F(0,v)$, then one has that $\mu > 0$. We have

$$J((0, V_T), \Omega_T) \geq \mu T. \tag{13}$$

For $\sigma \in (0,1)$, we define the following continuous functions

$$\hat{h}(x) := \begin{cases} \frac{4}{\sigma T}x, & x \in [0, \frac{\sigma T}{4}], \\ 1, & x \in [\frac{\sigma T}{4}, \frac{T}{2} - \frac{\sigma T}{4}], \\ \frac{2}{\sigma} - \frac{4}{\sigma T}x, & x \in [\frac{T}{2} - \frac{\sigma T}{4}, \frac{T}{2}], \end{cases}$$

and denote its odd extension to $[-\frac{T}{2}, \frac{T}{2}]$ as $h(x)$. Further we define

$$\psi(x,y) = \exp\left\{-\frac{y}{2^{b+1}}\right\} h(x),$$

where the parameter b will be determined later. Then $(\psi, 0) \in \Lambda_T$. We next compute the energy $J((\psi,0), \Omega_T)$. From conditions (2)–(3) of F, we have

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} F(\psi(x,0),0)dx = \int_{-\frac{T}{2}}^{\frac{T}{2}} F(h(x),0)dx < F(0,0)\sigma T. \tag{14}$$

For the other part of the energy, similar computation as in [17], we have

$$\int_{-\frac{T}{2}}^{\frac{T}{2}}\int_0^{\infty} y^a |\nabla\psi(x,y)|^2 dxdy$$
$$= \int_0^{\infty} y^a \exp\left\{-\frac{y}{2^b}\right\} dy \int_{-\frac{T}{2}}^{\frac{T}{2}} \left[\frac{h^2(x)}{2^{2b+2}} + (h'(x))^2\right] dx$$
$$\leq \Gamma(a+1)2^{b(a-1)}\left[\frac{T}{4} + 2^{2b}\frac{64}{\sigma T}\right].$$

Note that $a-1<0$, for the purpose that the term $2^{b(a-1)}\Gamma(a+1)$ is small, we can choose sufficiently large b. For chosen b, the other term $2^{2b}\frac{64}{\sigma T}$ is also small provided that T is large enough. Hence there exists $T_1 > 0$ such that for any $T > T_1$, the following estimate holds true

$$\int_{-\frac{T}{2}}^{\frac{T}{2}}\int_0^{\infty} y^a |\nabla\psi(x,y)|^2 dxdy < F(0,0)\sigma T. \tag{15}$$

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From (14)–(15), we have

$$J((\psi,0),\Omega_T)\le 2F(0,0)\sigma T.$$

From this and (13), if we choose $\sigma<\min\{\frac{\mu}{2F(0,0)},1\}$, then we have

$$J(\mathbf{U}_T,\Omega_T)\le J((\psi,0),\Omega_T)<J((0,V_T),\Omega_T),$$

which shows that $U_T\not\equiv 0$.

From (12) and the condition that $F(u,v)$ is even in u, by using Hopf Lemma we obtain that $U_T(x,y)>0$ for $x\in(0,\frac{T}{2}),\ y\ge 0$. For general smooth functions $\eta_1,\eta_2\in C_c^\infty((0,\frac{T}{2})\times[0,+\infty))$($\eta_1$ is not necessary non-negative), we extend η_1,η_2 from $(0,\frac{T}{2})\times[0,+\infty)$ to $(-\frac{T}{2},\frac{T}{2})\times[0,+\infty)$ oddly and evenly respectively. Then if $|\sigma|$ is sufficiently small, one has $\mathbf{U}_T+\sigma\eta\in\Lambda_T$. Hence we have

$$\begin{aligned}0&=\frac{d}{d\sigma}\varphi(\sigma)|_{\sigma=0}\\&=-2\int_0^{\frac{T}{2}}\int_0^\infty \eta_1\mathrm{div}(y^a\nabla U_T)dxdy+2\int_0^{\frac{T}{2}}\left[\frac{\partial U_T}{\partial\nu^a}+F_u(\mathbf{U}_T(x,0))\right]\eta_1(x,0)dx\\&\quad-2\int_0^{\frac{T}{2}}\int_0^\infty \eta_2\mathrm{div}(y^a\nabla V_T)dxdy+2\int_0^{\frac{T}{2}}\left[\frac{\partial V_T}{\partial\nu^a}+F_v(\mathbf{U}_T(x,0))\right]\eta_2(x,0)dx.\end{aligned}$$

which yields (11) and

$$\begin{cases}\mathrm{div}(y^a\nabla U_T)=0 & \text{in } [0,\frac{T}{2}]\times[0,+\infty),\\ \frac{\partial U_T}{\partial\nu^a}=-F_u(\mathbf{U}_T) & \text{on } [0,\frac{T}{2}].\end{cases}\tag{16}$$

Now we extend U_T oddly and V_T evenly respectively (with respect to x) from Ω_T to $[-\frac{T}{2},\frac{T}{2}]\times[0,+\infty)$. Further we extend it periodically (with respect to x again) from $[-\frac{T}{2},\frac{T}{2}]\times[0,+\infty)$ to the whole half space $\overline{\mathbb{R}^2_+}$, and we still denote them as U_T and V_T. Similar argument as in the proof of Theorem 1.1 in [14] shows that $\mathbf{U}_T=(U_T,V_T)$ is a weak solution of (5).

We set

$$\mathbf{u}_T(x):=\mathbf{U}_T(x,0),$$

then $\mathbf{u}_T=(u_T,v_T)$ is a periodic solution of (1), and its two components u_T,v_T are odd and even functions respectively. In view of $U_T|_{\Omega_T}\le 1$ and $U_T|_{\Omega_T}\not\equiv 1$, a Hopf principle in [7] shows that $U_T(x,0)=u_T(x)<1$, hence $u_T(x)\in(0,1)$ for all $x\in(0,\frac{T}{2})$. □

3. Proof of Theorem 1.2

Proof of Theorem 1.2. (i). From condition (6), we deduce that $J(U,0)\le J(U,V)$ for any $(U,V)\in\Lambda_T$. Furthermore, if $V\not\equiv 0$, one has the strict inequality $J(U,0)<J(U,V)$. Denote $\mathbf{U}_T(x,y)=(U_T,V_T)$ as the s-harmonic extension of periodic solution $\mathbf{u}_T(x)$. To the end that $J(\mathbf{U}_T)=\inf_{\mathbf{U}\in\Lambda_T}J(\mathbf{U},\Omega_T)$, we obtain that $V_T\equiv 0$, which yields the desired result.

(ii). Denote $M_T:=\sup_{\Omega_T}|U_T(x,y)|$. We define

$$\varepsilon_T:=\min\left\{\inf\{\delta(u):|u|\le M_T\},\sqrt{1-M_T^2}\right\},$$

where δ is the value defined in (7). We note that $\varepsilon_T>0$ is only dependent on M_T.

Note that $-1<a<0$, since $1/2<s<1$. We introduce the following continuous function

$$\hat\zeta(x):=\begin{cases}-\varepsilon_T, & x\in[-\frac{T}{2},-\frac{T}{2}+\frac{T^{\frac{a}{2}}}{4}],\\ \varepsilon_T+\frac{4\varepsilon_T}{T-T^{\frac{a}{2}}}\left(x+\frac{T^{\frac{a}{2}}}{4}\right), & x\in[-\frac{T}{2}+\frac{T^{\frac{a}{2}}}{4},-\frac{T^{\frac{a}{2}}}{4}],\\ \varepsilon_T, & x\in[-\frac{T^{\frac{a}{2}}}{4},0].\end{cases}$$

Denote ζ as the even extension of the function $\hat{\zeta}$ from $(-\frac{T}{2},0)$ onto $(-\frac{T}{2},\frac{T}{2})$. We choose test function

$$\eta(x,y) := (U_T(x,y), \exp\left\{-\frac{y}{2^{b+1}}\right\}\zeta(x)),$$

where the parameter b will be determined later. Clearly $\eta(x,y)\in \Lambda_T$.

We compute

$$\begin{aligned}
&\int_{-\frac{T}{2}}^{\frac{T}{2}}\int_0^\infty y^a\left|\nabla[\exp\left\{-\frac{y}{2^{b+1}}\right\}\zeta(x)]\right|^2 dxdy\\
&= \int_0^\infty y^a \exp\left\{-\frac{y}{2^b}\right\}dy \int_{-\frac{T}{2}}^{\frac{T}{2}}\left[\frac{\zeta^2(x)}{2^{2b+2}}+(\zeta'(x))^2\right]dx\\
&\le \left[\frac{\varepsilon_T^2}{2^{2b}}\frac{T}{4}+\frac{16\varepsilon_T^2}{T-T^{\frac{a}{2}}}\right]\int_0^\infty y^a\exp\left\{-\frac{y}{2^b}\right\}dy\\
&= 2^{b(a+1)}\left[\frac{\varepsilon_T^2}{2^{2b}}\frac{T}{4}+\frac{16\varepsilon_T^2}{T-T^{\frac{a}{2}}}\right]\int_0^\infty z^a e^{-z}dz\\
&= \Gamma(a+1)\varepsilon_T^2\left[2^{b(a-1)}\frac{T}{4}+2^{b(a+1)}\frac{16}{T-T^{\frac{a}{2}}}\right]\\
&\le \Gamma(a+1)\varepsilon_T^2\left[2^{b(a-1)}\frac{T}{4}+2^{b(a+1)}\frac{32}{T}\right],
\end{aligned} \tag{17}$$

where in the final inequality we used the relation that $T-T^{\frac{a}{2}} > \frac{T}{2}$ provided that $T\ge 2$, since $a<0$. From condition (7), we know that there exists $\beta>0$ such that $F_{vv}(u,0)\le -\frac{\lambda}{2}$ for any $|u|\le\beta$. By the a priori estimate of Proposition 2.9 in [21], we have that there exists a constant C independent of T such that $\|u_T\|_{C^{1,\alpha}(\mathbb{R})}\le C$ for any $\alpha<2s-1$.

Note that the image of η lies in the rectangular region $[-M_T,M_T]\times[-\varepsilon_T,\varepsilon_T]$ where $F_{vv}(u,v)\le 0$ by (7). Hence we have

$$\begin{aligned}
&\int_{-\frac{T}{2}}^{\frac{T}{2}}[F(u_T(x),\zeta(x))-F(u_T(x),0)]dx\\
&= \left\{\int_{-\frac{T}{2}}^{-\frac{T^{\frac{a}{2}}}{4}}+\int_{-\frac{T^{\frac{a}{2}}}{4}}^{\frac{T^{\frac{a}{2}}}{4}}+\int_{\frac{T^{\frac{a}{2}}}{4}}^{\frac{T}{2}}\right\}[F(u_T(x),\zeta(x))-F(u_T(x),0)]dx\\
&\le \int_{-\frac{T^{\frac{a}{2}}}{4}}^{\frac{T^{\frac{a}{2}}}{4}}[F(u_T(x),\zeta(x))-F(u_T(x),0)]dx\\
&\le -\frac{\lambda}{4}\varepsilon_T^2 T^{\frac{a}{2}},
\end{aligned} \tag{18}$$

where the first inequality follows from the definitions of ε_T, $\zeta(x)$ and the fact that $F(u_T,0)>F(u_T,v)$ for any $|v|\le\inf\{\delta(u),|u|\le M_T\}$. The reason of the final inequality is that $F_{vv}(u_T(x),0)\le-\frac{\lambda}{2}$ for any $x\in(-\frac{T^{\frac{a}{2}}}{4},\frac{T^{\frac{a}{2}}}{4})$, since $|u_T(x)-u_T(0)|=|u_T(x)|\le\beta$ owing to $\|u_T\|_{C^{1,\alpha}(\mathbb{R})}\le C$ and $T^{\frac{a}{2}}$ is small enough, provided that T is large enough. Choose b such that $2^{b(a+1)}=T^{1+a}$. Then $2^{b(a-1)}=T^{a-1}$. Substitute these into (17), we obtain

$$\int_{-\frac{T}{2}}^{\frac{T}{2}}\int_0^\infty y^a\left|\nabla[\exp\left\{-\frac{y}{2^{b+1}}\right\}\zeta(x)]\right|^2 dxdy\le\frac{129}{4}\Gamma(a+1)\varepsilon_T^2T^a.$$

From this and (18), it is easy to see that for large T we have

$$J((U_T,0),\Omega_T)>J(\eta(x,y),\Omega_T),$$

which gives the desired result. □

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4. Proof of Theorem 1.3

Proof of Theorem 1.3. We introduce the Hilbert space

$$\mathcal{H} := \{\mathbf{U} : \|\mathbf{U}\|_{\mathcal{H}}^2 := \int_{\Omega_T} y^a |\nabla \mathbf{U}(x,y)|^2 dxdy + \int_{-\frac{T}{2}}^{\frac{T}{2}} |\mathbf{U}(x,0)|^2 dx < \infty\},$$

where $\Omega_T = [-\frac{T}{2}, \frac{T}{2}] \times [0, +\infty)$. We consider the corresponding energy functional

$$J(\mathbf{U}, \Omega_T) = \frac{1}{2} \int_{\Omega_T} y^a |\nabla \mathbf{U}(x,y)|^2 dxdy + \int_{-\frac{T}{2}}^{\frac{T}{2}} F(\mathbf{U}(x,0))dx.$$

For simplicity we denote $J(\mathbf{U}, \Omega_T)$ as $J(\mathbf{U})$.

Plainly, $J \in C^1(\mathcal{H}, \mathbb{R})$, since F is smooth. Next we verify the Palais–Smale condition. Namely, for any sequence $\mathbf{U}_k \in \mathcal{H}$ with

$$J(\mathbf{U}_k) \text{ bounded}$$

and

$$J'(\mathbf{U}_k) \to 0 \quad \text{in } \mathcal{H},$$

it contains a convergent subsequence. Estimates similar to that (9) and (10) yield that there exists a subsequence of $\{\mathbf{U}_k\}$, still denoted as $\{\mathbf{U}_k\}$, converging weakly to a function $\bar{\mathbf{U}}$ in $(H^1(\Omega_T, y^a))^2$. In view of

$$(H^1(\Omega_T, y^a))^2 \hookrightarrow \left(H^s(-\frac{T}{2}, \frac{T}{2})\right)^2 \hookrightarrow\hookrightarrow \left(L^2(-\frac{T}{2}, \frac{T}{2})\right)^2,$$

we have

$$\mathbf{U}_k(x,0) \to \bar{\mathbf{U}}(x,0) \text{ in } \left(L^2(-\frac{T}{2}, \frac{T}{2})\right)^2. \tag{19}$$

Note that

$$\begin{aligned}
&\int_{\Omega_T} y^a |\nabla(\mathbf{U}_k(x,y) - \bar{\mathbf{U}}(x,y))|^2 dxdy \\
&= \langle J'(\mathbf{U}_k, \Omega_T) - J'(\bar{\mathbf{U}}, \Omega_T), \mathbf{U}_k - \bar{\mathbf{U}} \rangle \\
&- \int_{-\frac{T}{2}}^{\frac{T}{2}} [\nabla F(\mathbf{U}_k(x,0)) - \nabla F(\bar{\mathbf{U}}(x,0))] \cdot (\mathbf{U}_k(x,0) - \bar{\mathbf{U}}(x,0))dx,
\end{aligned}$$

where

$$\begin{aligned}
\langle J'(\mathbf{W}), \boldsymbol{\eta} \rangle = &\int_{\Omega_T} y^a \nabla \mathbf{W}(x,y) : \nabla \boldsymbol{\eta}(x,y) dxdy \\
&+ \int_{-\frac{T}{2}}^{\frac{T}{2}} \nabla F(\mathbf{W}(x,0)) \cdot \boldsymbol{\eta}(x,0)dx.
\end{aligned}$$

Clearly

$$\langle J'(\mathbf{U}_k) - J'(\bar{\mathbf{U}}), \mathbf{U}_k - \bar{\mathbf{U}} \rangle \to 0.$$

We also have

$$\begin{aligned}
&\left| \int_{-\frac{T}{2}}^{\frac{T}{2}} [\nabla F(\mathbf{U}_k(x,0)) - \nabla F(\bar{\mathbf{U}}(x,0))] \cdot (\mathbf{U}_k(x,0) - \bar{\mathbf{U}}(x,0))dx \right| \\
&\le C \int_{-\frac{T}{2}}^{\frac{T}{2}} |\mathbf{U}_k(x,0) - \bar{\mathbf{U}}(x,0)|^2 dx \to 0,
\end{aligned}$$

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where the convergence result follows from (19). Hence

$$\int_{\Omega_T} y^a |\nabla(\mathbf{U}_k(x,y) - \bar{\mathbf{U}}(x,y))|^2 dxdy \to 0.$$

This and (19) give that

$$\mathbf{U}_k \to \bar{\mathbf{U}} \text{ in } \mathcal{H}.$$

We have obtained the Palais–Smale condition.

Let

$$\boldsymbol{\Gamma} := \{g \in C([0,1];\mathcal{H}) : g(0) = \mathbf{b}_2,\ g(1) = \mathbf{b}_1\}.$$

Note that

$$J(\mathbf{b}_1) = J(\mathbf{b}_2) = 0 < J(\mathbf{W}), \quad \forall\ \mathbf{W} \in \mathcal{H}, \mathbf{W} \neq \mathbf{b}_i, i = 1,2.$$

Hence we have

$$\delta_T := \inf_{g\in\boldsymbol{\Gamma}} \sup_{t\in[0,1]} J(g(t)) > 0.$$

We set

$$J(\mathbf{U}_T) = \delta_T, \quad \text{where } \mathbf{U}_T = g(t_0) \text{ for some } g \in \boldsymbol{\Gamma} \text{ and some } t_0 \in (0,1).$$

Now we extend $\mathbf{U}_T$ periodically (with respect to x) from Ω_T to the whole half space $\overline{\mathbb{R}^2_+}$, and we still denote it as $\mathbf{U}_T$. An argument similar to that the proof of Theorem 1.1 in [14] shows that $\mathbf{U}_T$ is a weak solution of (5).

Next we show that $\mathbf{U}_T \not\equiv 0$. Choose a test function $(\psi, 0) \in \mathcal{H}$, where

$$\psi(x,y) = \exp\left\{-\frac{y}{2^{b+1}}\right\} h(x).$$

Here b is a parameter to be determined later and h is the odd function defined in Section 2. We construct a path as

$$\bar{g}(t) = \begin{cases} 2t\psi + (1-2t)(-1), & \text{for } 0 \le t \le \frac{1}{2}, \\ (2-2t)\psi + (2t-1), & \text{for } \frac{1}{2} \le t \le 1. \end{cases}$$

Clearly $(\bar{g}, 0) \in \boldsymbol{\Gamma}$. We have

$$\int_{\Omega_T} y^a |\nabla \bar{g}|^2 dxdy \le \int_{\Omega_T} y^a |\nabla \psi(x,y)|^2 dxdy.$$

Note that

$$\bar{g}(t) = \begin{cases} 2t\psi + (1-2t)(-1) \in [-1, \psi], & \text{for } 0 \le t \le \frac{1}{2}, \\ (2-2t)\psi + (2t-1) \in [\psi, 1], & \text{for } \frac{1}{2} \le t \le 1. \end{cases}$$

We denote $\bar{g}(t)$ as $\bar{g}_t(x,y)$ to emphasize the dependence of $\bar{g}$ on (x,y). Then for $0 \le t \le \frac{1}{2}$, we have

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} F(\bar{g}_t(x,0),0)dx = \int_{-\frac{T}{2}}^{0} F(\bar{g}_t(x,0),0)dx + \int_0^{\frac{T}{2}} F(\bar{g}_t(x,0),0)dx$$
$$\le \int_{-\frac{T}{2}}^{0} F(\psi(x,0),0)dx + F(0,0)\frac{T}{2}.$$

Similarly, for $\frac{1}{2} \le t \le 1$, we have

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} F(\bar{g}_t(x,0),0)dx \le \int_0^{\frac{T}{2}} F(\psi(x,0),0)dx + F(0,0)\frac{T}{2}.$$

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Hence for any $t \in [0,1]$, one has

$$J(\bar{g}_t, 0) \le F(0,0)\frac{T}{2} + \frac{1}{2}\int_{\Omega_T} y^a |\nabla\psi(x,y)|^2 dxdy \qquad (20)$$
$$+\max\left\{\int_{-\frac{T}{2}}^{0} F(\psi(x,0),0)dx, \int_0^{\frac{T}{2}} F(\psi(x,0),0)dx\right\}.$$

Then similar computation as in [17] shows that there exists $T_2 > 0$ such that for $T > T_2$ we have

$$J(\bar{g}_t, 0) < F(0,0)T = J(0,0), \quad \text{for } \forall\ t \in [0,1].$$

Hence

$$J(\mathbf{U}_T) = \delta_T \le \max_{t\in[0,1]} J(\bar{g}_t, 0) < J(0,0),$$

which gives that $\mathbf{U}_T \not\equiv 0$. Then $\mathbf{u}_T(x) := \mathbf{U}_T(x,0)$ is the desired solution of system (1) with $\mathbf{u}_T(x) \not\equiv 0$. □

5. Hamiltonian estimates

Proof of Theorem 1.4. By Lemma 5.1 in [7], we have $\int_0^{+\infty} y^a |\nabla \mathbf{U}(x,y)|^2 dy < \infty$. Hence $\lim_{y\to+\infty} y^a \mathbf{U}_y(x,y)\cdot\mathbf{U}_x(x,y) = 0$.

We introduce the function

$$w(x) := \frac{1}{2}\int_0^{\infty} [|\mathbf{U}_x(x,y)|^2 - |\mathbf{U}_y(x,y)|^2] y^a dy.$$

A suitable regularity result (see Lemma 5.1 in [7]) allows us to differentiate within the integral in the above equality to get

$$w'(x) = \int_0^{\infty} y^a [\mathbf{U}_x\cdot\mathbf{U}_{xx} - \mathbf{U}_y\cdot\mathbf{U}_{xy}](x,y)dy.$$

Note that

$$(y^a\mathbf{U}_y)_y + y^a\mathbf{U}_{xx} = 0.$$

Using integration by parts, we have

$$w'(x) = -[y^a\mathbf{U}_y(x,y)\cdot\mathbf{U}_x(x,y)]|_{y=0}^{+\infty} = \lim_{y\to0^+} y^a\mathbf{U}_y(x,y)\cdot\mathbf{U}_x(x,y).$$

Since $\mathbf{U}$ is the s-harmonic extension of solution $\mathbf{u}$ of (1), we have

$$\lim_{y\to0^+} y^a\mathbf{U}_y(x,y)\cdot\mathbf{U}_x(x,y) = (-\partial_{xx})^s\mathbf{u}(x)\cdot\mathbf{u}'(x) = \frac{d}{dx}F(\mathbf{U}(x,0)).$$

Hence

$$w'(x) = \frac{d}{dx}F(\mathbf{U}(x,0)),$$

which gives the result of this lemma. □

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