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Calculus of Variations

Some energy estimates for stable solutions to fractional Allen–Cahn equations

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Abstract

In this paper we study stable solutions to the fractional equation

$$(-\Delta)^s u = f(u), \quad |u| < 1 \quad \text{in } \mathbb{R}^d, \tag{0.1}$$

where $0 < s < 1$ and $f : [-1, 1] \to \mathbb{R}$ is a $C^{1,\alpha}$ function for $\alpha > \max\{0, 1-2s\}$. We obtain sharp energy estimates for $0 < s < 1/2$ and rough energy estimates for $1/2 \leq s < 1$. These lead to a different proof from literature of the fact that when $d = 2$, $0 < s < 1$, entire stable solutions to (0.1) are 1-D solutions. The scheme used in this paper is inspired by Cinti–Serra–Valdinoci [16] which deals with stable nonlocal sets, and Figalli–Serra [26] which studies stable solutions to (0.1) for the case $s = 1/2$.

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1 Introduction

1.1 Nonlocal stable De Giorgi conjeture

It is well known that for $0 < s < 1$, the fractional s-Laplacian is defined as

$$(-\Delta)^s u(x) := C(d, s)(P.V.) \int_{\mathbb{R}^d} \frac{u(x) - u(y)}{|x-y|^{d+2s}} dy \tag{1.1}$$

$$= \frac{C(d, s)}{2} \int_{\mathbb{R}^d} \frac{2u(x) - u(x+y) - u(x-y)}{|x-y|^{d+2s}} dy, \tag{1.2}$$

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where $C(d, x)$ is a constant such that

$$\widehat{(-\Delta)^s} u(\xi) = |\xi|^{2s} \hat{u}(\xi). \tag{1.3}$$

For $\Omega \subset \mathbb{R}^d$, we consider the fractional Allen–Cahn type equation

$$(-\Delta)^s u = f(u), \quad |u| < 1 \quad \text{in } \Omega,$$

which is the vanishing condition for the first variation of the energy

$$\begin{aligned}\mathcal{J}(u, \Omega) &= \mathcal{J}^s(u, \Omega) + \mathcal{J}^P(u, \Omega) \\ &:= \frac{1}{2} \int\int_{\mathbb{R}^d \times \mathbb{R}^d \setminus (\Omega^c \times \Omega^c)} \frac{|u(x) - u(y)|^2}{|x - y|^{d+2s}} dydx \\ &\quad + \int_\Omega F(u(x))dx,\end{aligned}$$

up to normalization constants that we omitted for simplicity.

Throughout the paper we assume that F is the primitive function of a given $C^{1,\alpha}$ function $f : [-1, 1] \to \mathbb{R}$, where $\alpha > \max\{0, 1 - 2s\}$. The regularity of f is to guarantee that any solution u to (0.1) is in $C^2(\mathbb{R}^d)$ so that the fractional Laplacian is well defined, see for example [8, Lemma 4.4] for the proof. We also throughout the paper assume that $F : \mathbb{R} \to [0, \infty)$ is a double well potential with two minima -1 and 1. This is the sufficient and necessary condition to guarantee the existence of 1-D layer solutions to (0.1), see [9, Theorem 2.4]. Recall that *layer solutions* are solutions that are monotone in one variable and have limits ± 1 at $\pm\infty$.

In this paper, we study stable solutions to the fractional Allen–Cahn equation (0.1). Recall that u is a *stable solution* to (0.1), if the second local variation of $\mathcal{J}(\cdot, \mathbb{R}^d)$ at u is nonnegative. Or equivalently,

$$\int_{\mathbb{R}^d} \left((-\Delta)^s v + f'(u)v\right) v \geq 0, \quad \forall v \in C_0^2(\mathbb{R}^d).$$

Note that stable solutions include local minimizers or monotone stationary solutions of $\mathcal{J}(\cdot, \mathbb{R}^d)$. Also it is known that 1-D stable solutions are layer solutions, which is a consequence of [18, Lemma 3.1] and [9, Theorem 2.12].

We would like to study the symmetry results for stable entire solutions to (0.1), which is related to the nonlocal version of De Giorgi Conjecture for stable solutions:

Conjecture 1 (Nonlocal Stable De Giorgi Conjecture) *Let* $0 < s < 1$ *and* u *be a stable solution to* (0.1)*, then* u *is a* 1*-D solution for* $d \leq 7$.

1.2 Background and motivation of Conjecture 1

In 1979, De Giorgi made the following conjecture on the entire solutions to classical Allen–Cahn equations:

Conjecture 2 (Classical De Girogi Conjecture) *If* u *is a solution to the classical Allen–Cahn equation*

$$-\Delta u = u - u^3, \quad |u| < 1 \quad \text{in } \mathbb{R}^d, \tag{1.4}$$

with $\partial_{x_d} u > 0$*, then* u *is a* 1*-D solution if* $d \leq 8$.

The classical De Giorgi conjecture is closely related to minimal surface theory. If u is a local minimizer to the associated energy funtional

$$\mathcal{E}(u, \Omega) := \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{4}\int_\Omega (1-u^2)^2 dx, \tag{1.5}$$

where $\Omega = \mathbb{R}^d$, then $u_\epsilon(x) := u(x/\epsilon)$ is a minimizer to

$$\mathcal{E}_\epsilon(v, \epsilon\Omega) = \int_{\epsilon\Omega} \frac{\epsilon}{2}|\nabla v|^2 dx + \frac{1}{4\epsilon}\int_{\epsilon\Omega} (1-v^2)^2 dx.$$

Scaling and energy estimates for minimizers imply

$$\mathcal{E}_\epsilon(u_\epsilon, B_1) = \epsilon^{d-1}\mathcal{E}(u, B_{1/\epsilon}) \le C(d).$$

By Modica-Mortola Gamma convergence result [29], $u_\epsilon \to \chi_E - \chi_{E^c}$ in L^1_{loc} for a subsequence $\epsilon_k \to 0$, and E is a perimeter minimizer in $\mathbb{R}^d$. If $2 \le d \le 8$ and ∂E is a graph, then the classification of entire minimal graphs in $\mathbb{R}^d$ implies that E must be a half space, and thus $\{u_\epsilon > t\}$ converge to a half space locally in L^1 for $-1 < t < 1$. Since $\{u_\epsilon > t\} = \epsilon\{u > t\}$, De Giorgi conjectured that $\{u > t\}$ itself should also be a half space for any t, even for u to be monotone in direction without having to be a minimizer.

The case when $d = 2$ was proved by Ghoussoub and Gui in [28], and the case when $d = 3$ was proved by Ambrosio and Cabré in [1]. For $d \ge 9$, counterexamples were given by Del Pino, Kowalczyk and Wei [17]. The case $4 \le d \le 8$ was proved by Savin [32] under the additional assumption that

$$\lim_{x_d \to \pm\infty} u(x', x_d) = \pm 1, \quad \text{for any } x' \in \mathbb{R}^{d-1}. \tag{1.6}$$

The conjecture remains open for $4 \le d \le 8$ without the limit condition (1.6). We remark that in [32], only the minimality of u is used, which is guaranteed by the the monotone condition and (1.6). We also remark that if the limit in (1.6) is uniform, then Conjecture 2 is true for any dimension d without the monotone assumption. This is proved in [2,3,22] independently.

This conjecture in its full generality remains open. As far as we know, at this moment the most general results can be found in [23,24] and references therein.

In the fractional analogue, if a solution u is a minimizer to the associated energy, then $u_\epsilon(x) := u(x/\epsilon)$ is a minimizer to

$$\mathcal{J}_{s,\epsilon}(u, \Omega) := \begin{cases} \epsilon^{2s-1}\mathcal{J}^s(u, \epsilon\Omega) + \frac{1}{\epsilon}\mathcal{J}^P(u, \epsilon\Omega), & \text{if} \quad 1/2 < s < 1, \\ \frac{1}{|\log\epsilon|}\mathcal{J}^s(u, \epsilon\Omega) + \frac{1}{\epsilon|\log\epsilon|}\mathcal{J}^P(u, \epsilon\Omega), & \text{if} \quad s = 1/2, \\ \mathcal{J}^s(u, \epsilon\Omega) + \frac{1}{\epsilon^{2s}}\mathcal{J}^P(u, \epsilon\Omega), & \text{if} \quad 0 < s < 1/2. \end{cases}$$

In [38], Savin and Valdinoci proved that if $\sup_{0<\epsilon<1} J_{s,\epsilon}(u_\epsilon, \Omega) < \infty$, then $u_\epsilon \to \chi_E - \chi_{E^c}$ in L^1 up to a subsequence, where E is a perimeter minimizer in Ω for $s \in [1/2, 1)$ and an s-perimeter minimizer in Ω for $s \in (0, 1/2)$. The classification for global s-minimal graphs is the following, which is a combination of several works due to Cafarelli, Figalli, Valdinoci and Savin, see [14], [27] and [33].

Let E be an s-perimeter graph. Assume that either

- $d = 2, 3$,
- or $d \le 8$ and $\frac{1}{2} - s \le \epsilon_0$ for some $\epsilon_0 > 0$ sufficiently small.

Then E must be a half space.

It is not known whether the above classification result is optimal, since there are no known examples of s-minimal graphs other than hyperplanes, as far as we are aware.

These results motivate the following De Giorgi conjecture in the nonlocal case:

Conjecture 3 (Nonlocal De Giorgi Conjecture) *Let* $0 < s < 1$ *and* u *be a solution to* (0.1) *with*

$$\partial_{x_d} u > 0, \tag{1.7}$$

then u *is a* 1*-D solution for* $d \leq 8$.

Conjecture 3 has been validated in different cases, according to the following result:

Theorem 1.1 *Let* u *be an entire solution to* (0.1) *satisfying* (1.7)*. Suppose that either* $d = 2, 3,\ s \in (0, 1)$ *or* $d = 4,\ s = 1/2$*, then* u *is* 1*-D.*

Theorem 1.1 is due to [11] when $d = 2,\ s = 1/2$, [9] and [35] when $d = 2,\ 0 < s < 1$, [5] when $d = 3,\ s = 1/2$, [6] when $d = 3,\ 1/2 < s < 1$, [18] when $d = 3,\ 0 < s < 1/2$ and [26] when $d = 4,\ s = 1/2$.

Concerning the nonlocal De Giorgi Conjecture in higher dimensions with the additional limit condition (1.6) or with minimality condition, the best known results are the following two theorems, which were proved in [36] when $s \in (1/2, 1)$, [37] when $s = 1/2$ and [19] when $s \in (1/2 - \epsilon_0, 1]$.

Theorem 1.2 *Let* $d \leq 8$*. Then, there exists* $\epsilon_0 \in (0, 1/2]$ *such that for any* $s \in (1/2 - \epsilon_0, 1]$*, the following statement holds true:*

Let u *be an entire solution to* (0.1) *satisfying* (1.6) [1] *and* (1.7)*, then* u *is* 1*-D.*

Theorem 1.3 *Let* $d \leq 7$*. Then, there exists* $\epsilon_0 \in (0, 1/2]$ *such that for any* $s \in (1/2 - \epsilon_0, 1]$*, the following statement holds true:*

Let u *be an entire solution to* (0.1) *which is a minimizer of* $\mathcal{J}(\cdot, \mathbb{R}^d)$*, then* u *is* 1*-D.*

A counterexample for $d = 9,\ 1/2 < s < 1$ is announced by H. Chan, J. Davila, M. del Pino, Y. Liu and J. Wei, see the comments after [10, Theorem 1.3]. The other cases remain open. We also refer the reader to the very nice survey paper [21] for a summary of recent results on the nonlocal De Giorgi conjecture.

Motivated by Conjecture 3, it is natural to study the stable De Giorgi Conjecture, that is, Conjecture 1. This is because, on the one hand, it is well known that monotone solutions to (0.1) are stable solutions. On the other hand, a further relation between stable solutions and monotone solutions to (0.1) is given in the following remark, whose proof can be found in [18].

Remark 1.4 If any entire stable solution to (0.1) in $\mathbb{R}^{d-1}$ is 1-D, then any monotone solution to (0.1) in $\mathbb{R}^d$ is also 1-D for $d \leq 3, s \in (0, 1)$ and for $4 \leq d \leq 8,\ s \in (1/2 - \epsilon_0, 1)$, where $\epsilon_0 \in (0, 1/2]$ is some constant.

Because of the connection between monotone solutions and stable solutions as revealed in Remark 1.4, it is important to study Conjecture 1.

1.3 Previous results on Conjecture 1

For $d = 2$, Conjecture 1 was validated by Cabré and Solá-Morales in [11] for $s = 1/2$, and by Cabré and Sire in [9] and by Sire and Valdinoci in [35] for every fractional power

[1] This condition can be replaced by the more general condition that the limits of u when $x_d \to \pm\infty$ are 2D , see [18, Theorem 8.1]

$0 < s < 1$ with different proofs, all of which require Cafarelli–Silvestre extension [12] and the stability of s-harmonic extension U in $\mathbb{R}^d \times (0, \infty)$. The stability condition used in these references is the following:

Remark 1.5 In [9,11,35], the stability of solution u to (0.1) was understood in the sense that the second local variation of the extension energy

$$\mathcal{E}(U; \mathbb{R}^{d+1}_+) = \frac{1}{2} \int_0^\infty \int_{\mathbb{R}^d} z^{1-2s} |\nabla U|^2 dx dt + \int_{\mathbb{R}^d} F(u(x)) dx$$

is nonnegative at U, where U is the Cafarelli–Silvestre extension of u which solves

$$\begin{cases} (i)\, \mathrm{div}(t^{1-2s} U(x,t)) = 0 & \text{in} \quad \mathbb{R}^d \times (0, \infty) \\ (ii)\, c_s \lim_{t \to 0} t^{1-2s} \partial_t U(x,t) = f(U(x,0)) & \text{on} \quad \partial \mathbb{R}^{d+1}_+ \end{cases}$$

with boundary condition $U(x, 0) = u(x)$, where c_s is a constant which is discussed in [8, Remark 3.11]. It appears that this stable assumption is stronger than ours which just considers local variations on $\mathbb{R}^d$ instead of $\mathbb{R}^d \times (0, \infty)$. Later it was shown in [18, Proposition 2.3] that the two stable definitions are equivalent for every fractional power $0 < s < 1$.

For $3 \le d \le 8$ and $0 < s < 1$, Conjecture 1 remains open except for the case $d = 3$ and $s = 1/2$. In fact, it has been recently validated by Figalli and Serra in [26] without using extension results in [12]. Figalli and Serra utilized the local BV estimates scheme originally developed by Cinti et al. [16] for stable sets (see Definition 1.6 there), together with the following sharp interpolation inequality

$$\mathcal{J}^{1/2}(u, B_1) \le C(d) \log L_0 \left(1 + \int_{B_2} |\nabla u| dx \right), \tag{1.8}$$

where $L_0 \ge 2$ is an upper bound for $\|\nabla u\|_\infty$, to prove the following energy estimates in any dimension d and $s = 1/2$, which is a key ingredient to validate Conjecture 1.

Proposition 1.6 [26, Proposition 1.7] *If u is a stable solution to* (0.1)*, then*

$$\int_{B_R} |\nabla u| \le C R^{d-1} \log(M_0 R) \tag{1.9}$$

and

$$\mathcal{J}^{1/2}(u, B_R) \le C R^{d-1} \log^2(M_0 R), \tag{1.10}$$

where C is a universal constant depending only on d and α, and $M_0 \ge 2$ is an upper bound for the Hölder norm of f.

With (1.9) and (1.10) being applied in the local BV estimate scheme, and by a bootstrap argument, Figalli and Serra were able to prove Conjecture 1 for $d = 3$ and $s = 1/2$.

1.4 Our contribution in this paper

Proving energy estimates like (1.9) and (1.10) for stable solutions to (0.1) for every fractional power $s \in (0, 1)$ is definitely a decisive step to solve Conjecture 1.

We have observed that actually suitable adaptation of the local BV estimate scheme used in [26] together with a generalized form of (1.8) can produce energy estimates for stable solutions in arbitrary dimension d and energy $0 < s < 1$. We prove:

Proposition 1.7 *Let $u \in C^2(\mathbb{R}^d)$ be a stable solution to*

$$(-\Delta)^s u = f(u), \quad |u| \le 1 \ \ in\ \mathbb{R}^d, \tag{1.11}$$

then there exists constant $C_1 = C_1(d,s)$ and $C_2 = C_2(d,s,f)$ such that for any ball $B_R \subset \mathbb{R}^d$, $R \ge 1$, we have

$$\int_{B_R} |\nabla u| \le \begin{cases} C_1 R^{d-1} & 0 < s < \frac{1}{2} \\ C_2 R^{d+2s-2} \log(M_0 R) & \frac{1}{2} \le s < 1 \end{cases} \tag{1.12}$$

and

$$\mathcal{J}^s(u, B_R) \le \begin{cases} C_1 R^{d-2s} & 0 < s < \frac{1}{2} \\ C_2 R^{d+2s-2} \log^2(M_0 R) & \frac{1}{2} \le s < 1, \end{cases} \tag{1.13}$$

where $M_0 \ge 2$ is an upper bound for L^∞ norm of f.

Note that it is easy to see that for a bounded Lipschitz function u, the natural growth for fractional energy is

$$\mathcal{J}^s(u, B_R) \le C R^d,$$

see for example Lemma 1.8 below. Such estimate is too rough. It is with the stability condition of u that we can derive a sharper fractional energy growth estimate (1.13) than the natural one.

(1.12) and (1.13) are sharp for the case $0 < s < 1/2$, in the sense that the local minimizers do satisfy same estimates, which are optimal, see [34] and [30]. Although for the case $1/2 \le s < 1$, our energy estimates are not optimal, the adaptation of local BV estimates scheme in [16] and [26] together with our energy estimates can also give a different proof to validate Conjecture 1 for the case $d = 2$, $0 < s < 1$, see Theorem 3.7.

We remark that when $s = 1/2$, C_2 does not depend on f by keeping track of the constant in our proof. Thus in this case, the second inequalities in (1.12) and (1.13) coincide with (1.9) and (1.10) in Proposition 1.6.

We also remark that the key of proving (1.8) is by [25, Lemma 2.1] (or [26, Theorem 2.4]), whose proof was based on by Plancherel formula plus some delicate estimates. The proof seems to work only for the case $s = 1/2$. We give a different proof in this paper that actually works for all cases $1/2 \le s < 1$. In fact, we can prove the following result, which might have independent interest.

Lemma 1.8 *For any ball $B_R \subset \mathbb{R}^d$ and u which belongs to appropriate space with $|u| \le 1$, and let $s \in (0, 1/2)$, there exists universal constant $C = C(d,s) > 0$ such that for any $R \ge 1$,*

$$\mathcal{J}^s(u, B_R) \le C\left(\int_{B_{2R}} |\nabla u| dx + R^{d-2s} + R^d\right). \tag{1.14}$$

If $1/2 \le s < 1$ and u is assumed to be a Lipschitz function with $\|\nabla u\|_{L^\infty(B_R)} \le L_0$, $L_0 \ge 2$, then there exists $C = C(d,s) > 0$ such that

$$\mathcal{J}^s(u, B_R) \le C\left(R^{d-2s} + L_0^{2s-1} \log(2L_0 R) \int_{B_{2R}} |\nabla u|\right). \tag{1.15}$$

Note that when $R = 1$ and $s = 1/2$, (1.15) is exactly (1.8).

It is with Lemma 1.8 and the adaptation of local BV estimate for arbitary fractional powers $s \in (0, 1)$, we can prove Proposition 1.7.

Remark 1.9 Only after this work was completed, we have noticed that Cinti has mentioned in her survey article [15] that she, Cabré and Serra are carrying out a careful study on nonlocal stable phase transitions in [7], which has not been posted yet. As Cinti mentioned, they will state energy estimates, density estimates, convergence of blow-down and some new classification results for stable solutions for fractional powers $0 < s < 1/2$. While our focus in this paper is to exploit the ideas in [16,26] to prove energy estimates for all fractional powers $0 < s < 1$, as best as we can do at this moment.

1.5 Outline of this paper

In Sect. 2 we prove Lemma 1.8. In Sect. 3, we validate the BV estimate scheme for any fractional power $s \in (0, 1)$ and use it to prove Proposition 1.7, and then as an application we validate Conjecture 1 for the case $d = 2,\ s \in (0, 1)$.

2 Proof of Lemma 1.8

In this section we prove Lemma 1.8. We first recall the fractional Sobolev embedding result:

Proposition 2.1 (see [21, Proposition 2.2]) *For $s \in (0, 1)$, $p \geq 1$ and $B_R \subset \mathbb{R}^d$, we have*

$$\|u\|_{W^{s,p}(B_R)} \leq C(d, p, s)\|u\|_{W^{1,p}(B_R)} \tag{2.1}$$

In order to prove Lemma 1.8, we also need to prove:

Lemma 2.2 *Assume $|u| \leq 1$ and $\|\nabla u\|_{L^\infty(B_1)} \leq L_0$, where $L_0 \geq 2$, then for $s \in [1/2, 1)$,*

$$\int_{B_1}\int_{B_1} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}dxdy \leq \frac{1}{1-s}d\omega_d L_0^{2s-1}\Big((2-2s)\log(2L_0)+1\Big)\int_{B_1}|\nabla u(x)|dx \tag{2.2}$$

$$= C(d, s)L_0^{2s-1}\log(L_0)\int_{B_1}|\nabla u(x)|dx, \tag{2.3}$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$.

Proof We estimate

$$\begin{aligned}
&\int_{B_1}\int_{B_1} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}dydx \\
&= \int_{B_1}\int_{B_2} \chi_{\{z:\, x+z\in B_1\}}|u(x+z)-u(x)|^{2-2s}\frac{|u(x+z)-u(x)|^{2s}}{|z|^{d+2s}}dzdx \\
&= \int_{B_1}\int_0^2\int_{B_2} \chi_{\{z:\, |u(x+z)-u(x)|^{2-2s}>t,\, x+z\in B_1\}}\frac{|u(x+z)-u(x)|^{2s}}{|z|^{d+2s}}dzdtdx \\
&\leq \int_{B_1}\int_0^2\int_{B_2} \chi_{\{z:|z|>\frac{t^{\frac{1}{2-2s}}}{M_0},\, x+z\in B_1\}}\frac{|u(x+z)-u(x)|^{2s}}{|z|^{d+2s}}dzdtdx \\
&= \int_{B_1}\int_0^2\int_{B_2} \chi_{\{z:|z|>\frac{t^{\frac{1}{2-2s}}}{M_0},\, x+z\in B_1\}}\frac{|u(x+z)-u(x)|}{|z|^{d+1}}\frac{|u(x+z)-u(x)|^{2s-1}}{|z|^{2s-1}}dzdtdx
\end{aligned}$$

$$
\begin{aligned}
&\le M_0^{2s-1}\int_{B_1}\int_0^2\int_{B_2}\chi_{\{z:|z|>\frac{t^{\frac{1}{2-2s}}}{M_0},\,x+z\in B_1\}}\frac{\int_0^1|\nabla u(x+rz)|dr}{|z|^d}dzdtdx\\
&= M_0^{2s-1}\int_0^1\int_0^2\int\chi_{\{z\in B_2:|z|>\frac{t^{\frac{1}{2-2s}}}{M_0}\}}\int_{B_1}\chi_{\{x\in B_1:\,x+z\in B_1\}}\frac{|\nabla u(x+rz)|}{|z|^d}dxdzdtdr\\
&\le M_0^{2s-1}\int_{B_1}|\nabla u(x)|dx\int_0^1\int_0^2\int\chi_{\{z\in B_2:|z|>\frac{t^{\frac{1}{2-2s}}}{M_0}\}}\frac{1}{|z|^d}dzdtdr\\
&= M_0^{2s-1}\int_{B_1}|\nabla u(x)|dx\int_0^1\int_0^{(2M_0)^{2-2s}\wedge 2}\int\chi_{\{z\in B_2:|z|>\frac{t^{\frac{1}{2-2s}}}{M_0}\}}\frac{1}{|z|^d}dzdtdr\\
&= dw_d M_0^{2s-1}\int_{B_1}|\nabla u(x)|dx\int_0^1\int_0^{(2M_0)^{2-2s}\wedge 2}\left(\log(2M_0)-\frac{1}{2-2s}\log t\right)dtdr\\
&= 2\wedge(2M_0)^{2-2s}d\omega_d M_0^{2s-1}\int_{B_1}|\nabla u(x)|dx\left(\log(2M_0)+\frac{1-log\left(2\wedge(2M_0)^{2-2s}\right)}{2-2s}\right)\\
&\le \frac{1}{1-s}d\omega_d M_0^{2s-1}\Big((2-2s)\log(2M_0)+1\Big)\int_{B_1}|\nabla u(x)|dx,
\end{aligned}
$$

where in the above we have used that the layer-cake formula for nonnegative function $g \in L^1(d\lambda)$, λ being a Radon measure,

$$
\int g(x)H(x)d\lambda = \int_0^{\|g\|_\infty}\int_{\{x:g(x)>t\}}H(x)d\lambda dt,
$$

that for $s \in [1/2, 1)$,

$$
\{z : |u(x+z)-u(x)|^{2-2s} > t\} \subset \{z : |z| > \frac{t^{\frac{1}{2-2s}}}{M_0}\},
$$

and that $x \in B_1,\ x+z \in B_1$ implies

$$
x + rz = r(x+z) + (1-r)x \in B_1, \quad \text{by convexity of } B_1.
$$

□

The following corollary can be obtained by modifying the proof of Lemma 2.2, and it might have some independent interest.

Corollary 2.3 *Let $L_0 \ge 2$. then for any $|u| \le 1$, $\|\nabla u\|_{L^\infty(B_1)} \le L_0$ and any $p > 1$, the following estimate holds:*

$$
\int_{B_1}\int_{B_1}\frac{|u(x)-u(y)|^p}{|x-y|^{d+1}}dxdy \le C(d,p)\log(L_0)\int_{B_1}|\nabla u(x)|dx.
$$

We omit the proof of this corollary.

Now we prove Lemma 1.8.

Proof of Lemma 1.8 For $0 < s < 1/2$, we estimate

$$
\mathcal{J}^s(u, B_R) = \int\int_{\mathbb{R}^d\times\mathbb{R}^d\setminus(B_R^c\times B_R^c)}\frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}dydx
$$

$$
\begin{aligned}
&\le \int\int_{B_{2R}\times B_{2R}} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}dydx + 2\int\int_{B_R\times B_{2R}^c} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}dydx\\
&\le 2\int\int_{B_{2R}\times B_{2R}} \frac{|u(x)-u(y)|}{|x-y|^{d+2s}}dydx + C(d,s)R^{d-2s}, \quad \text{since } |u|\le 1\\
&= 2[u]_{W^{2s,1}(B_{2R})} + C(d,s)R^{d-2s}\\
&\le C(d,s)\|u\|_{W^{1,1}(B_{2R})} + C(d,s)R^{d-2s}, \quad \text{by Proposition 2.1}\\
&\le C(d,s)\left(\int_{B_{2R}} |\nabla u|dx + R^{d-2s} + R^d\right).
\end{aligned}
$$

This concludes (1.14).

Let us now prove the lemma for the case $1/2 \le s < 1$. For any ball $B_R(x_0) \subset \mathbb{R}^d$, we let $u_R := u(x_0 + Rx)$, and thus $\|\nabla u_R\|_{L^\infty} = R\|\nabla u\|_{L^\infty} \le RL_0$.

By applying Lemma 2.2 to u_R and using the scaling properties

$$
\mathcal{J}^s(u_R, B_1) = R^{2s-d}\mathcal{J}(u, B_R(x_0)) \quad \text{and} \quad \int_{B_1} |\nabla u_R|dx = R^{1-d}\int_{B_R(x_0)} |\nabla u|dx,
$$

we thus derive

$$
\int\int_{B_R\times B_R} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}dydx \le C(d,s)L_0^{2s-1}\log(RL_0)\int_{B_R} |\nabla u(x)|dx. \tag{2.4}
$$

Therefore, (1.15) is obtained by the following straightforward computation

$$
\begin{aligned}
\mathcal{J}^s(u, B_R) &= \int\int_{\mathbb{R}^d\times\mathbb{R}^d\setminus(B_R^c\times B_R^c)} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}dydx\\
&\le \int\int_{B_{2R}\times B_{2R}} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}dydx + 2\int\int_{B_R\times B_{2R}^c} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}dydx\\
&\le \int\int_{B_{2R}\times B_{2R}} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}dydx + C(d,s)R^{d-2s}, \quad \text{since } |u|\le 1\\
&\le C(d,s)\left(R^{d-2s} + L_0^{2s-1}\log(2L_0R)\int_{B_{2R}} |\nabla u(x)|dx\right), \quad \text{by (2.4)}.
\end{aligned}
$$

□

3 Local BV estimate scheme for any power $0 < s < 1$ and Proof of Proposition 1.7

As we mentioned in introduction, the local BV estimate scheme was first developed in [16] and adapted by Figalli and Serra in [26] for the study of stable solutions to (0.1) when $s = 1/2$. In this section we show that thanks to Lemma 1.8, the scheme can be applied to give certain energy estimates for every fractional power $0 < s < 1$, as stated in Proposition 1.7.

First, to utilize the stability condition of solution u to (0.1), following [26], see also [4, Lemma 4.3], we construct suitable variations of energy with respect to a direction $\mathbf{v}$, where $\mathbf{v}$ is a fixed unit vector in $\mathbb{R}^d$.

Let $R \ge 1$ and

$$
\psi_{t,\mathbf{v}}(x) := x + t\phi(x)\mathbf{v},
$$

where

$$\phi(x) = \begin{cases} 1, & |x| \le \frac{R}{2} \\ 2 - 2\frac{|x|}{R}, & \frac{R}{2} \le |x| \le R \\ 0, & |x| \ge R. \end{cases} \tag{3.1}$$

It is clear that when $|t|$ small, $\psi_{t,\mathbf{v}}$ is a Lipschitz diffeomorphism, and thus it has an inverse. Define

$$P_{t,\mathbf{v}}u(x) := u\left(\psi_{t,\mathbf{v}}^{-1}(x)\right).$$

Remark 3.1 It is clear that for $x \in B_{1/2}$, if $|t|$ is small, then $P_{t,\mathbf{v}}u(x) = u(x - t\mathbf{v})$.

To simplify notation, we define the second variation operator $\Delta^t_{\mathbf{vv}}$ with respect to $\mathbf{v}$ on any functional $\mathcal{J}$ to be as

$$\Delta^t_{\mathbf{vv}}\mathcal{J}(u,\Omega) := \mathcal{J}(P_{t,\mathbf{v}}u,\Omega) + \mathcal{J}(P_{-t,\mathbf{v}}u,\Omega) - 2\mathcal{J}(u,\Omega).$$

The following estimate for the second variation of fractional energy is proved in [4, Lemma 4.3] and [26, Lemma 2.1]. For the courtesy of reader, we include a proof.

Lemma 3.2

$$\Delta^t_{\mathbf{vv}}\mathcal{J}^s(u,B_R) \le C(d,s)t^2\frac{\mathcal{J}^s(u,B_R)}{R^2}, \quad \forall R \ge 1.$$

Proof We start with more general domain variations as follows. We consider the map

$$F_t(x) := x + t\eta(x), \tag{3.2}$$

where η is a smooth vector field vanishing outside B_R. We set

$$P_t u(x) := u(F_t^{-1}(x)). \tag{3.3}$$

We estimate

$$\Delta^t\mathcal{J}^s(u,B_R) := \mathcal{J}^s(P_t u,B_R) + \mathcal{J}^s(P_{-t}u,B_R) - 2\mathcal{J}^s(u,B_R).$$

We use $\tilde{B_R}$ to denote $\mathbb{R}^d \times \mathbb{R}^d \setminus (B_R \times B_R)$. In the following computation, $z = x - y$ and $\epsilon(x,y) := \frac{\eta(x)-\eta(y)}{|x-y|}$. Since the Taylor expansion of the Jacobian of F_t is

$$JF_t = 1 + t\mathrm{div}\eta + t^2A(\eta) + O(t^3),$$

where

$$A(\eta) = \frac{(\mathrm{div}\eta)^2 - tr(\nabla\eta)^2}{2},$$

we can compute

$$\begin{aligned}\Delta^t\mathcal{J}^s(u,B_R) = \int\int_{\tilde{B_R}} |u(x)-u(y)|^2\Big(&K(z+t\epsilon|z|)(1+t\mathrm{div}\eta(x) \\ &+ A(\eta(x))t^2)(1+t\mathrm{div}\eta(y)+A(\eta(y))t^2) \\ &+ K(z-t\epsilon|z|)(1-t\mathrm{div}\eta(x)+A(\eta(x))t^2)(1-t\mathrm{div}\eta(y) \\ &+ A(\eta(y))t^2) - 2K(z)\Big)dydx\end{aligned}$$

$$:= \int\int_{\tilde{B_R}} |u(x)-u(y)|^2 e(x,y,\eta,R)dydx,$$

where $K(z) = \frac{1}{|z|^{d+2s}}$. Using that

$$K(az) = |a|^{-d-2s}K(z), \quad \forall a \in \mathbb{R},$$

we have

$$\begin{aligned}
& e(x,y,\eta,R) \\
&= K(z)\Big(K(\frac{z}{|z|}+t\epsilon)(1+t\mathrm{div}\eta(x)+A(\eta(x))t^2)(1+t\mathrm{div}\eta(y)+A(\eta(y))t^2) \\
&\quad + K(\frac{z}{|z|}-t\epsilon)(1-t\mathrm{div}\eta(x)+A(\eta(x))t^2)(1-t\mathrm{div}\eta(y)+A(\eta(y))t^2) - 2K(\frac{z}{|z|})\Big) \\
&= K(z)\Big(\Big(K(z/|z|)+t\nabla K(z/|z|)\epsilon + \frac{t^2}{2} < \nabla^2 K(z/|z|)\epsilon, \epsilon > + O(t^3)\Big) \\
&\quad (1+t\mathrm{div}\eta(x)+A(\eta(x))t^2) \\
&\quad \cdot (1+t\mathrm{div}\eta(y)+A(\eta(y))t^2) + (K(z/|z|)-t\nabla K(z/|z|)\epsilon \\
&\qquad + \frac{t^2}{2} < \nabla^2 K(z/|z|)\epsilon, \epsilon > + O(t^3)\Big) \\
&\quad \cdot (1-t\mathrm{div}\eta(x)+A(\eta(x))t^2)(1-t\mathrm{div}\eta(y)+A(\eta(y))t^2) - 2K(z/|z|)\Big) \\
&= 2K(z)t^2\Big(A(\eta(x))+A(\eta(y))+\mathrm{div}\eta(x)\mathrm{div}\eta(y)+(\mathrm{div}\eta(x) \\
&\quad + \mathrm{div}\eta(y))\nabla K(z/|z|)\epsilon + < \nabla^2 K(z/|z|)\epsilon, \epsilon > \Big) \\
&\quad + O(t^3) \\
&\le C(d,s)\|\nabla\eta\|^2_{L^\infty(B_R)}K(z)t^2
\end{aligned}$$

In particular, if we choose $\eta(x) = \phi(x)\mathbf{v}$, where ϕ is given as (3.1) and $\mathbf{v} \in S^{d-1}$, then we have

$$\Delta^t_{\mathbf{v},\mathbf{v}}\mathcal{J}^s(u,B_R) \le C(d,s)t^2\frac{\mathcal{J}^s(u,B_R)}{R^2}.$$

□

Next, we prove the following identity related to nonlocal fractional energy, which was implicitly used in the proof of [26, Lemma 2.2].

Lemma 3.3 *Let $\Omega \subset \mathbb{R}^d$. For any functions u, v in appropriate spaces, let $u \vee v := \max\{u, v\}$ and $u \wedge v := \min\{u, v\}$, we have the identity*

$$\begin{aligned}
&\mathcal{J}^s(v,\Omega)+\mathcal{J}^s(u,\Omega)-\mathcal{J}^s(u\vee v,\Omega)-\mathcal{J}^s(u\wedge v,\Omega) \\
&\quad = 2\int\int_{\mathbb{R}^d\times\mathbb{R}^d\setminus(\Omega^c\times\Omega^c)}(v-u)_+(x)(v-u)_-(y)K(x-y)dydx,
\end{aligned} \tag{3.4}$$

where $K(z) = \frac{1}{|z|^{d+2s}}$, $(v-u)_+ = (v-u)\vee 0$ and $(v-u)_- = (v-u)\wedge 0$.

Proof Define sets

$$A := \{x \in \mathbb{R}^d : v(x) > u(x)\}$$

and

$$\tilde{\Omega} := \mathbb{R}^d \times \mathbb{R}^d \setminus (\Omega^c \times \Omega^c).$$

Then we calculate

$$\begin{aligned}
&\mathcal{J}^s(v,\Omega) + \mathcal{J}^s(u,\Omega) - \mathcal{J}^s(u \vee v,\Omega) - \mathcal{J}^s(u \wedge v,\Omega) \\
&= \int\int_{(A\times A^c)\cap\tilde{\Omega}} \left(|v(x)-v(y)|^2 - |v(x)-u(y)|^2\right) K(x-y)dydx \\
&\quad + \int\int_{(A^c\times A)\cap\tilde{\Omega}} \left(|v(x)-v(y)|^2 - |u(x)-v(y)|^2\right) K(x-y)dydx \\
&\quad + \int\int_{(A\times A^c)\cap\tilde{\Omega}} \left(|u(x)-u(y)|^2 - |u(x)-v(y)|^2\right) K(x-y)dydx \\
&\quad + \int\int_{(A^c\times A)\cap\tilde{\Omega}} \left(|u(x)-u(y)|^2 - |v(x)-u(y)|^2\right) K(x-y)dydx \\
&= 2\int\int_{(A\times A^c)\cap\tilde{\Omega}} ((v(x)-u(x))(u(y)-v(y)))\, K(x-y)dydx \\
&\quad + 2\int\int_{(A^c\times A)\cap\tilde{\Omega}} ((u(x)-v(x))(v(y)-u(y)))\, K(x-y)dydx \\
&= 2\int\int_{\mathbb{R}^d\times\mathbb{R}^d\setminus(\Omega^c\times\Omega^c)} (v-u)_+(x)(v-u)_-(y)K(x-y)dydx.
\end{aligned}$$

□

Remark 3.4 The above lemma implies

$$\mathcal{J}^s(v,\Omega) + \mathcal{J}^s(u,\Omega) \geq \mathcal{J}^s(u \vee v,\Omega) + \mathcal{J}^s(u \wedge v,\Omega),$$

and " $=''$ holds only if either $v \leq u$ or $v \geq u$ in Ω. To our knowledge this was first used in [30, Corollary 3], and it is really reveals the nonlocal feature of fractional energies.

By using the matrix determinant lemma

$$det(I + \alpha \otimes \beta) = 1 + \alpha \cdot \beta$$

where I is the identity matrix and α, β are two vectors, one can also check that

$$\Delta^t_{\mathbf{vv}} \mathcal{J}^P(u, B_1) = 0. \tag{3.5}$$

This together with lemma 3.2 immediately yields

Lemma 3.5 *There exists universal constant $C = C(d,s) > 0$ such that*

$$\Delta^t_{\mathbf{vv}} \mathcal{J}(u, B_R) \leq Ct^2 \mathcal{J}^s(u, B_R)/R^2.$$

For the rest, unless otherwise specified, we write C as various universal constants depending on d and s.

The next lemma, which is from [26, Lemma 2.2], dealing with the case $s = 1/2$ and in the same spirit of [16, Lemma 2.5], gives upper bound for the interior BV-norm of u by the s-fractional energy in a larger ball. Again, the proof in [26, Lemma 2.2] works for all fractional powers $0 < s < 1$. We state the result and include the proof as courtesy to the readers.

Lemma 3.6 *Let u be a stable solution to (1.11), then there exists a universal constant $C = C(d,s)$ such that for any $R \geq 1$,*

$$\left(\int_{B_{1/2}} (\partial_{\mathbf{v}} u(x))_+ dx\right)\left(\int_{B_{1/2}} (\partial_{\mathbf{v}} u(y))_- dy\right) \leq C\mathcal{J}^s(u, B_R)/R^2 \tag{3.6}$$

and

$$\int_{B_{1/2}} |\nabla u(x)| dx \leq C\left(1 + \sqrt{\mathcal{J}^s(u, B_1)}\right). \tag{3.7}$$

Proof Let $\bar{u} = \max\{P_{t,\mathbf{v}}u, u\}$ and $\underline{u} = \min\{P_{t,\mathbf{v}}u, u\}$. By Lemma 3.3 and Remark 3.1, we have

$$\begin{aligned}&\mathcal{J}^s(\bar{u}, B_R) + \mathcal{J}^s(\underline{u}, B_R) + 2\int_{B_{1/2}}\int_{B_{1/2}} \frac{(u(x - t\mathbf{v}) - \mathbf{u}(\mathbf{x}))_+ (u(y - t\mathbf{v}) - \mathbf{u}(\mathbf{y}))_-}{|x-y|^{d+2s}} dydx \\ &\quad\leq \mathcal{J}^s(P_{t,\mathbf{v}}u, B_R) + \mathcal{J}^s(u, B_R).\end{aligned}$$

We also have

$$\begin{aligned}&\mathcal{J}^P(\bar{u}, B_R) + \mathcal{J}^P(\underline{u}, B_R) \\ &= \int_{\{P_{t,\mathbf{v}}u > u\}\cap B_R} F(P_{t,\mathbf{v}}u) + \int_{\{P_{t,\mathbf{v}}u < u\}\cap B_R} F(u) + \int_{\{P_{t,\mathbf{v}}u < u\}\cap B_R} F(P_{t,\mathbf{v}}u) \\ &\quad + \int_{\{P_{t,\mathbf{v}}u > u\}\cap B_R} F(u) \\ &= \mathcal{J}^P(P_{t,\mathbf{v}}u, B_R) + \mathcal{J}^P(u, B_R).\end{aligned}$$

Since $|x-y| < 1$ when $x, y \in B_{1/2}$, we have

$$\begin{aligned}&\mathcal{J}(\bar{u}, B_R) + \mathcal{J}(\underline{u}, B_R) + 2\int_{B_{1/2}}\int_{B_{1/2}} (u(x - t\mathbf{v}) - \mathbf{u}(\mathbf{x}))_+ (u(y - t\mathbf{v}) - \mathbf{u}(\mathbf{y}))_- \\ &\quad\leq \mathcal{J}(P_{t,\mathbf{v}}u, B_R) + \mathcal{J}(u, B_R).\end{aligned} \tag{3.8}$$

Using this and the stability condition of u, and by adding $\mathcal{J}(P_{-t,\mathbf{v}}u, B_R) - 3\mathcal{J}(u, B_R)$ to both sides of (3.8), we have:

$$\begin{aligned}&\int_{B_{1/2}}\int_{B_{1/2}} (u(x - t\mathbf{v}) - \mathbf{u}(\mathbf{x}))_+ (u(y - t\mathbf{v}) - \mathbf{u}(\mathbf{y}))_- \, dydx \leq o(t^2) + \Delta^t_{\mathbf{vv}}\mathcal{J}(u, B_R) \\ &\leq o(t^2) + Ct^2\mathcal{J}^s(u, B_R)/R^2, \quad \text{by Lemma 3.5.}\end{aligned}$$

Dividing t^2 on both sides and passing to limit as $t \to 0$, we can conclude (3.6).

Define $A^{\pm}_{\mathbf{v}} := \int_{B_{1/2}} (\partial_{\mathbf{v}} u(x))_{\pm} \, dx$, then by (3.6) we have

$$\min\{A^+_{\mathbf{v}}, A^-_{\mathbf{v}}\} \leq \sqrt{A^+_{\mathbf{v}} A^-_{\mathbf{v}}} \leq \sqrt{C\mathcal{J}^s(u, B_1)}. \tag{3.9}$$

In addition, since $|u| \leq 1$ and divergence theorem,

$$|A^+_{\mathbf{v}} - A^-_{\mathbf{v}}| = \left|\int_{B_{1/2}} \partial_{\mathbf{v}} u(x)\right| \leq C. \tag{3.10}$$

Therefore, (3.9) and (3.10) yield

$$\int_{B_{1/2}} |\partial_{\mathbf{v}} u(x)| dx = A^+_{\mathbf{v}} + A^-_{\mathbf{v}} = |A^+_{\mathbf{v}} - A^-_{\mathbf{v}}| + 2\min\{A^+_{\mathbf{v}}, A^-_{\mathbf{v}}\} \leq C\left(1 + \sqrt{\mathcal{J}^s(u, B_1)}\right).$$

This proves (3.7). □

Now we are in a position to prove Proposition 1.7.

Proof of Proposition 1.7 For $0 < s < 1/2$, combining (1.14) for $R = 1$ and Lemma 3.6, we have

$$\int_{B_1} |\nabla u| \leq C\left(1+\sqrt{C\left(1+\int_{B_4}|\nabla u|\right)}\right). \tag{3.11}$$

By AM–GM inequality and Young's inequality, for $0 < \delta < 1$, whose choice depends on d and s which will be specified later on, there exists $C > 0$ such that

$$\int_{B_1} |\nabla u| \leq \delta \int_{B_4} |\nabla u| + C/\delta. \tag{3.12}$$

Now we do the scaling argument. For any $x_0 \in \mathbb{R}^d$ and $\rho > 0$ with $B_\rho(x_0) \subset B_1$, let $w(x) := u(x_0 + \frac{\rho}{4}x)$, then w is also a stable solution to (0.1) with $f(x)$ replaced by $\frac{\rho^{2s}}{4^{2s}} f(x_0 + \frac{\rho}{4}x)$. Since the estimate above does not depend on f, by (3.12) we have

$$\int_{B_1} |\nabla w| \leq \delta \int_{B_4} |\nabla w| + C/\delta.$$

that is,

$$\rho^{1-d} \int_{B_{\rho/4}(x_0)} |\nabla u| \leq \delta\rho^{1-d} \int_{B_\rho(x_0)} |\nabla u| + C/\delta. \tag{3.13}$$

Then by Simon's Lemma proved in [31], see also (see [16, Lemma 3.1] and [26, Lemma 2.3]), we can choose universal constant δ depending on d and s such that from (3.13), we conclude that

$$\int_{B_{1/2}} |\nabla u| \leq C, \tag{3.14}$$

where C depends only on d and s.

Note that (3.14) is true for any stable solution u to (0.1), hence we can apply (3.14) for $u(x_0 + 2Rx)$, which is also a stable solution to (1.11), instead of $u(x)$, and thus we have

$$\int_{B_R(x_0)} |\nabla u| \leq CR^{d-1}, \quad \forall x_0 \in \mathbb{R}^d. \tag{3.15}$$

By (3.14) and (1.14), we have that for any stable solution u to (0.1),

$$\mathcal{J}^s(u, B_{1/4}) \leq C. \tag{3.16}$$

Also by scaling property

$$\mathcal{J}^s(u(x_0 + 4Rx), B_{1/4}) = (4R)^{2s-d} \mathcal{J}^s(u, B_R(x_0)).$$

Thus from (3.16) we conclude

$$\mathcal{J}^s(u, B_R(x_0)) \leq CR^{d-2s}, \quad \forall x_0 \in \mathbb{R}^d \tag{3.17}$$

These conclude (1.12) and (1.13) for the case $0 < s < 1/2$.

Next, we consider the case $1/2 \le s < 1$. By (1.15) and (3.7) we have

$$\int_{B_{1/2}} |\nabla u(x)|dx \le C\left(1+\sqrt{C\left(1+L_0^{2s-1}\log(2L_0)\int_{B_2}|\nabla u|\right)}\right),$$

where $L_0 \ge 2$ is an upper bound for $\|\nabla u\|_{L^\infty(B_1)}$. Then similar to the argument (3.11)-(3.14), we have

$$\int_{B_{1/2}} |\nabla u(x)|dx \le CL_0^{2s-1}\log(2L_0), \tag{3.18}$$

For any $x_0 \in \mathbb{R}^d$, since $u_R(x) := u(2Rx + x_0)$ is also a stable solution to (0.1) with f replaced by $R^{2s}f$, by (3.18) we have

$$\int_{B_{1/2}} |\nabla u_R(x)|dx \le CL_R^{2s-1}\log(2L_R), \tag{3.19}$$

where $L_R \ge 2$ is an upper bound for $\|\nabla u_R\|_{L^\infty(B_1)} = 2R\|\nabla u\|_{L^\infty(B_{2R}(x_0))}$. By [13, Proposition 5.2] and since $|u| \le 1$, $\|\nabla u\|_{L^\infty(\mathbb{R}^d)} \le C(d,s)M_0$, and thus we can choose $L_R \le CM_0R$. Hence by (3.19) and scaling property we can conclude (1.12) for the case $1/2 \le s < 1$. Then by (1.15), and elliptic estimate $L_0 \le CM_0$, we derive (1.13) for the case $1/2 \le s < 1$. Note that the constant C_2 in (1.13) for the case $1/2 < s < 1$ does depend on f. However, when $s = 1/2$, the constant in (1.13) does not depend on f. □

To the end, we recall the following result, which validates Conjecture 1 for the case $d = 2,\ 0 < s < 1$. This was proved in [9] and [35] using different approaches. Applying the energy estimates obtained in this paper, we can give a new proof.

Theorem 3.7 *If u is a stable solution to (0.1) in $\mathbb{R}^2$, then u is 1-D.*

Proof By Proposition 1.7, the RHS of (3.6) goes to zero as $R \to \infty$, and hence

$$\left(\int_{B_{1/2}} (\partial_{\mathbf{v}}u)_+(x)dx\right)\left(\int_{B_{1/2}} (\partial_{\mathbf{v}}u)_-(y)dy\right) = 0. \tag{3.20}$$

Then u is monotone in $B_{1/2}$ along direction $\mathbf{v}$. Since (3.20) is true for any fixed direction $\mathbf{v}$ and any half ball, by the continuity of u we conclude that u is 1-D. □

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