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# Uniqueness and Symmetry for the Mean Field Equation on Arbitrary Flat Tori

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We study the following mean field equation on a flat torus $T := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$:

$$\Delta u + \rho\left(\frac{e^u}{\int_T e^u} - \frac{1}{|T|}\right) = 0,$$

where $\tau \in \mathbb{C}, \mathrm{Im}\ \tau > 0$, and $|T|$ denotes the total area of the torus. We first prove that the solutions are evenly symmetric about any critical point of $u$ provided that $\rho \leq 8\pi$. Based on this crucial symmetry result, we are able to establish further the uniqueness of the solution if $\rho \leq \min\{8\pi, \lambda_1(T)|T|\}$. Furthermore, we also classify all one-dimensional solutions by showing that the level sets must be closed geodesics.

## 1 Introduction

In this paper, we consider a mean field equation on a flat torus $T := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, that is,

$$\Delta u + \rho\left(\frac{e^u}{\int_T e^u} - \frac{1}{|T|}\right) = 0, \tag{1.1}$$

where $\tau \in \mathbb{C}, \mathrm{Im}\ \tau > 0$, and $|T|$ denotes the total area of the torus.

Equation of type (1.1) arises in many areas of mathematics and physics. It is related to the prescribed gauss curvature problem from the geometric point of view (see,

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e.g., [4, 5, 12–14, 16, 20, 25, 26, 29, 30, 33]). Motivations also come from the vortex theory of two-dimensional turbulence (see, e.g., [3, 8–11, 18, 22, 27]) and Chern–Simons–Higgs gauge field theory (see, e.g., [7, 17, 19, 28, 36, 37]).

In particular, various authors have contributed an extensive literature on the mean field equation posed on flat tori. Ricciardi and Tarantello [34] showed that non-constant one-dimensional solutions to (1.1) on a square torus exist if and only if $\rho > 4\pi^2$ and the solutions are evenly symmetric. Concerning the nontrivial "two-dimensional" solutions, Struwe and Tarantello [35] used the Min-Max scheme to establish the existence for $\rho \in (8\pi, 4\pi^2)$. Both their results could be extended to a rectangular torus by a simple scaling argument while the bound $4\pi^2$ is replaced by $\lambda_1(T)|T|$ where $\lambda_1(T)$ denotes the 1st eigenvalue of $-\Delta$ on $T$. We would also like to point out that it is impossible to generalize Struwe and Tarantello's result [35] if $\lambda_1(T)|T| \le 8\pi$. On the other hand, it is recently found in a paper by Cheng *et al.* [15] that "two-dimensional" solutions exist for $\rho > 8\pi$ but close to $8\pi$ on a rectangular torus. Bartolucci *et al.* [2] generalized the result to any flat torus. One may wonder whether all solutions to (1.1) are one-dimensional when $\rho \le 8\pi$. A lot of progress has been made along this direction; Cabré *et al.* [6] proved a one-dimensional symmetry result for $\rho$ up to an upper bound, which could be written explicitly in terms of the maximum conformal radius of a rectangular torus. Later, Lin and Lucia [31] proved the uniqueness of constant solution provided that $\rho \le \min\{8\pi, 32\frac{l^2}{|T|}\}$ where $l$ denotes the length of the shortest closed geodesic of an arbitrary torus. One note that the result is sharp if $\frac{l^2}{|T|} \ge \frac{\pi}{4}$.

Similar questions can also be asked for the minimizers of the energy functional $J_\rho$ where

$$J_\rho(u) = \frac{1}{2}\int_T |\Delta u|^2 - \rho \log\left(\frac{1}{|T|}\int_T e^u\right), \quad u \in \mathcal{H}^1(T) = \{v \in H^1(T) : \int_T v = 0\}. \tag{1.2}$$

Lin and Lucia [32] derived a sharp uniqueness and symmetry result for rectangular tori. They actually obtained the one-dimensional symmetry of any global minimizer of $J_\rho$ by first showing that the minimizers must be Steiner-symmetric and then applying a careful nodal line analysis to the directional derivatives of the Steiner-symmetric solutions when $\rho \le 8\pi$ is assumed. They also stated a conjecture.

**Conjecture A.** ([32]) Suppose $T$ is an arbitrary flat torus and let $u$ be a solution of (1.1). Then, $u$ is constant if $\rho \le \min\{8\pi, \lambda_1(T)|T|\}$.

The conjecture is validated by the 2nd author and Moradifam [24] for the case of rectangular tori. They also obtained an optimal symmetry result that solutions to (1.1) must be one-dimensional if $\rho \leq 8\pi$. Their proof consists of two steps: step one is to exploit a newly developed tool named "sphere covering inequality" (which will be stated in details later in Section 2) to show that the solutions are evenly symmetric about both axes if the origin is a critical point of solutions; step two is to prove that symmetric solutions about $x$- and $y$-axis must be Steiner-symmetric on some "sub-torus" and thus taking [32, Theorem 1.2(b)] into account they conclude that the solutions are one dimensional.

The main achievement of this article is a complete resolution of Conjecture A. The results in the present work can be applied to any flat torus, not necessarily a rectangular torus, which was already settled in the paper of the 2nd author and Moradifam [24]. Our 1st main result is the even symmetry of the solutions.

**Theorem 1.1.** Suppose $T$ is an arbitrary flat torus and let $u$ be a solution of (1.1) with the origin being a critical point of $u$. Then $u$ is evenly symmetric, that is, $u(z) = u(-z)$ for any $z \in \mathbb{C}$ provided that $\rho \leq 8\pi$.

Note that $u$ is a smooth function on $T$, so $u$ achieve its maximum and minimum somewhere on the torus. Due to the translation invariance of (1.1), one can always shift the maximum point or the minimum point to the origin so that the assumption of Theorem 1.1 is fulfilled.

Our 2nd main result shows the one-dimensional symmetry for evenly symmetric solutions.

**Theorem 1.2.** Suppose $T$ is an arbitrary flat torus and let $u$ be an evenly symmetric solution of (1.1). Then $u$ is one dimensional provided that $\rho \leq 8\pi$.

We remark here that $u$ is "one-dimensional", which means either $u$ is constant or the level sets of $u$ are parallel straight lines. This is different from the notion Lin and Lucia [32] used. When referring to "one-dimensional" solutions, they actually talked about such solutions that are only dependent on either $x$ variable or $y$ variable on a rectangular torus where $(x, y) \in \mathbb{R}^2$. In their situation, the restriction of Steiner-symmetry of the solutions actually rules out all other possibilities.

We also present some results on one-dimensional solutions of the mean field equation, which is a natural generalization of Ricciardi and Tarantello's classic result [34] to arbitrary flat tori in Section 4.

Combining Theorem 1.1, Theorem 1.2, and Corollary 4.3, we provide a complete answer to Conjecture A.

**Theorem 1.3.** Equation (1.1) only admits trivial solutions if $\rho \leq \min\{8\pi, \lambda_1(T)|T|\}$ for any flat torus $T$.

We note that Theorem 1.2 also leads to a classification of nontrivial solutions when $\lambda_1(T)|T| < 8\pi$ and $\rho \in (\lambda_1(T)|T|, 8\pi)$, in addition to Theorem 1.3. Indeed, Theorem 1.2 together with Theorem 1.1 implies that the only nontrivial solutions are one-dimensional solutions as classified in Section 4. It is well known that the trivial solution $u \equiv 0$ is no longer a local minimizer of $J_\rho$ if $\rho \in (\lambda_1(T)|T|, 8\pi)$ (see [34]). However, $J_\rho$ is coercive when $\rho$ lies in the above range so it admits a non-zero global minimizer in $\mathcal{H}^1(T)$, which is one dimensional by Theorem 1.2. Finally, we see that Theorem 1.2 also extends [32, Theorem 1.1] to its maximum generality.

## 2 Proof of Theorem 1.1 by Sphere Covering Inequality

In this section, our main focus is to prove Theorem 1.1. The key ingredient in the proof is the sphere covering inequality recently discovered by the 2nd author and Moradifam [23] to establish the even symmetry of the solutions. It is worthwhile to mention that the uniqueness of solution of the mean field equation on the sphere was among the problems which motivated the study of the sphere covering inequality and were successfully addressed by this powerful new tool. We state the sphere covering inequality as well as a limiting version of the inequality for the convenience of the reader.

**Theorem A.** *(The sphere covering inequality [23])* Let $\Omega$ be a simply connected region of $\mathbb{R}^2$ and assume $v_i \in C^2(\overline{\Omega})$, $i = 1, 2$ are the classic solutions of the following equations:

$$\Delta v_i + e^{v_i} = f_i(y),$$

where $f_2 \geq f_1 \geq 0$ in $\Omega$. Assume further $v_2 \geq v_1$, $v_2 \not\equiv v_1$ and $v_2 = v_1$ on $\partial\Omega$, then

$$\int_\Omega \left(e^{v_1} + e^{v_2}\right) \mathrm{d}y \geq 8\pi.$$

Moreover, the equality holds only when $f_2 \equiv f_1 \equiv 0$ and $(\Omega, e^{v_i} dy)$ are isometric to two complimentary spherical caps on the sphere with radius $\sqrt{2}$.

Consider the limiting case $v_1 \equiv v_2 \equiv v$, one has the following variant of Alexandrov–Bol's inequality:

**Theorem B.** *[1, 21, 32])* Let $\Omega$ be a simply connected region of $\mathbb{R}^2$ and $v \in C^2(\overline{\Omega})$ such that

$$\Delta v + e^v > 0 \text{ in } \Omega.$$

Assume there exists $\varphi \in C^2(\overline{\Omega})$ such that

$$\Delta\varphi + e^v\varphi = 0, \quad \varphi = 0 \text{ on } \partial\Omega, \quad \varphi \not\equiv 0.$$

Then $\int_\Omega e^v > 4\pi$.

Assume that we have two sequences of solutions $v_i^k$ to the equations in Theorem A with corresponding $f_i^k$ such that other conditions in the theorem also hold and $v_i^k \to v, f_i^k \to f$ as $k \to \infty$. Then, it is straightforward to verify that the limit of the "normalized difference" solves the linearized equation at $v$, that is,

$$\varphi := \lim_{k\to\infty} \frac{v_2^k - v_1^k}{\|v_2^k - v_1^k\|_{L^\infty(\Omega)}} > 0$$

is a solution to the equation in Theorem B. Hence, one can deduce the conclusion of Theorem B by the sphere covering inequality and sending $k \to \infty$. To see a more general and rigorous argument, one can refer to [23, Section 3]. Note that the domain of equations can be an open subset $\omega$ of the simply connected region $\Omega$ and the integral conditions $\int_\Omega e^{v_i} \leq 8\pi$ and $\int_\Omega e^v \leq 8\pi$ are imposed.

**Proof of Theorem 1.1** Let $v(z) = u(z) + \log\rho - \log\left(\int_T e^u\right)$ and we have

$$\Delta v + e^v = \frac{\rho}{|T|} > 0, \quad z \in T. \tag{2.1}$$

We claim that

$$u(z) = u(-z), \quad \forall z \in T.$$

Consider

$$w(z) = v(z) - v(-z),$$

if one can show that $w \equiv 0$ then the even symmetry of $u$ is proved.

Assume to the contrary that $w \not\equiv 0$. Clearly, $w$ is a solution of the following linear elliptic equation

$$\Delta w + c(z)w = 0, \tag{2.2}$$

for some smooth $c(z) > 0$. Since origin is a critical point of $u$ and $w$ is an odd function, that is, $w(-z) = -w(z)$, we have the following expansion of $w$ at $z = 0$:

$$w(z) = a_m P_m(z) + o(|z|^{m+\epsilon}),$$

where $a_m \neq 0$, $P_m(z)$ is a harmonic polynomial of degree $m$, $m \geq 3$, and $\epsilon > 0$. Hence, there are at least three nodal lines intersecting at the origin. One may also argue by the Hopf lemma that it is impossible to have only one nodal line or two nodal lines originating from $z = 0$. Since the number of simply connected regions formed by nodal lines will not decrease if one increases the number of nodal lines, we assume without loss of generality that there are three nodal lines passing through the origin.

Note that all three half periods ($\omega_1 = \frac{1}{2}$, $\omega_2 = \frac{\tau}{2}$ and $\omega_3 = \frac{1+\tau}{2}$) lie on the nodal lines. It suffices to discuss the cases that there is only one nodal line passing through each $\omega_k$ for $k = 1, 2, 3$, since otherwise there must be at least three nodal lines passing through some $\omega_k$ due to the argument in the previous paragraph and hence at least the same number of nodal regions will be obtained to lead to a contradiction as below. Also taking into consideration that $T$ is a closed surface, we shall discuss the following several cases. Although we treat the nodal lines as closed loops on torus, we visualize them as curves on $\mathbb{R}^2$.

**Case I:** two of three closed loops starting from 0 are topologically trivial. Because $w$ is odd so if we have one trivial loop then we get two automatically. In this case, there are at least two simply connected region $\Omega_1$ and $\Omega_2$ such that both $v(z)$ and $v(-z)$ are solutions to (2.1) on $\Omega_i$ and $v(z) = v(-z)$ on $\partial\Omega_i$ for $i = 1, 2$. Then, by the sphere covering inequality (Theorem B), we have

$$\rho = \int_T e^v \geq \sum_{i=1}^{2} \int_{\Omega_i} e^v = \int_{\Omega_1} \left(e^{v(z)} + e^{v(-z)}\right) > 8\pi, \tag{2.3}$$

which leads to a contradiction. One can see Figure 1 for the illustration of this case. One shall note here that the two closed loops may spread outside of the fundamental parallelogram. However, one can still claim the existence of at least two simply connected regions sitting inside the torus by translating the outside parts into the fundamental domain. Please see Figure 2.

**Case II:** all three loops are topologically nontrivial but they have intersections other than origin and three half periods $\omega_k$s. Then similarly as in Case I there are at least two simply connected regions $\Omega_i$ such that $v(z)$ and $v(-z)$ are two distinct solutions to (2.1)

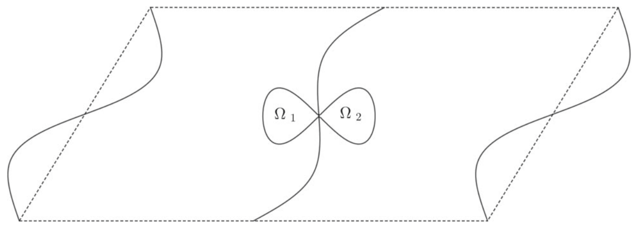


**Fig. 1.** The nodal curves include two topologically trivial loops inside the fundamental domain.

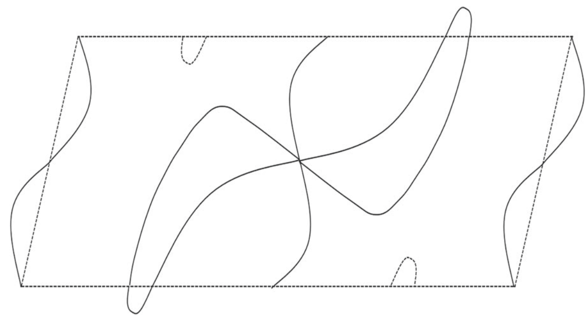

**Fig. 2.** The nodal curves include two topologically trivial loops extending out of the fundamental domain.

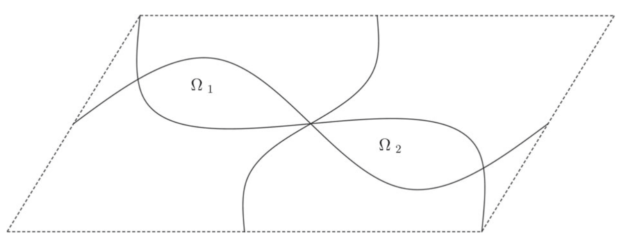


**Fig. 3.** The nodal curves consist of three nontrivial loops intersecting at more than one points in the fundamental domain.

and they agree on the boundary of $\Omega_i$ for $i = 1, 2$. We still get the contradiction for the same reason as (2.3). Please see Figure 3.

**Case III:** all three loops are nontrivial and they have no intersections other than origin and $\omega_k$s. We can split this case further into two different situations.

(a) All three loops are topologically different; please see Figure 4. There are exactly two simply connected region $\Omega_i$ such that $\cap_{i=1}^{2}\Omega_i = \emptyset$, $\cup_{i=1}^{2}\overline{\Omega_i} = T$, $w(z) > 0$ in $\Omega_1$, $w(z) < 0$ in $\Omega_2$, and $w = 0$ on $\partial\Omega_1 \cup \partial\Omega_2$. Again Theorem B yields a contradiction.

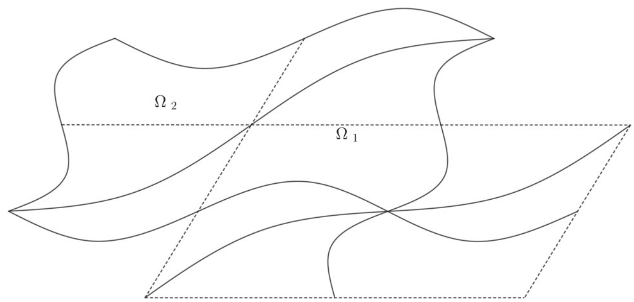


**Fig. 4.** The nodal curves consist of three nontrivial loops intersecting at only one point in the fundamental domain.

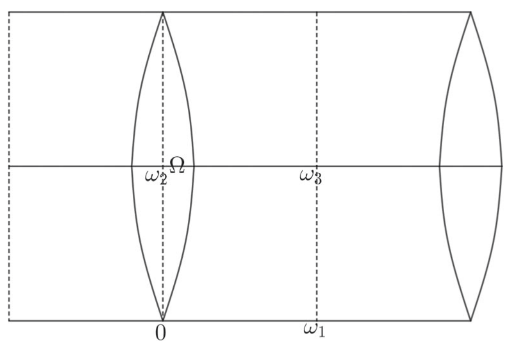


**Fig. 5.** The nodal curves separate the torus into four simply connected regions.

(b) At least two loops are topologically equivalent. One also note that if we have two equivalent loops passing through $\omega_2$ at the same time, then the two loops form two simply connected regions. One can refer to Figure 5 for the illustration of one simply connected region $\Omega$ formed by the two nodal lines, which are symmetric about $\omega_2$. Moreover, if the other nodal line is also an equivalent loop, then it passes through $\omega_2$ and separate $\Omega$ into two simply connected regions. Therefore, we just need to consider the remaining case when the 3rd loop is not equivalent as the 1st two. One can see from Figure 5 that these nodal lines separate the whole torus into at least four simply connected regions.

Finally, we conclude that $w \equiv 0$ and thus $u(z) = u(-z)$ for $\rho \leq 8\pi$. ■

Concerning the more general mean field equation on flat tori, we have the following corollary.

**Corollary 2.1.** Let $T$ be an arbitrary flat torus and $\rho \le 8\pi$. Suppose the positive potential $h \in C^2(T)$ satisfies the condition $\Delta(\log h) + \frac{\rho}{|T|} > 0$ everywhere on $T$ and $u$ solves the following mean field equation on torus:

$$\Delta u + \rho\left(\frac{he^u}{\int_T he^u} - \frac{1}{|T|}\right) = 0.$$

Moreover, if $u + \log h$ has a critical point at $(x_0, y_0)$, then $u + \log h$ is evenly symmetric about the point $(x_0, y_0)$.

## 3 Proof of Theorem 1.2

In this section, we study the one-dimensional symmetry of solution $u$ of (1.1) when $\rho \le 8\pi$. Keep in mind now $u$ is evenly symmetric by Theorem 1.1. We would like to apply some homotopic argument together with Theorem B as Lin and Lucia did in [32] to show the one-dimensional symmetry.

**Proof of Theorem 1.2** Since $u$ is evenly symmetric, it is fairly easy to check that 0 and $\omega_k$s lie on the nodal lines of any directional derivative $\frac{\partial u}{\partial \nu}$ for $k = 1, 2, 3$. Clearly, $\frac{\partial u}{\partial \nu}$ is a solution to the linearized equation:

$$\Delta\varphi + e^{v}\varphi = 0 \text{ in } T, \tag{3.1}$$

where $v = u + \log\rho - \log\left(\int_T e^u\right)$. One notes that if there are three nodal lines of $\frac{\partial u}{\partial \nu}$ passing through one of the points in the set $\mathcal{A} = \{0, \omega_1, \omega_2, \omega_3\}$, then we can conclude just like in the proof of Theorem 1.1 that there are at least two simply connected regions $\Omega_1$ and $\Omega_2$ such that $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega_1 \cup \partial\Omega_2$. Therefore, by Theorem B, we have $\frac{\partial u}{\partial \nu} \equiv 0$. Furthermore, the Hopf lemma with the fact $\frac{\partial u}{\partial \nu}$ is odd yield that we cannot have two nodal lines intersecting at any point in $\mathcal{A}$. So from now on, we assume that there is only one nodal line originating from each point in $\mathcal{A}$. We can also exclude the case when the single nodal line forms a closed loop. In this case, we would have at least two simply connected regions since $\frac{\partial u}{\partial \nu}$ is odd function with respect to any point from $\mathcal{A}$.

Assume that $\frac{\partial u}{\partial \nu} \not\equiv 0$ for all directions $\nu \in \mathbb{S}^1$. Otherwise, $u$ is one dimensional.

We define a family of doubly periodic functions $\varphi_t$ on $\mathbb{R}^2$ for $t \in [0, 1]$ as follows:

$$\varphi_t = \Delta u \cdot \nu(t),$$

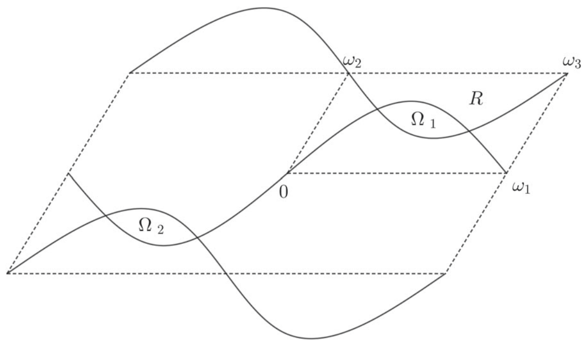


**Fig. 6.** Two nodal curves intersect at two points in $R$.

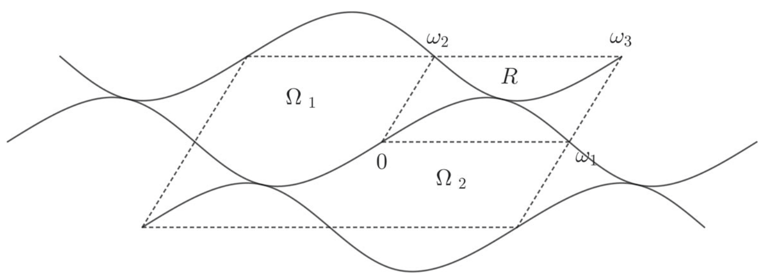


**Fig. 7.** Two nodal curves touch once in $R$.

where $\nu(t)$ is a smooth map from $[0,1]$ to $\mathbb{S}^1$ such that $\nu(0)=\nu_0$ and $\nu(1)=-\nu_0$ where $\nu_0$ is any fixed direction, which belongs to $\mathbb{S}^1$. It is readily checked that $\varphi_t$ is a solution to (3.1) for all $t\in[0,1]$. Let

$$\mathcal{Z}_t=\{x\in\mathbb{R}^2:\varphi_t(x)=0\}.$$

If two nodal curves in $\mathcal{Z}_t$ touch for some $t_0\in[0,1]$, then either of the following two cases happens.

**Case I:** two curves intersect at two or more points in a closed "quarter parallelogram" $R$. Without loss of generality, we can assume $R$ is the parallelogram with vertices $0$, $\omega_1$, $\omega_2$, and $\omega_3$. Then there exist at least two simply connected domains $\Omega_1$ and $\Omega_2$ such that $\varphi_{t_0}=0$ on $\partial\Omega_1\cup\partial\Omega_2$. The $4\pi$ lower bound in Theorem 2 for each domain $\Omega_i$ yields the contradiction. Please see Figure 6.

**Case II:** two curves touch once in $R$. Then there exist exactly two simply connected domains $\Omega_1$ and $\Omega_2$ such that $\cap_{i=1}^2\Omega_i=\emptyset$, $\cup_{i=1}^2\overline{\Omega_i}=T$, $\varphi_{t_0}>0$ in $\Omega_1$, $\varphi_{t_0}<0$ in $\Omega_2$ and $\varphi_{t_0}=0$ on $\partial\Omega_1\cup\partial\Omega_2$. Again, we have the same contradiction. Please see Figure 7.

Now we only need to consider the situation that nodal lines do not intersect each other. We can assume without loss of generality that 0 and $\omega_1$ are in the same connected component of $\mathcal{Z}_0$. Then, $\omega_2$ and $\omega_3$ lie on another nodal line. Let us introduce a set

$$\mathcal{T} := \{t \in [0,1] : 0 \text{ and } \omega_1 \text{are in the same connected component of } \mathcal{Z}_t\}.$$

Since the nodal lines do not touch each other, then by the homotopic invariance of the connectedness (see [32, Lemma A.1, Lemma A.2]), we conclude that the set $\mathcal{T}$ is open and also closed in $[0,1]$. Thus, 0 and $\omega_1$ are always connected by a nodal line for all $t \in [0,1]$. Similarly, one can also show that $\omega_2$ and $\omega_3$ lie on the same nodal line for all $t \in [0,1]$. We distinguish two possibilities.

(a) There exist two or more simple closed loops not touching any other nodal lines for some $t \in [0,1]$. The number of simple closed loops is at least two because $\varphi_t$ is an odd function. We easily get the contradiction as previously argued.

(b) We have only two families of nodal lines modulo the doubly periodicity: the curves $\gamma_t^1$ connecting $\omega_2$ and $\omega_3$ and the curves $\gamma_t^2$ connecting 0 and $\omega_1$ for all $t \in [0,1]$. Note here the initial nodal line $\gamma_0^1$ ($\gamma_0^2$) cannot deform to any nodal line connecting 0 and $\omega_1$ ($\omega_2$ and $\omega_3$) during the evolution as $t$ varies from 0 to 1 because we assume that there is one single nodal line passing through each point of $\mathcal{A}$. Since

$$\Delta\varphi_t \neq 0 \text{ for all } x \in \mathcal{Z}_t,$$

we have the following equation governing the evolution of curve $\gamma_t^1$ ($\gamma_t^2$):

$$\frac{\partial \gamma_t^1}{\partial t} = F\frac{\Delta\varphi_t}{|\Delta\varphi_t|},$$

where

$$F = -\frac{\Delta u \cdot \frac{\partial v(t)}{\partial t}}{|\Delta\varphi_t|}.$$

Therefore, we can define a homotopy $H : \mathbb{R}^1 \times [0,1] \to T \times \mathbb{R}^2$ such that

$$H(s,t) = (\gamma_t^1(s), n(\gamma_t^1(s), t)),$$

where $n = \frac{\Delta\varphi_t}{|\Delta\varphi_t|}$ is the unit "outward" normal of a point on the curve induced by $\varphi_t$ and $\gamma_t^1(s)$ is the natural arclength parametrization compatible with the "outward" normal $n$. One can treat every $H(\cdot,t)$ as an oriented loop on the flat torus $T$. However, $H(\cdot,0)$ and $H(\cdot,1)$ are two identical loops but with opposite orientations by the definition of $\varphi_t$. Thus, one can identify $H(\cdot,0)$ as a generator $e \in \pi_1(T,\omega_2)$ where $\pi_1(T,\omega_2)$ denotes the loop space with base point $\omega_2$ while $H(\cdot,1)$ is identified as $-e$. We arrive at contradictories since $e$ and $-e$ are not equivalent in $\pi_1(T,\omega_2)$. ■

## 4 One-Dimensional Solutions

In this section, we aim to extend Ricciardi and Tarantello's result on one-dimensional solutions of (1.1) to the greatest generality. We first present a lemma for one-dimensional $C^1$ functions on $T := \mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$ where $\tau = \alpha + i\beta$ with $\beta = |T| > 0$.

**Lemma 4.1.** Suppose $u$ is one dimensional and $u \in C^1(T)$ and $u$ is not a constant. Then the level sets of $u$ are parallel closed geodesics of $T$.

**Proof.** Since $u$ is one dimensional, we have

$$\Delta u \cdot \nu = 0$$

for some direction $\nu \neq 0$. Let $\nu = (s,t)$. Then we introduce the following affine transformation

$$\begin{cases} x' = x - \frac{\alpha}{\beta} y \\ y' = \frac{1}{\beta} y \end{cases},$$

so that $u = u(x,y)$ can be rewritten in terms of $(x',y')$ and has period 1 in both $x'$ and $y'$. Let $d = \frac{tx-sy}{\sqrt{s^2+t^2}}$. Then $u$ is a periodic function of $d$ where $d \in \mathbb{R}$. Since $u(x'+1,y') = u(x',y')$, we conclude that $u(d)$ has a period $\frac{t}{\sqrt{s^2+t^2}}$. Similarly, by the periodicity of $u(x',y')$ on $y'$, we also get that $u(d)$ has another period $\frac{\alpha t-\beta s}{\sqrt{s^2+t^2}}$. Recall that a non-constant continuous periodic function on $\mathbb{R}$ cannot have two periods whose quotient is irrational. Therefore, there exist two integers $m$ and $n$ such that

$$mt = n(\alpha t - \beta s).$$

Then, $\nu = C(n\tau - m)$ for a non-zero real constant $C$, that is, $\nu$ denotes the direction of a closed geodesic. ■

In particular, if $u$ is a nontrivial classic solution of (1.1) and $u$ is one-dimensional, then the level sets of $u$ must be closed geodesics of $T$. It is natural to

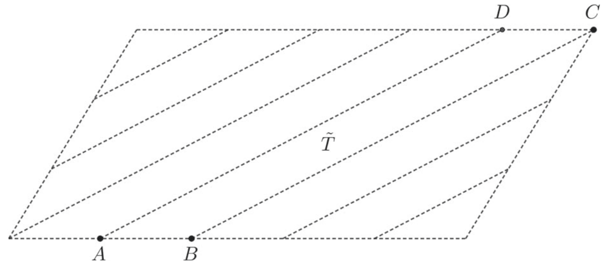


**Fig. 8.** Parallelogram $ABCD$ is the fundamental domain of a "$\frac{1}{5}$-subtorus" $\tilde{T}$.

consider whether non-constant one-dimensional solutions exist or not given a direction of closed geodesic. In the spirit of Ricciardi and Tarantello [34], we state the following theorem for an arbitrary torus $T$ as defined at the beginning of this article.

**Theorem 4.2.** Given a direction $\nu$ of a closed geodesic $\Gamma$, non-constant one-dimensional solutions $u$ of (1.1) with $\Delta u \cdot \nu = 0$ exist if and only if $\rho > 4\pi^2 \frac{L^2}{|T|}$ where $L$ denotes the length of the closed geodesic $\Gamma$.

**Proof.** If the level sets of $u$ are parallel to either 1 or $\tau$ or $1+\tau$ or $1-\tau$, then by a simple scaling argument, the conclusion in Theorem 4.2 follows from [34].

Let $\nu = m + n\tau$. Without loss of generality, one may assume that $m$ and $n$ are positive integers such that $(m, n) = 1$ and $m < n$. Then $u$ is a one-dimensional function on $T$ if and only if $u$ is one-dimensional on a "$\frac{1}{n}$-subtorus" $\tilde{T}$. One can see Figure 8 for the typical example $m = 3$ and $n = 5$. From here we see that the "period" in $\mathbb{R}$ of the one-dimensional solution is $\frac{|T|}{L}$. Finally, one can get the lower bound $4\pi^2 \frac{L^2}{|T|}$ from the lower bound $4\pi^2$ of the closed interval $[0, 1]$. ■

We would like to emphasize a classic result relating the length spectrum with the spectrum of $-\Delta$ on any flat torus $T$. Let $\Lambda$ be the lattice generated by 1 and $\tau$ in $\mathbb{C}$. One can consider the dual lattice

$$\Lambda' = \{\lambda' \in \mathbb{R}^2 : \lambda \cdot \lambda' \in \mathbb{Z}, \quad \forall \lambda \in \Lambda\}.$$

Let us define a family of functions as follows:

$$f_{\lambda'}(z) = e^{2\pi i(\lambda' \cdot z)}, \quad \lambda' \in \Lambda'.$$

Straightforward computation yields

$$-\Delta f_{\lambda'} = 4\pi^2 |\lambda'|^2 f_{\lambda'}.$$

One can also verify that the above family exhausts all eigenfunctions of $-\Delta$ on $T = \mathbb{C}/\Lambda$ and these functions span $L^2(T)$. The argument can be made rigorous with the help of a complex version of Stone–Weierstrass theorem. Let $\omega_1' = 1 - i\frac{\alpha}{\beta}$ and $\omega_2' = i\frac{1}{\beta}$. Then, $\Lambda'$ can be represented by $\mathbb{Z}\omega_1' + \mathbb{Z}\omega_2'$. It can be shown that $T$ is isometric to $T' = \mathbb{C}/\Lambda'$ by a factor $\frac{1}{|T|}$. Thus, given the $k$-th length spectrum $L_k$ on $T$, we have a corresponding eigenvalue $\lambda_k = 4\pi^2\frac{L_k^2}{|T|^2}$. We conclude the section by a corollary of Lemma 4.1 and Theorem 4.2.

**Corollary 4.3.** Non-constant one-dimensional solutions to (1.1) exist if and only if $\rho > 4\pi^2\frac{l^2}{|T|} = \lambda_1(T)|T|$ where $l$ denotes the length of shortest closed geodesic on $T$.

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