Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 269 (2020) 4332–4352
Journal of Differential Equations
www.elsevier.com/locate/jde

A reaction network approach to the theory of acoustic wave turbulence

Minh-Binh Tran [a,*], Gheorghe Craciun [b], Leslie M. Smith [c], Stanislav Boldyrev [d]

[a] *Department of Mathematics, Southern Methodist University, United States of America*
[b] *Department of Mathematics and Department of Biomolecular Chemistry, University of Wisconsin-Madison, United States of America*
[c] *Department of Mathematics and Department of Engineering Physics, University of Wisconsin-Madison, United States of America*
[d] *Department of Physics, University of Wisconsin-Madison, United States of America*

Received 12 February 2019; revised 20 March 2020; accepted 21 March 2020
Available online 27 March 2020

Abstract

We propose a new approach to study the long time dynamics of the wave kinetic equation in the statistical description of acoustic turbulence. The approach is based on rewriting the discrete version of the wave kinetic equation in the form of a chemical reaction network, then employing techniques used to study the Global Attractor Conjecture to investigate the long time dynamics of the newly obtained chemical system. We show that the solution of the chemical system converges to an equilibrium exponentially in time. In addition, a resonance broadening modification of the acoustic wave kinetic equation is also studied with the same technique. For the near-resonance equation, if the resonance broadening frequency is larger than a threshold, the solution of the system goes to infinity as time evolves.

MSC: 35Q20; 45A05; 47G10; 82B40; 37N25; 92C42; 37C10; 80A30; 92D25

Keywords: Weak turbulence; Rate of convergence to equilibrium; Global attractor conjecture; Biochemical networks; Acoustic waves

---

[*] Corresponding author.
E-mail addresses: minhbinht@mail.smu.edu (M.-B. Tran), craciun@wisc.edu (G. Craciun), lsmith@math.wisc.edu (L.M. Smith), boldyrev@wisc.edu (S. Boldyrev).

https://doi.org/10.1016/j.jde.2020.03.034
0022-0396/Published by Elsevier Inc.

Contents

---

1. Introduction

Describing the behavior of a spatially homogeneous field of random, weakly interacting dispersive waves, the theory of wave turbulence has been very successful in explaining the processes of spectral energy transfer in several areas of modern science, such as oceanography, plasmas, planetary waves, acoustic turbulence etc. [63]. The central ingredient of the theory is the derivation of the kinetic-wave equations, which describe the spectral energy transfer via n-wave resonant processes, and which are in one to one correspondence with the spectral moments [42]. Without making a priori assumptions on the statistics of the processes, these equations are closed. An important feature of these equations is that their exact equilibrium solutions have constant spectral fluxes of one of the conserved energy densities and the number density.

The derivation of kinetic equations to describe how weakly interacting waves share their energies in anharmonic crystal lattices in solid state physics go back to Peierls, in the early 30's [46,47]. Indeed, to our knowledge, Peirels' model is the first wave turbulence kinetic equation derived. The modern theory has been then developed based on the pioneering works of Hasselmann [31], Benney and Saffmann [8], Kadomtsev [37], Zakharov [63], Benney and Newell [7]. A great breakthough in the theory is the discovery of the Zakharov-Kolmogorov solution [61,62] by using the scaling symmetries of the dispersion relation and the coupling coefficient via the Zakharov transformation [61–63].

In [39], the authors develop expressions for the nonlinear wave damping and frequency correction of a field of random, spatially homogeneous, acoustic waves. They derived the 3-wave kinetic equation of acoustic waves, describing the evolution of the density distribution function f of the waves, in which the distribution $f(t, p)$ is a function of time t and wave number p. If we denote

$$f_1 = f(t, p_1),\ f_2 = f(t, p_2),\ f_3 = f(t, p_3),$$

then f_1 satisfies

$$\frac{\partial f_1}{\partial t} = Q[f_1], \tag{1}$$

where

$$\begin{aligned}Q[f_1] := &\int_{\mathbb{R}^6} \mathcal{V}_{p_1,p_2,p_3}\delta(p_1-p_2-p_3)\delta(\omega_{p_1}-\omega_{p_2}-\omega_{p_3})[f_2f_3-f_1(f_2+f_3)]dp_2dp_3\\ &-2\int_{\mathbb{R}^6} \mathcal{V}_{p_1,p_2,p_3}\delta(p_2-p_1-p_3)\delta(\omega_{p_2}-\omega_{p_1}-\omega_{p_3})[f_1f_3-f_2(f_1+f_3)]dp_2dp_3.\end{aligned} \tag{2}$$

The collision kernels $\mathcal{V}_{p_1,p_2,p_3} \geqslant 0$ are radially symmetric, and symmetric with respect to the permutation of p_1, p_2, p_3:

$$\mathcal{V}_{p_1,p_2,p_3} = \lambda|p_1||p_2||p_3|,$$

where $|p|$ denotes the length of the vector p, and λ is a positive constant that we can assume without loss of generality to be 1. In the case of acoustic waves, the dispersion relation $\omega_p = \omega(p)$ is given by the phonon dispersion law:

$$\omega(p) = |p|. \tag{3}$$

In this paper, besides the exact resonance equation (1), we also consider the following near-resonance turbulence kinetic equation for acoustic waves

$$\frac{\partial f_1}{\partial t} = Q^{NR}[f_1], \tag{4}$$

where

$$\begin{aligned}Q^{NR}[f_1] := &\int_{\{p_1=p_2+p_3,|\omega_{p_1}-\omega_{p_2}-\omega_{p_3}|\leqslant\Lambda\}} \mathcal{V}_{p_1,p_2,p_3}[f_2f_3-f_1(f_2+f_3)]dp_2dp_3\\ &-2\int_{\{p_2=p_1+p_3,|\omega_{p_2}-\omega_{p_1}-\omega_{p_3}|\leqslant\Lambda\}} \mathcal{V}_{p_1,p_2,p_3}[f_1f_3-f_2(f_1+f_3)]dp_2dp_3.\end{aligned} \tag{5}$$

Indeed, equation (1) has an explicit expression for the Kolmogorov-type spectrum of acoustic turbulence which is just a stationary solution of the equation [39]. Such a solution has not been found for the resonance broadening equation (4). In the current paper, we are interested in a different mathematical question on the time-dependent solutions rather than the stationary solution.

In general, the 3-wave kinetic equation plays an important role in the theory of weak turbulence, and has been rigorously studied in [43] for capillary waves, in [1,14,19,24] for the phonon interactions in anharmonic crystal lattices, in [58] for acoustic waves, and in [24] for stratified flows in the ocean.

It is the goal of our work to make the connection between equations (1), (4) and chemical reaction systems. Understanding the qualitative behavior of deterministically modeled chemical reaction systems has been a subject of great interests during the past four decades [60]. The main questions include the existence of positive equilibria, stability properties of equilibria, and the non-extinction, or persistence, of species [2–5,16,22,23,28,33,60]. Used to describe an important class of chemical kinetics, toric dynamical systems or complex-balanced systems [16,34] are among the most important models of chemical reaction systems. First introduced by Boltzmann [10] for modeling collisions in kinetic gas theory, the complex-balanced condition was used by Horn and Jackson [21,30,32,34,60] to show that a complex-balanced system has a unique locally stable equilibrium within each linear invariant subspace. Later on, the name "toric dynamical system" was proposed in [16] to underline the tight connection to the algebraic study of toric varieties. The Global Attractor Conjecture, the most important problem for toric dynamic systems, states that the complex balanced equilibrium of each system is also a globally attracting point within each linear invariant subspace. The Global Attractor Conjecture has been proved in [18] for small dimensional systems, and a solution has been proposed in [15] for the general case.

In this paper, *we discover, for the first time, the connection between the wave kinetic equation* (1) *and chemical reaction networks*. We prove that the discrete version of (1) can be associated with a chemical reaction network which takes the form

$$\begin{aligned} \mathrm{A}_{\mathrm{k}_2} + \mathrm{A}_{\mathrm{k}_3} &\longrightarrow \mathrm{A}_{\mathrm{k}_1} \\ \mathrm{A}_{\mathrm{k}_2} + \mathrm{A}_{\mathrm{k}_1} &\longrightarrow 2\mathrm{A}_{\mathrm{k}_2} + \mathrm{A}_{\mathrm{k}_3} \end{aligned}$$

and will be described in detail in Section 2.3. As a consequence, techniques that have been used to study the Global Attractor Conjecture in chemical reaction network theory can be applied to study the long time behavior of the wave kinetic equation (1). We prove that as time evolves, the solution of the discrete version of (1) converges to a steady state exponentially in time.

The Dirac-delta functions in (1) imply that the spectral energy transfer occurs on the resonant manifold, which is a set of wave vectors p, p_1, p_2 satisfying

$$p = p_1 + p_2, \qquad \omega_p = \omega_{p_1} + \omega_{p_2}. \tag{6}$$

However, in related systems, it is shown that exact resonances with $\omega_p = \omega_{p_1} + \omega_{p_2}$ do not capture some important physical effects [6,27,29,48,59]. Therefore one needs to include more physics by adding near-resonant interactions [9,13,35,38–41,52–55], which satisfy

$$p = p_1 + p_2, \quad |\omega_p - \omega_{p_1} - \omega_{p_2}| < \Lambda, \tag{7}$$

where Λ is the resonance broadening frequency.

By extending the chemical reaction network approach used to study the discrete versions of (1) and (4), we prove that:

- There exists a positive constant Λ_0 such that when $0 \leqslant \Lambda < \Lambda_0$, the solution of the discrete version of (4) converges to a unique equilibrium exponential having the form $|p|^{-1}$ in time, similar to the exact resonance case, which is the same as the equilibrium solution of the exact resonance case. Note that when $\Lambda = 0$ the equation (4) becomes (1).
- There exists a positive constant Λ_1 such that when $\Lambda \geqslant \Lambda_1$, the solution of the discrete version of (4) exits any compact set as t tends to infinity.

Besides the 3-wave kinetic equation, the rigorous study of 4-wave kinetic equation is a very important subject (see [11,12,20,25,26] and references therein). Finally, we note that wave-turbulence kinetic equations have very similar form with the quantum Boltzmann equations, used to describe the evolution of diluted bose gases at high and low temperature. We refer to the book [49] or the papers [36,44,45,50,51,56,57] and the references therein for more discussions on the latter topic.

The plan of our paper is the following:

- In section 2, we show that the discrete version of equation (1) can be rewritten as a chemical reaction network. By using an approach inspired by the theory of toric dynamical system, we prove in Theorem 2 that the solution of the discrete version of (1) converges to the equilibrium exponentially in time.
- In section 3, we generalize Theorem 2 to the near resonance case (4). We prove that depending on the resonance broadening frequency Λ, the solution of the discrete version of the equation may converge to the equilibrium exponentially or go to infinity as time evolves.

2. A reaction network approach to the exact resonance equation

2.1. The dynamical system associated with the exact resonance equation

Let us consider the discrete version of (1), which is described below.

Let $\mathcal{L}_R$ denote the lattice of integer points

$$\mathcal{L}_R = \{p \in \mathbb{Z}^3, |p| < R\}.$$

The discrete version of the quantum Boltzmann equation (1) reads

$$\begin{aligned} \dot{f}_{p_1} = & \sum_{\substack{p_2,p_3\in\mathcal{L}_R,\\ p_1-p_2-p_3=0,\\ \omega(p_1)-\omega(p_2)-\omega(p_3)=0}} \mathcal{V}_{p_1,p_2,p_3}\left[f_{p_2}f_{p_3} - f_{p_1}(f_{p_2}+f_{p_3})\right] \\ & -2\sum_{\substack{p_2,p_3\in\mathcal{L}_R,\\ p_1+p_2-p_3=0,\\ \omega(p_1)+\omega(p_2)-\omega(p_3)=0}} \mathcal{V}_{p_1,p_2,p_3}\left[f_{p_1}f_{p_2} - f_{p_3}(f_{p_1}+f_{p_2})\right], \end{aligned} \tag{8}$$

for all p_1 in $\mathcal{L}_R$, where $\omega(p)$ is defined in (3).

2.2. Decoupling the exact resonance equation

Observe that when $p_1 = 0$, $\mathcal{V}_{p_1,p_2,p_3}$ is also 0, and therefore,

$$\dot{f}_0 = 0, \tag{9}$$

which implies that $f_0(t)$ is a constant for all $t \geqslant 0$. Moreover, f_{p_1} does not depend on f_0 for all $p_1 \neq 0$. Therefore, without loss of generality, we can suppose that $f_0(0) = 0$, which leads to $f_0(t) = 0$ for all t.

Taking into account the fact $\omega(p) = c|p|$, note that if $p_1, p_2, p_3 \in \mathcal{L}_R$ are different from 0, and for $p_3 = p_1 + p_2$ and $|p_3| = |p_1| + |p_2|$ (like in the second sum of (8)), then p_1, p_2, p_3 must be collinear and on the same side of the origin. Therefore, we infer that there exists a vector P and $k_1, k_2, k_3 > 0$, $k_1, k_2, k_3 \in \mathbb{Z}$ such that

$$p_1 = k_1 P; \quad p_2 = k_2 P; \quad p_3 = k_3 P, \quad k_1 + k_2 = k_3.$$

Since $\mathcal{L}_R$ is bounded, it follows that k_1, k_2, k_3 belong to a finite set of integer indices $\mathbb{I} = \{1, \ldots, I\}$. Arguing similarly for the first sum in (8), we deduce that (8) is equivalent to the following system for $k_1 \in \mathbb{I}$

$$\begin{aligned} \dot{f}_{Pk_1} = & \sum_{\substack{k_2, k_3 \in \mathbb{I}, \\ k_1 - k_2 - k_3 = 0}} \mathcal{V}_{Pk_1, Pk_2, Pk_3} \left[f_{Pk_2} f_{Pk_3} - f_{Pk_1} (f_{Pk_2} + f_{Pk_3}) \right] \\ & - 2 \sum_{\substack{k_2, k_3 \in \mathbb{I}, \\ k_1 + k_2 - k_3 = 0}} \mathcal{V}_{Pk_1, Pk_2, Pk_3} \left[f_{Pk_1} f_{Pk_2} - f_{Pk_3} (f_{Pk_1} + f_{Pk_2}) \right]. \end{aligned} \tag{10}$$

Note that the system of equations (10) shows a *decoupling* of the system of equations (8) along a ray $\{kP_0\}$ with $k > 0$ (see Fig. 1). As a consequence, it is sufficient to study the system of equations (10) for a fixed value of P_0, instead of the system of equations (8).

If we denote $f_{k_1 P_0}$ by f_{k_1} (with $k_1 \in \mathbb{I}$) and $\mathcal{V}^{12}_{k_1 P_0, k_2 P_0, k_3 P_0}$ by $\mathcal{V}^{12}_{k_1, k_2, k_3}$, with an abuse of notation, we obtain the following new system for the ray $\{k_1 P_0 | k_1 > 0\}$:

$$\begin{aligned} \dot{f}_{k_1} = & \sum_{\substack{k_2, k_3 \in \mathbb{I}, \\ k_1 = k_2 + k_3}} \mathcal{V}_{k_1, k_2, k_3} [f_{k_2} f_{k_3} - f_{k_1} (f_{k_2} + f_{k_3})] \\ & - 2 \sum_{\substack{k_2, k_3 \in \mathbb{I}, \\ k_1 + k_2 = k_3}} \mathcal{V}_{k_1, k_2, k_3} [f_{k_1} f_{k_2} - f_{k_3} (f_{k_1} + f_{k_2})], \ \forall k_1 \in \mathbb{I}. \end{aligned} \tag{11}$$

The *conservation of energy* then follows

$$\sum_{k=1}^{I} k \dot{f}_k = 0, \tag{12}$$

or equivalently

$$\sum_{k=1}^{I} k f_k = \text{const.} \tag{13}$$

These conservation relations also follow easily from the rewriting of equations (11) as a mass-action system, as described in the next section. Also by abuse of notation, we denote this discrete version of Q by

$$\begin{aligned} \mathcal{Q}[f_{k_1}] := & \sum_{k_2 + k_3 = k_1} \mathcal{V}_{k_1, k_2, k_3} [f_{k_2} f_{k_3} - f_{k_1} (f_{k_2} + f_{k_3})] \\ & - 2 \sum_{k_1 + k_2 = k_3} \mathcal{V}_{k_3, k_1, k_2} [f_{k_1} f_{k_2} - f_{k_3} (f_{k_1} + f_{k_1})]. \end{aligned} \tag{14}$$

2.3. The chemical reaction network associated with the exact resonance equation

For $X \in \mathbb{R}^n_{>0}$ and $\alpha \in \mathbb{R}^n_{\geqslant 0}$, we denote by X^α the monomial $\Pi_{i=1}^n X_i^{\alpha_i}$.

Definition 1. Consider a chemical reaction of the form

$$\alpha_1 \mathrm{A}_1 + \alpha_2 \mathrm{A}_2 + \ldots + \alpha_\mathrm{n} \mathrm{A}_\mathrm{n} \xrightarrow{\mathcal{V}} \beta_1 \mathrm{A}_1 + \beta_2 \mathrm{A}_2 + \ldots + \beta_\mathrm{n} \mathrm{A}_\mathrm{n},$$

where $\mathcal{V}$ is a positive parameter, called reaction rate constant. Then the mass-action dynamical system generated by this reaction is

$$\dot{X} = \mathcal{V} X^\alpha (\beta - \alpha), \tag{15}$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)^T$, $\beta = (\beta_1, \cdots, \beta_n)^T$, $\alpha_i, \beta_i \geqslant 0$ and $X = (X_1, \cdots, X_n)^T$, in which X_i is the concentration of the chemical species A_i. For the case of a network that contains several reactions

$$\alpha_1^\mathrm{j} \mathrm{A}_1^\mathrm{j} + \alpha_2^\mathrm{j} \mathrm{A}_2^\mathrm{j} + \ldots + \alpha_\mathrm{n}^\mathrm{j} \mathrm{A}_\mathrm{n}^\mathrm{j} \xrightarrow{\mathcal{V}_\mathrm{j}} \beta_1 \mathrm{A}_1^\mathrm{j} + \beta_2^\mathrm{j} \mathrm{A}_2^\mathrm{j} + \ldots + \beta_\mathrm{n}^\mathrm{j} \mathrm{A}_\mathrm{n}^\mathrm{j},$$

for $1 \leqslant j \leqslant m$, its associated mass-action dynamical system is given by

$$\dot{X} = \sum_{j=1}^m \mathcal{V}_j X^{\alpha^j} (\beta^j - \alpha^j). \tag{16}$$

In this section, we will show that the system (11) has the form (16) for a well-chosen set of reactions. We will derive the system (11) from the network of chemical reactions of the form:

$$\mathrm{A}_{\mathrm{k}_2} + \mathrm{A}_{\mathrm{k}_3} \longrightarrow \mathrm{A}_{\mathrm{k}_1} \tag{17}$$

$$\mathrm{A}_{\mathrm{k}_2} + \mathrm{A}_{\mathrm{k}_1} \longrightarrow 2\mathrm{A}_{\mathrm{k}_2} + \mathrm{A}_{\mathrm{k}_3}, \tag{18}$$

for all k_1, k_2, k_3 in $\mathbb{I}$ such that $k_2 + k_3 = k_1$. If we denote by F_k the concentration of the species X_k, we will show that, *for appropriate choices of the reaction rate constants* in (17) and (18), the differential equations satisfied by F_k according the mass-action kinetics are exactly the same as (11).

In order to describe the connection between the mass-action system given by reactions of the form (17)-(18) and our system (11), we need to consider several cases.

Case 1: For $k_2 + k_3 = k_1$, $k_2 \neq k_3$, $k_1, k_2, k_3 \in \mathbb{I}$, we consider

$$\mathrm{A}_{\mathrm{k}_2} + \mathrm{A}_{\mathrm{k}_3} \xrightarrow{2\mathcal{V}_{\mathrm{k}_1,\mathrm{k}_2,\mathrm{k}_3}} \mathrm{A}_{\mathrm{k}_1} \tag{19}$$

$$\mathrm{A}_{\mathrm{k}_2} + \mathrm{A}_{\mathrm{k}_1} \xrightarrow{2\mathcal{V}_{\mathrm{k}_1,\mathrm{k}_2,\mathrm{k}_3}} 2\mathrm{A}_{\mathrm{k}_2} + \mathrm{A}_{\mathrm{k}_3}, \tag{20}$$

and for the reaction (19)-(20), we choose the reaction rate constants of the three reactions $A_{k_2} + A_{k_3} \to A_{k_1}$, $A_{k_2} + A_{k_1} \to 2A_{k_2} + A_{k_3}$ to be $2\mathcal{V}_{k_1,k_2,k_3}$. For example, consider the reaction (19):

in this reaction, A_{k_1} is created from $A_{k_2}+A_{k_3}$ with the rate $2\mathcal{V}_{k_1,k_2,k_3}F_{k_2}F_{k_3}$. Therefore, the rate of change of the species A_{k_1} due to this reaction is $2\mathcal{V}_{k_1,k_2,k_3}F_{k_2}F_{k_3}$. For the reaction (20), A_{k_1} is lost with the rate $-2\mathcal{V}_{k_1,k_2,k_3}F_{k_2}F_{k_1}$ to create $2A_{k_2}+A_{k_3}$. Therefore the rate of change of the species A_{k_1} due to this reaction is $-2\mathcal{V}_{k_1,k_2,k_3}F_{k_2}F_{k_1}$. By exchanging the roles of A_{k_2} and A_{k_3} in (20), we obtain the rate $-2\mathcal{V}_{k_1,k_2,k_3}[F_{k_2}F_{k_1}+F_{k_3}F_{k_1}]$. Therefore, the total rate of change of A_{k_1} due to the reactions in (19)-(20) is

$$2\mathcal{V}_{k_1,k_2,k_3}[F_{k_2}F_{k_3}-F_{k_2}F_{k_1}-F_{k_3}F_{k_1}]. \tag{21}$$

Case 2: For $k_2+k_3=k_1$, $k_2=k_3$, we obtain $2k_2=k_1$, $k_1,k_2\in\mathbb{I}$, and we consider

$$2\mathrm{A}_{\mathrm{k}_2}\xrightarrow{\mathcal{V}_{\mathrm{k}_1,\mathrm{k}_2,\mathrm{k}_3}}\mathrm{A}_{\mathrm{k}_1} \tag{22}$$

$$\mathrm{A}_{\mathrm{k}_2}+\mathrm{A}_{\mathrm{k}_1}\xrightarrow{2\mathcal{V}_{\mathrm{k}_1,\mathrm{k}_2,\mathrm{k}_3}}3\mathrm{A}_{\mathrm{k}_2}. \tag{23}$$

We choose the reaction rate constant of $2A_{k_2}\to A_{k_1}$ to be $\mathcal{V}_{k_1,k_2,k_3}$. Also, we choose the reaction rate constant of $A_{k_2}+A_{k_1}\to 3A_{k_2}$ to be $2\mathcal{V}_{k_1,k_2,k_3}$. Consider the first reaction (22): In this reaction, A_{k_1} is created from $2A_{k_2}$ with the rate $\mathcal{V}_{k_1,k_2,k_2}F_{k_2}^2$. The rate of change of the species A_{k_1} is $\mathcal{V}_{k_1,k_2,k_2}F_{k_2}^2$. For the second reaction (23): A_{k_1} is lost with the rate $-2\mathcal{V}_{k_1,k_2,k_2}F_{k_2}F_{k_1}$ to create $3A_{k_2}$. As a result, the rate of change of A_{k_1} due to the reactions (22)-(23) is

$$\mathcal{V}_{k_1,k_2,k_3}[F_{k_2}^2-2F_{k_2}F_{k_1}]. \tag{24}$$

In order to compute the total rate of change of A_{k_1}, we need the combination of (19)-(20), (22)-(23) and

$$\mathrm{A}_{\mathrm{k}_1}+\mathrm{A}_{\mathrm{k}_3}\xrightarrow{2\mathcal{V}_{\mathrm{k}_2,\mathrm{k}_1,\mathrm{k}_3}}\mathrm{A}_{\mathrm{k}_2} \tag{25}$$

$$\mathrm{A}_{\mathrm{k}_2}+\mathrm{A}_{\mathrm{k}_1}\xrightarrow{2\mathcal{V}_{\mathrm{k}_2,\mathrm{k}_1,\mathrm{k}_3}}2\mathrm{A}_{\mathrm{k}_1}+\mathrm{A}_{\mathrm{k}_3}, \tag{26}$$

which, by (21), (24), implies

$$\begin{aligned}\dot F_{k_1} &= \sum_{k_2+k_3=k_1,k_2<k_3} 2\mathcal{V}_{k_1,k_2,k_3}[F_{k_2}F_{k_3}-F_{k_1}(F_{k_2}+F_{k_3})]\\ &\quad+\sum_{2k_2=k_1}\mathcal{V}_{k_1,k_2,k_2}[F_{k_2}F_{k_2}-F_{k_1}(F_{k_2}+F_{k_2})]\\ &\quad-\sum_{k_1+k_3=k_2}2\mathcal{V}_{k_2,k_1,k_3}[F_{k_1}F_{k_3}-F_{k_2}(F_{k_1}+F_{k_3})],\end{aligned} \tag{27}$$

which can be written as

$$\begin{aligned}\dot F_{k_1} &= \sum_{k_2+k_3=k_1}\mathcal{V}_{k_1,k_2,k_3}[F_{k_2}F_{k_3}-F_{k_1}(F_{k_2}+F_{k_3})]\\ &\quad-2\sum_{k_1+k_3=k_2}\mathcal{V}_{k_2,k_1,k_3}[F_{k_1}F_{k_3}-F_{k_2}(F_{k_1}+F_{k_3})].\end{aligned} \tag{28}$$

Equation (28) shows that the system of differential equations satisfied by the concentrations F_k is exactly the same as the system of differential equations (11) satisfied by the densities f_k.

2.4. A Lyapunov function inspired by the associated reaction network

The observation above is very interesting because it shows a very strong connection between our system (11) *and the reaction network model* (17)-(18). As a consequence, the techniques developed to study the Global Attractor Conjecture [15] in reaction network theory can be applied to study (11). One of the key ingredients in proving the convergence to global attractors of detailed balanced or complex balanced reaction network models is that these networks have a specific type of Lyapunov functions. Then our system of interest (11) may also have a similar Lyapunov function. By using a change of variable similar to the approach in [14], we can create a global Lyapunov function (related to Boltzmann's original H-theorem calculations), as follows.

To illustrate this idea in a very simple way, we select in the system above three values k_1, k_2, k_3 such that $k_1 + k_2 = k_3$, and suppose $\mathcal{V}_{k_1,k_2,k_3} = 1$. Then this simplified "sub-system" can be rewritten as

$$\frac{d}{dt}\begin{pmatrix} F_{k_1} \\ F_{k_2} \\ F_{k_3} \end{pmatrix} = (F_{k_1}F_{k_3} + F_{k_2}F_{k_3} - F_{k_1}F_{k_2})\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right), \tag{29}$$

which can be developed into

$$\frac{d}{dt}\begin{pmatrix} F_{k_1} \\ F_{k_2} \\ F_{k_3} \end{pmatrix} = F_{k_1}F_{k_2}F_{k_3}\left(\frac{1}{F_{k_2}} + \frac{1}{F_{k_1}} - \frac{1}{F_{k_3}}\right)\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right). \tag{30}$$

Consider the change of variables

$$G_k = \exp\left(\frac{1}{F_k}\right), \quad F_k = \frac{1}{\log(G_k)},$$

then we have

$$\frac{1}{F_{k_1}} + \frac{1}{F_{k_2}} - \frac{1}{F_{k_3}} = \log(G_{k_1}G_{k_2}) - \log(G_{k_3}).$$

We then get

$$\frac{d}{dt}\begin{pmatrix} G_{k_1} \\ G_{k_2} \\ G_{k_3} \end{pmatrix} = \operatorname{diag}\begin{pmatrix} -G_{k_1}\log^2(G_{k_1}) \\ -G_{k_2}\log^2(G_{k_2}) \\ -G_{k_3}\log^2(G_{k_3}) \end{pmatrix} \frac{1}{\log G_{k_1}\log G_{k_2}\log G_{k_3}}[\log(G_{k_1}G_{k_2}) - \log(G_{k_3})]. \tag{31}$$

By following the approach in [14], we construct a Lyapunov function for the system in variables G_k given by

$$L(G) = \sum_{k=1}^{I} \log\log(G_k),$$

which, when translated back into the F_k coordinates has a very simple form:

$$L(F) = \sum_{k=1}^{I} \log\left(\frac{1}{F_k}\right) = -\sum_{k=1}^{I} \log(F_k).$$

In the next section, we will see that this construction does indeed give rise to a global strict Lyapunov function for our (full) dynamical system of interest.

2.5. *Convergence to equilibrium*

Theorem 2. *For any positive initial condition, the solution*

$$f(t) = (f_p(t))_{p\in\mathcal{L}_R}$$

of the discrete wave turbulence equation (8) *converges to an equilibrium state* $f^* = (f_p^*)_{p\in\mathcal{L}_R}$. *For each ray* $\{kP_0\}_{k\geqslant 1}$ *there exists a positive constant* $\rho(P_0)$ *such that if* $p = kP_0$ *then*

$$f_p^* = \frac{1}{k\rho(P_0)}.$$

Moreover, the solution $f(t)$ *of* (8) *converges to* f^* *exponentially fast, in the following sense: there exist positive constants* C_1, C_2 *such that*

$$\max_{p\in\mathcal{L}_R} |f_p(t) - f_p^*| < C_1 e^{-C_2 t}.$$

Proof. By using the decoupling and the change of variables discussed in the previous sections, for each ray $\{kP_0\}_{k\geqslant 1}$, we can reduce the study of f to F, which satisfies (28).

Step 1: The Lyapunov function. We define the function

$$L(F) = -\sum_{k=1}^{I} \log(F_k), \tag{32}$$

and we will show that L is a Lyapunov function for (28). Indeed, we have

$$\nabla L = \begin{pmatrix} -\frac{1}{F_1} \\ \cdots \\ -\frac{1}{F_I} \end{pmatrix}. \tag{33}$$

By defining

$$e_k = \begin{pmatrix} 0 \\ \cdots \\ 1 \\ \cdots \\ 0 \end{pmatrix}, \tag{34}$$

in which the number 1 stands for the k-th coordinate, we then have

$$\begin{aligned}\dot{F}\cdot\nabla L &= \sum_{k_1+k_2=k}\mathcal{V}_{k_1,k_2,k}[F_{k_1}F_{k_2}-F_kF_{k_1}-F_kF_{k_2}][e_k-e_{k_1}-e_{k_2}]\cdot\begin{pmatrix}-\frac{1}{F_1}\\ \cdots\\ -\frac{1}{F_I}\end{pmatrix}\\ &= \sum_{k_1+k_2=k}\mathcal{V}_{k_1,k_2,k}F_kF_{k_1}F_{k_2}\left[\frac{1}{F_k}-\frac{1}{F_{k_1}}-\frac{1}{F_{k_2}}\right][e_k-e_{k_1}-e_{k_2}]\cdot\begin{pmatrix}-\frac{1}{F_1}\\ \cdots\\ -\frac{1}{F_I}\end{pmatrix}\\ &= -\sum_{k_1+k_2=k}\mathcal{V}_{k_1,k_2,k}F_kF_{k_1}F_{k_2}\left[\frac{1}{F_k}-\frac{1}{F_{k_1}}-\frac{1}{F_{k_2}}\right]^2\\ &\leqslant 0.\end{aligned} \tag{35}$$

In this case the vector $\beta-\alpha$ in (15) takes the form $e_k-e_{k_1}-e_{k_2}$. Also, note that the above inequality is strict unless

$$\frac{1}{F_k}=\frac{1}{F_{k_1}}+\frac{1}{F_{k_2}}, \tag{36}$$

for all $k=k_1+k_2$. Equation (36) implies that at equilibrium $F_k^*=\frac{1}{\rho k}$, for some positive constant ρ. By the conservation relation

$$\sum_{k=1}^{I}kF_k=\sum_{k=1}^{I}kF_k^*,$$

we deduce that ρ is unique, i.e.,

$$F^*=\begin{pmatrix}\frac{1}{\rho}\\ \cdots\\ \frac{1}{\rho I}\end{pmatrix} \tag{37}$$

is the only equilibrium point that satisfies the same conservation relation as the initial condition.

Step 2: Differential inclusions and persistence.

Since the Lyapunov function L is infinite on the boundary, we conclude that the system is persistent. Therefore, by using the existence of the globally defined strict Lyapunov function L, and the LaSalle invariance principle, it follows that all trajectories converge to the unique positive equilibrium F^* that we discussed in Step 1.

Step 3: Exponential rate of convergence.

Despite the fact that the system (19)-(20) is irreversible, in this step, we will use methods that work well for reversible systems. To this end, we will introduce a change of variable technique, to convert (19)-(20) into the system for the reaction $A_{k_1}+A_{k_2}\leftrightarrow A_{k_3}$, which is indeed reversible, but with reaction rate functions different from mass action kinetics. In other words, by converting the irreversible system (19)-(20) into a reversible one, we can then employ techniques originally designed for reversible systems.

Let us start with our change of variable, which takes the following form

$$G_k = \exp\left(\frac{1}{F_k}\right),$$

then

$$F_k = \frac{1}{\ln(G_k)},$$

and

$$\begin{aligned} F_{k_1}F_{k_2} - F_{k_1}F_{k_3} - F_{k_2}F_{k_3} &= \frac{\ln(G_{k_1}) - \ln(G_{k_2}G_{k_3})}{\ln(G_{k_1})\ln(G_{k_2})\ln(G_{k_3})} \\ &= \frac{\ln(G_{k_1}) - \ln(G_{k_2}G_{k_3})}{\ln(G_{k_1})\ln(G_{k_2})\ln(G_{k_3})(G_{k_1} - G_{k_2}G_{k_3})}(G_{k_1} - G_{k_2}G_{k_3}). \end{aligned}$$

Notice that $0 < F_k < \infty$ and $1 < G_k < \infty$. Moreover,

$$\frac{\ln(G_{k_1}) - \ln(G_{k_2}G_{k_3})}{G_{k_1} - G_{k_2}G_{k_3}} > 0, \forall k_1, k_2, k_3 \in I.$$

We will now study G_k instead of F_k. To do this, we convert the system (28) into

$$\begin{aligned} &\frac{\dot{G}_{k_1}}{(\ln(G_{k_1}))^2} = \widetilde{\mathcal{Q}}[G](k_1) \\ :=&2\sum_{k_1+k_2=k_3} \mathcal{V}_{k_1,k_2,k_3} \frac{\ln(G_{k_3}) - \ln(G_{k_2}G_{k_1})}{\ln(G_{k_1})\ln(G_{k_2})\ln(G_{k_3})(G_{k_3} - G_{k_2}G_{k_1})}(G_{k_3} - G_{k_1}G_{k_2}) \\ &+ \sum_{k_1=k_2+k_3} \mathcal{V}_{k_1,k_2,k_3} \frac{\ln(G_{k_1}) - \ln(G_{k_2}G_{k_3})}{\ln(G_{k_1})\ln(G_{k_2})\ln(G_{k_3})(G_{k_1} - G_{k_2}G_{k_3})}(-G_{k_1} + G_{k_2}G_{k_3}), \forall k_1 \in \mathbb{I}. \end{aligned} \tag{38}$$

Suppose that G represents the column vector $(G_1, \dots, G_I)^T$. Let us also denote by A_k, with an abuse of notation, the vector

$$\begin{pmatrix} 0 \\ \cdots \\ 1 \\ \cdots \\ 0 \end{pmatrix},$$

in which the only element that different from 0 is the k-th one.

As discussed above, we convert F_k into G_k. This technique allows us to changes the irreversible system (19)-(20) into the reversible one $A_{k_1} + A_{k_2} \leftrightarrow A_{k_3}$. Let us now compute the reaction rate functions, which is quite different from mass action kinetics

$$V_{A_{k_1}+A_{k_2}\to A_{k_3}}(G) := 2\mathcal{V}_{k_1,k_2,k_3}\frac{\ln(G_{k_3})-\ln(G_{k_2}G_{k_1})}{\ln(G_{k_1})\ln(G_{k_2})\ln(G_{k_3})(G_{k_3}-G_{k_2}G_{k_1})}G_{k_1}G_{k_2},$$

$$V_{A_{k_3}\to A_{k_1}+A_{k_2}}(G) := 2\mathcal{V}_{k_1,k_2,k_3}\frac{\ln(G_{k_3})-\ln(G_{k_2}G_{k_1})}{\ln(G_{k_1})\ln(G_{k_2})\ln(G_{k_3})(G_{k_3}-G_{k_2}G_{k_1})}G_{k_3},$$

$$\mathcal{V}_{A_{k_1}+A_{k_2}\leftrightarrow A_{k_3}} := 2\mathcal{V}_{k_1,k_2,k_3}.$$

Otherwise, if $k_1 = k_2$, we write

$$V_{2A_{k_1}\to A_{k_3}}(G) := \mathcal{V}_{k_1,k_1,k_3}\frac{\ln(G_{k_3})-2\ln(G_{k_1})}{\ln(G_{k_3})\ln(G_{k_1})^2(G_{k_3}-G_{k_1}^2)}G_{k_1}G_{k_2},$$

$$V_{A_{k_3}\to 2A_{k_1}}(G) := \mathcal{V}_{k_1,k_1,k_3}\frac{\ln(G_{k_3})-2\ln(G_{k_1})}{\ln(G_{k_3})\ln(G_{k_1})^2(G_{k_3}-G_{k_1}^2)}G_{k_3},$$

$$\mathcal{V}_{2A_{k_1}\leftrightarrow A_{k_3}} := 2\mathcal{V}_{k_1,k_1,k_3}.$$

Using these reaction rate functions, the system (38) could be converted into

$$\begin{aligned}\dot{G} = \operatorname{diag}\begin{pmatrix}(\ln(G_1))^2\\ \cdots\\ (\ln(G_I))^2\end{pmatrix} \times \\ \times \sum_{k_1+k_2=k_3}\Big[V_{A_{k_1}+A_{k_2}\to A_{k_3}}(G) - V_{A_{k_3}\to A_{k_1}+A_{k_2}}(G)\Big](A_{k_3}-A_{k_1}-A_{k_2}).\end{aligned} \tag{39}$$

Equivalently, we can also write the following equation for the new reversible system $A_{k_1}+A_{k_2}\leftrightarrow A_{k_3}$

$$\dot{G} = \operatorname{diag}\begin{pmatrix}(\ln(G_1))^2\\ \cdots\\ (\ln(G_I))^2\end{pmatrix}\sum_{y\leftrightarrow y'}\big[V_{y\to y'}(G) - V_{y'\to y}(G)\big](y'-y), \tag{40}$$

where $y \leftrightarrow y'$ belongs to the set of reversible reactions

$$A_{k_1} + A_{k_2} \longleftrightarrow A_{k_3}, \tag{41}$$

with $k_1 + k_2 = k_3$.

Since we have converted (19)-(20) into the reversible system $A_{k_1} + A_{k_2} \leftrightarrow A_{k_3}$, we can now employ classical techniques for reversible systems, starting with the definitions of the two functionals $\mathcal{R}(G)$ and $\mathcal{S}(G)$

$$\begin{aligned}\mathcal{R}(G) &= \operatorname{diag}\begin{pmatrix}(\ln(G_1))^2\\ \cdots\\ (\ln(G_I))^2\end{pmatrix}\sum_{y\leftrightarrow y'}\big[V_{y\to y'}(G) - V_{y'\to y}(G)\big](y'-y)\\ &= \operatorname{diag}\begin{pmatrix}(\ln(G_1))^2\\ \cdots\\ (\ln(G_I))^2\end{pmatrix}\sum_{y\leftrightarrow y'}[\mathcal{V}_{y\leftrightarrow y'}G^y - \mathcal{V}_{y\leftrightarrow y'}G^{y'}]\mathcal{H}_{y\leftrightarrow y'}(G)(y'-y),\end{aligned} \tag{42}$$

and

$$\mathcal{S}(G) = \sum_{y\leftrightarrow y'} [\mathcal{V}_{y\leftrightarrow y'} G^{y} - \mathcal{V}_{y\leftrightarrow y'} G^{y'}]\mathcal{H}_{y\leftrightarrow y'}(G)(y'-y).$$

Next, we will follow the techniques introduced in [17] for reversible systems, by computing the Jacobian of $\mathcal{S}$ at the equilibrium point G^*, applied to an arbitrary vector $\delta \neq 0$ that belongs to the span of the vectors $y'-y$

$$\mathrm{Jac}(\mathcal{S}(G^*))\delta = \sum_{y\leftrightarrow y'} \mathcal{V}_{y\leftrightarrow y'}(G^*)^{y}((y-y')*\delta)\mathcal{H}_{y\leftrightarrow y'}(G^*)(y-y'), \tag{43}$$

since $\mathcal{V}_{y\leftrightarrow y'}(G^*)^{y} - \mathcal{V}_{y\leftrightarrow y'}(G^*)^{y'} = 0$ in which the inner product $*$ is defined as

$$y*\delta = \sum_{1}^{I} \frac{y_k\delta_k}{G_k}.$$

Therefore

$$\begin{aligned} &[\mathrm{Jac}(\mathcal{S}(G^*))\delta]*\delta = \\ &= \sum_{y\leftrightarrow y'} \mathcal{V}_{y\leftrightarrow y'}(G^*)^{y}\mathcal{H}_{y\leftrightarrow y'}(G^*)[(y-y')*\delta][(y'-y)*\delta] < 0. \end{aligned} \tag{44}$$

Now, we compute the Jacobian of $\mathcal{R}$ at the equilibrium point G^*,

$$\begin{aligned} \mathrm{Jac}(\mathcal{R}(G^*)) &= \mathrm{diag}\begin{bmatrix} \partial_{G_1}(\ln(G_1^*))^2\mathcal{S}(G^*)_1 \\ \cdots \\ \partial_{G_I}(\ln(G_I^*))^2\mathcal{S}(G^*)_I \end{bmatrix} + \mathrm{diag}\begin{bmatrix} (\ln(G_I^*))^2 \\ \cdots \\ (\ln(G_I^*))^2 \end{bmatrix}\mathrm{Jac}(\mathcal{S}(G^*)) \\ &= \mathrm{diag}\begin{bmatrix} (\ln(G_1^*))^2 \\ \cdots \\ (\ln(G_I^*))^2 \end{bmatrix}\mathrm{Jac}(\mathcal{S}(G^*)), \end{aligned}$$

where the second equality is due to the fact that since G^* is an equilibrium we have that $\mathcal{S}(G^*) = 0$. Since

$$\mathfrak{D} := \mathrm{diag}\begin{bmatrix} (\ln(G_1^*))^2 \\ \cdots \\ (\ln(G_I^*))^2 \end{bmatrix}$$

is a diagonal matrix and $\mathfrak{J} := \mathrm{Jac}(\mathcal{S}(G^*))$ is negative definite, then $\mathfrak{D}^{1/2}\mathfrak{J}\mathfrak{D}^{1/2}$ is also negative definite with respect to this inner product. Since

$$\det(\mathfrak{D}\mathfrak{J} - \Lambda Id) = \det(\mathfrak{D}^{1/2}\mathfrak{J}D^{1/2} - \Lambda Id), \quad \forall \Lambda \in \mathbb{R},$$

it follows that $\mathfrak{D}^{1/2}\mathfrak{J}\mathfrak{D}^{1/2}$ and $\mathfrak{D}\mathfrak{J}$ have the same eigenvectors, so $\mathfrak{D}\mathfrak{J}$ is negative definite. In other words, $\mathrm{Jac}(\mathcal{R}(G^*))$ is negative definite. The exponential rate of convergence

$$\max\{|G_1(t)-G_1^*|,\cdots,|G_I(t)-G_I^*|\}\leqslant C_1 e^{-C_2 t},$$

then follows from the fact that the Jacobian above is negative definite. This leads to the conclusion of the theorem. □

Remark 3. In the proof above we could have used the Lyapunov function

$$L(F)=-\Pi_{k=1}^{I}F_k, \tag{45}$$

and all the computations remain the same.

3. A reaction network approach to the near resonance equation

3.1. The dynamical system associated to the near resonance equation

Similar as in Section 2, let us consider the discrete version of (4), which is described below. Let $\mathcal{L}_R$ denote the lattice of integer points

$$\mathcal{L}_R=\{p\in\mathbb{Z}^3,|p|<R\}.$$

The discrete version of the near-resonance equation (4) reads

$$\begin{aligned}\dot f_{p_1} = & \sum_{\substack{p_2,p_3\in\mathcal{L}_R,\\ p_1-p_2-p_3=0,\\ |\omega(p_1)-\omega(p_2)-\omega(p_3)|<\Lambda}} \mathcal{V}_{p_1,p_2,p_3}\left[f_{p_2}f_{p_3}-f_{p_1}(f_{p_2}+f_{p_3})\right]\\ & -2\sum_{\substack{p_2,p_3\in\mathcal{L}_R,\\ p_1+p_2-p_3=0,\\ |\omega(p_1)+\omega(p_2)-\omega(p_3)|<\Lambda}} \mathcal{V}_{p_1,p_2,p_3}\left[f_{p_1}f_{p_2}-f_{p_3}(f_{p_1}+f_{p_2})\right],\end{aligned} \tag{46}$$

for all p_1 in $\mathcal{L}_R$.

Since $\mathcal{L}_R$ contains a finite number of grid points, then there exists a positive number Λ_* satisfying

$$\Lambda_*=\min_{\substack{p_1,p_2,p_3\in\mathcal{L}_R,\\ p_1-p_2-p_3=0,\\ |\omega(p_1)-\omega(p_2)-\omega(p_3)|\neq 0}} |\omega(p_1)-\omega(p_2)-\omega(p_3)| \tag{47}$$

See Fig. 1 for a geometric illustration of the construction of Λ_*. The red triangle in Fig. 1 shows an example of a triple (p_1,p_2,p_3) such that we have $p_1-p_2-p_3=0$ but $|\omega(p_1)-\omega(p_2)-\omega(p_3)|\neq 0$. Moreover, we also define

$$\Lambda^*=\min_{p\in\mathcal{L}_R\setminus\{0\}}|p|. \tag{48}$$

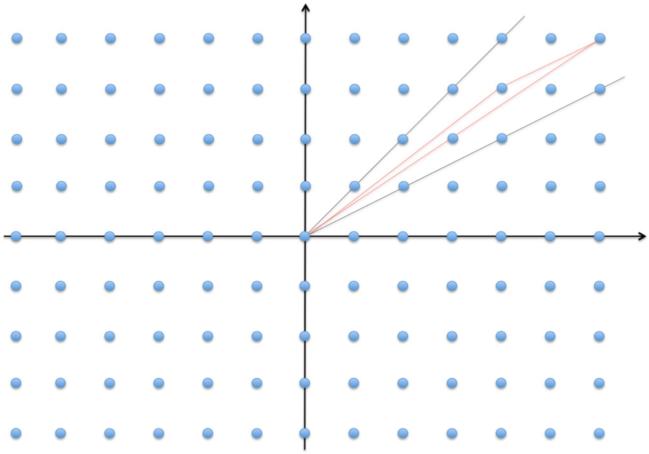

Fig. 1. The system of equations (8) can be decoupled alongs rays, as shown in (10). Two examples of such rays are shown in black above. The red triangle illustrates the definition of Λ_* in (47). (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

3.2. Convergence to equilibrium

Theorem 4. *The solution of the discrete near resonance equation* (46) *has different behaviors for different values of the resonance broadening frequency* Λ.

- *If* $\Lambda < \Lambda_*$, *where* Λ_* *is defined in* (47)*: For any positive initial condition, the solution*

  $$f(t) = (f_p(t))_{p\in\mathcal{L}_R}$$

  of the discrete wave turbulence equation (46) *converges to an equilibrium state* $f^* = (f_p^*)_{p\in\mathcal{L}_R}$. *For each ray* $\{kP_0\}_{k\geqslant 1}$ *there exists a positive constant* $\rho(P_0)$ *such that if* $p = kP_0$ *then*

  $$f_p^* = \frac{1}{k\rho(P_0)}.$$

  Moreover, the solution $f(t)$ *of* (46) *converges to* f^* *exponentially fast, in the following sense: there exist positive constants* C_1, C_2 *such that*

  $$\max_{p\in\mathcal{L}_R} |f_p(t) - f_p^*| < C_1 e^{-C_2 t}. \tag{49}$$

  The reason for (49) *to hold is that when* $\Lambda < \Lambda_*$, *the resonance broadening equation becomes the exact resonance one.*
- *If* $2\Lambda^* \leqslant \Lambda$, *where* Λ^* *is defined in* (48)*: For any positive initial condition, the solution*

  $$f(t) = (f_p(t))_{p\in\mathcal{L}_R}$$

  of the discrete wave turbulence equation (46) *exits any compact set* $K \subset (0,\infty)^{|\mathcal{L}_R|}$ *as* t *tends to infinity, and moreover we have* $\liminf_{t\to\infty} ||f(t)|| = \infty$.

Proof. *Step 1: The Lyapunov function.* We define the function

$$L(f) = -\sum_{p\in\mathcal{L}_R}^{I} \log(f_p), \tag{50}$$

and we will show that L is a Lyapunov function for (46). Similar as the proof of Theorem 4, we also have

$$\begin{aligned} &\dot{f}\cdot\nabla L = \\ &= -\sum_{p_2+p_3=p_1,||p_2|+|p_3|-|p_1||\leqslant\Lambda} \mathcal{V}_{p_1,p_2,p_3} f_{p_1} f_{p_2} f_{p_3}\left[\frac{1}{f_{p_1}}-\frac{1}{f_{p_2}}-\frac{1}{f_{p_3}}\right]^2 \\ &\leqslant 0. \end{aligned} \tag{51}$$

Note that the above inequality is strict unless

$$\frac{1}{f_{p_1}} = \frac{1}{f_{p_2}} + \frac{1}{f_{p_3}}, \tag{52}$$

for all $p_1 = p_2 + p_3$ and $||p_2| + |p_3| - |p_1|| \leqslant \Lambda$.

Step 2: The two cases of Λ.

Case 1: $\Lambda < \Lambda_*$.

Since $\Lambda < \Lambda_*$, the system $p_1 = p_2 + p_3$ and $||p_2| + |p_3| - |p_1|| \leqslant \Lambda < \Lambda_*$ becomes exactly the system $p_1 = p_2 + p_3$ and $||p_2| + |p_3| - |p_1|| = 0$. Therefore, equation (52) implies that at equilibrium $f_k^* = \rho k$, for some positive constant ρ. By the conservation relation

$$\sum_{p\in\mathcal{L}_R} p f_p = \sum_{p\in\mathcal{L}_R} p f_p^*,$$

we deduce that ρ is unique and f^* is the only equilibrium point that satisfies the same conservation relation as the initial condition. The same argument as in Theorem 2 can be applied and the conclusion of Theorem 4 follows.

Case 2: $2\Lambda^* \leqslant \Lambda$.

Define $Z_p = \frac{1}{F_p}$. The equilibrium set of (51) satisfies the following system of linear equations.

$$Z_{p_1} = Z_{p_2} + Z_{p_3}, \tag{53}$$

for all $p_1 = p_2 + p_3$ and $||p_2| + |p_3| - |p_1|| \leqslant \Lambda$. We will show that the system (53) has no solution by contradiction.

The system (53) contains the exact resonance one as a subsystem

$$Z_{p_1} = Z_{p_2} + Z_{p_3}, \tag{54}$$

for all $p_1 = p_2 + p_3$ and $||p_2| + |p_3| - |p_1|| = 0$.

For each ray p, we denote by e_p the closet point of $\mathcal{L}^R$ to the origin on this ray. Suppose that p is chosen such that $|e_p| = \Lambda^*$. Let us consider all of the points of $\mathcal{L}^R$, which is a set of the form $\{e_p, 2e_q, 3e_q, \cdots, Je_q\}$. On this set, the exact resonance equation (54) is the classical one

$$Z_{(i+j)e_p} = Z_{ie_p} + Z_{je_p}$$

and has a unique solution

$$Z_p = \lambda|p|,$$

where λ is a positive constant.

By a similar argument, let us consider the ray $-e_p$; then $-e_p$ is also the closest point of $\mathcal{L}^R$ to the origin on this ray. Thus, we know

$$Z_q = \lambda'|q|,$$

where λ' is a positive constant, for all q on the ray of $-e_p$.

Consider the vectors $p_1 = 3e_p$, $p_2 = 4e_p$ and $p_3 = -e_p$, then $p_1 = p_2 + p_3$ and $||p_2| + |p_3| - |p_1|| = 2\Lambda_* \leqslant \Lambda$. Therefore for these choices

$$Z_{p_1} = Z_{p_2} + Z_{p_3},$$

which is equivalent to

$$\lambda 3|e_p| = \lambda 4|e_p| + \lambda'|e_p|.$$

This leads to a contradiction.

Since the system (53) has no solution, for any compact set $K \subset (0,\infty)^{|\mathcal{L}_R|}$, there exists $\Delta_K > 0$ such that

$$\begin{aligned} \dot{f} \cdot \nabla L = & - \sum_{p_2+p_3=p_1, ||p_2|+|p_3|-|p_1||\leqslant\Lambda} \mathcal{V}_{p_1,p_2,p_3} f_{p_1} f_{p_2} f_{p_3} \left[\frac{1}{f_{p_1}} - \frac{1}{f_{p_2}} - \frac{1}{f_{p_3}}\right]^2 \\ \leqslant & -\Delta_K. \end{aligned} \tag{55}$$

Therefore f exits any compact set $K \subset (0,\infty)^{|\mathcal{L}_R|}$ as t tends to infinity. In particular, consider the compact sets of the form

$$\{x \in (0,\infty)^{|\mathcal{L}_R|} \mid \prod_{p\in\mathcal{L}_R} x_p \geqslant \varepsilon_0 \text{ and } ||x|| \leqslant M\}. \tag{56}$$

Note that the set

$$\{x \in (0,\infty)^{|\mathcal{L}_R|} \mid \prod_{p\in\mathcal{L}_R} x_p = \varepsilon_0\} \tag{57}$$

is a *level set* of the Lyapunov function $L(f)$ (see also Remark 5 below).

Consider now a solution $f(t)$ of equation (46) and some $\varepsilon_0 > 0$ such that for $f^0 = f(0)$ we have $\prod_{p\in\mathcal{L}_R} f_p^0 > \varepsilon_0$.

Then, since $f(t)$ must exit the compact set given by (56) for any $M > 0$, and the Lyapunov function property prevents it from exiting through the boundary of the form (57), it follows that $\liminf_{t\to\infty} ||f(t)|| = \infty$. $\square$

Remark 5. In the proof of Theorem 4 we could also have used the Lyapunov function

$$L(f) = -\prod_{p\in\mathcal{L}_R} f_p, \tag{58}$$

and all the computations remain the same.

4. Perspectives

The explicit expression for the Kolmogorov-type spectrum of acoustic turbulence of equation (1) is (cf. [39])

$$f(p) = |p|^{-3/2}.$$

An open question is if this is also a spectrum of the discrete system. In our work, we prove that $|p|^{-1}$ is a global attractor for the discrete system. In a future work, we will try to compare these two solutions. We suspect that it may be not possible to find a Lyapunov function for $|p|^{-3/2}$ despite the fact that it is possible to find a Lyapunov function for $|p|^{-1}$, as shown above. Moreover, we will also plan to study the behavior of the system when $\Lambda_* \leqslant \Lambda \leqslant 2\Lambda^*$.

Acknowledgments

G. Craciun was supported by NSF grants DMS-1412643 and DMS-1816238. M.-B. Tran was partially supported by NSF Grant DMS-1814149, NSF Grant DMS-1854453, SMU URC Grant 2020, Linking Fellowship of Dedman College of Humanities and Sciences, and Alexander von Humboldt Research Fellowship of Alexander von Humboldt Foundation.

References

[1] R. Alonso, I.M. Gamba, M.-B. Tran, The Cauchy problem and BEC stability for the quantum Boltzmann-condensation system for bosons at very low temperature, preprint, arXiv:1609.07467, 2018.

[2] D.F. Anderson, A proof of the global attractor conjecture in the single linkage class case, SIAM J. Appl. Math. 71 (4) (2011) 1487–1508.

[3] D.F. Anderson, G. Craciun, T.G. Kurtz, Product-form stationary distributions for deficiency zero chemical reaction networks, Bull. Math. Biol. 72 (8) (2010) 1947–1970.

[4] D. Angeli, P. De Leenheer, E.D. Sontag, Persistence results for chemical reaction networks with time-dependent kinetics and no global conservation laws, SIAM J. Appl. Math. 71 (1) (2011) 128–146.

[5] M. Banaji, G. Craciun, Graph-theoretic approaches to injectivity and multiple equilibria in systems of interacting elements, Commun. Math. Sci. 7 (4) (2009) 867–900.

[6] P. Bartello, Geostrophic adjustment and inverse cascades in rotating stratified turbulence, J. Atmos. Sci. 52 (24) (1995) 4410–4428.

[7] D.J. Benney, A.C. Newell, The propagation of nonlinear wave envelopes, Stud. Appl. Math. 46 (1–4) (1967) 133–139.

[8] D.J. Benney, P.G. Saffman, Nonlinear interactions of random waves in a dispersive medium, Proc. R. Soc. Lond. A 289 (1418) (1966) 301–320.

[9] S. Boldyrev, J.C. Perez, V. Zhdankin, Residual energy in mhd turbulence and in the solar wind, in: AIP Conference Proceedings, vol. 1436, AIP, 2012, pp. 18–23.

[10] L. Boltzmann, Neuer Beweis zweier Satze uber das Warmegleichgewicht unter mehratomigen Gas-molekulen, Sitzungsb. Kais. Akad. Wiss. Wien (95) (1887) 153–164.

[11] T. Buckmaster, P. Germain, Z. Hani, J. Shatah, Analysis of the (CR) equation in higher dimensions, preprint, arXiv:1610.05736, 2016.

[12] T. Buckmaster, P. Germain, Z. Hani, J. Shatah, Effective dynamics of the nonlinear Schrödinger equation on large domains, preprint, arXiv:1610.03824, 2016.

[13] C. Connaughton, S. Nazarenko, A. Pushkarev, Discreteness and quasiresonances in weak turbulence of capillary waves, Phys. Rev. E 63 (4) (2001) 046306.

[14] G. Craciun, M.-B. Tran, A reaction network approach to the convergence to equilibrium of quantum Boltzmann equations for Bose gases, preprint, arXiv:1608.05438, 2017.

[15] G. Craciun, Toric differential inclusions and a proof of the global attractor conjecture, preprint, arXiv:1501.02860v2, 2016.

[16] G. Craciun, A. Dickenstein, A. Shiu, B. Sturmfels, Toric dynamical systems, J. Symb. Comput. 44 (11) (2009) 1551–1565.

[17] G. Craciun, M. Feinberg, Multiple equilibria in complex chemical reaction networks. I. The injectivity property, SIAM J. Appl. Math. 65 (5) (2005) 1526–1546 (electronic).

[18] G. Craciun, F. Nazarov, C. Pantea, Persistence and permanence of mass-action and power-law dynamical systems, SIAM J. Appl. Math. 73 (1) (2013) 305–329.

[19] M. Escobedo, M.-B. Tran, Convergence to equilibrium of a linearized quantum Boltzmann equation for bosons at very low temperature, Kinet. Relat. Models 8 (3) (2015) 493–531.

[20] E. Faou, P. Germain, Z. Hani, The weakly nonlinear large-box limit of the 2d cubic nonlinear Schrödinger equation, J. Am. Math. Soc. 29 (4) (2016) 915–982.

[21] M. Feinberg, Lectures on Chemical Reaction Networks, 1979, written version of lectures given at the Mathematical Research Center, University of Wisconsin, Madison WI, available at http://www.crnt.osu.edu/LecturesOnReactionNetworks.

[22] M. Feinberg, Complex balancing in general kinetic systems, Arch. Ration. Mech. Anal. 49 (1972/1973) 187–194.

[23] M. Feinberg, The existence and uniqueness of steady states for a class of chemical reaction networks, Arch. Ration. Mech. Anal. 132 (4) (1995) 311–370.

[24] I.M. Gamba, L.M. Smith, M.-B. Tran, On the wave turbulence theory for stratified flows in the ocean, M3AS: Math. Models Methods Appl. Sci. 30 (1) (2020) 105–137.

[25] P. Germain, Z. Hani, L. Thomann, On the continuous resonant equation for NLS. I. Deterministic analysis, J. Math. Pures Appl. 105 (1) (2016) 131–163.

[26] P. Germain, A.D. Ionescu, M.-B. Tran, Optimal local well-posedness theory for the kinetic wave equation, J. Funct. Anal. (2020), in press, arXiv:1711.05587v2.

[27] A.E. Gill, Resonant interactions between planetary waves, Proc. R. Soc. Lond. A 299 (1456) (1967) 120–144.

[28] M. Gopalkrishnan, E. Miller, A. Shiu, A geometric approach to the global attractor conjecture, SIAM J. Appl. Dyn. Syst. 13 (2) (2014) 758–797.

[29] H.P. Greenspan, On the non-linear interaction of inertial modes, J. Fluid Mech. 36 (2) (1969) 257–264.

[30] J. Gunawardena, Chemical reaction network theory for in-silico biologists, Lecture notes available online at http://vcp.med.harvard.edu/papers.html, 2003.

[31] K. Hasselmann, On the non-linear energy transfer in a gravity-wave spectrum. I. General theory, J. Fluid Mech. 12 (1962) 481–500.

[32] F. Horn, Necessary and sufficient conditions for complex balancing in chemical kinetics, Arch. Ration. Mech. Anal. 49 (1972/1973) 172–186.

[33] F. Horn, The dynamics of open reaction systems, in: Mathematical Aspects of Chemical and Biochemical Problems and Quantum Chemistry, Proc. SIAM-AMS Sympos. Appl. Math., New York, 1974, in: SIAM–AMS Proceedings, vol. VIII, Amer. Math. Soc., Providence, R.I., 1974, pp. 125–137.

[34] F. Horn, R. Jackson, General mass action kinetics, Arch. Ration. Mech. Anal. 47 (1972) 81–116.

[35] H.-P. Huang, B. Galperin, S. Sukoriansky, Anisotropic spectra in two-dimensional turbulence on the surface of a sphere, Phys. Fluids 13 (2000) 225–240.

[36] S. Jin, M.-B. Tran, Quantum hydrodynamic approximations to the finite temperature trapped Bose gases, Phys. D: Nonlinear Phenom. 380 (2018) 45–57.

[37] B.B. Kadomtsev, Collective Phenomena in Plasma, vol. 2, Nauka, Moscow, 1988.
[38] Y. Lee, L.M. Smith, On the formation of geophysical and planetary zonal flows by near-resonant wave interactions, J. Fluid Mech. 576 (2007) 405–424.
[39] V.S. Lvov, Y. Lvov, A.C. Newell, V. Zakharov, Statistical description of acoustic turbulence, Phys. Rev. E 56 (1) (1997) 390.
[40] Y.V. Lvov, K.L. Polzin, N. Yokoyama, Resonant and near-resonant internal wave interactions, J. Phys. Oceanogr. 42 (5) (2012) 669–691.
[41] A.C. Newell, Rossby wave packet interactions, J. Fluid Mech. 35 (2) (1969) 255–271.
[42] A.C. Newell, B. Rumpf, Wave turbulence, Annu. Rev. Fluid Mech. 43 (2011) 59–78.
[43] T.T. Nguyen, M.-B. Tran, On the kinetic equation in Zakharov's wave turbulence theory for capillary waves, SIAM J. Math. Anal. 50 (2) (2018) 2020–2047.
[44] T.T. Nguyen, M.-B. Tran, Uniform in time lower bound for solutions to a quantum Boltzmann equation of bosons, Arch. Ration. Mech. Anal. 231 (2019) 63–89.
[45] M.-B. Tran, Y. Pomeau, Boltzmann-type collision operators for Bogoliubov excitations of Bose-Einstein condensates: a unified framework, Phys. Rev. E 101 (2020) 032119.
[46] R. Peierls, Zur kinetischen theorie der warmeleitung in kristallen, Ann. Phys. 395 (8) (1929) 1055–1101.
[47] R.E. Peierls, Quantum theory of solids, in: Theoretical Physics in the Twentieth Century (Pauli Memorial Volume), Interscience, New York, 1960, pp. 140–160.
[48] O.M. Phillips, The interaction trapping of internal gravity waves, J. Fluid Mech. 34 (2) (1968) 407–416.
[49] Y. Pomeau, M.-B. Tran, Statistical Physics of Non Equilibrium Quantum Phenomena, Lecture Notes in Physics, vol. 967, Springer Nature, 2019.
[50] L.E. Reichl, M.-B. Tran, A kinetic model for very low temperature dilute Bose gases, J. Phys. A, Math. Theor. 52 (6) (2019) 063001.
[51] L. Saint-Raymond, Kinetic models for superfluids: a review of mathematical results, C. R. Phys. 5 (1) (2004) 65–75.
[52] L.M. Smith, Numerical study of two-dimensional stratified turbulence, in: Contemporary Mathematics: Advances in Wave Interaction and Turbulence, 2001, pp. 91–106.
[53] L.M. Smith, Y. Lee, On near resonances and symmetry breaking in forced rotating flows at moderate Rossby number, J. Fluid Mech. 535 (2005) 111–142.
[54] L.M. Smith, F. Waleffe, Transfer of energy to two-dimensional large scales in forced, rotating three-dimensional turbulence, Phys. Fluids 11 (6) (1999) 1608–1622.
[55] L.M. Smith, F. Waleffe, Generation of slow large scales in forced rotating stratified turbulence, J. Fluid Mech. 451 (2002) 145–168.
[56] A. Soffer, M.-B. Tran, On coupling kinetic and Schrodinger equations, J. Differ. Equ. 265 (5) (2018) 2243–2279.
[57] A. Soffer, M.-B. Tran, On the dynamics of finite temperature trapped Bose gases, Adv. Math. 325 (2018) 533–607.
[58] A. Soffer, M.-B. Tran, On the energy cascade of 3-wave kinetic equations: beyond Kolmogorov-Zakharov solutions, Commun. Math. Phys. (19 December 2019), https://doi.org/10.1007/s00220-019-03651-w, Published Online.
[59] T. Warn, Statistical mechanical equilibria of the shallow water equations, Tellus, Ser. A Dyn. Meteorol. Oceanogr. 38 (1) (1986) 1–11.
[60] P.Y. Yu, G. Craciun, Mathematical analysis of chemical reaction systems, Isr. J. Chem. 58 (6–7) (2018) 733–741.
[61] V.E. Zakharov, Weak turbulence in media with a decay spectrum, J. Appl. Mech. Tech. Phys. 6 (4) (1965) 22–24.
[62] V.E. Zakharov, The instability of waves in nonlinear dispersive media, Sov. Phys. JETP 24 (4) (1967) 740–744.
[63] V.E. Zakharov, V.S. L'vov, G. Falkovich, Kolmogorov Spectra of Turbulence I: Wave Turbulence, Springer Science & Business Media, 2012.