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ENERGY DECAYING PHASE-FIELD MODEL FOR FLUID-PARTICLE INTERACTION IN TWO-PHASE FLOW*

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Abstract. In this paper, we study a phase-field model for the dynamics of a solid particle in two-phase flow. The governing system in our model is a coupled system of Navier–Stokes equations, Cahn–Hillard equations for the multiphase flow, and Newton's law for the motion of the particle. The effect of the wettability of the particle and the motion of the contact line are modeled by the generalized Navier boundary condition. To show that our model is physically consistent, we show that the model can be derived from the principle of minimum energy dissipation (entropy production) and has the energy decaying property. Using the method of matched asymptotic expansions, we also derive the sharp interface limit for our model.

Key words. fluid-particle interaction, two-phase flow, phase-field model, principle of minimum energy dissipation, sharp interface limit

AMS subject classifications. 35Q30, 76D05, 76T10

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1. Introduction.

The two-phase fluid-particle interaction problem has wide applications in scientific and engineering areas such as materials separation, crude oil emulsions, slurry transport, etc. There have been many works on modeling and simulation of the two-phase fluid-particle interaction problems. The numerical approach of fluid-particle systems may be classified into two types: the continuum approach and the direct numerical simulation (DNS) approach. In the continuum approach, solid particles and fluids are viewed as interpenetrating mixtures with different viscosities that are governed by conservation laws [19, 32, 38, 39]. The continuum approach is efficient and flexible. However, the false response from the viscous material used to mimic the rigid objects might produce undesirable hydrodynamic effects, thus causing potential difficulties in the continuous approach when the particle concentration is dense, or when there are particle-wall and particle-particle interactions in the problem. On the other hand, the DNS approach [15, 16, 17] takes on a fundamental approach with Navier–Stokes equations for fluids and Newton's law for particles. The DNS method gives a clear understanding of the mechanisms between fluid and particle and

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is well designed for many complicated problems involving nonlinear and geometrically complicated phenomena.

For the DNS approach to fluid-particle interaction in two-phase flow, extra difficulties arise from the discontinuities of field variables near the fluid-fluid interface. In order to overcome these problems, it is necessary to model the fluid-fluid interface while conserving the conservative quantities at a discrete level even with discontinuities. On this aspect, mathematical modeling of the two-phase flow may be classified into the sharp interface method and the diffuse interface method. In the sharp interface method, the fluid-fluid interface is of zero thickness and the variables near the interface may be discontinuous. The sharp interface method has been successfully applied to a wide range of physical problems; some of the best-known examples of the sharp interface method include the marker and cell method [13], the volume of fluid method [14], the front tracking method [12, 33], and the level set method [24]. Meanwhile, the diffuse interface method, which is also known as the phase-field method, assumes that the interface between different fluids has a finite thickness and the variables change smoothly across the interface. The earliest diffuse interface method may be traced back to van der Waals [34], which is based on the thermodynamic consideration of the free energy of a binary system, with a hypothesis that the equilibrium interface profiles can be obtained by minimizing the free energy functional. In the work of Cahn and Hilliard [6], the free energy is derived from a multivariable Taylor expansion about the free energy per molecule. The diffusive interface method has been further developed in [3, 21, 37, 28].

On the problem of two-phase flows, another difficulty comes from the dynamics near the moving contact line (MCL). The MCL is defined as the intersection of the fluid-fluid interface with the solid wall and particle surface. In order to describe the dynamics near the MCL, proper boundary conditions are required on the solid wall and the particle surface. Unlike single-phase flows where the no-slip boundary condition is widely used in application, such a no-slip condition is incompatible with the MCL in two-phase flows [8, 9, 11, 18, 22, 23]. In [26, 27] a generalized Navier boundary condition (GNBC) is introduced to model the effect of the wettability and the MCL. It is demonstrated that the GNBC can quantitatively reproduce the MCL slip velocity profiles obtained from molecular dynamics simulations. Moreover, it has been shown that a phase-field model with GNBC may be derived by the principle of minimum dissipation [27].

In this paper, we develop a phase-field model for the two-phase fluid particle interaction problem. An example of such a problem is a solid sphere falling through a water surface; see Figure 1. Our model uses the DNS approach, which consists of the Cahn–Hillard–Navier–Stokes equations for the dynamics of the two-phase fluid flow and Newton's second law for the particle motion. The effect of the wettability of the particle and the motion of the contact line are modeled by the GNBC. Unlike the previous models (e.g., [7]), the contribution of the capillary force to the particle motion is also taken into account. The model and the boundary conditions are properly set up so that they are physically consistent.

In order to describe practical problems, we consider two-phase flow with unequal density. Constructing a physically consistent phase-field model for the unequal density case is very challenging. One of the reasons is that when fluid densities are not equal, mass conservation is not a direct consequence of incompressibility any more. As a result, when describing the unequal density case, one should either start from mass conservation or start from incompressibility; none of the current models satisfy both of them. According to such choice, there are mainly two kinds of

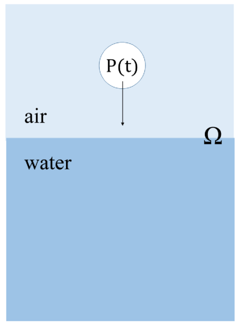

Fig. 1. *Solid sphere falling through water surface.*

approaches: the quasi-compressible approach and the incompressible approach. In the quasi-compressible approach [1, 21, 31], the fluid is assumed to be slightly compressible near the fluid-fluid interface, and as a result the governing system conserves mass. In the incompressible approach [2, 4, 20, 29, 30], fluid is still incompressible while the equation system is modified by physical properties of the problem, such as the energy law. In this paper, we use the incompressible approach since we want to verify our model by a variational point of view and derive the sharp interface limit. We show that our model can be derived variationally through the principle of minimum energy dissipation and has an energy decaying property.

For phase-field models, to capture the fluid-fluid interface, we introduce a diffuse interface with nonzero thickness. However, in practical simulation, limited by, e.g., the computation resources, the thickness of the diffuse interface usually cannot be chosen as small as the physical size. Therefore, it is important to study the influence of interface thickness in the equation system, especially the limit of the governing system when interface thickness goes to 0. Such a limit is called the sharp interface limit of the governing system. Using the method of asymptotic expansion for MCL problems [25, 35, 36], we also derive the sharp interface limit of the governing equations. We show that the leading order problem is a coupled system of the Hale–Shaw equations, the Navier–Stokes equations, and Newton's law, with the fluid-fluid interface being a free boundary, and the leading order dynamic contact angle is the same as the static contact angle in the Young's equation. Moreover, the sharp interface limit also satisfies the energy decaying property. Compared with the previous results of equal density two-phase flow, when densities of the two fluids are unequal, additional terms will appear in the form of chemical potential and the pressure jump condition on the fluid-fluid interface. These terms balance the change of kinetic energy caused by unequal densities of fluids and thus are very important to the energy decaying property of the sharp interface limit.

The paper is organized as follows. In section 2, we give the governing equations for a two-dimensional (2D) model problem. In section 3 we present a variational derivation of our model, which is based on the principle of minimum dissipation. The energy decaying property of the model is shown in section 4. In section 5 we derive the sharp interface limit of our model. Final discussions are given in section 6.

2. A phase-field model of the two-phase fluid-particle system.

2.1. Governing equations. In this section we present the phase-field governing equations for the dynamics of a particle in two-phase flow. For simplicity we consider a 2D case, while extension of our model to the 3D case is straightforward. Let Ω denote the entire computational domain, including fluid and the particle, which is time-independent. The particle is moving inside Ω. The region of the particle is denoted by $P(t)$. Ω and $P(t)$ are open sets. $\partial P(t)$ and $\partial\Omega$ stand for the boundaries of the particle and the computation domain. The particle is a rigid body and is homogeneous with equal density. Two-phase flow in this system is a mixture of two immiscible, incompressible fluids. Densities and viscosities of the two fluids are denoted by ρ_1, ρ_2 and η_1, η_2. In the phase-field model, we introduce a variable ϕ such that

$$\begin{cases} \phi = \sqrt{r/u} & \text{fluid 1,} \\ \phi = -\sqrt{r/u} & \text{fluid 2,} \end{cases}$$

with a thin transition layer near the fluid-fluid interface. Here r and u are interface thickness related parameters. In this paper we assume $r = u$. Using the phase variable ϕ, fluid density and viscosity may be described by volume average:

$$\rho(\phi) = \left(\frac{1+\phi}{2}\right)\rho_1 + \left(\frac{1-\phi}{2}\right)\rho_2, \quad \eta(\phi) = \left(\frac{1+\phi}{2}\right)\eta_1 + \left(\frac{1-\phi}{2}\right)\eta_2.$$

In this paper we may abbreviate $\rho(\phi)$ and $\eta(\phi)$ by ρ and η if there is no ambiguity.

Using the phase-field model, we may derive a phase-field model for the two-phase fluid-particle system. Governing equations for the fluid is a coupled system of Cahn–Hilliard equations and Navier–Stokes equations,

$$\begin{aligned}
&\frac{\partial\phi}{\partial t} + \mathbf{u}\cdot\nabla\phi = M\Delta\mu && \text{in} \quad \Omega \setminus \overline{P(t)}, \\
&\mu = -K\Delta\phi - r\phi + u\phi^3 + \frac{1}{2}\rho'(\phi)|\mathbf{u}|^2 && \text{in} \quad \Omega \setminus \overline{P(t)}, \\
&\rho\left(\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = \nabla\cdot(-p\mathbf{I} + \eta\boldsymbol{\sigma} - K\nabla\phi\otimes\nabla\phi) + \rho\mathbf{g} && \text{in} \quad \Omega \setminus \overline{P(t)}, \\
&\nabla\cdot\mathbf{u} = 0 && \text{in} \quad \Omega \setminus \overline{P(t)},
\end{aligned}$$

where μ is called chemical potential, $\mathbf{u}$ denotes the fluid velocity, and p stands for pressure. The term $\nabla\phi\otimes\nabla\phi$ denotes the Kronecker product of $\nabla\phi$ and its transpose $(\nabla\phi)^T$, and $\boldsymbol{\sigma}$ is defined as $\boldsymbol{\sigma} := \nabla\mathbf{u} + (\nabla\mathbf{u})^T$. M is a phenomenological mobility coefficient, and K is a material-related parameter.

By introducing the particle velocity $\mathbf{U}_s$ and particle angular velocity $\boldsymbol{\omega}_s$, and denoting $\mathbf{r}$ the vector from the particle mass center to the current position, we may define $\mathbf{u}_s := \mathbf{U}_s + \boldsymbol{\omega}_s \times \mathbf{r}$ as the pointwise velocity of the current position on the particle surface. Denote by $\mathbf{n}$ the outward normal on $\partial\Omega$ and $\partial P(t)$, in which the outward direction is w.r.t. the fluid domain $\Omega \setminus \overline{P(t)}$.

The equations of particle motion and rotation are given by Newton's law:

$$\begin{aligned}
M_s\frac{d\mathbf{U}_s}{dt} &= -\int_{\partial P(t)} (-p\mathbf{I} + \eta\boldsymbol{\sigma} - K\nabla\phi\otimes\nabla\phi)\cdot\mathbf{n}ds + M_s\mathbf{g}, \\
I_s\frac{d\boldsymbol{\omega}_s}{dt} &= -\int_{\partial P(t)} \mathbf{r}\times((-p\mathbf{I} + \eta\boldsymbol{\sigma} - K\nabla\phi\otimes\nabla\phi)\cdot\mathbf{n})ds.
\end{aligned}$$

Here M_s stands for the mass and I_s stands for the inertia tensor of the particle. For a particle with density ρ_s, M_s and I_s can be written as

$$M_s = \rho_s \int_{P(t)} dx, \quad I_s = \rho_s \int_{P(t)} [(\mathbf{r}\cdot\mathbf{r})\mathbf{I} - \mathbf{r}\otimes\mathbf{r}]dx.$$

On the particle surface $\partial P(t)$, we apply the GNBC to describe the dynamics of the MCL. Let $\boldsymbol{\tau}$ denote the tangent direction on $\partial\Omega$ and $\partial P(t)$, define $u_\tau = \mathbf{u}\cdot\boldsymbol{\tau}$ as the tangent component of fluid velocity field, and define $u_\tau^{slip} := (\mathbf{u}-\mathbf{u}_s)\cdot\boldsymbol{\tau}$ as the slip velocity of fluid on the particle surface. On particle surface $\partial P(t)$ the GNBC in the governing system is given by

$$\begin{aligned}
&\beta u_\tau^{slip} = -\eta(\boldsymbol{\sigma}\cdot\mathbf{n})\cdot\boldsymbol{\tau} + L(\phi)\frac{\partial\phi}{\partial\tau} && \text{on} \quad \partial P(t),\\
&\frac{\partial\phi}{\partial t} + \nabla\phi\cdot\mathbf{u} = -\lambda L(\phi) && \text{on} \quad \partial P(t),\\
&L(\phi) = K\frac{\partial\phi}{\partial n} + \frac{\partial\gamma(\phi)}{\partial\phi} && \text{on} \quad \partial P(t),\\
&(\mathbf{u}-\mathbf{u}_s)\cdot\boldsymbol{n} = 0 && \text{on} \quad \partial P(t),\\
&\frac{\partial\mu}{\partial n} = 0 && \text{on} \quad \partial P(t).
\end{aligned}$$

Denote $u_{\tau,w}^{slip} := (\mathbf{u}-\mathbf{u}_w)\cdot\boldsymbol{\tau}$ the slip velocity of fluid on $\partial\Omega$, where $\mathbf{u}_w$ is the velocity of the solid wall $\partial\Omega$. GNBC on $\partial\Omega$ is given by

$$\begin{aligned}
&\beta u_{\tau,w}^{slip} = -\eta(\boldsymbol{\sigma}\cdot\mathbf{n})\cdot\boldsymbol{\tau} + L(\phi)\frac{\partial\phi}{\partial\tau} && \text{on} \quad \partial\Omega,\\
&\frac{\partial\phi}{\partial t} + \nabla\phi\cdot\mathbf{u} = -\lambda L(\phi) && \text{on} \quad \partial\Omega,\\
&L(\phi) = K\frac{\partial\phi}{\partial n} + \frac{\partial\gamma}{\partial\phi} && \text{on} \quad \partial\Omega,\\
&\mathbf{u}\cdot\mathbf{n} = 0 && \text{on} \quad \partial\Omega,\\
&\frac{\partial\mu}{\partial n} = 0 && \text{on} \quad \partial\Omega.
\end{aligned}$$

In GNBC the interfacial tension $\gamma(\phi)$ is defined as $\gamma(\phi) := -\frac{1}{2}\gamma_{12}\cos\theta\sin(\frac{\pi}{2}\phi)$. γ_{12} is defined as $\gamma_{12} := \frac{2\sqrt{2}}{3}\frac{r^2\xi}{u}$. θ is the static contact angle. $\xi := \sqrt{K/r}$ denotes the interfacial thickness. $L(\phi)$ represents the uncompensated Young stress. $\beta(\phi) := \frac{1+\phi}{2}\beta_1 + \frac{1-\phi}{2}\beta_2$ is the slip coefficient, and λ is a positive phenomenological parameter.

2.2. Dimensionless form. In numerical simulation, it is convenient to introduce a dimensionless form of the governing equations for the two-phase fluid-particle system. We scale length by a characteristic length L_0, velocity by a characteristic velocity V_0, angular velocity by V_0/L_0, time by L_0/V_0, density by ρ_1, pressure by $\eta_1 V_0/L_0$, and external body force density by V_0^2/L_0. Then, we may derive a dimensionless form of the governing equations in our model.

In the dimensionless form, since density ρ is scaled by ρ_1, and viscosity η is scaled by η_1, $\rho(\phi)$ and $\eta(\phi)$ are defined as

$$\rho(\phi) := \frac{1+\phi}{2} + \frac{1-\phi}{2}\lambda_\rho, \quad \eta(\phi) := \frac{1+\phi}{2} + \frac{1-\phi}{2}\lambda_\eta, \tag{1}$$

where $\lambda_\rho := \rho_2/\rho_1$ and $\lambda_\eta := \eta_2/\eta_1$ stand for density and viscosity ratios. Governing equations of the fluid read

$$\frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi = \mathcal{L}_d \Delta \mu \qquad \text{in } \Omega \setminus \overline{P(t)}, \tag{2}$$

$$\mu = -\epsilon \Delta \phi + \frac{1}{\epsilon}(\phi^3 - \phi) + \frac{1}{2}\mathcal{B}^{-1}\mathcal{R}e\rho'(\phi)|\mathbf{u}|^2 \qquad \text{in } \Omega \setminus \overline{P(t)}, \tag{3}$$

$$\begin{aligned} &\mathcal{R}e\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) \\ &\qquad = \nabla \cdot (-p\mathbf{I} + \eta\boldsymbol{\sigma} - \mathcal{B}\epsilon\nabla\phi \otimes \nabla\phi) + \mathcal{R}e\rho\mathbf{g} \qquad \text{in } \Omega \setminus \overline{P(t)}, \end{aligned} \tag{4}$$

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{in } \Omega \setminus \overline{P(t)}. \tag{5}$$

Since particle mass M_s and inertia I_s are also scaled by ρ_1, which is

$$M_s = (\rho_s/\rho_1)\int_{P(t)} dx, \quad I_s = (\rho_s/\rho_1)\int_{P(t)} [(\mathbf{r}\cdot\mathbf{r})\mathbf{I} - \mathbf{r}\otimes\mathbf{r}]dx,$$

the dimensionless equations of particle motion and rotation are

$$\mathcal{R}eM_s\frac{\mathrm{d}\mathbf{U}_s}{\mathrm{d}t} = -\int_{\partial P(t)} (-p\mathbf{I} + \eta\boldsymbol{\sigma} - \mathcal{B}\epsilon\nabla\phi\otimes\nabla\phi)\cdot\boldsymbol{n}ds + \mathcal{R}eM_s\mathbf{g}, \tag{6}$$

$$\mathcal{R}eI_s\frac{\mathrm{d}\boldsymbol{\omega}_s}{\mathrm{d}t} = -\int_{\partial P(t)} \mathbf{r}\times((-p\mathbf{I} + \eta\boldsymbol{\sigma} - \mathcal{B}\epsilon\nabla\phi\otimes\nabla\phi)\cdot\boldsymbol{n})ds. \tag{7}$$

In GNBC, $\gamma(\phi)$ is defined as $\gamma(\phi) := -\frac{\sqrt{2}}{3}\cos\theta\sin(\frac{\pi}{2}\phi)$. Introduce slip length l_s by $l_{s1} := \eta_1/\beta_1,\ l_{s2} := \eta_2/\beta_2$, then slip length $l_s(\phi) := \frac{1+\phi}{2} + \frac{1-\phi}{2}\lambda_{l_s}$, where $\lambda_{l_s} = l_{s2}/l_{s1}$. We also abbreviate $l_s(\phi)$ by l_s if there is no ambiguity. The dimensionless boundary conditions on $\partial P(t)$ are

$$\frac{\eta}{\mathcal{L}_s l_s}u_\tau^{slip} = -\eta(\boldsymbol{\sigma}\cdot\mathbf{n})\cdot\boldsymbol{\tau} + \mathcal{B}L(\phi)\frac{\partial\phi}{\partial\tau} \qquad \text{on } \partial P(t), \tag{8}$$

$$\frac{\partial\phi}{\partial t} + \nabla\phi\cdot\mathbf{u} = -\mathcal{V}_s L(\phi) \qquad \text{on } \partial P(t), \tag{9}$$

$$L(\phi) = \epsilon\frac{\partial\phi}{\partial n} + \frac{\partial\gamma(\phi)}{\partial\phi} \qquad \text{on } \partial P(t), \tag{10}$$

$$(\mathbf{u} - \mathbf{u}_s)\cdot\boldsymbol{n} = 0 \qquad \text{on } \partial P(t), \tag{11}$$

$$\frac{\partial\mu}{\partial n} = 0 \qquad \text{on } \partial P(t), \tag{12}$$

while the dimensionless boundary conditions on $\partial\Omega$ are

$$\frac{\eta}{\mathcal{L}_s l_s}u_{\tau,w}^{slip} = -\eta(\boldsymbol{\sigma}\cdot\mathbf{n})\cdot\boldsymbol{\tau} + \mathcal{B}L(\phi)\frac{\partial\phi}{\partial\tau} \qquad \text{on } \partial\Omega, \tag{13}$$

$$\frac{\partial\phi}{\partial t} + \nabla\phi\cdot\mathbf{u} = -\mathcal{V}_s L(\phi) \qquad \text{on } \partial\Omega, \tag{14}$$

$$L(\phi) = \epsilon\frac{\partial\phi}{\partial n} + \frac{\partial\gamma(\phi)}{\partial\phi} \qquad \text{on } \partial\Omega, \tag{15}$$

$$\mathbf{u}\cdot\mathbf{n} = 0 \qquad \text{on } \partial\Omega, \tag{16}$$

$$\frac{\partial\mu}{\partial n} = 0 \qquad \text{on } \partial\Omega. \tag{17}$$

Definitions and meanings of other dimensionless parameters are listed as follows:

$$\begin{cases} \epsilon := \xi/L_0 & \text{Cahn number,} \\ \mathcal{L}_d := 3\gamma_{12}M/2\sqrt{2}V_0L_0^2 & \text{Diffusion coefficient,} \\ \mathcal{R}e := \rho_1 V_0 L_0/\eta_1 & \text{Reynolds number,} \\ \mathcal{B} := 3\gamma_{12}/2\sqrt{2}\eta_1 V_0 & \text{Inverse capillary number,} \\ \mathcal{L}_s := \eta_1/\beta_1 L_0 & \text{Slip length,} \\ \mathcal{V}_s := 3\gamma_{12}\lambda L_0/2\sqrt{2}V_0 & \text{Mobility coefficient.} \end{cases}$$

For the dimensionless model, we may define the total energy F of the governing system:

$$\begin{aligned} F &:= \mathcal{R}eF_k + \mathcal{B}F_b + \mathcal{R}eF_{pm} + \mathcal{R}eF_{pr} + \mathcal{B}F_{\partial P(t)} + \mathcal{B}F_{\partial\Omega}, \\ F_k &:= \int_{\Omega\setminus P(t)} \frac{1}{2}\rho|\mathbf{u}|^2 dx, \quad F_{pm} := \frac{1}{2}M_s|\mathbf{U}_s|^2, \quad F_{pr} := \frac{1}{2}I_s|\boldsymbol{\omega}_s|^2, \\ F_b &:= \int_{\Omega\setminus P(t)} f_b dx, \qquad F_{\partial P(t)} := \int_{\partial P(t)} \gamma ds, \quad F_{\partial\Omega} := \int_{\partial\Omega} \gamma ds, \end{aligned} \tag{18}$$

where $f_b := \frac{\epsilon}{2}|\nabla\phi|^2 + \frac{1}{4\epsilon}(\phi^2-1)^2$.

Remark 2.1. In the original form and the dimensionless form of the model, we use the same notation, such as $\mathbf{u}, \phi, M_s, \ldots$. This is because we would like to avoid introducing too many different notational symbols in the paper. In the rest of this paper, definitions of such symbols always follow the definitions in dimensionless form.

3. Variational derivation of the governing equations for the two-phase flow. In this section we show that our model may be derived by the principle of minimum energy dissipation. First of all, we should restrict the variables in the governing system, such that they fit some basic physical properties of the system. First, since the two-phase fluid is incompressible and impermeable, we have

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega\setminus\overline{P(t)}, \tag{19}$$

$$(\mathbf{u}-\mathbf{u}_s)\cdot\mathbf{n} = 0 \quad \text{on } \partial P(t), \tag{20}$$

$$\mathbf{u}\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega. \tag{21}$$

Moreover, defining material derivative $\frac{D}{Dt}$ as

$$\frac{Df}{Dt} := \frac{\partial f}{\partial t} + \nabla f\cdot\mathbf{u},$$

we may define the diffusive current $\mathbf{J}$, such that $\frac{D\phi}{Dt} = -\nabla\cdot\mathbf{J}$. We require that $\mathbf{J}$ satisfies the following boundary condition:

$$\mathbf{J}\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega\cup\partial P(t). \tag{22}$$

Boundary conditions (20)–(22) are called the impermeability boundary conditions. In the derivation by variation, we assume that (19)–(22) hold. We also assume that the wall velocity $\mathbf{u}_w$ is $\mathbf{0}$.

For incompressible two-phase flows, the governing model system may be derived from a minimum dissipation theorem [27] by minimizing the functional $(\Phi + \frac{d}{dt}F)$ for prescribed ϕ, where Φ is the dissipation function. For the fluid-particle interaction problem, the dissipation function Φ may be defined by $\Phi := R_1 + R_b$, where R_1 stands for the dissipation caused by fluid motion, and R_b denotes the dissipation due to the displacement from the two-phase equilibrium. The two dissipation terms R_1 and R_b may be given by

$$R_1 := \frac{1}{4}\int_{\Omega\setminus P(t)} \eta|\boldsymbol{\sigma}|_F^2 dx + \frac{1}{2}\int_{\partial P(t)} \frac{\eta}{\mathcal{L}_s l_s}|u_\tau^{slip}|^2 ds + \frac{1}{2}\int_{\partial\Omega} \frac{\eta}{\mathcal{L}_s l_s}|u_\tau^{slip}|^2 ds,$$

$$R_b := \frac{1}{2}\mathcal{B}L_d^{-1}\int_{\Omega\setminus P(t)} |\mathbf{J}|^2 dx + \frac{1}{2}\mathcal{B}\mathcal{V}_s^{-1}\int_{\partial P(t)} \left|\frac{D\phi}{Dt}\right|^2 ds + \frac{1}{2}\mathcal{B}\mathcal{V}_s^{-1}\int_{\partial\Omega} \left|\frac{D\phi}{Dt}\right|^2 ds,$$

where $|\boldsymbol{\sigma}|_F$ is the Fronbenius norm of $\boldsymbol{\sigma}$.

Given the definition of $(\Phi + \frac{d}{dt}F)$, according to the principle of minimum energy dissipation (see [27, Appendix A]), for prescribed phase variable ϕ, we derive the governing equations by minimizing $(\Phi + \frac{\partial}{\partial t}F)$ w.r.t. perturbations to velocity field $\mathbf{u} \to \mathbf{u} + \delta\mathbf{u}$, particle velocity $\mathbf{U}_s \to \mathbf{U}_s + \delta\mathbf{U}_s$, particle angular velocity $\boldsymbol{\omega}_s \to \boldsymbol{\omega}_s + \delta\boldsymbol{\omega}_s$, diffusive current $\mathbf{J} \to \mathbf{J} + \delta\mathbf{J}$, and $\frac{D\phi}{Dt} \to \frac{D\phi}{Dt} + \delta\frac{D\phi}{Dt}$. Note that for the original velocity field, we have the incompressibility condition (19) and impermeability boundary condition (20)–(22), and the perturbed velocity field also needs to satisfy the same conditions. Therefore, we require that the perturbation $\delta\mathbf{J}$, $\delta\mathbf{u}$, $\delta\mathbf{U}_s$, and $\boldsymbol{\omega}_s$ satisfy

$$\nabla\cdot\delta\mathbf{u} = 0 \quad \text{in } \Omega\setminus\overline{P(t)}, \tag{23}$$

$$(\delta\mathbf{u} - \delta\mathbf{u}_s)\cdot\mathbf{n} = 0 \quad \text{on } \partial P(t), \tag{24}$$

$$\delta\mathbf{u}\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega, \tag{25}$$

$$\delta\mathbf{J}\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega\cup\partial P(t). \tag{26}$$

Here $\delta\mathbf{u}_s := \delta\mathbf{U}_s + \delta\boldsymbol{\omega}_s\times\mathbf{r}$. Moreover, similar to the discussion in section 3 of [20], we choose perturbations that satisfy

$$\frac{D(\delta\mathbf{u})}{dt} = 0 \;\text{ in } \Omega\setminus\overline{P(t)}, \quad \frac{d(\delta\mathbf{U}_s)}{dt} = 0, \; \frac{d(\delta\boldsymbol{\omega}_s)}{dt} = 0. \tag{27}$$

It is easy to see that such additional constraints do not affect the value at current time; thus (19)–(22) are still well-defined.

For an arbitrary functional $G(\phi, \frac{D\phi}{Dt}, \mathbf{u}, \mathbf{U}_s, \boldsymbol{\omega}_s, \mathbf{J})$, we introduce an operator δ to denote the variation for prescribed ϕ, while the perturbation of variables is subject to constraints (23)–(27). More precisely, we define δG by

$$\delta G := \left[\delta G/\left(\delta\frac{D\phi}{Dt}\right)\right]\delta\frac{D\phi}{Dt} + \frac{\delta G}{\delta\mathbf{u}}\cdot\delta\mathbf{u} + \frac{\delta G}{\delta\mathbf{U}_s}\cdot\delta\mathbf{U}_s + \frac{\delta G}{\delta\boldsymbol{\omega}_s}\cdot\delta\boldsymbol{\omega}_s + \frac{\delta G}{\delta\mathbf{J}}\cdot\delta\mathbf{J},$$

where the variations above (e.g., $\delta G/(\delta\frac{D\phi}{Dt})$) are taken by viewing ϕ as given data. We may present the theorem on variational derivation.

THEOREM 3.1. *Given incompressibility condition* (19) *and impermeability boundary conditions* (20)–(22)*, governing equations* (2)–(17) *may be derived by minimizing the functional* $\Phi + \frac{d}{dt}F$ *w.r.t. velocity field* $\mathbf{u} \to \mathbf{u} + \delta\mathbf{u}$*, particle velocity* $\mathbf{U}_s \to \mathbf{U}_s + \delta\mathbf{U}_s$*, particle angular velocity* $\boldsymbol{\omega}_s \to \boldsymbol{\omega}_s + \delta\boldsymbol{\omega}_s$*, diffusive current* $\mathbf{J} \to \mathbf{J} + \delta\mathbf{J}$*, and* $\frac{D\phi}{Dt} \to \frac{D\phi}{Dt} + \delta\frac{D\phi}{Dt}$ *for prescribed variable* ϕ*, where the perturbations satisfy* (23)–(27)*.*

Proof. We start from calculating the time derivative of energy F. Note that the integration area $\Omega \setminus P(t)$ changes over the motion of $P(t)$; it follows from (19)–(20)–(21) that

$$\begin{aligned}\frac{d}{dt}F_b &= \frac{d}{dt}\int_{\Omega\setminus P(t)} f_b dx = \int_{\Omega\setminus P(t)} \frac{\partial f_b}{\partial t}dx + \int_{\partial P(t)} f_b(\mathbf{u}_s\cdot\mathbf{n})ds \\ &= \int_{\Omega\setminus P(t)} \frac{\partial f_b}{\partial t}dx + \int_{\partial P(t)} f_b(\mathbf{u}\cdot\mathbf{n})ds + \int_{\partial\Omega} f_b(\mathbf{u}\cdot\mathbf{n})ds \\ &= \int_{\Omega\setminus P(t)} \frac{\partial f_b}{\partial t} + \nabla f_b\cdot\mathbf{u} + f_b(\nabla\cdot\mathbf{u})dx = \int_{\Omega\setminus P(t)} \frac{D}{Dt}f_b dx. \end{aligned} \tag{28}$$

According to (3), define μ as

$$\mu := -\epsilon\Delta\phi + \frac{1}{\epsilon}(\phi^3-\phi) + \frac{1}{2}\mathcal{B}^{-1}\mathcal{R}e\rho'(\phi)|\mathbf{u}|^2.$$

Since $f_b := \frac{\epsilon}{2}|\nabla\phi|^2 + \frac{1}{4\epsilon}(\phi^2-1)^2$, we have

$$\begin{aligned}&\int_{\Omega\setminus P(t)} \frac{\partial f_b}{\partial t}dx = \int_{\Omega\setminus P(t)} \left(\epsilon\nabla\phi\nabla\frac{\partial\phi}{\partial t} + \frac{1}{\epsilon}(\phi^3-\phi)\frac{\partial\phi}{\partial t}\right)dx \\ &= \int_{\Omega\setminus P(t)} \left(-\epsilon\Delta\phi + \frac{1}{\epsilon}(\phi^3-\phi)\right)\frac{\partial\phi}{\partial t}dx + \epsilon\int_{\partial P(t)} \frac{\partial\phi}{\partial n}\frac{\partial\phi}{\partial t}ds + \epsilon\int_{\partial\Omega}\frac{\partial\phi}{\partial n}\frac{\partial\phi}{\partial t}ds \\ &= \int_{\Omega\setminus P(t)} \left(\mu - \frac{1}{2}\mathcal{B}^{-1}\mathcal{R}e\rho'(\phi)|\mathbf{u}|^2\right)\frac{\partial\phi}{\partial t}dx + \epsilon\int_{\partial P(t)} \frac{\partial\phi}{\partial n}\frac{\partial\phi}{\partial t}ds + \epsilon\int_{\partial\Omega}\frac{\partial\phi}{\partial n}\frac{\partial\phi}{\partial t}ds. \end{aligned} \tag{29}$$

Using the well-known identity (cf., e.g., [10])

$$\nabla\cdot(\nabla\phi\otimes\nabla\phi) - \frac{1}{2}\nabla(|\nabla\phi|^2) = \Delta\phi\nabla\phi,$$

we have

$$\nabla f_b = \left(\mu - \frac{1}{2}\mathcal{B}^{-1}\mathcal{R}e\rho'(\phi)|\mathbf{u}|^2\right)\nabla\phi + \epsilon\nabla\cdot(\nabla\phi\otimes\nabla\phi).$$

Using the above equation and Green's formula, we may derive that

$$\begin{aligned}\int_{\Omega\setminus P(t)} \nabla f_b\cdot\mathbf{u}dx &= \int_{\Omega\setminus P(t)} (\mu - \frac{1}{2}\mathcal{B}^{-1}\mathcal{R}e\rho'(\phi)|\mathbf{u}|^2)(\nabla\phi\cdot\mathbf{u})dx \\ &\quad + \epsilon\int_{\Omega\setminus P(t)} (\nabla\cdot(\nabla\phi\otimes\nabla\phi))\cdot\mathbf{u}dx.\end{aligned} \tag{30}$$

Here $\mathbf{A}:\mathbf{B} = \sum_{i,j} a_{ij}b_{ij}$ for two matrices $\mathbf{A} = \{a_{ij}\}$ and $\mathbf{B} = \{b_{ij}\}$.

Plugging (30) and (29) into (28), and multiplying $\mathcal{B}$ to both sides of the equation, we may derive that

$$\begin{aligned}\mathcal{B}\frac{d}{dt}F_b &= \mathcal{B}\int_{\Omega\setminus P(t)} \left(\mu - \frac{1}{2}\mathcal{B}^{-1}\mathcal{R}e\rho'(\phi)|\mathbf{u}|^2\right)\frac{D\phi}{Dt}dx \\ &\quad + \mathcal{B}\epsilon\int_{\Omega\setminus P(t)} (\nabla\cdot(\nabla\phi\otimes\nabla\phi))\cdot\mathbf{u}dx + \mathcal{B}\epsilon\int_{\partial P(t)} \frac{\partial\phi}{\partial n}\frac{\partial\phi}{\partial t}ds + \mathcal{B}\epsilon\int_{\partial\Omega}\frac{\partial\phi}{\partial n}\frac{\partial\phi}{\partial t}ds.\end{aligned} \tag{31}$$

By the impermeability boundary condition (22), we have

$$\int_{\Omega\setminus P(t)} \mu \frac{D\phi}{Dt} dx = -\int_{\Omega\setminus P(t)} \mu \nabla\cdot \mathbf{J} dx = \int_{\Omega\setminus P(t)} \nabla\mu\cdot\mathbf{J} dx. \tag{32}$$

Combining (31) and (32), we have

$$\begin{aligned}\mathcal{B}\frac{d}{dt}F_b = {}& \mathcal{B}\int_{\Omega\setminus P(t)} \nabla\mu\cdot\mathbf{J}dx + \mathcal{B}\epsilon\int_{\Omega\setminus P(t)} (\nabla\cdot(\nabla\phi\otimes\nabla\phi))\cdot\mathbf{u}dx \\ &+ \mathcal{B}\epsilon\int_{\partial P(t)} \frac{\partial\phi}{\partial n}\frac{\partial\phi}{\partial t}ds + \mathcal{B}\epsilon\int_{\partial\Omega}\frac{\partial\phi}{\partial n}\frac{\partial\phi}{\partial t}ds - \frac{1}{2}\mathcal{R}e\int_{\Omega\setminus P(t)} \rho'(\phi)\frac{D\phi}{Dt}|\mathbf{u}|^2dx.\end{aligned} \tag{33}$$

On the time derivative of the kinetic energy F_k, similar to the derivation in (28),

$$\begin{aligned}\mathcal{R}e\frac{d}{dt}F_k &= \mathcal{R}e\frac{d}{dt}\int_{\Omega\setminus P(t)}\frac{1}{2}\rho(\phi)|\mathbf{u}|^2dx = \mathcal{R}e\int_{\Omega\setminus P(t)}\frac{1}{2}\frac{D(\rho(\phi)|\mathbf{u}|^2)}{Dt}dx. \\ &= \mathcal{R}e\int_{\Omega\setminus P(t)}\rho(\phi)\frac{D\mathbf{u}}{Dt}\cdot\mathbf{u}dx + \frac{1}{2}\mathcal{R}e\int_{\Omega\setminus P(t)}\rho'(\phi)\frac{D\phi}{Dt}|\mathbf{u}|^2dx.\end{aligned} \tag{34}$$

Moreover, it is straightforward that

$$\mathcal{R}e\frac{d}{dt}F_{pm} = \mathcal{R}eM_s\frac{d\mathbf{U}_s}{dt}\cdot\mathbf{U}_s, \quad \mathcal{R}e\frac{d}{dt}F_{pr} = \mathcal{R}eI_s\frac{d\boldsymbol{\omega}_s}{dt}\cdot\boldsymbol{\omega}_s dt. \tag{35}$$

According to GNBC (10) and (15), we define $L(\phi)$ as $L(\phi) := \epsilon\frac{\partial\phi}{\partial n} + \frac{\partial\gamma(\phi)}{\partial\phi}$. Using impermeability boundary condition (20), we have

$$\mathbf{u} - \mathbf{u}_s = ((\mathbf{u}-\mathbf{u}_s)\cdot\boldsymbol{\tau})\boldsymbol{\tau} + ((\mathbf{u}-\mathbf{u}_s)\cdot\mathbf{n})\mathbf{n} = u_\tau^{slip}\boldsymbol{\tau} \quad \text{on} \quad \partial P(t).$$

Thus we have

$$\begin{aligned}\mathcal{B}\frac{d}{dt}F_{\partial P(t)} &= \mathcal{B}\int_{\partial P(t)}\frac{\partial\gamma(\phi)}{\partial\phi}\left(\frac{\partial\phi}{\partial t}+\nabla\phi\cdot\mathbf{u}_s\right)ds \\ &= \mathcal{B}\int_{\partial P(t)}\left(L(\phi)-\epsilon\frac{\partial\phi}{\partial n}\right)\left(\frac{\partial\phi}{\partial t}+\nabla\phi\cdot\mathbf{u}_s\right)ds \\ &= \mathcal{B}\int_{\partial P(t)}L(\phi)\left(\frac{D\phi}{Dt}-\frac{\partial\phi}{\partial\tau}u_\tau^{slip}\right)ds - \mathcal{B}\epsilon\int_{\partial P(t)}\frac{\partial\phi}{\partial n}\left(\frac{\partial\phi}{\partial t}+\nabla\phi\cdot\mathbf{u}_s\right)ds.\end{aligned} \tag{36}$$

Similarly, using impermeability boundary condition (21), we have

$$\begin{aligned}\mathcal{B}\frac{d}{dt}F_{\partial\Omega} &= \mathcal{B}\int_{\partial\Omega}\frac{\partial\gamma(\phi)}{\partial\phi}\frac{\partial\phi}{\partial t} = \mathcal{B}\int_{\partial\Omega}\left(L(\phi)-\epsilon\frac{\partial\phi}{\partial n}\right)\frac{\partial\phi}{\partial t}ds \\ &= \mathcal{B}\int_{\partial\Omega}L(\phi)\left(\frac{D\phi}{Dt}-\frac{\partial\phi}{\partial\tau}u_\tau\right)ds - \mathcal{B}\epsilon\int_{\partial\Omega}\frac{\partial\phi}{\partial n}\frac{\partial\phi}{\partial t}ds.\end{aligned} \tag{37}$$

Summing up (33), (34), (35), (36), and (37), we come to

$$\begin{aligned}\frac{d}{dt}F = {}&\mathcal{B}\int_{\Omega\setminus P(t)}\nabla\mu\cdot\mathbf{J}dx + \mathcal{B}\epsilon\int_{\Omega\setminus P(t)}(\nabla\cdot(\nabla\phi\otimes\nabla\phi))\cdot\mathbf{u}dx \\ &\mathcal{B}\int_{\partial P(t)}L(\phi)\left(\frac{D\phi}{Dt}-\frac{\partial\phi}{\partial\tau}u_\tau^{slip}\right)ds - \mathcal{B}\epsilon\int_{\partial P(t)}\frac{\partial\phi}{\partial n}\nabla\phi\cdot\mathbf{u}_s ds\end{aligned} \tag{38}$$

$$+\mathcal{B}\int_{\partial\Omega} L(\phi)\left(\frac{D\phi}{Dt}-\frac{\partial\phi}{\partial\tau}u_\tau\right)ds+\mathcal{R}e\int_{\Omega\setminus P(t)}\rho(\phi)\frac{D\mathbf{u}}{Dt}\cdot\mathbf{u}dx$$
$$+\mathcal{R}e\frac{d\mathbf{U}_s}{dt}\cdot\mathbf{U}_s+\mathcal{R}e\frac{d\boldsymbol{\omega}_s}{dt}\cdot\boldsymbol{\omega}_s.$$

On the variation of Φ, it is straightforward to see that

$$\delta\left(\int_{\Omega\setminus P(t)}\frac{\eta}{4}|\boldsymbol{\sigma}|_F^2dx\right)=-\int_{\Omega\setminus P(t)}\delta\mathbf{u}\cdot(\eta\nabla\cdot\boldsymbol{\sigma})dx+\int_{\partial P(t)}\delta\mathbf{u}\cdot(\eta\boldsymbol{\sigma}\cdot\mathbf{n})ds+\int_{\partial\Omega}\delta\mathbf{u}\cdot(\eta\boldsymbol{\sigma}\cdot\mathbf{n})ds. \tag{39}$$

Moreover, define

$$\delta u_\tau^{slip}:=(\delta\mathbf{u}-(\delta\mathbf{U}_s+\delta\boldsymbol{\omega}_s\times\mathbf{r}))\cdot\boldsymbol{\tau}\text{ on }\partial P(t),\quad \delta u_\tau:=\delta\mathbf{u}\cdot\boldsymbol{\tau}\text{ on }\partial\Omega;$$

using (24)–(25), we have

$$\begin{aligned}\delta\left(\frac{1}{4}\int_{\Omega\setminus P(t)}|\boldsymbol{\sigma}|_F^2dx\right)=&-\int_{\Omega\setminus P(t)}\delta\mathbf{u}\cdot(\eta\nabla\cdot\boldsymbol{\sigma})dx+\delta\mathbf{U}_s\cdot\int_{\partial P(t)}\eta\boldsymbol{\sigma}\cdot\mathbf{n}ds\\&+\delta\boldsymbol{\omega}_s\cdot\int_{\partial P(t)}\mathbf{r}\times\eta(\boldsymbol{\sigma}\cdot\mathbf{n})ds+\int_{\partial P(t)}\delta u_\tau^{slip}\eta(\boldsymbol{\sigma}\cdot\mathbf{n})\cdot\boldsymbol{\tau}ds\\&+\int_{\partial\Omega}\delta u_\tau\eta(\boldsymbol{\sigma}\cdot\mathbf{n})\cdot\boldsymbol{\tau}ds.\end{aligned}\tag{40}$$

Recall the definition of Φ that

$$\begin{aligned}\Phi=&\frac{1}{4}\int_{\Omega\setminus P(t)}\eta|\boldsymbol{\sigma}|_F^2dx+\frac{1}{2}\int_{\partial P(t)}\frac{\eta}{\mathcal{L}_sl_s}|u_\tau^{slip}|^2ds+\frac{1}{2}\int_{\partial\Omega}\frac{\eta}{\mathcal{L}_sl_s}|u_\tau^{slip}|^2ds\\&+\frac{1}{2}\mathcal{B}L_d^{-1}\int_{\Omega\setminus P(t)}|\mathbf{J}|^2dx+\frac{1}{2}\mathcal{B}\mathcal{V}_s^{-1}\int_{\partial P(t)}\left|\frac{D\phi}{Dt}\right|^2ds+\frac{1}{2}\mathcal{B}\mathcal{V}_s^{-1}\int_{\partial\Omega}\left|\frac{D\phi}{Dt}\right|^2ds,\end{aligned}$$

we may derive the variation of Φ:

$$\begin{aligned}\delta\Phi=&-\int_{\Omega\setminus P(t)}\delta\mathbf{u}\cdot(\eta\nabla\cdot\boldsymbol{\sigma})dx+\delta\mathbf{U}_s\cdot\int_{\partial P(t)}\eta\boldsymbol{\sigma}\cdot\mathbf{n}ds\\&+\delta\boldsymbol{\omega}_s\cdot\int_{\partial P(t)}\mathbf{r}\times\eta(\boldsymbol{\sigma}\cdot\mathbf{n})ds+\int_{\partial P(t)}\delta u_\tau^{slip}\eta(\boldsymbol{\sigma}\cdot\mathbf{n})\cdot\boldsymbol{\tau}ds\\&+\int_{\partial\Omega}\delta u_\tau\eta(\boldsymbol{\sigma}\cdot\mathbf{n})\cdot\boldsymbol{\tau}ds+\mathcal{B}L_d^{-1}\int_{\Omega\setminus P(t)}\delta\mathbf{J}\cdot\mathbf{J}dx\\&+\int_{\partial P(t)}\delta u_\tau^{slip}\frac{\eta}{\mathcal{L}_sl_s}u_\tau^{slip}ds+\mathcal{B}\mathcal{V}_s^{-1}\int_{\partial P(t)}\delta\left(\frac{D\phi}{Dt}\right)\frac{D\phi}{Dt}ds\\&+\int_{\partial\Omega}\delta u_\tau\frac{\eta}{\mathcal{L}_sl_s}u_\tau ds+\mathcal{B}\mathcal{V}_s^{-1}\int_{\partial\Omega}\delta\left(\frac{D\phi}{Dt}\right)\frac{D\phi}{Dt}ds.\end{aligned}\tag{41}$$

For the variation of $\frac{d}{dt}F$, using (23)–(27), we have

$$\begin{aligned}\delta\left(\frac{d}{dt}F\right)=&\mathcal{B}\int_{\Omega\setminus P(t)}\delta\mathbf{J}\cdot\nabla\mu dx+\mathcal{B}\epsilon\int_{\Omega\setminus P(t)}\delta\mathbf{u}\cdot(\nabla\cdot(\nabla\phi\otimes\nabla\phi))dx\\&+\mathcal{B}\int_{\partial P(t)}\delta\left(\frac{D\phi}{Dt}\right)L(\phi)ds-\mathcal{B}\int_{\partial P(t)}\delta u_\tau^{slip}L(\phi)\frac{\partial\phi}{\partial\tau}ds\end{aligned}\tag{42}$$

$$
\begin{aligned}
&- \mathcal{B}\epsilon\delta\mathbf{U}_s \cdot \int_{\partial P(t)} \frac{\partial\phi}{\partial n}\nabla\phi ds - \mathcal{B}\epsilon\delta\boldsymbol{\omega}_s \cdot \int_{\partial P(t)} \mathbf{r} \times \left(\frac{\partial\phi}{\partial n}\nabla\phi\right) ds \\
&+ \mathcal{B}\int_{\partial\Omega} \delta\left(\frac{D\phi}{Dt}\right) L(\phi) ds - \mathcal{B}\int_{\partial\Omega} \delta u_\tau L(\phi)\frac{\partial\phi}{\partial\tau} ds \\
&+ \mathcal{R}e\int_{\Omega\setminus P(t)} \delta\mathbf{u}\cdot\rho\frac{D\mathbf{u}}{Dt}dx + \mathcal{R}e\delta\mathbf{U}_s\cdot M_s\frac{d\mathbf{U}_s}{dt} + \mathcal{R}e\delta\boldsymbol{\omega}_s\cdot I_s\frac{d\boldsymbol{\omega}_s}{dt}.
\end{aligned}
$$

Finally, using the incompressibility of the perturbated velocity field $\delta\mathbf{u}$ and (24)–(25), we have

$$
0 = -\int_{\Omega\setminus P(t)} (\nabla\cdot\delta\mathbf{u}) p dx = \int_{\Omega\setminus P(t)} \delta\mathbf{u}\cdot\nabla p dx - \int_{\partial P(t)} \delta\mathbf{u}_s \cdot (p\mathbf{I}\cdot\mathbf{n}) ds. \tag{43}
$$

Summing up (42), (41), and (43), we come to

(44)

$$
\begin{aligned}
&\delta\left(\frac{d}{dt}F + \Phi\right) \\
&= -\int_{\Omega\setminus P(t)} \delta\mathbf{u}\cdot\left(\mathcal{R}e\rho\frac{D\mathbf{u}}{Dt} + \nabla p - \eta\nabla\cdot\boldsymbol{\sigma} + \mathcal{B}\epsilon\nabla\cdot(\nabla\phi\otimes\nabla\phi)\right) dx \\
&+ \mathcal{B}\int_{\partial P(t)} \delta\left(\frac{D\phi}{Dt}\right)\left(\mathcal{V}_s^{-1}\frac{D\phi}{Dt} + L(\phi)\right) ds + \mathcal{B}\int_{\partial\Omega} \delta\left(\frac{D\phi}{Dt}\right)\left(\mathcal{V}_s^{-1}\frac{D\phi}{Dt} + L(\phi)\right) ds \\
&+ \mathcal{B}\int_{\Omega\setminus P(t)} \delta\mathbf{J}\cdot(L_d^{-1}\mathbf{J} + \nabla\mu) dx \\
&+ \delta\mathbf{U}_s\cdot\left[\mathcal{R}eM_s\frac{d\mathbf{U}_s}{dt} + \int_{\partial P(t)} (-p\mathbf{I} + \eta\boldsymbol{\sigma} - \mathcal{B}\epsilon(\nabla\phi\otimes\nabla\phi))\cdot\mathbf{n} ds\right] \\
&+ \delta\boldsymbol{\omega}_s\cdot\left[\mathcal{R}eI_s\frac{d\boldsymbol{\omega}_s}{dt} + \int_{\partial P(t)} \mathbf{r}\times((-p\mathbf{I} + \eta\boldsymbol{\sigma} - \mathcal{B}\epsilon(\nabla\phi\otimes\nabla\phi))\cdot\mathbf{n}) ds\right] \\
&+ \int_{\partial P(t)} \delta u_\tau^{slip}\left(-\mathcal{B}L(\phi)\frac{\partial\phi}{\partial\tau} + \frac{\eta}{\mathcal{L}_s l_s}u_\tau^{slip} + \eta(\boldsymbol{\sigma}\cdot\mathbf{n})\cdot\boldsymbol{\tau}\right) ds \\
&+ \int_{\partial\Omega} \delta u_\tau\left(-\mathcal{B}L(\phi)\frac{\partial\phi}{\partial\tau} + \frac{\eta}{\mathcal{L}_s l_s}u_\tau^{slip} + \eta(\boldsymbol{\sigma}\cdot\mathbf{n})\cdot\boldsymbol{\tau}\right) ds.
\end{aligned}
$$

In order to minimize $\frac{d}{dt}F + \Phi$ for prescribed ϕ, variation w.r.t. each variable in the above equality should vanish respectively. First, variation w.r.t. $\delta\mathbf{J}$ gives

$$
\frac{\partial\phi}{\partial t} + \mathbf{u}\cdot\nabla\phi = -\nabla\cdot\mathbf{J} = \mathcal{L}_d\Delta\mu \qquad \text{in } \Omega\setminus\overline{P(t)},
$$

while the definition of μ gives

$$
\mu = -\epsilon\Delta\phi + \frac{1}{\epsilon}(\phi^3 - \phi) + \frac{1}{2}\mathcal{B}^{-1}\mathcal{R}e\rho'(\phi)|\mathbf{u}|^2 \qquad \text{in } \Omega\setminus\overline{P(t)}.
$$

The impermeability boundary condition (22) is equivalent to

$$
\mathbf{J}\cdot\mathbf{n} = \nabla\mu\cdot\mathbf{n} = 0 \qquad \text{on } \partial\Omega\cup\partial P(t).
$$

Using the variation w.r.t. the velocity variables $\mathbf{u}$ in (44), we have

$$\mathcal{R}e\rho\frac{D\mathbf{u}}{Dt} = \nabla\cdot(-p\mathbf{I}+\eta\boldsymbol{\sigma}-\mathcal{B}\epsilon\nabla\phi\otimes\nabla\phi) \qquad \text{in } \ \Omega\setminus\overline{P(t)}.$$

On the boundary of the fluid and the particle surface, variation w.r.t. u_τ and u_τ^{slip} in (44) gives

$$\begin{aligned}
\frac{\eta}{\mathcal{L}_s l_s}u_\tau^{slip} &= -\eta(\boldsymbol{\sigma}\cdot\mathbf{n})\cdot\boldsymbol{\tau}+\mathcal{B}L(\phi)\frac{\partial\phi}{\partial\tau} && \text{on } \ \partial P(t),\\
\frac{\eta}{\mathcal{L}_s l_s}u_\tau &= -\eta(\boldsymbol{\sigma}\cdot\mathbf{n})\cdot\boldsymbol{\tau}+\mathcal{B}L(\phi)\frac{\partial\phi}{\partial\tau} && \text{on } \ \partial\Omega.
\end{aligned}$$

By the variation w.r.t. $\mathbf{U}_s$ and $\boldsymbol{\omega}_s$ in (44), we have

$$\begin{aligned}
\mathcal{R}eM_s\frac{d\mathbf{U}_s}{dt} &= -\int_{\partial P(t)}(-p\mathbf{I}+\eta\boldsymbol{\sigma}-\mathcal{B}\epsilon\nabla\phi\otimes\nabla\phi)\cdot\mathbf{n}ds,\\
\mathcal{R}eI_s\frac{d\boldsymbol{\omega}_s}{dt} &= -\int_{\partial P(t)}\mathbf{r}\times((-p\mathbf{I}+\eta\boldsymbol{\sigma}-\mathcal{B}\epsilon\nabla\phi\otimes\nabla\phi)\cdot\mathbf{n})ds.
\end{aligned}$$

Finally, variation w.r.t. $\frac{D\phi}{Dt}$ in (44) gives

$$\frac{\partial\phi}{\partial t}+\nabla\phi\cdot\mathbf{u} = -\mathcal{V}_sL(\phi) \qquad \text{on } \ \partial P(t)\cup\partial\Omega,$$

while definition of $L(\phi)$ is

$$L(\phi) = \epsilon\frac{\partial\phi}{\partial n}+\frac{\partial\gamma(\phi)}{\partial\phi} \qquad \text{on } \ \partial P(t)\cup\partial\Omega.$$

Recall the incompressible and impermeability boundary conditions, we have

$$\begin{aligned}
&\nabla\cdot\mathbf{u} = 0 && \text{in } \ \Omega\setminus\overline{P(t)},\\
&(\mathbf{u}-\mathbf{u}_s)\cdot\boldsymbol{n} = 0 && \text{on } \ \partial P(t),\\
&\mathbf{u}\cdot\mathbf{n} = 0 && \text{on } \ \partial\Omega.
\end{aligned}$$

Collecting all the above equations, we have recovered the governing system introduced in this paper. ∎

4. Energy decaying property. In the fluid-particle interaction system, if there is no energy inflow/outflow or external force, total energy of the system should decay over time. We now show that the energy decaying property can be derived from our phase-field model, The goal of this section is to prove the following theorem.

THEOREM 4.1. *Suppose that wall speed* $\mathbf{u}_w = 0$*, and that external force* $\mathbf{g} = \mathbf{0}$*; governing system* (2)–(17) *satisfies the following energy decaying property:*

$$\begin{aligned}
\frac{dF}{dt} = &-\int_{\Omega\setminus P(t)}\frac{\eta}{2}|\boldsymbol{\sigma}|_F^2dx-\mathcal{B}L_d\int_{\Omega\setminus P(t)}|\nabla\mu|^2dx-\int_{\partial P(t)}\frac{\eta}{\mathcal{L}_s l_s}|u_\tau^{slip}|^2ds\\
&-\mathcal{B}\mathcal{V}_s\int_{\partial P(t)}|L(\phi)|^2ds-\int_{\partial\Omega}\frac{\eta}{\mathcal{L}_s l_s}|u_\tau|^2ds-\mathcal{B}\mathcal{V}_s\int_{\partial\Omega}|L(\phi)|^2ds.
\end{aligned} \tag{45}$$

Proof. Recall from (34) that

$$\mathcal{R}e\frac{d}{dt}F_k = \mathcal{R}e\int_{\Omega\setminus P(t)}\rho(\phi)\frac{D\mathbf{u}}{Dt}\cdot\mathbf{u}dx+\frac{1}{2}\mathcal{R}e\int_{\Omega\setminus P(t)}\rho'(\phi)\frac{D\phi}{Dt}|\mathbf{u}|^2dx. \tag{46}$$

Taking the inner product of (4) with $\mathbf{u}$ and integrating in $\Omega \setminus P(t)$, we have

$$\begin{aligned}\mathcal{R}e\int_{\Omega\setminus P(t)}\rho\left(\frac{D\mathbf{u}}{Dt}\right)\cdot\mathbf{u}dx &= \int_{\Omega\setminus P(t)}\mathbf{u}\cdot(-\nabla p+\eta\nabla\cdot\boldsymbol{\sigma}-\mathcal{B}\epsilon\nabla\cdot(\nabla\phi\otimes\nabla\phi))dx\\ &= -\int_{\Omega\setminus P(t)}\frac{\eta}{2}|\boldsymbol{\sigma}|_F^2dx+\mathcal{B}\epsilon\int_{\Omega\setminus P(t)}(\nabla\phi\otimes\nabla\phi):\nabla\mathbf{u}dx\\ &\quad+\int_{\partial P(t)}((-p\mathbf{I}+\eta\boldsymbol{\sigma}-\mathcal{B}\epsilon(\nabla\phi\otimes\nabla\phi))\cdot\mathbf{n})\cdot\mathbf{u}ds\\ &\quad+\int_{\partial\Omega}((-p\mathbf{I}+\eta\boldsymbol{\sigma}-\mathcal{B}\epsilon(\nabla\phi\otimes\nabla\phi))\cdot\mathbf{n})\cdot\mathbf{u}ds.\end{aligned}\tag{47}$$

Plugging (46) into (47), we have

$$\begin{aligned}\frac{d}{dt}F_k = &-\int_{\Omega\setminus P(t)}\frac{\eta}{2}|\boldsymbol{\sigma}|_F^2dx+\mathcal{B}\epsilon\int_{\Omega\setminus P(t)}(\nabla\phi\otimes\nabla\phi):\nabla\mathbf{u}dx\\ &+\int_{\partial P(t)}((-p\mathbf{I}+\eta\boldsymbol{\sigma}-\mathcal{B}\epsilon(\nabla\phi\otimes\nabla\phi))\cdot\mathbf{n})\cdot\mathbf{u}ds\\ &+\int_{\partial\Omega}((-p\mathbf{I}+\eta\boldsymbol{\sigma}-\mathcal{B}\epsilon(\nabla\phi\otimes\nabla\phi))\cdot\mathbf{n})\cdot\mathbf{u}ds\\ &+\frac{1}{2}\mathcal{R}e\int_{\Omega\setminus P(t)}\rho'(\phi)\frac{D\phi}{Dt}|\mathbf{u}|^2dx.\end{aligned}\tag{48}$$

Using Green's formula on the right-hand side of (31), we have

$$\begin{aligned}\mathcal{B}\frac{d}{dt}F_b = &\,\mathcal{B}\int_{\Omega\setminus P(t)}\left(\mu-\frac{1}{2}\mathcal{B}^{-1}\mathcal{R}e\rho'(\phi)|\mathbf{u}|^2\right)\frac{D\phi}{Dt}dx+\mathcal{B}\epsilon\int_{\Omega\setminus P(t)}(\nabla\cdot(\nabla\phi\otimes\nabla\phi))\cdot\mathbf{u}dx\\ &+\mathcal{B}\epsilon\int_{\partial P(t)}\frac{\partial\phi}{\partial n}\frac{\partial\phi}{\partial t}ds+\mathcal{B}\epsilon\int_{\partial\Omega}\frac{\partial\phi}{\partial n}\frac{\partial\phi}{\partial t}ds\\ = &\,\mathcal{B}\int_{\Omega\setminus P(t)}\mu\frac{D\phi}{Dt}dx-\frac{1}{2}\mathcal{R}e\int_{\Omega\setminus P(t)}\rho'(\phi)|\mathbf{u}|^2\frac{D\phi}{Dt}dx+\mathcal{B}\epsilon\int_{\partial P(t)}\frac{\partial\phi}{\partial n}\frac{D\phi}{Dt}ds\\ &+\mathcal{B}\epsilon\int_{\partial\Omega}\frac{\partial\phi}{\partial n}\frac{D\phi}{Dt}ds-\mathcal{B}\epsilon\int_{\Omega\setminus P(t)}(\nabla\phi\otimes\nabla\phi):\nabla\mathbf{u}dx.\end{aligned}$$

Multiplying (2) by $\mathcal{B}\mu$, integrating in $\Omega \setminus P(t)$, and using boundary conditions (12) and (17), we have

$$\mathcal{B}\int_{\Omega\setminus P(t)}\mu\frac{D\phi}{Dt}dx=\mathcal{B}L_d\int_{\Omega\setminus P(t)}\mu\Delta\mu dx=-\mathcal{B}L_d\int_{\Omega\setminus P(t)}|\nabla\mu|^2dx,$$

and thus we have

$$\begin{aligned}\mathcal{B}\frac{d}{dt}F_b = &-\mathcal{B}L_d\int_{\Omega\setminus P(t)}|\nabla\mu|^2dx-\frac{1}{2}\mathcal{R}e\int_{\Omega\setminus P(t)}\rho'(\phi)|\mathbf{u}|^2\frac{D\phi}{Dt}dx\\ &+\mathcal{B}\epsilon\int_{\partial P(t)}\frac{\partial\phi}{\partial n}\frac{D\phi}{Dt}ds+\mathcal{B}\epsilon\int_{\partial\Omega}\frac{\partial\phi}{\partial n}\frac{D\phi}{Dt}ds-\mathcal{B}\epsilon\int_{\Omega\setminus P(t)}(\nabla\phi\otimes\nabla\phi):\nabla\mathbf{u}dx.\end{aligned}\tag{49}$$

Taking inner product to (6) by $\mathbf{U}_s$, we have

$$\mathcal{R}eM_s\frac{d}{dt}\left(\frac{1}{2}|\mathbf{U}_s|^2\right)=-\int_{\partial P(t)}((-p\mathbf{I}+\eta\boldsymbol{\sigma}-\mathcal{B}\epsilon(\nabla\phi\otimes\nabla\phi))\cdot\mathbf{n})\cdot\mathbf{U}_sds.$$

Taking inner product to (7) by $\boldsymbol{\omega}_s$, we have

$$\mathcal{R}eI_s\frac{d}{dt}(\frac{1}{2}|\boldsymbol{\omega}_s|^2) = -\int_{\partial P(t)}((-p\mathbf{I}+\eta\boldsymbol{\sigma}-\mathcal{B}\epsilon(\nabla\phi\otimes\nabla\phi))\cdot\mathbf{n})\cdot(\boldsymbol{\omega}_s\times\mathbf{r})ds.$$

Summing up the two equations above, we come to

$$\frac{d}{dt}(\mathcal{R}eF_{pm}+\mathcal{R}eF_{pr}) = -\int_{\partial P(t)}((-p\mathbf{I}+\eta\boldsymbol{\sigma}-\mathcal{B}\epsilon(\nabla\phi\otimes\nabla\phi))\cdot\mathbf{n})\cdot\mathbf{u}_s ds. \tag{50}$$

Now we deal with the surface energy $F_{\partial P(t)}$. Recall from (36) that

$$\begin{aligned}\mathcal{B}\frac{d}{dt}F_{\partial P(t)}ds =&\mathcal{B}\int_{\partial P(t)}L(\phi)\frac{D\phi}{Dt}ds-\mathcal{B}\epsilon\int_{\partial P(t)}\frac{\partial\phi}{\partial n}\frac{D\phi}{Dt}ds\\ &-\mathcal{B}\int_{\partial P(t)}L(\phi)\frac{\partial\phi}{\partial\tau}u_\tau^{slip}ds+\mathcal{B}\epsilon\int_{\partial P(t)}\frac{\partial\phi}{\partial n}\frac{\partial\phi}{\partial\tau}u_\tau^{slip}ds.\end{aligned}$$

Using the boundary condition (9), we have

$$\int_{\partial P(t)}L(\phi)\frac{D\phi}{Dt}ds = -\int_{\partial P(t)}\mathcal{V}_s|L(\phi)|^2ds.$$

Equation (8) also gives

$$-\int_{\partial P(t)}L(\phi)\frac{\partial\phi}{\partial\tau}u_\tau^{slip}ds = -\int_{\partial P(t)}\frac{\eta}{\mathcal{L}_sl_s}|u_\tau^{slip}|^2ds-\int_{\partial P(t)}\eta((\boldsymbol{\sigma}\cdot\mathbf{n})\cdot\boldsymbol{\tau})u_\tau^{slip}ds,$$

thus we have

$$\begin{aligned}\mathcal{B}\frac{d}{dt}F_{\partial P(t)} =&-\mathcal{B}\int_{\partial P(t)}\mathcal{V}_s|L(\phi)|^2ds-\mathcal{B}\epsilon\int_{\partial P(t)}\frac{\partial\phi}{\partial n}\frac{D\phi}{Dt}ds-\mathcal{B}\int_{\partial P(t)}\frac{\eta}{\mathcal{L}_sl_s}|u_\tau^{slip}|^2ds\\ &-\mathcal{B}\int_{\partial P(t)}\eta((\boldsymbol{\sigma}\cdot\mathbf{n})\cdot\boldsymbol{\tau})u_\tau^{slip}ds+\mathcal{B}\epsilon\int_{\partial P(t)}\frac{\partial\phi}{\partial n}\frac{\partial\phi}{\partial\tau}u_\tau^{slip}ds.\end{aligned} \tag{51}$$

Similarly the time derivative of the surface energy $\mathcal{B}F_{\partial\Omega}$ is

$$\begin{aligned}\mathcal{B}\frac{d}{dt}F_{\partial\Omega} =&-\mathcal{B}\int_{\partial\Omega}\mathcal{V}_s|L(\phi)|^2ds-\mathcal{B}\epsilon\int_{\partial\Omega}\frac{\partial\phi}{\partial n}\frac{D\phi}{Dt}ds-\mathcal{B}\int_{\partial\Omega}\frac{\eta}{\mathcal{L}_sl_s}|u_\tau|^2ds\\ &-\mathcal{B}\int_{\partial\Omega}\eta((\boldsymbol{\sigma}\cdot\mathbf{n})\cdot\boldsymbol{\tau})u_\tau ds+\mathcal{B}\epsilon\int_{\partial\Omega}\frac{\partial\phi}{\partial n}\frac{\partial\phi}{\partial\tau}u_\tau ds.\end{aligned} \tag{52}$$

Energy decaying property (45) may be derived by summing up (48), (49), (50), (51), and (52). ∎

5. Sharp interface limit. Since our model is a diffusive interface model, the interface between two fluids is assumed to have a finite thickness of $O(\epsilon)$. Using the method of matched asymptotic expansion, we study the limit of the solutions of our model as the interface thickness $\epsilon\to 0$. In this paper, we consider the case that mobility constants L_d and $\mathcal{V}_s$ are constant. The cases when L_d and $\mathcal{V}_s$ depend on ϵ are discussed in [36]. We assume that the external force $\mathbf{g}$ and wall speed u_w are zero. For simplicity of derivation, we shift the pressure p in the governing system by $p\to p-\mathcal{B}f_b$ to get equivalent forms of (4), (6), and (7):

$$\mathcal{R}e\rho\left(\frac{\partial\mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right) = \nabla\cdot(-p\mathbf{I}+\eta\boldsymbol{\sigma})+\mathcal{B}\mu\nabla\phi \quad\text{in}\ \ \Omega\setminus\overline{P(t)}, \tag{53}$$

$$\mathcal{R}eM_s\frac{\mathrm{d}\mathbf{U}_s}{\mathrm{d}t} = -\int_{\partial P(t)}(-p\mathbf{I}+\mathcal{B}f_b\mathbf{I}+\boldsymbol{\sigma})\cdot\boldsymbol{n}ds, \tag{54}$$

$$\mathcal{R}eI_s\frac{\mathrm{d}\boldsymbol{\omega}_s}{\mathrm{d}t} = -\int_{\partial P(t)}\mathbf{r}\times((-p\mathbf{I}+\mathcal{B}f_b\mathbf{I}+\boldsymbol{\sigma})\cdot\boldsymbol{n})ds. \tag{55}$$

Other governing equations remain unchanged since there are no pressure terms in these equations. In this section, we replace (4), (6), and (7) by (53), (54), and (55) and study the sharp interface limit of the equivalent system.

Since we study the limit of solution as $\epsilon \to 0$, we denote $(\phi^\epsilon, \mu^\epsilon, \mathbf{u}^\epsilon, \mathbf{U}_s^\epsilon, \boldsymbol{\omega}_s^\epsilon)$ the solution of the (pressure-shifted) equation system, which depends on ϵ. The two-phase interface is given by the zero level-set of the phase-field function,

$$\Gamma^\epsilon := \{\mathbf{x}\in\Omega\setminus P(t)|\phi^\epsilon(x)=0\}.$$

Let $d^\epsilon(\mathbf{x},t)$ be the signed distance function to Γ^ϵ, which satisfies $|\nabla d^\epsilon|=1$. Suppose that d^ϵ has the expansion

$$d^\epsilon = \sum_{i=0}^{\infty}\epsilon^i d^i(\mathbf{x},t);$$

then we also have $|\nabla d^0|=1$. Using definition of d^0, we may also define

$$\Gamma^0 := \{(\mathbf{x},t)|d^0(\mathbf{x},t)=0\},$$
$$\Omega_0^\pm := \{(\mathbf{x},t)\in\Omega\setminus P(t)|\pm d^0(\mathbf{x},t)>0\}.$$

Using the method of asymptotic expansion, we may derive governing equations when $\epsilon\to 0$, with Γ^0 being the fluid-fluid interface.

5.1. Outer expansion. First we consider the asymptotic expansion away from the fluid-fluid interface, which is called the outer expansion. We seek an expansion of the variables in $\{\Omega_0^\pm\}$, respectively, which are in the form

$$\phi^\epsilon = \sum_{i=0}^{\infty}\epsilon^i\phi_i^\pm,\ \mu^\epsilon = \sum_{i=0}^{\infty}\epsilon^i\mu_i^\pm,\ \mathbf{u}^\epsilon = \sum_{i=0}^{\infty}\epsilon^i\mathbf{u}_i^\pm,\ \mathbf{U}_s^\epsilon = \sum_{i=0}^{\infty}\epsilon^i\mathbf{U}_i,\ \boldsymbol{\omega}_s^\epsilon = \sum_{i=0}^{\infty}\epsilon^i\boldsymbol{\omega}_i.$$

Define $\{\phi_0^\pm, \mu_0^\pm, \mathbf{u}_0^\pm, p_0^\pm, \boldsymbol{\sigma}_0^\pm\}$ the corresponding variables in $\Omega_0^\pm$. It is straightforward to derive the leading order equations in $\Omega_0^\pm$:

$$\phi_0^\pm = \pm 1, \tag{56}$$

$$\Delta\mu_0^\pm = 0, \tag{57}$$

$$\mathcal{R}e\rho\left(\frac{\partial\mathbf{u}_0^\pm}{\partial t}+(\mathbf{u}_0^\pm\cdot\nabla)\mathbf{u}_0^\pm\right) = -\nabla p_0^\pm + \eta\nabla\cdot\boldsymbol{\sigma}_0^\pm, \tag{58}$$

$$\nabla\cdot\mathbf{u}_0^\pm = 0. \tag{59}$$

The boundary conditions on $\partial\Omega$ are

$$\frac{\partial\mu_0^\pm}{\partial n}=0,\ \mathbf{u}_0^\pm\cdot\mathbf{n}=0,\ \frac{1}{\mathcal{L}_s l_s}\mathbf{u}_0^\pm\cdot\boldsymbol{\tau} = -\eta(\boldsymbol{\sigma}_0^\pm\cdot\mathbf{n})\cdot\boldsymbol{\tau}, \tag{60}$$

while the boundary conditions on $\partial P(t)$ are

$$\frac{\partial\mu_0^\pm}{\partial n}=0,\ (\mathbf{u}_0^\pm-\mathbf{u}_{s,0})\cdot\mathbf{n}=0,\ \frac{1}{\mathcal{L}_s l_s}(\mathbf{u}_0^\pm-\mathbf{u}_{s,0})\cdot\boldsymbol{\tau} = -\eta(\boldsymbol{\sigma}_0^\pm\cdot\mathbf{n})\cdot\boldsymbol{\tau}. \tag{61}$$

Here $\mathbf{u}_{s,0}=\mathbf{U}_0+\boldsymbol{\omega}_0\times\mathbf{r}$.

Since the above equations are defined in $\Omega_0^{\pm}$, respectively, in order to close the equation system, we also need boundary conditions on Γ^0, which will be derived by the inner expansion.

5.2. Inner expansion. We study the asymptotic behavior of solutions to the governing system in a neighborhood of Γ^0. In order to do that, we examine the inner expansion of the solution of our governing equations near the interface. Define $\xi = \frac{d^\epsilon}{\epsilon}$ as the scaled distance from the interface, and consider the inner expansion of the following form:

$$(\phi^\epsilon, \mu^\epsilon, \mathbf{u}^\epsilon, p^\epsilon)(\mathbf{x}, t) = (\widetilde{\phi}^\epsilon, \widetilde{\mu}^\epsilon, \widetilde{\mathbf{u}}^\epsilon, \widetilde{p}^\epsilon)(\mathbf{x}, t, \xi),$$

$$(\widetilde{\phi}^\epsilon, \widetilde{\mu}^\epsilon, \widetilde{\mathbf{u}}^\epsilon, \widetilde{p}^\epsilon)(\mathbf{x}, t, \xi) = \sum_{i=0}^{\infty} \epsilon^i (\widetilde{\phi}^i, \widetilde{\mu}^i, \widetilde{\mathbf{u}}^i, \widetilde{p}^i)(\mathbf{x}, t, \xi). \tag{62}$$

Given the inner and outer expansions, we need the matching conditions for the inner and outer expansions. Following [5], we match the expansions by requiring that

$$(\widetilde{\phi}^\epsilon, \widetilde{\mu}^\epsilon, \widetilde{\mathbf{u}}^\epsilon, \widetilde{p}^\epsilon)(\mathbf{x}, t, \xi) \approx (\phi^\epsilon, \mu^\epsilon, \mathbf{u}^\epsilon, p^\epsilon)(\mathbf{x}_\xi, t) \quad \text{as} \quad \xi \to \pm\infty, \tag{63}$$

where $\mathbf{x}_\xi = \mathbf{x} + \epsilon\xi\nabla d^\epsilon$. Applying Taylor expansion at point $\mathbf{x}$ to (63), then the leading and the next order asymptotic expansions give

$$\lim_{\xi\to\pm\infty} (\widetilde{\phi}^0, \widetilde{\mu}^0, \widetilde{\mathbf{u}}^0, \widetilde{p}^0)(\mathbf{x}, t, \xi) = (\phi_0^\pm, \mu_0^\pm, \mathbf{u}_0^\pm, p_0^\pm)(\mathbf{x}, t) \tag{64}$$

and

$$\begin{aligned}\lim_{\xi\to\pm\infty} (\widetilde{\phi}^1, \widetilde{\mu}^1, \widetilde{\mathbf{u}}^1, \widetilde{p}^1)(\mathbf{x}, t, \xi) = {}& (\phi_1^\pm, \mu_1^\pm, \mathbf{u}_1^\pm, p_1^\pm)(\mathbf{x}, t) \\ &+ \xi\nabla d^0 \cdot (\nabla\phi_0^\pm, \nabla\mu_0^\pm, \nabla\mathbf{u}_0^\pm, \nabla p_0^\pm)(\mathbf{x}, t),\end{aligned} \tag{65}$$

which imply

$$\lim_{\xi\to\pm\infty} \partial_\xi (\widetilde{\phi}^0, \widetilde{\mu}^0, \widetilde{\mathbf{u}}^0, \widetilde{p}^0)(\mathbf{x}, t, \xi) = \mathbf{0}, \tag{66}$$

$$\lim_{\xi\to\pm\infty} \partial_\xi (\widetilde{\phi}^1, \widetilde{\mu}^1, \widetilde{\mathbf{u}}^1, \widetilde{p}^1)(\mathbf{x}, t, \xi) = (\nabla d^0 \cdot (\nabla\phi_0^\pm, \nabla\mu_0^\pm, \nabla\mathbf{u}_0^\pm, \nabla p_0^\pm))(\mathbf{x}, t). \tag{67}$$

We then study the inner expansion of our model. Applying expansion (62) into (2), (3), (53), and (5), we have

$$\begin{aligned}L_d\widetilde{\mu}^\epsilon_{\xi\xi} - \epsilon[L_d(\Delta d^\epsilon \widetilde{\mu}^\epsilon_\xi + 2\nabla d^\epsilon \cdot \widetilde{\nabla}\widetilde{\mu}^\epsilon_\xi) - \partial_t d^\epsilon \widetilde{\phi}^\epsilon_\xi - (\widetilde{\mathbf{u}}^\epsilon \cdot \nabla) d^\epsilon \widetilde{\phi}^\epsilon_\xi)] \\ + \epsilon^2 (L_d \widetilde{\Delta}\widetilde{\mu}^\epsilon - \partial_t \widetilde{\phi}^\epsilon - \widetilde{\mathbf{u}}^\epsilon \cdot \widetilde{\nabla}\widetilde{\phi}^\epsilon) = 0,\end{aligned} \tag{68}$$

$$\begin{aligned}\epsilon\left(\Delta d^\epsilon \widetilde{\phi}^\epsilon_\xi + 2\nabla d^\epsilon \cdot \widetilde{\nabla}\widetilde{\phi}^\epsilon_\xi + \widetilde{\mu}^\epsilon - \frac{1}{2}\mathcal{R}e\mathcal{B}^{-1}\rho'(\widetilde{\phi}^\epsilon)|\widetilde{\mathbf{u}}^\epsilon|^2\right) \\ + \epsilon^2 \widetilde{\Delta}\widetilde{\phi}^\epsilon + (\widetilde{\phi}^\epsilon_{\xi\xi} - ((\widetilde{\phi}^\epsilon)^3 - \widetilde{\phi}^\epsilon)) = 0,\end{aligned} \tag{69}$$

$$\begin{aligned}\epsilon[\mathcal{R}e\rho^\epsilon(\widetilde{\mathbf{u}}^\epsilon_\xi \partial_t d^\epsilon + (\widetilde{\mathbf{u}}^\epsilon \cdot \nabla d^\epsilon)\widetilde{\mathbf{u}}^\epsilon_\xi) - \eta\Delta d^\epsilon \widetilde{\mathbf{u}}^\epsilon_\xi - 2\eta(\nabla d^\epsilon \cdot \widetilde{\nabla})\widetilde{\mathbf{u}}^\epsilon_\xi + (\widetilde{p}^\epsilon_\xi - \mathcal{B}\widetilde{\mu}^\epsilon \widetilde{\phi}^\epsilon_\xi)\nabla d^\epsilon] \\ + \epsilon^2[\mathcal{R}e\rho(\partial_t \widetilde{\mathbf{u}}^\epsilon + (\widetilde{\mathbf{u}}^\epsilon \cdot \widetilde{\nabla})\widetilde{\mathbf{u}}^\epsilon) + \widetilde{\nabla}\widetilde{p}^\epsilon - \eta\widetilde{\Delta}\widetilde{\mathbf{u}}^\epsilon - \mathcal{B}\widetilde{\mu}^\epsilon \widetilde{\nabla}\widetilde{\phi}^\epsilon] - \eta\widetilde{\mathbf{u}}^\epsilon_{\xi\xi} = 0,\end{aligned} \tag{70}$$

$$\epsilon\widetilde{\nabla} \cdot \widetilde{\mathbf{u}}^\epsilon + \widetilde{\mathbf{u}}^\epsilon_\xi \cdot \nabla d^\epsilon = 0. \tag{71}$$

Here $\widetilde{\nabla} := (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$, $\widetilde{\Delta} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, $f_\xi := \frac{\partial f}{\partial \xi}$.

The leading order equations of the above equation system are

$$
\begin{cases}
L_d \widetilde{\mu}^0_{\xi\xi} = 0, \\
\widetilde{\phi}^0_{\xi\xi} - \left(\left(\widetilde{\phi}^0\right)^3 - \widetilde{\phi}^0 \right) = 0, \\
\eta \widetilde{\mathbf{u}}^0_{\xi\xi} = 0, \\
\widetilde{\mathbf{u}}^0_\xi \cdot \nabla d^0 = 0,
\end{cases}
\tag{72}
$$

while the next order equations are

$$
\begin{cases}
L_d \widetilde{\mu}^1_{\xi\xi} - L_d \left(\widetilde{\Delta} d^0 \widetilde{\mu}^0_\xi + 2\nabla d^0 \cdot \widetilde{\nabla} \widetilde{\mu}^0_\xi \right) - \partial_t d^0 \widetilde{\phi}^0_\xi - \left(\widetilde{\mathbf{u}}^0 \cdot \nabla \right) d^0 \widetilde{\phi}^0_\xi = 0, \\
\widetilde{\phi}^1_{\xi\xi} - 3\left(\left(\widetilde{\phi}^0\right)^2 - 1 \right) \widetilde{\phi}^1 + \Delta d^\epsilon \widetilde{\phi}^0_\xi + 2\nabla d^0 \cdot \widetilde{\nabla} \widetilde{\phi}^0_\xi \\
\qquad\qquad + \widetilde{\mu}^0 - \dfrac{1}{2} \mathcal{R}e \mathcal{B}^{-1} \rho' \left(\widetilde{\phi}^0 \right) |\widetilde{\mathbf{u}}^0|^2 = 0, \\
\mathcal{R}e \rho\left(\widetilde{\phi}^0 \right) \left(\widetilde{\mathbf{u}}^0_\xi \partial_t d^0 + \left(\widetilde{\mathbf{u}}^0 \cdot \nabla d^0 \right) \widetilde{\mathbf{u}}^0_\xi \right) - \eta \Delta d^0 \widetilde{\mathbf{u}}^0_\xi - 2\eta \left(\nabla d^0 \cdot \widetilde{\nabla} \right) \widetilde{\mathbf{u}}^0_\xi \\
\qquad\qquad + \left(\widetilde{p}^0_\xi - \mathcal{B} \widetilde{\mu}^0 \widetilde{\phi}^0_\xi \right) \nabla d^0 - \eta \widetilde{\mathbf{u}}^1_{\xi\xi} = 0, \\
\widetilde{\nabla} \cdot \widetilde{\mathbf{u}}^0 + \widetilde{\mathbf{u}}^1_\xi \cdot \nabla d^0 + \widetilde{\mathbf{u}}^0_\xi \cdot \nabla d^1 = 0.
\end{cases}
\tag{73}
$$

Given the leading order equations (72) and the matching condition (64), it follows immediately that $\widetilde{\mu}^0$ and $\widetilde{\mathbf{u}}^0$ are independent of ξ, thus

$$[\mathbf{u}_0] = 0 \text{ and } \nabla d^0 \cdot [\nabla \mathbf{u}_0] = 0 \quad \text{on } \Gamma^0,$$

where $[\mathbf{u}_0]$ stands for the jump between $\mathbf{u}_0^+$ and $\mathbf{u}_0^-$ on Γ_0.

From the second equation in (72), and the matching condition (64), we find that $\widetilde{\phi}^0$ satisfies the following equations:

$$\widetilde{\phi}^0_{\xi\xi} = (\widetilde{\phi}^0)^3 - \widetilde{\phi}^0, \ \widetilde{\phi}^0|_{\xi=0} = 0, \ \lim_{\xi \to \pm\infty} \widetilde{\phi}^0 = \pm 1. \tag{74}$$

The solution of the above equations is $\widetilde{\phi}^0 = \tanh(\frac{\sqrt{2}}{2}\xi)$.

Note that $\widetilde{\mu}^0_\xi = 0$, and the first equation in next order equations (73) gives

$$L_d \widetilde{\mu}^1_{\xi\xi} - \partial_t d^0 \widetilde{\phi}^0_\xi - (\widetilde{\mathbf{u}}^0 \cdot \nabla) d^0 \widetilde{\phi}^0_\xi = 0.$$

Integrating the above equation w.r.t. ξ over $(-\infty, +\infty)$, and using the matching condition (64), (67), we may derive that

$$\partial_t d^0 + (\mathbf{u}_0 \cdot \nabla) d^0 = \frac{L_d}{2} \nabla d^0 \cdot [\nabla \mu_0], \tag{75}$$

which is the evolution equation for the interface Γ^0.

Next, to derive the equation for μ^0 on Γ^0, we use (74) and the matching condition for $\widetilde{\phi}^0$ to get

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} (\widetilde{\phi}^1_{\xi\xi} - (3(\widetilde{\phi}^0)^2 - 1)\widetilde{\phi}_1) \widetilde{\phi}^0_\xi d\xi = \int_{-\infty}^{+\infty} \widetilde{\phi}^1_{\xi\xi} \widetilde{\phi}^0_\xi - \partial_\xi ((\widetilde{\phi}^0)^3 - \widetilde{\phi}^0) \widetilde{\phi}_1 d\xi \\
&= \int_{-\infty}^{+\infty} \widetilde{\phi}^1_{\xi\xi} \widetilde{\phi}^0_\xi - \widetilde{\phi}^0_{\xi\xi\xi} \widetilde{\phi}^1 d\xi = 0.
\end{aligned}
\tag{76}
$$

Similar to [25], we assume that

$$\nabla d^0 \cdot \widetilde{\nabla}\widetilde{\phi}^0_\xi = 0. \tag{77}$$

Multiplying $\widetilde{\phi}^0_\xi$ to the second equation in (73), integrating w.r.t. ξ over $(-\infty,+\infty)$, and plugging (76)–(77), we have

$$\int_{-\infty}^{+\infty} \left(\widetilde{\mu}^0 + \Delta d^0 \widetilde{\phi}^0_\xi - \frac{1}{2}\mathcal{R}e\mathcal{B}^{-1}\rho'(\widetilde{\phi}^0)|\widetilde{\mathbf{u}}^0|^2\right)\widetilde{\phi}^0_\xi d\xi = 0.$$

Recalling that $\widetilde{\mu}^0$ and $\widetilde{\mathbf{u}}^0$ are independent of ξ, using match condition (64), the above equation means that

$$\mu_0^\pm = -\frac{F_{Ca}}{2}\Delta d^0 + \frac{1}{4}\mathcal{R}e\mathcal{B}^{-1}[\rho(\phi_0)]|\mathbf{u}_0|^2 \quad \text{on } \Gamma^0. \tag{78}$$

Here F_{Ca} is defined as $F_{Ca} := \int_{-\infty}^{+\infty} |\widetilde{\phi}^0_\xi|^2 d\xi$.

From the last equation in (73), by noting that $\widetilde{\mathbf{u}}^0_\xi = 0$, we have

$$\widetilde{\nabla} \cdot \widetilde{\mathbf{u}}^0 + \widetilde{\mathbf{u}}^1_\xi \cdot \nabla d^0 = 0.$$

Moreover, since $\widetilde{\mathbf{u}}^0$ is independent of ξ, we have

$$\widetilde{\nabla} \cdot \widetilde{\mathbf{u}}^0 = \lim_{\xi\to\pm\infty} \widetilde{\nabla} \cdot \widetilde{\mathbf{u}}^0 = \lim_{\xi\to\pm\infty} \left(\widetilde{\nabla} \cdot \widetilde{\mathbf{u}}^0 + \epsilon^{-1}\nabla d^\epsilon \cdot \widetilde{\mathbf{u}}^0_\xi\right) = \nabla \cdot \mathbf{u}_0^\pm = 0,$$

and we have $\widetilde{\mathbf{u}}^1_\xi \cdot \nabla d^0 = 0$. The matching condition (67) then gives

$$\left((\nabla d^0 \cdot \nabla)\mathbf{u}_0^\pm\right) \cdot \nabla d^0 = 0 \quad \text{on } \Gamma^0. \tag{79}$$

From the third equation in (73), since $\widetilde{\mathbf{u}}^0_\xi = 0$, we have

$$\eta \widetilde{\mathbf{u}}^1_{\xi\xi} - \left(\widetilde{p}^0_\xi - \mathcal{B}\widetilde{\mu}^0\widetilde{\phi}^0_\xi\right)\nabla d^0 = 0.$$

By multiplying ∇d^0 to both sides of the equation above and noting that $|\nabla d^0| = 1$, we have

$$\eta \nabla d^0 \cdot \widetilde{\mathbf{u}}^1_{\xi\xi} = \widetilde{p}^0_\xi - \mathcal{B}\widetilde{\mu}^0\widetilde{\phi}^0_\xi.$$

After integrating the above equation w.r.t. ξ over $(-\infty,+\infty)$, it follows from the matching conditions (64)–(67) and (79)–(78) that

$$[p_0] = 2\mathcal{B}\mu_0^\pm = -\mathcal{B}F_{Ca}\Delta d^0 + \frac{1}{2}\mathcal{R}e[\rho(\phi_0)]|\mathbf{u}_0|^2 \quad \text{on } \Gamma^0.$$

Remark 5.1. Compared with the standard results in, e.g., [35], we introduced a new term $\frac{1}{2}\mathcal{R}e[\rho(\phi_0)]|\mathbf{u}_0|^2$. This term balanced the change of kinetic energy caused by the motion of the fluid-fluid interface. In an equal density case such a term will vanish; then our pressure jump will reduce to the original result.

Next we deal with the boundary conditions on $\partial\Omega$. As a relaxation parameter, $\mathcal{V}_s$ may take on different choices. In this paper, following [35], we assume that the mobility constant $\mathcal{V}_s \sim O(1)$. For other choices of $\mathcal{V}_s$, we refer to [36] for a detailed discussion.

Eliminating $L(\phi)$ in (15) and (14), then using inner expansion (62), we have

$$\left(\widetilde{\phi}_\xi^\epsilon \partial_t d^\epsilon + \widetilde{\phi}_\xi^\epsilon \nabla d^\epsilon \cdot \widetilde{\mathbf{u}}^\epsilon\right) + \epsilon\left(\partial_t \widetilde{\phi}^\epsilon + \widetilde{\nabla}\widetilde{\phi}^\epsilon \cdot \widetilde{\mathbf{u}}^\epsilon + \mathcal{V}_s \widetilde{\phi}_\xi^\epsilon \frac{\partial d^\epsilon}{\partial n} + \mathcal{V}_s \frac{\partial \gamma(\widetilde{\phi}^\epsilon)}{\partial \phi}\right) + \epsilon^2 \mathcal{V}_s \frac{\partial \widetilde{\phi}^\epsilon}{\partial n} = 0.$$

The leading order term of the above equation gives

$$\widetilde{\phi}_\xi^0 \partial_t d^0 + \widetilde{\phi}_\xi^0 \nabla d^0 \cdot \widetilde{\mathbf{u}}^0 = 0 \text{ on } \partial\Omega.$$

Integrating the above equality w.r.t. ξ over $(-\infty, +\infty)$, we have

$$\partial_t d^0 + \nabla d^0 \cdot \mathbf{u}_0 = 0 \quad \text{on } \partial\Omega.$$

The above equation and (75) also lead to

$$\nabla d^0 \cdot [\nabla \mu^0] = 0 \quad \text{on } \partial\Omega \cap \Gamma^0.$$

Eliminating $L(\phi)$ by (13) and (15), the inner expansion (62) leads to

$$\begin{aligned}\frac{\eta}{\mathcal{L}_s l_s} \widetilde{\mathbf{u}}^\epsilon \cdot \boldsymbol{\tau} = &-\eta\left(\widetilde{\boldsymbol{\sigma}}^\epsilon \cdot \mathbf{n} + \epsilon^{-1} \widetilde{\mathbf{u}}_\xi^\epsilon \frac{\partial d^\epsilon}{\partial n} + \epsilon^{-1}(\widetilde{\mathbf{u}}_\xi^\epsilon \cdot \mathbf{n}) \nabla d^\epsilon\right) \cdot \boldsymbol{\tau} \\ &+ \mathcal{B}\left(\epsilon \frac{\partial \widetilde{\phi}^\epsilon}{\partial n} + \widetilde{\phi}_\xi^\epsilon \frac{\partial d^\epsilon}{\partial n} + \frac{\partial \gamma(\widetilde{\phi}^\epsilon)}{\partial \phi}\right)\left(\frac{\partial \widetilde{\phi}^\epsilon}{\partial \tau} + \epsilon^{-1} \widetilde{\phi}_\xi^\epsilon \frac{\partial d^\epsilon}{\partial \tau}\right).\end{aligned}$$

Since $\widetilde{\mathbf{u}}^0$ is independent of ξ, $O(\epsilon^{-1})$ terms of the above equality give

$$\mathcal{B}\left(\widetilde{\phi}_\xi^0 \frac{\partial d^0}{\partial n} + \frac{\partial \gamma(\widetilde{\phi}^0)}{\partial \phi}\right) \widetilde{\phi}_\xi^0 \frac{\partial d^0}{\partial \tau} = 0.$$

Noting that $\frac{\partial d^0}{\partial \tau} \neq 0$ and $\widetilde{\phi}_\xi^0 \neq 0$ on $\partial\Omega$, we have

$$\widetilde{\phi}_\xi^0 \frac{\partial d^0}{\partial n} - \frac{\partial \gamma(\widetilde{\phi}^0)}{\partial \phi} = 0.$$

Multiplying the above equation by $\widetilde{\phi}_\xi^0$, integrating w.r.t. ξ over $(-\infty, +\infty)$, and defining $\gamma_\pm := \gamma(\phi_0^\pm)$, we have

$$\frac{\partial d^0}{\partial n} \int_{-\infty}^{+\infty} |\widetilde{\phi}_\xi^0|^2 d\xi = \gamma_- - \gamma_+.$$

We define $\alpha_0 := \arccos(\frac{\partial d^0}{\partial n})$ as the angle between interface Γ^0 and the fluid surface $\partial\Omega$; the above equation then equals to

$$\cos \alpha_0 = (\gamma_- - \gamma_+)/F_{Ca}.$$

Similarly, we may derive the boundary conditions on the particle surface $\partial P(t)$:

$$\begin{aligned}&\partial_t d^0 + \nabla d^0 \cdot u_0 = 0 && \text{on } \partial P(t), \\ &\nabla d^0 \cdot [\nabla \mu^0] = 0 && \text{on } \partial P(t) \cap \Gamma^0, \\ &\cos \alpha_0 = (\gamma_- - \gamma_+)/F_{Ca}.\end{aligned}$$

5.3. Forces on the particle. We study the sharp interface for (54) and (55). First we derive a sharp interface limit of (54), then we may deal with (55) in a similar way. In order to derive the leading order term of the above equations, we let $\epsilon \to 0$ on both sides of (54) and calculate the limit with the help of asymptotic expansion. According to the inner expansion, since $\widetilde{p}^\epsilon$ and $\widetilde{\boldsymbol{\sigma}}^\epsilon$ is bounded of ϵ near the fluid-fluid interface Γ^0, we have

$$\lim_{\epsilon \to 0} -\int_{\partial P(t)} (-p^\epsilon \mathbf{I} + \eta \boldsymbol{\sigma}^\epsilon) \cdot \boldsymbol{n} ds = -\int_{\partial P(t)} (-p_0 \mathbf{I} + \eta \boldsymbol{\sigma}_0) \cdot \boldsymbol{n} ds.$$

On the force term

$$-\int_{\partial P(t)} (\mathcal{B} f_b(\phi^\epsilon)\mathbf{I} - \mathcal{B}\epsilon \nabla \phi^\epsilon \otimes \nabla \phi^\epsilon) \cdot \boldsymbol{n} ds,$$

note that when $\epsilon \to 0$, $\nabla \phi^\epsilon$ have singularity near the fluid-fluid interface Γ^0, and we cannot simply take the limit inside the integral. In order to study the leading order behavior of

$$-\int_{\partial P(t)} -\mathcal{B}\epsilon (\nabla \phi^\epsilon \otimes \nabla \phi^\epsilon) \cdot \boldsymbol{n} ds,$$

given a fixed small parameter ϵ_0, we split $P(t)$ into $\Lambda^{\epsilon_0} := \{(\mathbf{x}, t) \in P(t) | d^0(\mathbf{x}, t) < \epsilon_0\}$ and $P(t) \setminus \Lambda^{\epsilon_0}$, and deal with the integration over Λ^{ϵ_0} and $P(t) \setminus \Lambda^{\epsilon_0}$ separately. When the integration area is away from the fluid-fluid interface, we have

$$\begin{aligned} &\lim_{\epsilon \to 0} -\int_{P(t)\setminus \Lambda^{\epsilon_0}} (f_b(\widetilde{\phi}^\epsilon)\mathbf{I} - \mathcal{B}\epsilon \nabla \phi^\epsilon \otimes \nabla \phi^\epsilon) \cdot \boldsymbol{n} ds \\ =& \lim_{\epsilon \to 0} -\int_{P(t)\setminus \Lambda^{\epsilon_0}} (f_b(\phi_0)\mathbf{I} - \mathcal{B}\epsilon \nabla \phi_0 \otimes \nabla \phi_0) \cdot \boldsymbol{n} ds = 0. \end{aligned}$$

Next we derive the sharp interface limit of

$$-\int_{\Lambda^{\epsilon_0}} (f_b(\widetilde{\phi}^\epsilon) - \mathcal{B}\epsilon \nabla \widetilde{\phi}^\epsilon \otimes \nabla \widetilde{\phi}^\epsilon) \cdot \boldsymbol{n} ds.$$

By defining $\xi^i = \frac{d^i}{\epsilon}$, we have $\xi = \sum_{i=0}^{\infty} \epsilon^i \xi^i$; then we may expand $\widetilde{\phi}^\epsilon$ by

$$\widetilde{\phi}^\epsilon(x, y, t, \xi) = \hat{\phi}^0(x, y, t, \xi^0) + \epsilon \hat{\phi}^1(x, y, t, \xi^0, \xi^1) + \epsilon^2 \hat{\phi}^2(x, y, t, \xi^0, \xi^1, \xi^2) \dots.$$

Using the above inner expansion, we may see that $\hat{\phi}^0$ also satisfies

$$\hat{\phi}^0_{\xi\xi} = (\hat{\phi}^0)^3 - \hat{\phi}^0, \ \hat{\phi}^0|_{\xi=0} = 0, \ \lim_{\xi \to \pm\infty} \hat{\phi}^0 = \pm 1, \tag{80}$$

which means that $\hat{\phi}^0 = \tanh(\frac{\sqrt{2}}{2}\xi^0)$.

To deal with the integration in Λ^{ϵ_0}, first we deal with the case that Λ^{ϵ_0} is straight. We introduce a local coordinate system. The $\widetilde{x}$ axis is parallel to Λ^{ϵ_0} and the origin is on the intersection of the fluid and the solid boundary, as shown in Figure 2. With the help of the local coordinates, we may express Λ^{ϵ_0} by $\widetilde{x} \in (-\epsilon_0, \epsilon_0)$.

Now we find the limit as $\epsilon \to 0$ of the integral below:

$$-\int_{\Lambda^{\epsilon_0}} -\mathcal{B}\epsilon \nabla \widetilde{\phi}^\epsilon \otimes \nabla \widetilde{\phi}^\epsilon \cdot \boldsymbol{n} ds.$$

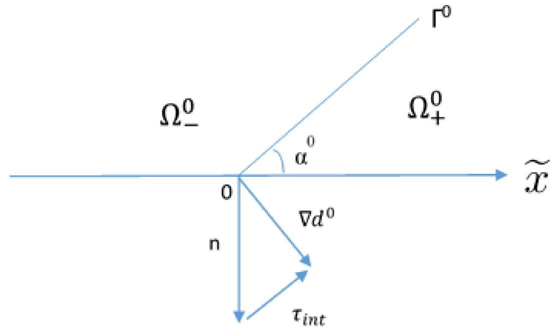

FIG. 2. *Local coordinates on straight boundary.*

Using the local coordinates, we have

$$\lim_{\epsilon\to 0} -\!\!\!\!\!\!\int_{\Lambda^{\epsilon_0}} -\mathcal{B}\epsilon\nabla\widetilde{\phi}^\epsilon \otimes \nabla\widetilde{\phi}^\epsilon \cdot \boldsymbol{n} ds = \lim_{\epsilon\to 0} -\!\!\!\!\!\!\int_{\Lambda^{\epsilon_0}} -\mathcal{B}\epsilon(\nabla\hat{\phi}^0 \otimes \nabla\hat{\phi}^0) \cdot \boldsymbol{n} ds$$

$$= \lim_{\epsilon\to 0} -\!\!\!\!\!\!\int_{-\epsilon_0}^{\epsilon_0} -\mathcal{B}\epsilon(\nabla\hat{\phi}^0 \otimes \nabla\hat{\phi}^0) \cdot \boldsymbol{n} d\widetilde{x} = \lim_{\epsilon\to 0} -\!\!\!\!\!\!\int_{-\epsilon_0/\epsilon}^{\epsilon_0/\epsilon} -\mathcal{B}\epsilon(\nabla\hat{\phi}^0 \otimes \nabla\hat{\phi}^0) \cdot \boldsymbol{n} \frac{\epsilon}{\sin\alpha_0} d\xi^0$$

$$= \mathcal{B}\frac{1}{\sin\alpha_0}\int_{-\epsilon_0/\epsilon}^{\epsilon_0/\epsilon} \left|\frac{\partial\hat{\phi}^0}{\partial\xi^0}\right|^2 \nabla d^0 \frac{\partial d^0}{\partial n} d\xi^0 = \mathcal{B}\frac{\cos\alpha_0}{\sin\alpha_0}\nabla d^0 F_{Ca}.$$

Next we find

$$\lim_{\epsilon\to 0} -\!\!\!\!\!\!\int_{\Lambda^{\epsilon_0}} \mathcal{B} f_b(\widetilde{\phi}^\epsilon)\mathbf{I} \cdot \boldsymbol{n} ds.$$

Since $\hat{\phi}^0 = \tanh(\frac{\sqrt{2}}{2}\xi^0)$, we have

$$\lim_{\epsilon\to 0} -\!\!\!\!\!\!\int_{\Lambda^{\epsilon_0}} \mathcal{B} f_b(\widetilde{\phi}^\epsilon)\mathbf{I} \cdot \boldsymbol{n} ds = \lim_{\epsilon\to 0} -\!\!\!\!\!\!\int_{\Lambda^{\epsilon_0}} \mathcal{B} f_b(\hat{\phi}^0)\mathbf{I} \cdot \boldsymbol{n} ds$$

$$= \lim_{\epsilon\to 0} -\!\!\!\!\!\!\int_{-\epsilon_0/\epsilon}^{\epsilon_0/\epsilon} \mathcal{B} f_b(\hat{\phi}^0)\mathbf{I} \cdot \mathbf{n}\frac{\epsilon}{\sin\alpha_0} d\xi = -\frac{\mathcal{B}}{\sin\alpha_0}\mathbf{n}\int_{-\epsilon_0/\epsilon}^{\epsilon_0/\epsilon} |\hat{\phi}^0_{\xi^0}|^2 d\xi = -\frac{\mathcal{B}}{\sin\alpha_0}\mathbf{n}F_{Ca}.$$

Summing the two leading order terms in the above two equations together, we may see that the leading order term of

$$-\!\!\!\!\!\!\int_{\Lambda^{\epsilon_0}} (f_b(\widetilde{\phi}^\epsilon)\mathbf{I} - \mathcal{B}\epsilon\nabla\widetilde{\phi}^\epsilon \otimes \nabla\widetilde{\phi}^\epsilon) \cdot \mathbf{n} ds$$

is given by

$$\mathcal{B}\frac{\cos\alpha_0}{\sin\alpha_0}\nabla d^0 F_{Ca} - \frac{\mathcal{B}}{\sin\alpha_0}\mathbf{n}F_{Ca} = \mathcal{B}F_{Ca}\boldsymbol{\tau}_I,$$

where the direction of the total force is $\boldsymbol{\tau}_I = \nabla d^0 \cos\alpha_0 - \mathbf{n}$, which is parallel to the fluid interface, as shown in Figure 2. We may see that leading order term of force

$$-\!\!\!\!\!\!\int_{\Lambda^{\epsilon_0}} (f_b(\widetilde{\phi}^\epsilon)\mathbf{I} - \mathcal{B}\epsilon\nabla\widetilde{\phi}^\epsilon \otimes \nabla\widetilde{\phi}^\epsilon) \cdot \mathbf{n} ds$$

is physically consistent to the capillary force given by the fluid interface.

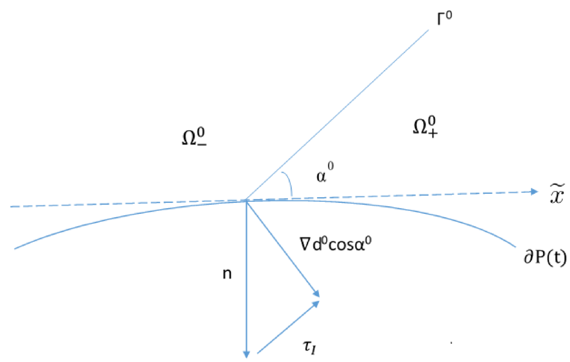

FIG. 3. *Local coordinates on curved boundary.*

Now we deal with the case that the particle boundary is curved (see Figure 3). We also introduce a local coordinate system, as shown in Figure 3. Let the $\widetilde{x}$ axis be parallel to the tangent line of the particle surface on the intersection of the particle surface and fluid-fluid interface, $\{\widetilde{x}=0\}$ being the intersection point. Representing the boundary of the particle by $\widetilde{y}=\widetilde{y}(x)$, then $\widetilde{y}'(0)=0$. Then we have

$$\begin{aligned}
&-\int_{\Lambda^{\epsilon_0}} (f_b(\widetilde{\phi}^\epsilon(x,y,t,\xi))\mathbf{I} - \mathcal{B}\epsilon\nabla\widetilde{\phi}^\epsilon(x,y,t,\xi)\otimes\nabla\widetilde{\phi}^\epsilon(x,y,t,\xi))\cdot\mathbf{n}ds\\
&=-\int_{-\epsilon_0}^{\epsilon_0}[(f_b(\widetilde{\phi}^\epsilon(\widetilde{x},\widetilde{y}(\widetilde{x}),t,\xi))\mathbf{I} - \mathcal{B}\epsilon\nabla\widetilde{\phi}^\epsilon(\widetilde{x},\widetilde{y}(\widetilde{x}),t,\xi)\otimes\nabla\widetilde{\phi}^\epsilon(\widetilde{x},\widetilde{y}(\widetilde{x}),t,\xi)]\cdot\mathbf{n}\\
&\sqrt{1+|\widetilde{y}'(\widetilde{x})|^2}]d\widetilde{x}.
\end{aligned}$$

Since $y'(0)=0$, we may expand $\sqrt{1+|\widetilde{y}'(\widetilde{x})|^2}$ by

$$\sqrt{1+|\widetilde{y}'(\widetilde{x})|^2}=1+\widetilde{y}''(0)x+O(\widetilde{x}^2).$$

Moreover when $\widetilde{y}=0$, we have $\xi=\frac{\widetilde{x}\cos\alpha_0}{\epsilon}$, thus we have

$$\sqrt{1+|\widetilde{y}'(\widetilde{x})|^2}=1+\epsilon\xi\left(\frac{1}{\cos\alpha_0}\widetilde{y}''(0)+O(1)\right)=1+\xi O(\epsilon).$$

Then it is straightforward to see that the curved boundary does not contribute to the leading order of the boundary force term.

To derive the sharp interface limit for each intersection point between the fluid interface and particle surface, we denote by $\sum_{A_I}$ the sum taken from each intersection and derive the sharp interface limit for (54):

$$\mathcal{R}eM_s\frac{\mathrm{d}\mathbf{U}_0}{\mathrm{d}t}=-\int_{\partial P(t)}(-p_0\mathbf{I}+\eta\boldsymbol{\sigma}_0)\cdot\boldsymbol{n}ds+\sum_{A_I}\mathcal{B}F_{Ca}\boldsymbol{\tau}_I.$$

The sharp interface limit of (55) may then be derived similarly:

$$\mathcal{R}eI_s\frac{d\boldsymbol{\omega}_0}{dt}=-\int_{\partial P(t)}\mathbf{r}\times((-p_0\mathbf{I}+\eta\boldsymbol{\sigma}_0)\cdot\boldsymbol{n})ds+\sum_{A_I}\mathbf{r}\times\mathcal{B}F_{Ca}\boldsymbol{\tau}_I.$$

We may see that leading order terms of the forces on the particle coincide with the physical concept of capillary force, whose direction is parallel to the fluid-fluid interface.

Collecting all the results in the above, we may derive the leading order behavior of the governing system. The leading profiles $(\phi_0^\pm, \mu_0^\pm, \mathbf{u}_0^\pm, p_0^\pm)$ satisfy the following coupled system of Hele–Shaw equations and incompressible Navier–Stokes equations in $\Omega_0^\pm$:

$$\begin{cases} \phi_0^\pm = \pm 1, \\ \Delta \mu_0^\pm = 0, \\ \mathcal{R}e\rho \left(\dfrac{\partial \mathbf{u}_0^\pm}{\partial t} + (\mathbf{u}_0^\pm \cdot \nabla)\mathbf{u}_0^\pm \right) = -\nabla p_0^\pm + \eta \Delta \mathbf{u}_0^\pm, \\ \nabla \cdot \mathbf{u}_0^\pm = 0. \end{cases}$$

The boundary conditions on $\partial\Omega$ are

$$\frac{\partial \mu_0^\pm}{\partial n} = 0,\ \mathbf{u}_0^\pm \cdot \mathbf{n} = 0,\ \frac{1}{\mathcal{L}_s l_s}\mathbf{u}_0^\pm \cdot \boldsymbol{\tau} = -(\boldsymbol{\sigma}_0^\pm \cdot \mathbf{n}) \cdot \boldsymbol{\tau},$$

while the boundary conditions on $\partial P(t)$ are

$$\frac{\partial \mu_0^\pm}{\partial n} = 0,\ (\mathbf{u}_0^\pm - \mathbf{u}_{s,0}) \cdot \mathbf{n} = 0,\ \frac{1}{\mathcal{L}_s l_s}(\mathbf{u}_0^\pm - \mathbf{u}_{s,0}) \cdot \boldsymbol{\tau} = -(\boldsymbol{\sigma}_0^\pm \cdot \mathbf{n}) \cdot \boldsymbol{\tau}.$$

Boundary conditions on the fluid-fluid interface Γ^0 are

$$\begin{aligned} &\mu_0^\pm = -\frac{F_{Ca}}{2}\Delta d^0 + \frac{1}{4}\mathcal{R}e\mathcal{B}^{-1}[\rho(\phi_0)]|\mathbf{u}_0|^2, \\ &[\mathbf{u}_0] = 0,\ (\nabla d^0 \cdot \nabla)\mathbf{u}_0^\pm \cdot \nabla d^0 = 0, \\ &[p_0] = -\mathcal{B}F_{Ca}\Delta d^0 + \frac{1}{2}\mathcal{R}e[\rho(\phi_0)]|\mathbf{u}_0|^2. \end{aligned}$$

The dynamics of the interface is given by

$$\partial_t d^0 + (\mathbf{u}_0 \cdot \nabla) d^0 = \frac{L_d}{2}\nabla d^0 \cdot [\nabla \mu_0],$$

with the contact angle satisfying

$$\cos\alpha_0 = (\gamma_- - \gamma_+)/F_{Ca}.$$

Moreover, on $\partial\Omega$ and $\partial P(t)$ we have

$$\partial_t d^0 + (\mathbf{u}_0 \cdot \nabla) d^0 = 0.$$

Equations for the particle motion are

$$\begin{aligned} \mathcal{R}eM_s \frac{\mathrm{d}\mathbf{U}_0}{\mathrm{dt}} &= -\int_{\partial P(t)} (-p_0\mathbf{I} + \eta\boldsymbol{\sigma}_0) \cdot \boldsymbol{n} ds + \sum_{A_I} \mathcal{B}F_{Ca}\boldsymbol{\tau}_I, \\ \mathcal{R}eI_s \frac{d\boldsymbol{\omega}_0}{dt} &= -\int_{\partial P(t)} \mathbf{r} \times ((-p_0\mathbf{I} + \eta\boldsymbol{\sigma}_0) \cdot \boldsymbol{n}) ds + \sum_{A_I} \mathbf{r} \times \mathcal{B}F_{Ca}\boldsymbol{\tau}_I, \end{aligned}$$

where $\sum_{A_I}$ is taken from each intersection point between Γ^0 and $\partial P(t)$.

Similar to the governing system (2)–(17), we can also derive the energy decaying property for the sharp interface limit. For an arbitrary curve Γ, let $|\Gamma|$ denote its length. The total energy F^0 of the sharp interface limit is defined as

$$F^0 := \mathcal{R}eF_k^0 + \mathcal{B}F_b^0 + \mathcal{R}eM_sF_{pm}^0 + \mathcal{R}eI_sF_{pr}^0 + \mathcal{B}F_{\partial P(t)}^0 + \mathcal{B}F_{\partial\Omega}^0,$$

$$F_k^0 := \int_{\Omega\setminus P(t)} \frac{1}{2}\rho(\phi_0)|\mathbf{u}_0|^2 dx, \quad F_b^0 := F_{ca}|\Gamma^0|, \quad F_{pm}^0 := \frac{1}{2}|\mathbf{U}_0|^2, \quad F_{pr}^0 := \frac{1}{2}|\boldsymbol{\omega}_0|^2,$$

$$F_{\partial P(t)}^0 := \gamma^+|\partial P(t)\cap\overline{\Omega_0^+}| + \gamma^-|\partial P(t)\cap\overline{\Omega_0^-}|, \quad F_{\partial\Omega}^0 := \gamma^+|\partial\Omega\cap\overline{\Omega_0^+}| + \gamma^-|\partial\Omega\cap\overline{\Omega_0^-}|.$$

If we assume that the wall speed $\mathbf{u}_w = 0$ and the external force $\mathbf{g} = 0$, similar to the phase-field model in this paper, the total energy of the sharp interface limit also decays over time, such that

$$\begin{aligned}
&\frac{d}{dt}(\mathcal{R}e\rho F_k^0 + \mathcal{B}F_b^0 + \mathcal{R}eM_sF_{pm}^0 + \mathcal{R}eI_sF_{pr}^0 + \mathcal{B}F_{\partial P(t)}^0 + \mathcal{B}F_{\partial\Omega}^0) \\
&\quad = -\int_{\Omega\setminus P(t)} \frac{\eta}{2}|\boldsymbol{\sigma}_0|_F^2 dx - \mathcal{B}L_d\int_{\Omega_0^+}|\nabla\mu_0^+|^2 dx - \mathcal{B}L_d\int_{\Omega_0^-}|\nabla\mu_0^-|^2 dx \\
&\qquad - \int_{\partial P(t)} \frac{\eta}{\mathcal{L}_s l_s}|(\mathbf{u}_0 - \mathbf{u}_{s,0})\cdot\boldsymbol{\tau}|^2 ds - \int_{\partial\Omega}\frac{\eta}{\mathcal{L}_s l_s}|\mathbf{u}_0\cdot\boldsymbol{\tau}|^2 ds.
\end{aligned}$$

6. Conclusions.

In this paper, a new phase-field model is constructed for the fluid-particle interaction problem in two-phase flows. Our model may be derived by the principle of minimal dissipation. Moreover our model satisfies the energy decaying property. We also derive the sharp interface limit of the governing equations. There are various future works to be done on the model introduced in this paper, such as developing efficient numerical schemes for our model which maintain the energy decaying property.

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