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Stochastic representation of solution to nonlocal-in-time diffusion☆

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Abstract

The aim of this paper is to derive a stochastic representation of the solution to a nonlocal-in-time evolution equation (with a historical initial condition), which serves a bridge between normal diffusion and anomalous diffusion. We first derive the Feynman–Kac formula by reformulating the original model into an auxiliary Caputo-type evolution equation with a specific forcing term subject to certain smoothness and compatibility conditions. After that, we confirm that the stochastic formula also provides the solution in the weak sense even though the problem data is nonsmooth. Finally, numerical experiments are presented to illustrate the theoretical results and the application of the stochastic formula.

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Keywords: Nonlocal evolution; Historical initial condition; Feynman–Kac formula

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1. Introduction

In this paper, we study the nonlocal-in-time evolution equation

$$\begin{cases} D_\infty^{(\rho)}u(t,x) - \Delta u(t,x) = f(t,x), & (t,x)\in(0,T]\times\Omega, \\ u(t,x)=0, & (t,x)\in(0,T]\times\partial\Omega, \\ u(t,x)=\phi(t,x), & (t,x)\in(-\infty,0]\times\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^d$ is a regular domain, the functions f and ϕ are given data, and $D_\infty^{(\rho)}$ denotes the nonlocal operator defined by

$$D_\infty^{(\rho)}u(t,x) := \int_0^\infty (u(t,x)-u(t-r,x))\rho(t,r)\,dr, \tag{1.2}$$

with the nonnegative kernel function $\rho(t,r)$ satisfying certain hypotheses (see details in Section 2). The nonlocal operator $-D_\infty^{(\rho)}$ is proved to be the Markovian generator of a $(-\infty,T]$-valued decreasing Lévy-type process, denoted by $-X^{t,(\rho)}$ when started at $t\in[0,T]$. We denote by B^x a d-dimensional Brownian motion started at $x\in\mathbb{R}^d$ generated by the Laplacian Δ. The processes $-X^{t,(\rho)}$ and B^x are always assumed to be independent.

The aim of the current work is to derive a stochastic representation for the solution to the problem (1.1) with the historical initial condition. Besides their theoretical importance, stochastic representations are extensively used in applications, e.g., to compute solutions through the particle tracking method (see, e.g., [43,45]). It is a deep and classical result that the solution to the diffusion equation

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x), & (t,x)\in(0,T]\times\mathbb{R}^d, \\ u(0,x)=\phi(0,x), & x\in\mathbb{R}^d, \end{cases}$$

allows the stochastic representation $u(t,x) = \mathbf{E}[\phi(0,B^x(t))]$. This model describes normal diffusion phenomena that exhibits homogeneity in both space and time. With the aid of single particle tracking, recent studies have provided many examples of anomalous diffusion. One typical example is the time-fractional (sub-)diffusion model,

$$\begin{cases} \partial_t^\alpha u(t,x) = \Delta u(t,x), & (t,x)\in(0,T]\times\mathbb{R}^d, \\ u(0,x)=\phi(0,x), & x\in\mathbb{R}^d, \end{cases} \tag{1.3}$$

where ∂_t^α denotes the Caputo fractional derivative of order $\alpha\in(0,1)$, defined by

$$\partial_t^\alpha u(t,x) = \partial_t \int_0^t \frac{(t-r)^{-\alpha}}{\Gamma(1-\alpha)}(u(r,x)-u(0,x))\,dr.$$

The sub-diffusion phenomena has attracted much attention in applications such as contaminant transport in groundwater [25], protein diffusion within cells [21], and thermal diffusion in fractal media [35]. The problem (1.3) has been extensively studied both analytically and numerically (see [34, Chapter 2.4] for an overview). Its solution can be expressed by $u(t,x) = \mathbf{E}[\phi(0,Y^x(t))]$ [32], where $Y^x(t) = B^x(\tau_0^\alpha(t))$ and $\tau_0^\alpha(t) = \inf\{s>0 : X^\alpha(s)\geq t\}$ is the inverse process of the α-stable subordinator X^α. The density of $Y^x(t)$ can be derived using a conditioning argument [3,37]

$$H_{t,x}(y) = \int_0^\infty p_s(x,y)\partial_s\mathbf{P}[X^\alpha(s)\geq t]\,ds, \tag{1.4}$$

where $\partial_s\mathbf{P}[X^\alpha(s)\geq t] = \alpha^{-1}ts^{-1-1/\alpha}g_\alpha(ts^{-1/\alpha})$, with g_α being the density of $X^\alpha(1)$ and $p_s(x)$ the density of $B^x(s)$. It is interesting to observe that the time-changed Brownian motion $Y^x(t)$

displays time heterogeneity, as the non-Markovian time change $t \mapsto \tau_0^\alpha(t)$ is constant precisely when the subordinator $t \mapsto X^\alpha(t)$ jumps [34]. This leads to the past-dependent diffusion Y^x being trapped, and in general spreading at a slower rate than B^x (see e.g. [30,36,44]). Let us recall that Y^x is sometimes called fractional kinetic and it enjoys surprising universality properties [5]. It is easy to see that the Caputo fractional derivative can be written in the form (1.2) by

$$\partial_t^\alpha u(t,x) = c_\alpha \int_0^\infty (u(t,x) - u(t-r,x)) r^{-\alpha-1}\, dr,$$

with the kernel $\rho(t,r) := c_\alpha r^{-\alpha-1}$, $c_\alpha = -\Gamma(-\alpha)^{-1}$, where we extend the function u to the negative real line by $u(t) \equiv u(0)$ for $t \in (-\infty, 0)$. On the other hand, under certain hypothesis, by taking $\rho(t,r)$ to be a compactly supported function in r (with the support measured by the so-called horizon parameter), one may show that the nonlocal operator could reproduce the first order derivative, as the horizon of nonlocal effects tends to zero with suitable normalization [16]. Therefore, it is actually an interesting intermediate case between infinite-horizon fractional derivatives and infinitesimal local derivatives. Moreover, it can be shown that the nonlocal setting also serves to bridge between a short-time anomalous diffusion and a long-time normal diffusion [17] which has been observed in many experiments [22]. More discussions on connections to nonlocal modeling can be found in [15].

In case of time-independent Lévy measures and initial data, i.e. $\rho(t,r)dr \equiv \rho(dr)$ and $\phi(t,x) \equiv \phi(x)$, there exist some pioneer works about wellposedness and probabilistic representations. In [26], by Laplace transform and the theory of complete Bernstein functions, Kochubei studied a Cauchy problem involving a general Caputo-type derivative, giving assumptions on the Laplace transform of the kernel function. After that, these techniques combined with semigroup theory were applied to a homogeneous time-fractional evolution equation involving both first-order and fractional derivatives in time [40]. Recently, in [11–13], the authors provided a general and explicit method to study Caputo-type diffusion models (possibly with a first-order time derivative and a source term). See also [31,33] for the particular case of distributed-order fractional time derivatives. Finally, we note that the Caputo-type problem and its Feynman–Kac formula can be generalized in different ways, e.g., by adding a time parameter [1]. In comparison with the aforementioned works, the nonlocal-in-time model (1.1) allows a time-dependent kernel function $\rho(t,r)$, and requires a historical initial condition. However, the probabilistic investigation of this model is still largely missing in the literature, apart from the fractional case [41]. This motivates us to study the model (1.1) and derive a clean stochastic explanation, which is the main contribution of this work.

Our technique first treats an equivalent inhomogeneous Caputo-type problem, essentially by inverting $(-D_\infty^{(\rho)} + \Delta)$ when understood as an abstract generator of a Markov process taking values in $(0,T] \times \Omega$ and absorbed at $\{0\} \times \Omega$. Secondly, after explicitly computing the dual of the abstract generator, we show that the above solution is indeed a weak solution. Uniqueness of the weak solution is proved for variable separable kernels, extending results from [16]. As an example, we show that the weak solution to the homogeneous problem (for $f = 0$) allows the stochastic representation

$$\begin{aligned} u(t,x) &= \mathbf{E}\left[\phi\left(-X^{t,(\rho)}(\tau_0(t)), B^x(\tau_0(t))\right) \mathbf{1}_{\{\tau_0(t) < \tau_\Omega(x)\}}\right] \\ &= \int_{-\infty}^0 \int_\Omega \phi(r,y) H_{t,x}(r,y)\, dr\, dy, \end{aligned} \tag{1.5}$$

where $\tau_0(t) = \inf\{s > 0 : -X^{t,(\rho)}(s) \le 0\}$, $\tau_\Omega(x) = \inf\{s > 0 : x + B(s) \notin \Omega\}$ and the heat kernel is given by

$$H_{t,x}(r, y) = \int_0^t \rho(z, z-r) \left(\int_0^\infty p_s^\Omega(x, y) \partial_z \mathbf{P}[-X^{t,(\rho)}(s) \le z]\, ds \right) dz.$$

Here we denote by $p_s^\Omega(x, y)$ the density of the killed Brownian motion $B^x(s)\mathbf{1}_{\{s<\tau_\Omega(x)\}}$. Note that for the standard fractional kernel $\rho(t,r) = c_\alpha r^{-\alpha-1}$, $-X^{t,(\rho)} = t - X^\alpha$ and $\tau_0(t) = \tau_0^\alpha(t)$. The representation (1.5) appears to be new, and it suggests an interesting interpretation. This is because the diffusion on Ω is still the anomalous diffusion $Y^x(t) = B^x(\tau_0(t))$, but the contribution in time of the initial condition $\phi(\cdot, Y^x(t))$ depends on the waiting/trapping time of $Y^x(t)$, which is indeed $W(t) = X^{t,(\rho)}(\tau_0(t))$. Let us stress that as a particular case we treat Caputo-type evolution equations.

The paper is organized as follows. In Section 2, we introduce some basic settings of the nonlocal-in-time model (1.1) as well as some probabilistic background. Some popular and concrete models will be provided as examples. In Section 3, after reformulating the model (1.1) into a Caputo-type fractional diffusion problem, we develop some general solution theory, provided additional smoothness and compatibility conditions on problem data. In Section 4, we show that the candidate stochastic representation provides a weak solution of (1.1) even though the data is weak. Finally, some numerical experiments will be presented in order to illustrate our theoretical findings. Throughout, the notation c denotes a generic positive constant, whose value may differ at each occurrence.

2. Preliminaries

2.1. General notation

We denote by $\mathbb{N}$, $\mathbb{R}^+$, $\mathbb{R}^d$, $a \wedge b$, $\Gamma(\cdot)$, $\mathbf{1}_E(\cdot)$ and a.e., the set of positive integers, the set of non-negative real numbers, the d-dimensional Euclidean space, the minimum between $a, b \in \mathbb{R}$, the gamma function, the indicator function of the set E and the statement almost everywhere with respect to Lebesgue measure, respectively. To ease notation, $F(I) = FI$ whenever $F(I)$ is a space of real-valued functions on an interval $I \subset \mathbb{R}$. We denote by $\|\cdot\|_B$ the norm of a Banach space B, and if $\mathcal{L}$ is a bounded linear operator between Banach spaces, we denote its operator norm by $\|\mathcal{L}\|$. We denote by $C(E)$ the space of real-valued continuous functions on $E \subset \mathbb{R}^d$, and by $B(E)$ the space of real-valued bounded and measurable functions on E. For any $T \ge 0$, we define the Banach spaces

$$\begin{aligned} C_\infty(-\infty, T] &= \{f \in C(-\infty, T] : f(t) \to 0 \text{ as } t \to -\infty\}, \\ C_0[0, T] &= \{f \in C[0, T] : f(0) = 0\}, \end{aligned}$$

both equipped with the supremum norm. We will also use the standard spaces $C^1[0, T] = \{f, f' \in C[0, T]\}$, $C^1_0[0, T] = \{f, f' \in C_0[0, T]\}$ and $C^1_\infty(-\infty, T] = \{f, f' \in C_\infty(-\infty, T]\}$. Then we reserve specific notation for several Banach spaces of continuous functions. Namely, for a bounded open set $\Omega \subset \mathbb{R}^d$ and $T \ge 0$ we write

$$\begin{aligned} C_{\partial\Omega}(\Omega) &= \{f \in C(\overline{\Omega}) : f = 0 \text{ on } \partial\Omega\}, \\ C_{\partial\Omega}([0, T] \times \Omega) &= \{f \in C([0, T] \times \overline{\Omega}) : f = 0 \text{ on } [0, T] \times \partial\Omega\}, \\ C_{0,\partial\Omega}([0, T] \times \Omega) &= \{f \in C_{\partial\Omega}([0, T] \times \Omega) : f = 0 \text{ on } \{0\} \times \overline{\Omega}\}, \\ C_{\infty,\partial\Omega}((-\infty, T] \times \Omega) &= \{f \in C((-\infty, T] \times \overline{\Omega}) : f = 0 \text{ on } \partial\Omega, \\ &\qquad f(\cdot, x) \in C_\infty(-\infty, T] \text{ for } x \in \overline{\Omega}\}. \end{aligned}$$

We denote by $L^p(E)$, $p \in [1,\infty]$ the usual Banach spaces of Lebesgue p-integrable real-valued functions on E. We define $L^p(a,b;B) = \{f : (a,b) \to B \text{ such that } t \mapsto \|f(t)\|_B \in L^p(a,b)\}$, for $p \in [1,\infty]$ and $(a,b) \subset \mathbb{R}$, and by $L^p_{loc}(a,b;B)$ the locally p-integrable version. If F and $\tilde{F}$ are two sets of real-valued functions, we define $F \cdot \tilde{F} := \{f\tilde{f} : f \in F, \tilde{f} \in \tilde{F}\}$, and we denote by $\mathrm{Span}\{F\}$ the set of all linear combination of elements in F.

The notation we use for an E-valued stochastic process started at $x \in E$ is $X^x = \{X^x(s)\}_{s\geq 0}$. Note that the symbol t will often be used to denote the starting point of a stochastic process with state space $E \subset \mathbb{R}$. By a *strongly continuous contraction semigroup* T we mean a collection of linear operators $T_s : B \to B$, $s \geq 0$, where B is a Banach space, such that $T_{s+r} = T_s T_r$, for every $s, r \geq 0$, T_0 is the identity operator, $\lim_{s\downarrow 0} T_s f = f$ in B, for every $f \in B$, and $\sup_s \|T_s\| \leq 1$. The generator of the semigroup T is defined as the pair $(\mathcal{L}, \mathrm{Dom}(\mathcal{L}))$, where $\mathrm{Dom}(\mathcal{L}) := \{f \in B : \mathcal{L}f := \lim_{s\downarrow 0} s^{-1}(T_s f - f) \text{ exists in } B\}$. We say that a set $C \subset \mathrm{Dom}(\mathcal{L})$ is a *core for* $(\mathcal{L}, \mathrm{Dom}(\mathcal{L}))$ if the generator equals the closure of the restriction of $\mathcal{L}$ to C. We say that a set $C \subset B$ is *invariant under* T if $T_s C \subset C$ for every $s > 0$. If a set C is invariant under T and a core for $(\mathcal{L}, \mathrm{Dom}(\mathcal{L}))$, then we say that C is an *invariant core for* $(\mathcal{L}, \mathrm{Dom}(\mathcal{L}))$. Recall that if C is a dense subspace of $\mathrm{Dom}(\mathcal{L})$ and C is invariant under T, then C is an invariant core for $(\mathcal{L}, \mathrm{Dom}(\mathcal{L}))$ (see [9, Lemma 1.34]), and that $\mathrm{Dom}(\mathcal{L})$ is invariant under T. For a given $\lambda \geq 0$ we define the *resolvent of* T by $(\lambda - \mathcal{L})^{-1} := \int_0^\infty e^{-\lambda s} T_s \, ds$, and recall that for $\lambda > 0$, $(\lambda - \mathcal{L})^{-1} : B \to \mathrm{Dom}(\mathcal{L})$ is a bijection and it solves the abstract resolvent equation

$$\mathcal{L}(\lambda - \mathcal{L})^{-1} f = \lambda(\lambda - \mathcal{L})^{-1} f - f, \quad f \in B,$$

see for example [18, Theorem 1.1]. By a *Feller semigroup* we mean a strongly continuous contraction semigroup T on any of the Banach spaces of continuous functions defined above such that T preserves non-negative functions. A Feller semigroup T is said to be *conservative* if the extension of T to bounded measurable functions preserves constants. Feller semigroups are in one-to-one correspondence with Feller processes, where a Feller process is a time-homogeneous sub-Markov process $\{X(s)\}_{s\geq 0}$ such that $s \mapsto T_s f(x) := \mathbf{E}[f(X(s)) | X(0) = x]$, $f \in B$ is a Feller semigroup [9, Chapter 1.2]. We recall that every Feller process admits a cádlág modification which enjoys the strong Markov property [9, Theorem 1.19 and Theorem 1.20], and we always work with such modification. For further discussions on these terminologies and notations, we refer to [9].

2.2. *Nonlocal operators and related stochastic processes*

Next, we review some basics on the nonlocal operators, along with some properties and related definitions.

(**H0**) The function $\rho : \mathbb{R} \times (0,\infty) \to [0,\infty)$ is continuous and continuously differentiable in the first variable. Furthermore,

$$\int_0^\infty (1 \wedge r) \sup_t \rho(t,r) \, dr < \infty, \quad \int_0^\infty (1 \wedge r) \sup_t \left| \partial_t \rho(t,r) \right| dr < \infty,$$

and

$$\lim_{\delta \to 0} \sup_t \int_{0<r\leq\delta} r\rho(t,r) \, dr = 0.$$

Moreover, there exist $\epsilon > 0$ and $\gamma > 0$, such that the function ρ satisfies $\rho(t,r) \geq \gamma > 0$ for all t and $|r| < \epsilon$.

Definition 2.1. For any kernel function ρ satisfying condition (H0), the *Marchaud-type derivative* $D_\infty^{(\rho)}$ and the *Caputo-type derivative* $D_0^{(\rho)}$ are respectively defined by

$$D_\infty^{(\rho)} u(t) := \int_0^\infty (u(t) - u(t-r))\rho(t,r)\,dr, \qquad t \in (-\infty, T], \tag{2.1}$$

$$D_0^{(\rho)} u(t) := \int_0^t (u(t) - u(t-r))\rho(t,r)\,dr + (u(t) - u(0)) \int_t^\infty \rho(t,r)\,dr, \qquad t \in (0, T], \tag{2.2}$$

and $D_0^{(\rho)} u(0) := \lim_{t\downarrow 0} D_0^{(\rho)} u(t)$.

Under the assumption (H0), the Marchaud-type derivative (2.1) (as well as (2.2)) is well-defined pointwise for regular functions, e.g., $u \in L^\infty(-\infty, T) \cap C^1(-\infty, T]$. In fact, it is easy to observe that for any $t \in (-\infty, T]$

$$\begin{aligned} |D_\infty^{(\rho)} u(t)| &\le \left| \int_0^1 \int_0^r u'(t-y)\,dy \rho(t,r)\,dr \right| + \left| \int_1^\infty (u(t) - u(t-r))\rho(t,r)\,dr \right| \\ &\le \|u'\|_{L^\infty(t-1,t)} \int_0^1 r\rho(t,r)\,dr + 2\|u\|_{L^\infty(-\infty,t)} \int_1^\infty \rho(t,r)\,dr \le c. \end{aligned}$$

Moreover, we can prove that $D_\infty^{(\rho)} u \in L^p(0,T)$ for any $u \in W^{1,p}(-\infty, T)$ (see details in Lemma 4.2). Also, the operator could be defined in weak sense, i.e. Definition 4.5, which is useful in the study of PDE theories.

The operator $D_\infty^{(\rho)}$ can be seen as the left-sided generalization of the Marchaud derivative [38, eq. (5.57) and (5.58)]. It is also known as the generator form of fractional derivatives [27,34], or a Lévy-type generator [9].

Example 2.2. We mention some concrete and popular examples of the nonlocal operators.

(i) By setting $\rho(t,r) = -r^{-\alpha-1}/\Gamma(-\alpha)$ with $\alpha \in (0,1)$, the nonlocal operator $D_0^{(\rho)}$ reproduces the Caputo fractional derivative [14], and $D_\infty^{(\rho)}$ the Marchaud fractional derivative [38].

(ii) The operator $\mathcal{G}_\delta$, defined in [16, formula (1.2)] with a finite horizon parameter δ, is a special case of the Marchaud-type derivative $D_\infty^{(\rho)}$ with a time-independent and compactly supported kernel function, see [15] for more discussions on nonlocal operators with a finite horizon.

(iii) Other particular cases include the fractional derivatives of variable order, which are obtained by taking ρ as the function $\rho(t,r) = -r^{-1-\alpha(t)}/\Gamma(-\alpha(t))$ with a suitable function $\alpha(t) : \mathbb{R} \to (0,1)$ [23], and tempered Lévy kernels $\rho(t,r) = -e^{-\lambda r} r^{-1-\alpha}/\Gamma(-\alpha)$, $\alpha \in (0,1)$, $\lambda > 0$, [10,42].

Remark 2.3. The nonlocal derivatives $-D_\infty^{(\rho)}$ and $-D_0^{(\rho)}$ have a clear probabilistic interpretation. The former tells us that the process at t makes a negative jump of size $|r|$ with intensity $\rho(t,r)$. The latter tells us that, as long as the jump does not cross 0, the process jumps from t to $t-r$ with intensity $\rho(t,r)$. Otherwise, it gets killed with rate/intensity $\int_t^\infty \rho(t,r)\,dr$ and regenerated at 0 with the same rate, where it remains absorbed. This will be made rigorous in Definition 2.4 and Proposition 2.6.

2.3. Probabilistic interpretation and preliminary results

In this section, we discuss three stochastic processes generated by the operators defined in (2.1) and (2.2) with kernel functions satisfying (H0).

Definition 2.4. Assume (H0).

(i) [27, Theorem 5.1.1]: Let $T^{(\rho),\infty} = \{T_s^{(\rho),\infty}\}_{s\geq 0}$ be the Feller semigroup on $C_\infty(-\infty, T]$ with the generator

$$\left(\mathcal{L}_\infty^{(\rho)}, \mathrm{Dom}(\mathcal{L}_\infty^{(\rho)})\right) \text{ being the closure of } \left(-D_\infty^{(\rho)}, C_\infty^1(-\infty, T]\right),$$

and recall that $C_\infty^1(-\infty, T]$ is invariant under $T^{(\rho),\infty}$.
We denote the induced Feller process by

$$-X^{t,(\rho)} = \{-X^{t,(\rho)}(s)\}_{s\geq 0}, \quad t \in (-\infty, T].$$

(ii) [28, Theorem 4.1]: Let $T^{(\rho)} = \{T_s^{(\rho)}\}_{s\geq 0}$ be the Feller semigroup on $C[0, T]$ with the generator

$$\left(\mathcal{L}^{(\rho)}, \mathrm{Dom}(\mathcal{L}^{(\rho)})\right) \text{ being the closure of } \left(-D_0^{(\rho)}, C^1[0, T]\right),$$

and recall that $C^1[0, T]$ is invariant under $T^{(\rho)}$.
We denote the induced Feller process by $-X_0^{t,(\rho)} = \{-X^{t,(\rho)}(s)\mathbf{1}_{\{s<\tau_0(t)\}}\}_{s\geq 0}$, $t \in [0, T]$.

(iii) We denote by $T^{(\rho),\mathrm{kill}} = \{T_s^{(\rho)}\}_{s\geq 0}$ the Feller semigroup on $C_0[0, T]$ with the generator

$$\left(\mathcal{L}_{\mathrm{kill}}^{(\rho)}, \mathrm{Dom}(\mathcal{L}_{\mathrm{kill}}^{(\rho)})\right) \text{ being the closure of } \left(-D_0^{(\rho)}, C_0^1[0, T]\right),$$

and $C_0^1[0, T]$ is invariant under $T^{(\rho),\mathrm{kill}}$.
We denote the induced Feller process by $-X_0^{t,(\rho),\mathrm{kill}} = \{-X_0^{t,(\rho),\mathrm{kill}}(s)\}_{s\geq 0}$, $t \in (0, T]$.

Remark 2.5. The next proposition justifies the notation for the stochastic processes $-X_0^{t,(\rho)}$ and Definition 2.4(iii). The proof of parts (i), (ii) and (iii) is given in [24, Proposition 2.7], and hence omitted here. Part (iv) can be proved by the same argument for Lemma 3.4.

Proposition 2.6.

(i) The processes $-X^{t,(\rho)}$, $-X_0^{t,(\rho)}$ and $-X_0^{t,(\rho),kill}$ are non-increasing and

$$\mathbf{P}[-X^{t,(\rho)}(s) \in (a, b)] = \mathbf{P}[-X_0^{t,(\rho)}(s) \in (a, b)] = \mathbf{P}[-X_0^{t,(\rho),kill}(s) \in (a, b)],$$

for every $t \in (0, T]$, $0 < a < b \leq T$, $s > 0$. In particular $\mathbf{P}[-X_0^{t,(\rho)}(s) \in \{0\}] = \mathbf{P}[-X^{t,(\rho)}(s) \leq 0]$, for every $t \in [0, T]$, $s > 0$.

(ii) The law of

$$\tau_0(t) := \inf\{s > 0 : -X^{t,(\rho)}(s) \leq 0\}, \quad t \in (-\infty, T],$$

equals the law of the first exit time from the interval $(0, T]$ of the processes $-X_0^{t,(\rho)}$ for each $t \in (0, T]$ (so that we will use indistinctly the same notation $\tau_0(t)$).

(iii) The expectation of $\tau_0(t)$ is uniformly bounded, i.e., $\sup_{t\in[0,T]} \mathbf{E}[\tau_0(t)] < \infty$.

(iv) It holds that $\left(\mathcal{L}_{kill}^{(\rho)}, Dom(\mathcal{L}_{kill}^{(\rho)})\right) = \left(\mathcal{L}^{(\rho)}, Dom(\mathcal{L}^{(\rho)}) \cap \{f(0) = 0\}\right)$.

Remark 2.7.

(i) It follows from Proposition 2.6 that the process $-X_0^{t,(\rho)}$ is obtained by absorbing at the point 0 the process $-X^{t,(\rho)}$ on its first attempt to leave the interval $(0, T]$.

(ii) Definition 2.4(ii) (Definition 2.4(iii)) could be a proposition derived from absorbing at 0 (killing on crossing 0) the process $-X^{t,(\rho)}$, $t > 0$.

(iii) If the Lévy kernel is independent of t, i.e. $\rho(t,r) = \rho(r)$, then $-X^{t,(\rho)}(s) = t - X^{(\rho)}(s)$ is the non-increasing Lévy process with generator $-D_\infty^{(\rho)}$ acting on $C_c^\infty(\mathbb{R})$, where $X^{(\rho)}$ is the subordinator with Lévy measure $\rho(r)dr$. This is a consequence of the fact that $\mathcal{L}_\infty^{(\rho)} = -D_\infty^{(\rho)}$ on $C_c^\infty(\mathbb{R}) \subset \mathrm{Dom}(\mathcal{L}_\infty^{(\rho)})$, and [9, Theorem 2.7].

(iv) If the kernel $\rho(t,r) = \rho(r)$ is integrable, then $-D_\infty^{(\rho)}$ is the generator of a non-increasing compound Poisson process.

Remark 2.8. The assumption (H0) could be replaced with an alternative one, as long as $-D_\infty^{(\rho)}$ generates a non-increasing Feller process with the first exit times from $(0, T]$ having finite expectation, along with the existence of invariant cores with the properties in Definition 2.4. Nevertheless, the assumption (H0) provides a satisfactory level of generality for most of the applications we have in mind.

Finally, we use one more assumption on the stochastic process $X^{t,(\rho)}$.

(**H1**) The law of $-X^{t,(\rho)}(s)$ is absolutely continuous with respect to Lebesgue measure for each $t \in [0, T]$, $s > 0$, and we denote such density by $p_s^{(\rho)}(t)$. Furthermore assume that $\mathbf{P}[-X^{t,(\rho)}(\tau_0(t)) \in \{0\}] = 0$, for each $t \in (0, T]$.

Remark 2.9. Assumption (H1) ensures the existence of the probability density function $p_s^{(\rho)}(t)$, which helps us handle the weak problem data (see Theorem 3.10(ii)). Otherwise, without (H1), we could assume that the problem data g in Theorem 3.10(ii) is a Baire class 1 function (Remark 3.11). This would allow us to handle several cases, such as ρ being integrable [39, Remark 27.3].

Remark 2.10. Assumption $\mathbf{P}[-X^{t,(\rho)}(\tau_0(t)) \in \{0\}] = 0$ is implied by the existence of a density $p_s^{(\rho)}(t)$ if $\rho(t,r)dr = \rho(dr)$. This is because the existence of a density implies that $\rho((0,\infty)) = \infty$, as $X^{(\rho)}$ cannot be a compound Poisson process. Then $\tau_0(t) = \inf\{s > 0 : X^{(\rho)}(s) > t\}$, the right inverse of $X^{(\rho)}$, and one can apply [6, III, Theorem 4]. Here $X^{(\rho)}$ is the increasing subordinator with Lévy measure $\rho(dr)$.

Example 2.11. We list some examples where the densities $p_s^{(\rho)}(t)$, $t, s > 0$, exist:

(i) kernels $\rho(t,r)dr = \rho(dr)$ such that $\rho(dr) \geq r^{-1-\alpha}dr$ for all small r [39, Proposition 28.3];

(ii) kernels $\rho(t,r) = \rho(r)$ such that $\int_0^\infty \rho(r)\,dr = \infty$ [39, Theorem 27.7];

(iii) kernels $\rho(t,r)$ such that the respective symbols satisfy the Hölder continuity-type conditions in [29, Theorem 2.14];

(iv) see [20] for another set of assumptions for kernels of the type $\rho(t,r) = p(t)q(r)$ and a literature discussion.

2.4. The spatial operator Δ

Definition 2.12. For a bounded open set $\Omega \subset \mathbb{R}^d$ we say that $z \in \partial\Omega$ *is a regular point for* Ω, if there exists a right circular finite cone with vertex at z, denoted by V_z, such that $V_z \subset \Omega^c$. We say a bounded open set $\Omega \subset \mathbb{R}^d$ *is regular* if every $z \in \partial\Omega$ is a regular point for Ω.

Remark 2.13. From now on, we always assume that $\Omega \subset \mathbb{R}^d$ is a regular set. In particular, every Lipschitz domain is regular.

Definition 2.14. Let $\Omega \subset \mathbb{R}^d$ be a regular set. Let $(\Delta_\Omega, \mathrm{Dom}(\Delta_\Omega))$ be the generator of the Feller semigroup $T^\Omega = \{T_s^\Omega\}_{\geq 0}$ on $C_{\partial\Omega}(\Omega)$, where $T_s^\Omega f(x) := \mathbf{E}[f(B^x(s))\mathbf{1}_{\{s<\tau_\Omega(x)\}}]$, $s \geq 0$, $x \in \Omega$, with $B^x(s) = x + B(2s)$, $s \geq 0$, $x \in \Omega$, $\{B(s)\}_{s\geq 0}$ being the standard d-dimensional Brownian motion, and define the first exit times

$$\tau_\Omega(x) := \inf\{s > 0 : B^x(s) \notin \Omega\}, \quad x \in \Omega.$$

Remark 2.15. Recall that $\mathrm{Dom}(\Delta_\Omega) = \{f \in C_{\partial\Omega}(\Omega) \cap C^2(\Omega) : \Delta f \in C_{\partial\Omega}(\Omega)\}$ (see, e.g., [4, Theorem 2.3]). We write $\Delta_\Omega = \Delta$ from now on. We denote the law of $B^x(s)\mathbf{1}_{\{s<\tau_\Omega(x)\}}$ by $p_s^\Omega(x, y)dy$, recalling that $(x, y) \mapsto p_s^\Omega(x, y)$ is continuous for each $s > 0$.

Remark 2.16. The arguments in Section 3 could be extended to the case where the Laplacian Δ is replaced by an operator whose semigroup on $C_{\partial\Omega}(\Omega)$ allows a density function $p_s^\Omega(x, y)$ with respect to Lebesgue measure for positive time (i.e. the respective version of the first part of assumption (H1)). The restricted fractional Laplacian is an example of such operator (see, e.g., [7,8]).

2.5. The inhomogeneous Caputo-type evolution equation

In order to study the stochastic representation and wellposedness of the solution to problem (1.1), we consider the following equivalent form

$$\begin{cases} (-D_0^{(\rho)} + \Delta)u(t,x) = -g(t,x), & \text{in } (0,T] \times \Omega, \\ u(t,x) = 0, & \text{in } (0,T] \times \partial\Omega, \\ u(0,x) = \phi(0,x), & \text{in } \Omega, \end{cases} \tag{2.3}$$

with the forcing term $g = f + f_\phi$, where we define

$$f_\phi(t,x) := \int_t^\infty (\phi(t-r,x) - \phi(t,x))\rho(t,r)\,dr, \quad \text{in } (0,T] \times \Omega.$$

Notice that $f_\phi = -D_\infty^{(\rho)}\phi$, for ϕ extended to $\phi(0)$ on $(0,T] \times \overline{\Omega}$, and $D_\infty^{(\rho)}u = D_0^{(\rho)}u - f_\phi$ for any smooth u such that $u = \phi$ on $(-\infty, 0]$. In the following section, we shall discuss the probabilistic representation of the solution to (1.1) with the help of the reformulation (2.3), provided certain hypothesis on problem data.

3. General theory

In this section, we study the solution theory of the nonlocal problem. To this end, we begin with the study of some time–space compound semigroups which are constructed using temporal semigroups and spatial ones. This allows us to treat the Caputo-type evolution equation (2.3) as an elliptic boundary value problem.

3.1. Time–space compound semigroups

The next lemma shows that $\{T_s^{(\rho)}T_s^{\Omega}\}_{s\geq 0}$ is a well-defined Feller semigroup on $C_{\partial\Omega}([0,T]\times\Omega)$ such that its generator is the closure of $-D_0^{(\rho)}+\Delta$.

Lemma 3.1. *With the notation of Definitions 2.4 and 2.14, the operators*

$$T^{(\rho),\Omega} := \{T_s^{(\rho)}T_s^{\Omega}\}_{s\geq 0}$$

form a Feller semigroup on $C_{\partial\Omega}([0,T]\times\Omega)$, whose generator $(\mathcal{L}_{\Omega}^{(\rho)}, Dom(\mathcal{L}_{\Omega}^{(\rho)}))$ is the closure of

$$\left(-D_0^{(\rho)}+\Delta,\ Span\left\{C^1[0,T]\cdot Dom(\Delta_\Omega)\right\}\right) \quad in \quad C_{\partial\Omega}([0,T]\times\Omega),$$

where $T^{(\rho)}$ and $-D_0^{(\rho)}$ act on the $[0,T]$-variable, and T^{Ω} and Δ act on the Ω-variable.

Proof. It is straightforward to show that $T^{(\rho),\Omega}$ is a Feller semigroup by observing that

$$T_s^{(\rho)}T_r^{\Omega} = T_r^{\Omega}T_s^{(\rho)}, \qquad \text{for every} \quad s,r\geq 0,$$

and the contraction property

$$\|T_s^{\Omega}f\|_{C([0,T]\times\overline{\Omega})} \leq \|f\|_{C([0,T]\times\overline{\Omega})} \quad \text{and} \quad \|T_s^{(\rho)}f\|_{C([0,T]\times\overline{\Omega})} \leq \|f\|_{C([0,T]\times\overline{\Omega})},$$

holds for every $f\in C_{\partial\Omega}([0,T]\times\Omega)$, $s>0$. We denote the generator of $T^{(\rho),\Omega}$ by $(\mathcal{L}_{\Omega}^{(\rho)}, \mathrm{Dom}(\mathcal{L}_{\Omega}^{(\rho)}))$. Let $f=pq$, where $p\in C^1[0,T]$ and $q\in\mathrm{Dom}(\Delta_\Omega)$. Then $\mathcal{L}^{(\rho)}p = -D_0^{(\rho)}p$ from Definition 2.4(ii), and by a standard triangle inequality argument, we obtain

$$\begin{aligned}
&\left|\frac{T_h^{(\rho)}T_h^{\Omega}f(t,x)-f(t,x)}{h} - (-D_0^{(\rho)}+\Delta)f(t,x)\right| \\
&\leq \|p\|_{C[0,T]}\left\|\frac{T_h^{\Omega}q-q}{h}-\Delta q\right\|_{C(\overline{\Omega})} + \|\Delta q\|_{C(\overline{\Omega})}\left\|T_h^{(\rho)}p-p\right\|_{C[0,T]} \\
&\quad + \|q\|_{C(\overline{\Omega})}\left\|\frac{T_h^{(\rho)}p-p}{h}+D_0^{(\rho)}p\right\|_{C[0,T]} \to 0
\end{aligned}$$

as $h\downarrow 0$. As a result, $\mathcal{L}_{\Omega}^{(\rho)} = (-D_0^{(\rho)}+\Delta)$ on $\mathrm{Span}\{C^1[0,T]\cdot\mathrm{Dom}(\Delta_\Omega)\}\subset\mathrm{Dom}(\mathcal{L}_{\Omega}^{(\rho)})$.

Next, we aim to show that $\mathrm{Span}\{C^1[0,T]\cdot\mathrm{Dom}(\Delta_\Omega)\}$ is dense in $C_{\partial\Omega}([0,T]\times\Omega)$. It is enough to show that $\mathrm{Span}\{C^\infty[0,T]\cdot C_c^\infty(\Omega)\}$ is dense in $C_{\partial\Omega}([0,T]\times\Omega)$ by the inclusion

$$\mathrm{Span}\{C^\infty[0,T]\cdot C_c^\infty(\Omega)\}\subset\mathrm{Span}\{C^1[0,T]\cdot\mathrm{Dom}(\Delta_\Omega)\}.$$

To this end, we notice that $\mathrm{Span}\{C^\infty[0,T]\cdot C^\infty(\overline{\Omega})\}$ is a sub-algebra of $C([0,T]\times\overline{\Omega})$ that contains constant functions and separates points. Hence $\mathrm{Span}\{C^\infty[0,T]\cdot C^\infty(\overline{\Omega})\}$ is dense in $C([0,T]\times\overline{\Omega})$ by Stone–Weierstrass Theorem for compact Hausdorff spaces. Then for $f\in C_{\partial\Omega}([0,T]\times\Omega)$ we take a sequence $\{f_n\}_{n\in\mathbb{N}}\subset\mathrm{Span}\{C^\infty[0,T]\cdot C^\infty(\overline{\Omega})\}$ such that $f_n\to f$. Pick functions $\{1_{K_n}\}_{n\in\mathbb{N}}\subset C_c^\infty(\Omega)$ such that $0\leq 1_{K_n}\leq 1$, $1_{K_n}(x)=1$ for $x\in K_n$, and $1_{K_n}(x)=0$ for $x\in\Omega\backslash K_{n+1}$, where K_n is compact and $K_n\subset K_{n+1}\subset\Omega$ for each n, and $\cup_n K_n=\Omega$. Define $\tilde{f}_n := 1_{K_n}f_n\in\mathrm{Span}\{C^\infty[0,T]\cdot C_c^\infty(\Omega)\}$ for each $n\in\mathbb{N}$. Then, as $n\to\infty$

$$\begin{aligned}
\|\tilde{f}_n - f\|_{C([0,T]\times\Omega)} \leq\ & \|\tilde{f}_n - f\|_{C([0,T]\times K_n)} + \|\tilde{f}_n - f\|_{C\left([0,T]\times K_{n+1}\backslash K_n\right)} \\
& + \|\tilde{f}_n - f\|_{C\left([0,T]\times\overline{\Omega}\backslash K_{n+1}\right)}
\end{aligned}$$

$$
\begin{aligned}
&= \|f_n - f\|_{C([0,T]\times K_n)} + \|\tilde{f}_n - f\|_{C([0,T]\times K_{n+1}\setminus K_n)} \\
&\quad + \|f\|_{C([0,T]\times\overline{\Omega}\setminus K_{n+1})} \\
&\to 0.
\end{aligned}
$$

Then the density of $\mathrm{Span}\{C^1[0,T]\cdot \mathrm{Dom}(\Delta_\Omega)\}$ in $C_{\partial\Omega}([0,T]\times\Omega)$ together with the fact that $\mathrm{Span}\{C^1[0,T]\cdot \mathrm{Dom}(\Delta_\Omega)\}$ is invariant under $T^{(\rho),\Omega}$ and a subspace of $\mathrm{Dom}(\mathcal{L}_\Omega^{(\rho)})$ completes the proof by [9, Lemma 1.34]. □

Then a similar argument shows the following corollary.

Corollary 3.2. *With the notation of Definitions 2.4 and 2.14, it holds that:*

(i) the operators $T^{(\rho),kill,\Omega} := \{T_s^{(\rho),kill}T_s^{\Omega}\}_{s\ge 0}$ form a Feller semigroup on $C_{0,\partial\Omega}([0,T]\times\Omega)$. The generator $(\mathcal{L}_\Omega^{(\rho),kill}, Dom(\mathcal{L}_\Omega^{(\rho),kill}))$ of $T^{(\rho),kill,\Omega}$ is the closure of

$$\left(-D_0^{(\rho)} + \Delta,\ Span\{C_0^1[0,T]\cdot Dom(\Delta_\Omega)\}\right) \quad in \quad C_{0,\partial\Omega}([0,T]\times\Omega),$$

where $T^{(\rho),kill}$ and $-D_0^{(\rho)}$ act on the $[0,T]$-variable, and T^{Ω} and Δ act on the Ω-variable.

(ii) The operators $T^{(\rho),\infty,\Omega} := \{T_s^{(\rho),\infty}T_s^{\Omega}\}_{s\ge 0}$ form a Feller semigroup on $C_{\infty,\partial\Omega}((-\infty,T]\times\Omega)$. The generator $(\mathcal{L}_\Omega^{(\rho),\infty}, Dom(\mathcal{L}_\Omega^{(\rho),\infty}))$ of $T^{(\rho),\infty,\Omega}$ is the closure of

$$\left(-D_\infty^{(\rho)} + \Delta,\ Span\{C_\infty^1(-\infty,T]\cdot Dom(\Delta_\Omega)\}\right) \quad in \quad C_{\infty,\partial\Omega}((-\infty,T]\times\Omega),$$

where $T^{(\rho),\infty}$ and $-D_\infty^{(\rho)}$ act on the $(-\infty,T]$-variable, and T^{Ω} and Δ act on the Ω-variable.

Remark 3.3. If the spatial generator is not the Laplacian, it could happen that $C_c^\infty(\Omega)$ is not contained in the domain of the spatial generator (as in the case of the restricted fractional Laplacian). In such case one can extend the proof of Lemma 3.1 as in [41, Appendix II].

Lemma 3.4. *With the notation of Definitions 2.4 and 2.14, it holds that*

$$T^{(\rho),\Omega} = T^{(\rho),kill,\Omega} \quad on \quad C_{0,\partial\Omega}([0,T]\times\Omega),$$

and

$$\mathcal{L}_\Omega^{(\rho)} = \mathcal{L}_\Omega^{(\rho),kill} \quad on \quad Dom(\mathcal{L}_\Omega^{(\rho),kill}) = Dom(\mathcal{L}_\Omega^{(\rho)}) \cap \{f(0)=0\}.$$

Proof. The first claim is an immediate consequence of the observation that $T^{(\rho),\mathrm{kill}} = T^{(\rho)}$ on $C_0[0,T]$. To prove the second claim, we first confirm that $\mathrm{Dom}(\mathcal{L}_\Omega^{(\rho),\mathrm{kill}}) \subset \mathrm{Dom}(\mathcal{L}_\Omega^{(\rho)})$ by the fact that $T_s^{(\rho),\Omega} = T^{(\rho),\mathrm{kill},\Omega}$ on $C_{0,\partial\Omega}([0,T]\times\Omega)$. Next, we show that

$$u - u(0) \in \mathrm{Dom}(\mathcal{L}_\Omega^{(\rho),\mathrm{kill}}) \quad \text{for all} \quad u \in \mathrm{Dom}(\mathcal{L}_\Omega^{(\rho)}).$$

In fact, let $u \in \mathrm{Dom}(\mathcal{L}_\Omega^{(\rho)})$ and consider its resolvent representation for some $\lambda > 0$ and $g \in C_{\partial\Omega}([0,T]\times\Omega)$

$$u(t,x) = \int_0^\infty e^{-\lambda s} T_s^{(\rho)} T_s^{\Omega} g(t,x)\, ds,$$

and hence

$$u(t,x)-u(0,x)=\int_0^\infty e^{-\lambda s}T_s^{\Omega}T_s^{(\rho)}(g-g(0))(t,x)\,ds$$
$$=\int_0^\infty e^{-\lambda s}T_s^{\Omega}T_s^{(\rho),\mathrm{kill}}(g-g(0))(t,x)\,ds\in \mathrm{Dom}(\mathcal{L}_\Omega^{(\rho),\mathrm{kill}}),$$

where we use the fact that $T^{(\rho),\mathrm{kill}}=T^{(\rho)}$ on $C_{0,\partial\Omega}([0,T]\times\Omega)$ and that $g-g(0)\in C_{0,\partial\Omega}([0,T]\times\Omega)$. □

Remark 3.5. Note that the resolvent representation yields that

$$\left(-\mathcal{L}_\Omega^{(\rho),\mathrm{kill}}\right)^{-1}g(t,x)=\int_0^\infty T_s^{(\rho),\Omega}g(t,x)\,ds$$
$$=\mathbf{E}\left[\int_0^{\tau_0(t)\wedge\tau_\Omega(x)} g\left(-X^{t,(\rho)}(s),B^x(s)\right)ds\right],$$

for $g\in C_{0,\partial\Omega}([0,T]\times\Omega)$, as

$$T_s^{(\rho),\Omega}g(t,x)=T_s^{(\rho)}T_s^{\Omega}g(t,x)=\mathbf{E}\left[g\left(-X^{t,(\rho)}(s)\mathbf{1}_{\{s<\tau_0(x)\}},B^x(s)\mathbf{1}_{\{s<\tau_\Omega(x)\}}\right)\right]$$
$$=\mathbf{E}\left[g\left(-X^{t,(\rho)}(s),B^x(s)\right)\mathbf{1}_{\{s<\tau_0(x)\}}\mathbf{1}_{\{s<\tau_\Omega(x)\}}\right].$$

Also, if $g=1$ then $(-\mathcal{L}_\Omega^{(\rho),\mathrm{kill}})^{-1}g(t,x)=\mathbf{E}[\tau_{t,x}]$, where we write $\tau_{t,x}=\tau_0(t)\wedge\tau_\Omega(x)$.

3.2. Notions of solutions

In order to discuss the stochastic representation of solutions to (1.1), we use the following two auxiliary notions of solutions to the variant problem (2.3), as in [24].

Definition 3.6. Let $g\in C_{\partial\Omega}([0,T]\times\Omega)$ and $\phi(0)\in\mathrm{Dom}(\Delta_\Omega)$ such that $g(0)=-\Delta\phi(0)$. We say that a function $u\in C_{\partial\Omega}([0,T]\times\Omega)$ is a *solution in the domain of the generator to problem* (2.3) if

$$\mathcal{L}_\Omega^{(\rho)}u=-g \text{ on } (0,T]\times\overline{\Omega},\quad u(0)=\phi(0),\quad \text{and } u\in\mathrm{Dom}(\mathcal{L}_\Omega^{(\rho)}). \tag{3.1}$$

The next solution concept for problem (2.3) is defined as a pointwise approximation of solutions in the domain of the generator.

Definition 3.7. Let $g\in B([0,T]\times\Omega)$ and $\phi(0)\in\mathrm{Dom}(\Delta_\Omega)$. We say that a function $u\in B([0,T]\times\overline{\Omega})$ is a *generalized solution to problem* (2.3) if

$$u=\lim_{n\to\infty}u_n\quad\text{pointwise},$$

where $\{u_n\}_{n\in\mathbb{N}}$ is a sequence of solutions in the domain of the generator for a corresponding sequence of data $\{g_n\}_{n\in\mathbb{N}}\subset C_{\partial\Omega}([0,T]\times\Omega)$ such that $g_n\to g$ a.e. on $(0,T]\times\Omega$, $\sup_n\|g_n\|_{C([0,T]\times\overline{\Omega})}<\infty$, and $g_n(0)=-\Delta\phi(0)$ for each $n\in\mathbb{N}$.

Remark 3.8. The generalized solution will retain the homogeneous Dirichlet boundary condition on $\partial\Omega$ and the initial condition $u(0)=\phi(0)$.

3.3. Well-posedness and Feynman–Kac formula for problem (2.3)

In order to study the Feynman–Kac stochastic formula, we use the following assumption on the initial data:

(**H2**) The initial data $\phi : (-\infty, 0] \times \overline{\Omega} \to \mathbb{R}$ is such that the extension of ϕ to $\phi(0)$ on $(0, T] \times \overline{\Omega}$ satisfies $\phi \in \mathrm{Dom}(\mathcal{L}_{\Omega}^{(\rho),\infty})$ and $\mathcal{L}_{\Omega}^{(\rho),\infty}\phi = (-D_{\infty}^{(\rho)} + \Delta)\phi$.

Remark 3.9. We have some observations on the assumption (H2):

(i) Assumption (H2) is satisfied for example by linear combinations of initial conditions in variables-separable form, that is, $\phi(t, x) = p(t)q(x)$, where $p \in C_{\infty}^{1}(-\infty, 0]$, $p'(0-) = 0$ and $q \in \mathrm{Dom}(\Delta_{\Omega})$. Such set of functions is dense in $C_{\infty,\partial\Omega}((-\infty, 0] \times \Omega)$ by a Stone–Weierstrass argument as mentioned in Remark 3.3. The problem (1.1) with such a kind of initial data has been analytically studied in [16].

(ii) Note that (H2) implies $\phi(0) \in \mathrm{Dom}(\Delta_{\Omega})$ and $f_{\phi} \in C([0, T] \times \Omega)$. This is because (H2) implies $\phi(0) \in C_{\partial\Omega}(\Omega)$, $\Delta\phi(t) = \Delta\phi(0) \in C_{\partial\Omega}(\Omega)$ for $t \in [0, T]$, observing that $D_{\infty}^{(\rho)}\phi(t) \in C_{\partial\Omega}(\Omega)$ for each $t > 0$ by Dominated Convergence Theorem, and then use $f_{\phi} = -D_{\infty}^{(\rho)}\phi$.

(iii) The case where (H2) no longer holds is to be discussed in the next section.

Theorem 3.10. *Assume (H0). Then*

(i) If $g + \Delta\phi(0) \in C_{0,\partial\Omega}([0, T] \times \Omega)$ for some $g \in C_{\partial\Omega}([0, T] \times \Omega)$ and $\phi(0) \in \mathrm{Dom}(\Delta_{\Omega})$, then there exists a unique solution in the domain of the generator to problem (2.3).

(ii) Assume (H1). If $g \in B([0, T] \times \Omega)$ and $\phi(0) \in \mathrm{Dom}(\Delta_{\Omega})$, then there exists a unique generalized solution to problem (2.3), and the generalized solution allows the stochastic representation for any $(t, x) \in (0, T] \times \Omega$

$$u(t, x) = \mathbf{E}\left[\phi(0, B^{x}(\tau_0(t)))\mathbf{1}_{\{\tau_0(t)<\tau_\Omega(x)\}}\right] + \mathbf{E}\left[\int_0^{\tau_0(t)\wedge\tau_\Omega(x)} g\left(-X^{t,(\rho)}(s), B^{x}(s)\right) ds\right]. \tag{3.2}$$

(iii) Assume (H1), (H2) and let $g = f + f_{\phi}$, for $f \in B([0, T] \times \overline{\Omega})$. Then both solutions in part (i) and (ii) allow the stochastic representation for any $(t, x) \in (0, T] \times \Omega$

$$u(t, x) = \mathbf{E}\left[\phi\left(-X^{t,(\rho)}(\tau_0(t)), B^{x}(\tau_0(t))\right)\mathbf{1}_{\{\tau_0(t)<\tau_\Omega(x)\}}\right] + \mathbf{E}\left[\int_0^{\tau_0(t)\wedge\tau_\Omega(x)} f\left(-X^{t,(\rho)}(s), B^{x}(s)\right) ds\right]. \tag{3.3}$$

Proof. (i) Recall that we write $\tau_{t,x} = \tau_0(t) \wedge \tau_\Omega(x)$. Then using Proposition 2.6(iii) with the inequality

$$\left|(-\mathcal{L}_{\Omega}^{(\rho),\mathrm{kill}})^{-1}w(t, x)\right| = \left|\mathbf{E}\left[\int_0^{\tau_{t,x}} w\left(-X^{t,(\rho)}(s), B^{x}(s)\right) ds\right]\right| \le \|w\|_{C([0,T]\times\overline{\Omega})}\mathbf{E}\left[\tau_{t,x}\right],$$

for any $w \in C_{0,\partial\Omega}([0, T] \times \Omega)$, we know that $(-\mathcal{L}_{\Omega}^{(\rho),\mathrm{kill}})^{-1}$ is bounded on $C_{0,\partial\Omega}([0, T] \times \Omega)$. Meanwhile, we observe that $T_s^{(\rho),\mathrm{kill},\Omega}w \in C_{0,\partial\Omega}([0, T] \times \Omega)$ if $w \in C_{0,\partial\Omega}([0, T] \times \Omega)$ for each $s > 0$, and it holds that

$$\int_0^{\infty} \left|T_s^{(\rho),\mathrm{kill},\Omega}w(t, x)\right| ds \le \|w\|_{C([0,T]\times\overline{\Omega})} \int_0^{\infty} \mathbf{P}[s < \tau_{t,x}]ds = \|w\|_{C([0,T]\times\overline{\Omega})}\mathbf{E}[\tau_{t,x}] < \infty.$$

Therefore we conclude that $(-\mathcal{L}_{\Omega}^{(\rho),\mathrm{kill}})^{-1}$ maps $C_{0,\partial\Omega}([0,T]\times\Omega)$ to itself. Then it follows by [18, Theorem 1.1'] that $\bar{u} := (-\mathcal{L}_{\Omega}^{(\rho),\mathrm{kill}})^{-1}(g+\Delta\phi(0))$ is the unique solution to

$$\mathcal{L}_{\Omega}^{(\rho),\mathrm{kill}}\bar{u} = -(g+\Delta\phi(0)) \text{ on } (0,T]\times\overline{\Omega}, \quad \bar{u}(0)=0, \quad \text{and } \bar{u}\in \mathrm{Dom}(\mathcal{L}_{\Omega}^{(\rho),\mathrm{kill}}). \tag{3.4}$$

It remains to show that u satisfies (3.1) if and only if $u-\phi(0)$ satisfies (3.4). For the 'if' direction, let $\bar{u}$ satisfy (3.4). Then $u := \bar{u}+\phi(0)\in \mathrm{Dom}(\mathcal{L}_{\Omega}^{(\rho)})$ and $\mathcal{L}_{\Omega}^{(\rho),\mathrm{kill}}\bar{u} = \mathcal{L}_{\Omega}^{(\rho)}\bar{u}$, both by Lemma 3.4. Also $\mathcal{L}_{\Omega}^{(\rho)}\phi(0) = \Delta\phi(0)$ by Lemma 3.1, using $\mathcal{L}^{(\rho)}1 = 0$. To conclude observe that by (3.4), $u(0)=\phi(0)$ and

$$\mathcal{L}_{\Omega}^{(\rho)}(\bar{u}+\phi(0)) = \mathcal{L}_{\Omega}^{(\rho),\mathrm{kill}}\bar{u}+\Delta\phi(0) = -g.$$

The 'only if' direction is similar and omitted.

(ii) Now we let $g\in B([0,T]\times\Omega)$ and $\phi(0)\in\mathrm{Dom}(\Delta_\Omega)$. Then we can take a sequence $\{g_n\}_{n\mathbb{N}}\in C_{0,\partial\Omega}([0,T]\times\Omega)$ such that $g_n\to g$ a.e., $\sup_n\|g_n\|_{C([0,T]\times\overline{\Omega})}<\infty$ and $g_n(0) = -\Delta\phi(0)$ as required by Definition 3.7. Now for each g_n, by Remark 3.5, we consider the stochastic representation of the respective solution in the domain of the generator

$$u_n(t,x) = \mathbf{E}\left[\int_0^{\tau_{t,x}} g_n\left(-X^{t,(\rho)}(s), B^x(s)\right)ds\right] + \mathbf{E}\left[\int_0^{\tau_{t,x}}\Delta\phi(0,B^x(s))ds\right]+\phi(0,x).$$

Then for any $(t,x)\in(0,T]\times\Omega$, we note that

$$\begin{aligned}
&\mathbf{E}\left[\int_0^{\tau_{t,x}} g_n\left(-X^{t,(\rho)}(s), B^x(s)\right)ds\right]\\
&= \int_0^\infty \mathbf{E}\left[g_n\left(-X^{t,(\rho)}(s), B^x(s\wedge\tau_\Omega(x))\right)\mathbf{1}_{\{s<\tau_0(t)\}}\right]ds\\
&= \int_0^\infty\left(\int_\Omega\int_{(0,t]} g_n(z,y)p_s^{(\rho)}(t,z)p_s^{\Omega}(x,y)\,dz\,dy\right)ds\\
&\le \sup_n\|g_n\|_{C([0,T]\times\overline{\Omega})}\mathbf{E}\left[\tau_{t,x}\right]<\infty,
\end{aligned}$$

where we use the first part of (H1) and the density p_s^Ω in the last equality. Hence we can apply the Dominated Convergence Theorem to obtain as $n\to\infty$,

$$\mathbf{E}\left[\int_0^{\tau_{t,x}} g_n\left(-X^{t,(\rho)}(s), B^x(s)\right)ds\right]\to\mathbf{E}\left[\int_0^{\tau_{t,x}} g\left(-X^{t,(\rho)}(s), B^x(s)\right)ds\right].$$

It follows that a generalized solution u exists and it is given by

$$\begin{aligned}
u(t,x) &= \mathbf{E}\left[\int_0^{\tau_{t,x}} g\left(-X^{t,(\rho)}(s), B^x(s)\right)ds\right]+\mathbf{E}\left[\int_0^{\tau_{t,x}}\Delta\phi(0,B^x(s))ds\right]+\phi(0,x)\\
&= \mathbf{E}\left[\int_0^{\tau_{t,x}} g\left(-X^{t,(\rho)}(s), B^x(s)\right)ds\right]+\mathbf{E}\left[\phi(0,B^x(\tau_{t,x}))\right],
\end{aligned}$$

where Dynkin formula [18, Theorem 5.1] with Lemma 3.1 is used in the last equality. Finally, the uniqueness of the generalized solution follows immediately from the independence of the approximating sequence.

(iii) Extend ϕ to $\phi(0)$ on $(0,T]\times\overline{\Omega}$, and denote it again by ϕ. Then by Dynkin formula ([18, Theorem 5.1]) and Corollary 3.2(ii) provided assumption (H2), we have

$$\begin{aligned}
&\mathbf{E}\left[\phi\left(-X^{t,(\rho)}(\tau_{t,x}), B^x(\tau_{t,x})\right)\right]-\phi(t,x)\\
&= \mathbf{E}\left[\int_0^{\tau_{t,x}}(-D_\infty^{(\rho)}+\Delta)\phi\left(-X^{t,(\rho)}(s), B^x(s)\right)ds\right].
\end{aligned}$$

Meanwhile, for $(t,x)\in(0,T]\times\overline{\Omega}$ the identities $f_\phi(t,x)=-D_\infty^{(\rho)}\phi(t,x)$, $\Delta\phi(0,x)=\Delta\phi(t,x)$ and

$$\int_0^t(\phi(t-r,x)-\phi(t,x))\rho(t,r)\,dr=\int_0^t(\phi(0,x)-\phi(0,x))\rho(t,r)\,dr=0$$

hold, and we can derive the equality

$$\begin{aligned}&\mathbf{E}\left[\int_0^{\tau_{t,x}}(-D_\infty^{(\rho)}+\Delta)\phi\left(-X^{t,(\rho)}(s),B^x(s)\right)ds\right]\\&\quad=\mathbf{E}\left[\int_0^{\tau_{t,x}}(f_\phi+\Delta\phi)\left(-X^{t,(\rho)}(s),B^x(s)\right)ds\right].\end{aligned}$$

Therefore, the generalized solution allows the following representation

$$\begin{aligned}u(t,x)&=\mathbf{E}\left[\int_0^{\tau_{t,x}}\Delta\phi\left(-X^{t,(\rho)}(s),B^x(s)\right)ds\right]+\phi(0,x)\\&\quad+\mathbf{E}\left[\int_0^{\tau_{t,x}}(f_\phi+f)\left(-X^{t,(\rho)}(s),B^x(s)\right)ds\right]\\&=\mathbf{E}\left[\int_0^{\tau_{t,x}}(-D_\infty^{(\rho)}+\Delta)\phi\left(-X^{t,(\rho)}(s),B^x(s)\right)ds\right]+\phi(0,x)\\&\quad+\mathbf{E}\left[\int_0^{\tau_{t,x}}f\left(-X^{t,(\rho)}(s),B^x(s)\right)ds\right]\\&=\mathbf{E}\left[\phi\left(-X^{t,(\rho)}(\tau_{t,x}),B^x(\tau_{t,x})\right)\right]+\mathbf{E}\left[\int_0^{\tau_{t,x}}f\left(-X^{t,(\rho)}(s),B^x(s)\right)ds\right]\\&\quad+\phi(0,x)-\phi(t,x)\\&=\mathbf{E}\left[\phi\left(-X^{t,(\rho)}(\tau_0(t)),B^x(\tau_0(t))\right)\mathbf{1}_{\{\tau_0(t)<\tau_\Omega(x)\}}\right]\\&\quad+\mathbf{E}\left[\int_0^{\tau_{t,x}}f\left(-X^{t,(\rho)}(s),B^x(s)\right)ds\right].\end{aligned}$$

for all $(t,x)\in(0,T]\times\Omega$. This completes the proof of the theorem. □

Remark 3.11. If assumption (H1) does not hold, one shall modify the definition of a generalized solution requiring pointwise convergence everywhere on $(0,T]\times\Omega$ of the approximating sequence. This allows to run the argument of Theorem 3.10(ii) as long as one such sequence exists. This means that our data g has to be a Baire class 1 function (which includes continuous functions but it is a smaller class than $B([0,T]\times\overline{\Omega})$).

Remark 3.12. Note that every generalized solution is the pointwise limit on $[0,T]\times\overline{\Omega}$ of a sequence of solutions in the domain of the generator $\{u_n\}_{n\in\mathbb{N}}$, and from the stochastic representation we can infer that $\sup_n\|u_n\|_{C([0,T]\times\overline{\Omega})}<\infty$. This implies the convergence $u_n\to u$ in $L^p((0,T)\times\Omega)$ for every $p\in[1,\infty)$.

We now give a more explicit formula for the heat kernel of the solution in (3.3) ($f=0$).

Proposition 3.13. *Let assumptions (H0) and (H1) hold true. Then*

$$\mathbf{E}\left[\phi\left(-X^{t,(\rho)}(\tau_0(t)),B^x(\tau_0(t))\right)\mathbf{1}_{\{\tau_0(t)<\tau_\Omega(x)\}}\right]=\int_{-\infty}^0\int_\Omega\phi(r,y)H_{t,x}(r,y)\,dr\,dy,\tag{3.5}$$

for every $(t,x) \in (0,T] \times \Omega$ *and* $\phi \in B((-\infty,0] \times \Omega)$, *where*

$$H_{t,x}(r,y) = \int_0^t \rho(z,z-r) \left(\int_0^\infty p_s^\Omega(x,y) p_s^{(\rho)}(t,z)\, ds \right) dz.$$

Proof. By (H1), it is enough to prove formula (3.5) on the set $\{-X^{t,(\rho)}(\tau_0(t)) < 0\}$. Fix $(t,x) \in (0,T] \times \Omega$. Let $\phi \in \mathrm{Span}\{C^1_\infty(-\infty,T] \cdot \mathrm{Dom}(\Delta_\Omega)\}$ such that $\phi = 0$ on $[-n^{-1},T]$ for $n \in \mathbb{N}$. By Remark 3.9(i) ϕ satisfies (H2). Then by Dynkin formula along with $\mathcal{L}_\Omega^{(\rho),\infty}\phi = (-D_\infty^{(\rho)} + \Delta)\phi$ by Corollary 3.2 and $\Delta\phi = 0$ on $(0,T]$, we have that

$$\begin{aligned} u(t,x) :&= \mathbf{E}\left[\phi\left(-X^{t,(\rho)}(\tau_0(t)), B^x(\tau_0(t))\right) \mathbf{1}_{\{\tau_0(t)<\tau_\Omega(x)\}}\right] \\ &= \mathbf{E}\left[\int_0^{\tau_{t,x}} -D_\infty^{(\rho)}\phi\left(-X^{t,(\rho)}(s), B^x(s)\right) ds\right] \\ &= \int_0^\infty \mathbf{E}\left[\mathbf{1}_{\{s<\tau_0(t)\}} \int_{-X^{t,(\rho)}(s)}^\infty \phi\left(-X^{t,(\rho)}(s) - r, B^x(s \wedge \tau_\Omega(x))\right) \right. \\ &\qquad \left. \times \rho\left(-X^{t,(\rho)}(s), r\right) dr \right] ds \end{aligned}$$

Next, using the independence of $-X^{t,(\rho)}(s \wedge \tau_0(t))$ and $B^x(s \wedge \tau_\Omega(x))$, $\{s < \tau_0(t)\} = \{0 < -X^{t,(\rho)}(s)\}$, Fubini's Theorem and standard change of variables, we obtain

$$\begin{aligned} u(t,x) &= \int_\Omega \int_0^\infty \left(\int_0^t \left(\int_z^\infty \phi(z-r,y)\, \rho(z,r)\, dr \right) p_s^{(\rho)}(t,z)\, dz \right) p_s^\Omega(x,y)\, ds\, dy \\ &= \int_\Omega \int_0^\infty \left(\int_0^t \left(\int_{-\infty}^0 \phi(r,y)\, \rho(z,z-r)\, dr \right) p_s^{(\rho)}(t,z)\, dz \right) p_s^\Omega(x,y)\, ds\, dy \\ &= \int_{-\infty}^0 \int_\Omega \phi(r,y) \left(\int_0^t \rho(z,z-r) \int_0^\infty p_s^{(\rho)}(t,z) p_s^\Omega(x,y)\, ds\, dz \right) dy\, dr. \end{aligned}$$

By a density argument the identity (3.5) holds for every $\phi \in B((-\infty,n^{-1}) \times \Omega) \cap C((-\infty,n^{-1}) \times \Omega)$ for every $n \in \mathbb{N}$. Considering the non-negative increasing sequence $\phi_n = \mathbf{1}_{(-\infty,n^{-1}) \times \Omega}$, $n \in \mathbb{N}$, by Monotone Convergence Theorem one can pass to the limit in both sides of (3.5), confirming that $H_{t,x}$ induces a finite measure on $(-\infty,0) \times \Omega$, as the right hand side of (3.5) is finite. By another density argument the equality (3.5) holds for every

$$\phi \in C((-\infty,0] \times \overline{\Omega}) \cap \{f = 0 \text{ on } \{0\} \times \overline{\Omega} \cup (-\infty,0] \times \partial\Omega\},$$

and we are done by Riesz–Markov–Kakutani representation Theorem [27, Theorem 1.7.3]. □

Remark 3.14. Suppose that (H0) and (H1) hold, and that $\phi_n, \phi \in B((-\infty,0] \times \overline{\Omega})$, for $n \in \mathbb{N}$, such that $\phi_n \to \phi$ a.e. on $(-\infty,0] \times \overline{\Omega}$, $\sup_n \|\phi_n\|_{B((-\infty,0]\times\overline{\Omega})} < \infty$, and $f \in B((0,T] \times \Omega)$. Then Proposition 3.13 and Dominated Convergence Theorem imply that $u_n \to u$ pointwise on $(0,T] \times \Omega$ and $\sup_n \|u_n\|_{B((-\infty,0]\times\overline{\Omega})} < \infty$. Here u_n is defined as (3.3) for ϕ_n, f, $n \in \mathbb{N}$, and u is defined as (3.3) for ϕ, f. This in turn implies the convergence $u_n \to u$ in $L^p((0,T) \times \Omega)$ for each $p \in [1,\infty)$.

4. Stochastic representation for solutions in weak sense

In Section 3, the stochastic representation of the solution to the nonlocal-in-time evolution model (1.1) is established in case that the data is smooth and compatible. The aim of this section is to show that the representation (3.3) still provides a solution of (1.1) in the weak sense, even though the data does not satisfy the smoothness and compatibility conditions required in Section 3. Now we denote by $W^{1,p}(\Omega)$ the standard Sobolev space of p-integrable functions on Ω with p-integrable weak first derivatives, $p \in [1,\infty]$. Denote by $H^{-1}(\Omega)$ the dual of $H_0^1(\Omega)$, where $H_0^1(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $W^{1,2}(\Omega)$.

In case that the kernel ρ is time-independent and compactly supported, the existence and uniqueness of the weak solution (4.3) has been confirmed in [16]. The uniqueness argument for the more general variables-separable kernel $\rho(t,r) = p(r)q(r)$ is similar, so we only present some useful results here and omit some similar detailed proof in order to avoid redundancy. We do not prove uniqueness of weak solutions for our general time-dependent kernel $\rho(t,r)$.

Lemma 4.1. *Suppose that $u \in B(-\infty,T) \cap L^1(-\infty,T)$, and $v \in C_c^\infty(0,T)$ with zero extension out of the interval $(0,T)$. Further, we suppose that*

$$\int_0^T \int_0^\infty |u(t) - u(t-r)|\rho(t,r)\,dr\,dt < \infty. \tag{4.1}$$

Then it holds that

$$\int_0^T D_\infty^{(\rho)} u(t) v(t)\,dt = -\int_{-\infty}^T u(t)(D_\infty^{(\rho),*} v)(t)\,dt$$

with

$$D_\infty^{(\rho),*} v(t) = -\int_0^\infty v(t)\rho(t,r) - v(t+r)\rho(t+r,r)\,dr. \tag{4.2}$$

The next lemma gives an upper bound of $D_\infty^{(\rho)}$ for smooth functions in Sobolev spaces.

Lemma 4.2. *Let the kernel ρ satisfy (H0). Then the operator $D_\infty^{(\rho)}$ defined by (1.2) satisfies*

$$\|D_\infty^{(\rho)} v\|_{L^p(-\infty,T)} \le C\|v\|_{W^{1,p}(-\infty,T)}, \quad v \in W^{1,p}(-\infty,T),$$

with $p \in [1,\infty]$.

Proof. We only prove the result for $p \in [1,\infty)$, as the case $p = \infty$ follows analogously. By Hölder's inequality and assumption (H0) we have that

$$\begin{aligned}
&\int_{-\infty}^T \left(\int_0^1 |u(t) - u(t-r)|\rho(t,r)\,dr\right)^p dt \\
&\le \int_{-\infty}^T \int_0^1 \frac{|u(t)-u(t-r)|^p}{r^p} r\rho(t,r)\,dr \left(\int_0^1 r\rho(t,r)\,dr\right)^{p-1} dt \\
&\le c\int_{-\infty}^T \int_0^1 \frac{|u(t)-u(t-r)|^p}{r^p} r\rho(t,r)\,ds\,dt \\
&\le c\int_0^1 r^{1-p}|\max_t \rho(t,r)| \int_{-\infty}^T |u(t)-u(t-r)|^p\,dt\,dr \\
&\le c\int_0^1 r|\max_t \rho(t,r)|\,dr\,\|u\|^p_{W^{1,p}(-\infty,T)} \le c\|u\|^p_{W^{1,p}(-\infty,T)},
\end{aligned}$$

where we apply the fact that $\int_{-\infty}^{T}|u(t)-u(t-r)|^p\,dt \le c|r|^p\|u\|^p_{W^{1,p}(-\infty,T)}$ in the second last inequality. On the other hand, we have the following estimate

$$\begin{aligned}
&\int_{-\infty}^{T}\left(\int_{1}^{\infty}|u(t)-u(t-r)|\rho(t,r)\,dr\right)^p dt\\
&\le \int_{-\infty}^{T}\int_{1}^{\infty}|u(t)-u(t-r)|^p\rho(t,r)\,dr\left(\int_{1}^{\infty}\rho(t,r)\,dr\right)^{p-1} dt\\
&\le c\int_{-\infty}^{T}\int_{1}^{\infty}|u(t)-u(t-r)|^p\rho(t,r)\,dr\,dt\\
&\le c\int_{1}^{\infty}\max_t\rho(t,r)\int_{-\infty}^{T}|u(t)-u(t-r)|^p\,dt\,dr\\
&\le c\int_{1}^{\infty}\max_t\rho(t,r)\,dr\,\|u\|^p_{L^p(-\infty,T)} \le c\|u\|^p_{W^{1,p}(-\infty,T)}.
\end{aligned}$$

Then we obtain the desired assertion. □

Similar argument yields the following a priori bound for the dual operator $D_\infty^{(\rho),*}$ given by (4.2).

Lemma 4.3. *Let the kernel ρ satisfy (H0) and let the operator $D_\infty^{(\rho),*}$ be defined by (4.2). Then for any $v\in W^{1,p}(\mathbb{R})$ with $p\in[1,\infty]$, it holds that*

$$\|D_\infty^{(\rho),*}v\|_{L^p(\mathbb{R})} \le C\|v\|_{W^{1,p}(\mathbb{R})}.$$

Proof. First, we use the following splitting

$$D_\infty^{(\rho),*}v(t) = \int_0^\infty (v(t+r)-v(t))\rho(t,r)\,dr + \int_0^\infty v(t+r)(\rho(t+r,r)-\rho(t,r))\,dr = I_1+I_2.$$

Now using the same argument as that in Lemma 4.2, we derive that for $p\in[1,\infty)$

$$\|I_1\|_{L^p(\mathbb{R})} \le C\|v\|_{W^{1,p}(\mathbb{R})}.$$

Therefore it suffices to bound I_2. For $p\in[1,\infty)$, by Hölder's inequality and assumption (H0) we have that

$$\begin{aligned}
&\int_{-\infty}^{\infty}\left(\int_0^1|v(t+r)||\rho(t,r)-\rho(t+r)|\,dr\right)^p dt\\
&\le \int_{-\infty}^{\infty}\int_0^1|v(t+r)|^p|\rho(t,r)-\rho(t+r)|\,dr\left(\int_0^1|\rho(t,r)-\rho(t+r,r)|\,dr\right)^{p-1} dt.
\end{aligned}$$

Then we observe that

$$\int_0^1|\rho(t,r)-\rho(t+r,r)|\,dr \le \int_0^1\int_t^{t+r}|\partial_y\rho(y,r)|\,dr \le \int_0^1 r\max_t|\partial_t\rho(t,r)|\,dr \le c,$$

and hence

$$\begin{aligned}
&\int_{-\infty}^{\infty}\left(\int_0^1|v(t+r)||\rho(t,r)-\rho(t+r,r)|\,dr\right)^p dt\\
&\le c\int_{-\infty}^{\infty}\int_0^1|v(t+r)|^p|\rho(t,r)-\rho(t+r,r)|\,dr\,dt
\end{aligned}$$

$$
\begin{aligned}
&\leq c \int_0^1 \int_{-\infty}^{\infty} |v(t+r)|^p \, dt \max_t |\rho(t,r) - \rho(t+r,r)| \, dr \\
&\leq c \|v\|_{L^p(\mathbb{R})} \int_0^1 r \max_t |\partial_t \rho(t,r)| \, dr \leq c \|v\|_{L^p(\mathbb{R})}.
\end{aligned}
$$

Meanwhile, applying the following observation

$$
\int_1^{\infty} |\rho(t,r) - \rho(t+r,r)| \, dr \leq \int_1^{\infty} |\rho(t,r)| + |\rho(t+r,r)| \, dr \leq c,
$$

we have the following estimate

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \left(\int_1^{\infty} |v(t+r)||\rho(t,r) - \rho(t+r,r)| \, ds \right)^p dt \\
&\leq c \int_{-\infty}^{\infty} \int_1^{\infty} |v(t+r)|^p |\rho(t,r) - \rho(t+r,r)| \, ds \, dt \\
&\leq c \int_1^{\infty} \int_{-\infty}^{\infty} |v(t+r)|^p \, dt \, (|\rho(t,r)| + |\rho(t+r,r)|) \, dr \\
&\leq c \|v\|_{L^p(\mathbb{R})},
\end{aligned}
$$

which yields that

$$
\|I_2\|_{L^p(\mathbb{R})} \leq C \|v\|_{W^{1,p}(\mathbb{R})}.
$$

This completes the proof for $p \in [1, \infty)$, and the case that $p = \infty$ follows analogously. □

Then we have the following result for a smooth function with compact support.

Corollary 4.4. *Let the kernel ρ satisfy (H0) and let the operator $D_{\infty}^{(\rho),*}$ be defined by (4.2). Then $D_{\infty}^{(\rho),*} v \in L^1(-\infty, T) \cap L^{\infty}(-\infty, T)$ for any $v \in C_c^1(0, T)$.*

Definition 4.5. We define the *weak Marchaud-type derivative* of a function $u \in L^1_{loc}(\mathbb{R}; B)$, for a Banach space B, to be a function $\widetilde{D_{\infty}^{(\rho)}} u \in L^1_{loc}(\mathbb{R}; B)$ that satisfies

$$
\int_{\mathbb{R}} \widetilde{D_{\infty}^{(\rho)}} u(t) v(t) \, dt = \int_{\mathbb{R}} u(t) (D_{\infty}^{(\rho),*} v)(t) \, dt, \quad \text{for every } v \in C_c^{\infty}(0, T),
$$

with the integral defined in the Bochner sense.

The following lemma gives the equivalence between the variational nonlocal operator and the strong one in the case that $B = \mathbb{R}$ and ρ is variables-separable.

Lemma 4.6. *Suppose that the kernel ρ satisfies (H0) and it is variables-separable, i.e., $\rho(t,r) = p(t)q(r)$ with $p(t) \in C^1[0,T]$ and $p(t) \geq c_1 > 0$. Moreover, we let $u \in L^{\infty}(\mathbb{R})$ and $D_{\infty}^{(\rho)} u \in L^2(0,T)$. Then $D_{\infty}^{(\rho)} u \in L^2(0,T)$ and*

$$
D_{\infty}^{(\rho)} u = \widetilde{D_{\infty}^{(\rho)}} u \quad \textit{almost everywhere},
$$

where $D_{\infty}^{(\rho)}$ is defined by (1.2).

Proof. First of all, we consider the case that the kernel function is translation preserved, i.e., $\rho(t,r) = \rho(r)$. To this end, we define the truncated nonlocal operator

$$
D_{\delta}^{(\rho)} u(t) = \int_0^{\delta} (u(t) - u(t-r)) \rho(r) \, dr
$$

as well as its adjoint operator $D_\delta^{(\rho),*}$ and the weak operator $\widetilde{D_\delta^{(\rho)}}$. Since for any $\delta > 0$, we have

$$\int_\delta^\infty (u(t) - u(t-r))\rho(r)\,dr = u(t)\int_\delta^\infty \rho(r)\,dr - \int_\delta^\infty u(t-r)\rho(r)\,dr \in L^2(0,T),$$

by assumption (H0). By the definition of the weak operator, one may deduce that

$$\widetilde{D_\delta^{(\rho)}}u(t) = \widetilde{D_\infty^{(\rho)}}u(t) - \int_\delta^\infty (u(t) - u(t-r))\rho(r)\,dr \in L^2(0,T)$$

Now by Lemma [16, Lemma 2.4] we have that $D_\delta^{(\rho)}u \in L^2(0,T)$ and $D_\delta^{(\rho)}u = \widetilde{D_\delta^{(\rho)}}u$. As a result, we derive that

$$D_\infty^{(\rho)}u(t) = \int_0^\infty (u(t) - u(t-r))\rho(r)\,dr = D_\delta^{(\rho)}u(t) + \int_\delta^\infty (u(t) - u(t-r))\rho(r)\,dr \in L^2(0,T),$$

and hence $D_\infty^{(\rho)}u = \widetilde{D_\infty^{(\rho)}}u$ almost everywhere.

Next, we consider the case that $\rho(t,r) = p(t)q(r)$ and define the operator

$$D_\infty^{(q)}u(t) = \int_0^\infty (u(t) - u(t-r))q(r)\,ds.$$

The same as before, we may define corresponding adjoint and weak operators. Define $\langle f, g\rangle_a^b := \int_a^b fg\,dt$, $b > a \geq -\infty$. Then we note that

$$\langle p\widetilde{D_\infty^{(q)}}u, v\rangle_0^T = \langle u, D_\infty^{(q),*}(pv)\rangle_{-\infty}^T = \langle u, D_\infty^{(\rho),*}v\rangle_{-\infty}^T = \langle \widetilde{D_\infty^{(\rho)}}u, v\rangle_0^T,$$

which together with the positivity assumption on $p(t)$ yields that

$$\widetilde{D_\infty^{(q)}}u(t) = \frac{1}{p(t)}\widetilde{D_\infty^{(\rho)}}u(t) \leq \frac{1}{c_1}\left|\widetilde{D_\infty^{(\rho)}}u(t)\right| \in L^2(0,T).$$

As a result, we obtain that $D_\infty^{(q)}u(t) = \widetilde{D_\infty^{(q)}}u(t) \in L^2(0,T)$ and

$$D_\infty^{(\rho)}u(t) = p(t)\widetilde{D_\infty^{(q)}}u(t) = \widetilde{D_\infty^{(\rho)}}u(t) \in L^2(0,T). \quad \square$$

Lemma 4.7. *Let $\tilde{u} \in B((-\infty,T] \times \Omega)$ be the function defined in (3.3) under the assumptions (H0) and (H1), for $\phi \in L^\infty(-\infty,0;H_0^1(\Omega))$ and $f \in L^\infty(0,T;H_0^1(\Omega))$. Then $\tilde{u} \in L^\infty(-\infty,T;H_0^1(\Omega))$.*

Proof. Consider (3.3) for $f = 0$ (the proof for $f \neq 0$ is similar and omitted). Fix $t > 0$. By [19, Chapter 7.1] we have $T_s^\Omega\phi(r,\cdot) = \mathbf{E}[\phi(r, B^\cdot(s))\mathbf{1}_{\{s<\tau_\Omega\}}] \in H_0^1(\Omega)$ for a.e. $r \in (-\infty,0)$ and $s \geq 0$. Consider the Borel probability space (Γ, μ_t), where $\Gamma = (-\infty,0) \times (0,\infty)$ and $\mu_t(dsdr) = \left(\int_0^t \rho(z, z-r)p_s^{(\rho)}(t,z)\,dz\right)dsdr$, so that formula (3.5) reads $u(t,x) = \int_\Gamma T_s^\Omega\phi(r,x)\,\mu_t(dsdr)$. Note that for a.e. $r \in (-\infty,0)$ and every $s \geq 0$

$$\|T_s^\Omega\phi(r)\|_{H^1(\Omega)} \leq \|\phi(r)\|_{H^1(\Omega)} \leq \|\phi\|_{L^\infty(-\infty,0;H_0^1(\Omega))} =: C,$$

where the first inequality holds by [19, Chapter 7.1, Theorem 5.(i)], as $\phi(r) \in H_0^1(\Omega)$ for a.e. $r \in (-\infty,0)$. We conclude that $\tilde{u}(t) \in H_0^1(\Omega)$, because the above bound proves that $T_\cdot^\Omega\phi(\cdot) : (\Gamma,\mu_t) \to H_0^1(\Omega)$ is Bochner integrable, which implies that $\tilde{u}(t) = \int_\Gamma T_\cdot^\Omega\phi(\cdot)\,\mu_t(d\cdot) = \lim_{n\to\infty} S_n$ in $H^1(\Omega)$, where each S_n is a linear combination of functions in $H_0^1(\Omega)$.

Formula (3.5) suggests the definition

$$\begin{aligned}\nabla\tilde{u}(t,x) :&= \int_{-\infty}^{0}\left(\int_0^t \rho(z,z-r)\left(\int_0^\infty \nabla T_s^{\Omega}\phi(r,x)p_s^{(\rho)}(t,z)\,ds\right)dz\right)dr\\ &= \int_\Gamma \nabla T_s^{\Omega}\phi(r,x)\,\mu_t(dsdr).\end{aligned}$$

Then $\nabla\tilde{u}(t)\in L^2(\Omega)$, because

$$\begin{aligned}\int_\Omega (\nabla\tilde{u}(t,x))^2\,dx &= \int_\Omega\left(\int_\Gamma \nabla T_s^{\Omega}\phi(r,x)\,\mu_t(dsdr)\right)\left(\int_\Gamma \nabla T_{s'}^{\Omega}\phi(r',x)\,\mu_t(ds'dr')\right)dx\\ &= \int_\Gamma\int_\Gamma\left(\int_\Omega \nabla T_s^{\Omega}\phi(r,x)\nabla T_{s'}^{\Omega}\phi(r',x)\,dx\right)\mu_t(dsdr)\mu_t(ds'dr')\\ &\le \int_\Gamma\int_\Gamma \|T_s^{\Omega}\phi(r)\|_{H^1(\Omega)}\|T_{s'}^{\Omega}\phi(r')\|_{H^1(\Omega)}\,\mu_t(dsdr)\mu_t(ds'dr')\\ &\le C^2\left(\int_\Gamma \mu_t(dsdr)\right)^2 = C^2.\end{aligned}$$

Applying Fubini's Theorem to the definition of weak derivative proves that $\nabla\tilde{u}(t)$ is indeed the weak derivative of $\tilde{u}(t)$. Finally, $\sup_{t\in(0,T)}\int_\Omega(\nabla\tilde{u}(t,x))^2\,dx \le C^2$ and the smoothness of ϕ implies that $\tilde{u}\in L^\infty(-\infty,T;H_0^1(\Omega))$, concluding the proof. □

Next we shall show that the stochastic representation (3.3) provides a weak solution of problem (1.1), whose definition is given as below.

Definition 4.8. A function u is called a *weak solution to problem* (1.1) if $u\in L^2(0,T;H_0^1(\Omega))$ and $\widetilde{D_\infty^{(\rho)}}u\in L^2(0,T;H^{-1}(\Omega))$, and for every $v\in L^2(0,T;H_0^1(\Omega))$ (with zero extension to $t<0$)

$$\begin{cases}\langle \widetilde{D_\infty^{(\rho)}}u, v\rangle = -\langle\nabla u,\nabla v\rangle + \langle f,v\rangle, & \text{and,}\\ \quad u(t)=\phi(t), & \text{for a.e. } t\in(-\infty,0),\end{cases}\tag{4.3}$$

where the notation $\langle\cdot,\cdot\rangle$ is defined by

$$\langle u,v\rangle = \int_{-\infty}^T\int_\Omega u(t,x)v(t,x)\,dx\,dt,$$

or the duality in case that $u\in L^2(0,T;H^{-1}(\Omega))$.

Remark 4.9. If u is the weak solution of (1.1) and $\widetilde{D_\infty^{(\rho)}}u\in L^2(0,T;L^2(\Omega))$, we have $\widetilde{D_\infty^{(\rho)}}u = D_\infty^{(\rho)}u$ by Lemma 4.6, provided that the kernel function is variables-separable, i.e., $\rho(t,s)=p(t)q(s)$ with $p(t)\in C^1[0,T]$ and $p(t)\ge c_1>0$. Then u satisfies Eq. (1.1) almost everywhere.

Theorem 4.10. *Assume (H0) and (H1). Let u be given by formula (3.3), where $\phi\in L^\infty(-\infty,0;H_0^1(\Omega))\cap L^\infty((-\infty,0)\times\Omega)$ and $f\in L^\infty(0,T;H_0^1(\Omega))\cap L^\infty((0,T)\times\Omega)$. Define the extension $\tilde{u}$ of u as*

$$\tilde{u} := \begin{cases} u, & \text{on } (0,T]\times\Omega,\\ \phi, & \text{on } (-\infty,0)\times\Omega.\end{cases}\tag{4.4}$$

Then $\tilde{u}$ is a weak solution to problem (1.1).

Proof. Assume for the first two steps that ϕ satisfies (H2).

Step 1: Let u be a solution in the domain of the generator to problem (2.3) for $g \equiv f + f_\phi$, and initial condition $\phi(0)$, for some $f \in C_{\partial\Omega}([0,T]\times\Omega)$. As $u \in \mathrm{Dom}(\mathcal{L}_\Omega^{(\rho)})$, by Lemma 3.4, $u - \phi(0) \in \mathrm{Dom}(\mathcal{L}_\Omega^{(\rho),\mathrm{kill}})$, and hence applying Corollary 3.2(i) there exists $\{\hat{u}_n\}_{n\in\mathbb{N}} \subset \mathrm{Dom}(\mathcal{L}_\Omega^{(\rho),\mathrm{kill}})$ such that

$$\hat{u}_n \to u - \phi(0), \quad \mathcal{L}_\Omega^{(\rho)}\hat{u}_n \to \mathcal{L}_\Omega^{(\rho)}(u-\phi(0)) \ \text{ and } \ \mathcal{L}_\Omega^{(\rho)}\hat{u}_n = (-D_0^{(\rho)} + \Delta)\hat{u}_n.$$

Then we apply Lemmas 3.4 and 3.1 to obtain that

$$u_n := \hat{u}_n + \phi(0) \in \mathrm{Dom}(\mathcal{L}_\Omega^{(\rho)}), \quad u_n \to u, \quad \mathcal{L}_\Omega^{(\rho)}u_n = \mathcal{L}_\Omega^{(\rho)}\hat{u}_n + \Delta\phi(0) \to \mathcal{L}_\Omega^{(\rho)}u$$

and $u_n(0) = \phi(0)$ for all $n \in \mathbb{N}$. Then using the fact that $D_\infty^{(\rho)}\tilde{u}_n = D_0^{(\rho)}u_n - f_\phi$ for $t \in [0,T]$, we have

$$(D_0^{(\rho)} - \Delta)u_n - f_\phi = D_\infty^{(\rho)}\tilde{u}_n - \Delta\tilde{u}_n, \quad \text{on } [0,T]\times\Omega,$$

where $\tilde{u}_n$ is defined for each $n \in \mathbb{N}$ by

$$\tilde{u}_n := \begin{cases} u_n, & \text{on } (0,T]\times\Omega,\\ \phi, & \text{on } (-\infty,0]\times\Omega. \end{cases} \tag{4.5}$$

Therefore, we have that

$$(-D_\infty^{(\rho)} + \Delta)\tilde{u}_n = (-D_0^{(\rho)} + \Delta)u_n + f_\phi \to \mathcal{L}_\Omega^{(\rho)}u + f_\phi = -f,$$

where the convergence is in $C_{\partial\Omega}([0,T]\times\Omega)$.

On the other hand, we apply Corollary 4.4 for any $v \in C_c^\infty((0,T)\times\Omega)$ to obtain as $n \to \infty$

$$\langle(-D_\infty^{(\rho)} + \Delta)\tilde{u}_n, v\rangle = \langle\tilde{u}_n, (-D_\infty^{(\rho),*} + \Delta)v\rangle \to \langle\tilde{u}, (-D_\infty^{(\rho),*} + \Delta)v\rangle,$$

where Corollary 4.4 guarantees that $(-D_\infty^{(\rho),*} + \Delta)v \in L^1((-\infty,0)\times\Omega) \cap L^\infty((0,T)\times\Omega)$, and hence

$$\langle u, (D_\infty^{(\rho),*} - \Delta)v\rangle = \langle f, v\rangle, \qquad \text{for any } \ v \in C_c^\infty((0,T)\times\Omega).$$

Step 2: Let now u be the generalized solution to problem (2.3) for $g = f + f_\phi$, where $f \in L^\infty((0,T)\times\Omega)$, and let $\tilde{u}$ be its extension with historical initial data ϕ. By the definition of the generalized solution, we pick a sequence $f_n \in C_{\partial\Omega}([0,T]\times\Omega)$ such that

$$f_n \to f \quad \text{a.e.}, \qquad f_n(0) = -(f_\phi(0) + \Delta\phi(0)) \qquad \text{and} \qquad \sup_n \|f_n\|_{C([0,T]\times\overline{\Omega})} < \infty.$$

Besides, we denote by u_n the respective solution in the domain of the generator and let $\tilde{u}_n$ be its extension by (4.5). Then by Step 1, we know that each $\tilde{u}_n$ satisfies

$$\langle\tilde{u}_n, (-D_\infty^{(\rho),*} + \Delta)v\rangle = \langle -f_n, v\rangle, \qquad \text{for any } \ v \in C_c^\infty((0,T)\times\Omega),$$

as well as the initial and boundary conditions in (1.1). Now the Dominated Convergence Theorem provided the uniform upper bound of f_n implies that

$$f_n \to f \qquad \text{in} \qquad L^2(0,T;L^2(\Omega)) \qquad \text{as} \quad n \to \infty.$$

On the other hand, we have $\tilde{u}_n \to \tilde{u}$ in $L^2(0,T;L^2(\Omega))$ by Remark 3.12. Meanwhile $(D_\infty^{(\rho),*} - \Delta)v \in L^1((-\infty,0)\times\Omega) \cap L^\infty((0,T)\times\Omega)$ for any $v \in C_c^\infty((0,T)\times\Omega)$ by Corollary 4.4. Therefore we obtain as $n \to \infty$

$$\langle\tilde{u}_n, (D_\infty^{(\rho),*} - \Delta)v\rangle \to \langle\tilde{u}, (D_\infty^{(\rho),*} - \Delta)v\rangle, \qquad \text{for any } \ v \in C_c^\infty((0,T)\times\Omega).$$

Step 3: Now we consider the case that $\phi \in L^\infty(-\infty, 0; H_0^1(\Omega)) \cap L^\infty((-\infty, 0) \times \Omega)$ and $f \in L^\infty(0, T; H_0^1(\Omega)) \cap L^\infty((0, T) \times \Omega)$. To this end, we set functions $\phi_K(t, x) = \phi(t, x)\mathbf{1}_{\{t<-K\}}$, for $K \in \mathbb{N}$. By the density of $\mathrm{Span}\{C_c^\infty(-K, 0) \cdot C_c^\infty(\Omega)\}$ in $B([-K, 0] \times \overline{\Omega})$ with respect to sequential convergence a.e., we choose $\phi_{K,j} \in \mathrm{Span}\{C_c^\infty(-K, 0) \cdot C_c^\infty(\Omega)\}$ such that

$$\phi_{K,j} \to \phi_K \quad \text{a.e.} \quad \text{and} \quad \sup_j \|\phi_{K,j}\|_{C([-K,0]\times\overline{\Omega})} < \infty.$$

By Remark 3.9(i), we know that $\phi_{K,j}$ satisfies assumption (H2) for each $j \in \mathbb{N}$. Denote by $u_{K,j}$ the generalized solution with the initial data $\phi_{K,j}$ and source term f, and denote by u_K the function given by formula (3.3) with $\phi \equiv \phi_K$ and source term f. By Remark 3.14 we conclude that

$$\sup_j \|\tilde{u}_{K,j}\|_{B([-K,T]\times\overline{\Omega})} < \infty \quad \text{and} \quad \tilde{u}_{K,j} \to \tilde{u}_K \quad \text{a.e. on } (-K, T] \times \Omega.$$

Then for any $v \in C_c^\infty((0, T)\times\Omega)$, we know that $(D_\infty^{(\rho)} - \Delta)^* v \in L^1((-K, 0)\times\Omega) \cap L^\infty((0, T)\times\Omega)$ by Corollary 4.4, and hence

$$\langle \tilde{u}_K, (D_\infty^{(\rho),*} - \Delta)v\rangle = \lim_{j\to\infty} \langle \tilde{u}_{K,j}, (D_\infty^{(\rho),*} - \Delta)v\rangle = \langle f, v\rangle, \tag{4.6}$$

and $\tilde{u}_K = \phi_K$ on $(-K, 0] \times \Omega$. We can now pass to the limit as $K \to \infty$ in (4.6), given that $\tilde{u}_K \to \tilde{u}$ a.e. on $(-\infty, T) \times \Omega$, with $\sup_K \|\tilde{u}_K\|_{B((-\infty,T]\times\overline{\Omega})} < \infty$, again by Remark 3.14, and $(D_\infty^{(\rho)} - \Delta)^* v \in L^1((-\infty, 0) \times \Omega) \cap L^\infty((0, T) \times \Omega)$ by Corollary 4.4. Here u is defined by (3.3) for ϕ and f, and $\tilde{u}$ by (4.4). Therefore

$$\langle \tilde{u}, (D_\infty^{(\rho),*} - \Delta)v\rangle = \langle f, v\rangle.$$

By Lemma 4.7 and the smoothness of the problem data f and ϕ we obtain $\tilde{u} \in L^2(0, T; H_0^1(\Omega))$ and $\tilde{u}$ satisfies the identities in (4.3). Also, for every $w \in H_0^1(\Omega)$, $v \in C_c^1(0, T)$, and properties of Bochner integrals

$$\int_{-\infty}^T (\tilde{u}(t), w) D_\infty^{(\rho),*} v(t)\, dt = \left(\int_0^T (\Delta\tilde{u}(t) + f(t))v(t)\, dt, w \right),$$

where $(\cdot, \cdot)$ is the dual pairing of $H_0^1(\Omega)$. Then by the smoothness of v, the left hand side satisfies

$$\int_{-\infty}^T (\tilde{u}(t), w) D_\infty^{(\rho),*} v(t)\, dt = \int_{-\infty}^T (\tilde{u}(t) D_\infty^{(\rho),*} v(t), w)\, dt = \left(\int_{-\infty}^T \tilde{u}(t) D_\infty^{(\rho),*} v(t)\, dt, w \right).$$

Therefore, we derive that

$$\int_{-\infty}^T \tilde{u}(t) D_\infty^{(\rho),*} v(t)\, dt = \int_0^T (\Delta\tilde{u}(t) + f(t))v(t)\, dt.$$

This confirms that $\widetilde{D_\infty^{(\rho)}} \tilde{u} = \Delta\tilde{u} + f \in L^2(0, T; H^{-1}(\Omega))$, and we proved that u is a weak solution to problem (1.1). □

Remark 4.11. The uniqueness of the weak solution can be derived straightforwardly, provided that the kernel function is variables-separable and satisfies the assumption given in Lemma 4.6. Here we let $f = 0$ and $\phi = 0$, and consider the eigenvalue problem

$$-\Delta\varphi = \lambda\varphi \text{ in } \Omega \qquad \text{and} \qquad \varphi_n = 0 \text{ on } \partial\Omega \tag{4.7}$$

By the spectrum theorem of the Laplacian, this eigenvalue problem (4.7) admits a nondecreasing sequence $\{\lambda_n\}_{n=1}^{\infty}$ of positive eigenvalues, which tend to ∞ with $n \to \infty$, and a corresponding sequence $\{\varphi_n\}_{n=1}^{\infty}$ of eigenfunctions which form an orthonormal basis in $L^2(\Omega)$. Then for any $\psi \in C_c^{\infty}(0,T)$, we have

$$\int_0^T \widetilde{D_{\infty}^{(\rho)}}(u(t),\varphi_n)\psi(t)\,dt + \int_0^T \lambda_n(u(t),\varphi_n)\psi(t)\,dt = 0$$

As a result, $(u(t),\varphi_n)$ is the solution of the initial value problem

$$\widetilde{D_{\infty}^{(\rho)}}(u(t),\varphi_n) + \lambda_n(u(t),\varphi_n) = 0 \qquad \text{with } (u(t),\varphi_n) = 0 \text{ for all } t < 0.$$

Then Lemma 4.6 and the uniqueness of the solution [16, Section 3][1] yields that $(u(t),\varphi_n) = 0$ for all n, and hence $u(t) \equiv 0$. See [2] for a discussion of uniqueness of weak solutions in the time-fractional case.

Remark 4.12. If $\phi(t,x) \equiv \phi_0(x) \in H_0^1(\Omega)$ in Theorem 4.10, then one recovers the weak solution to the (inhomogeneous) Caputo-type fractional diffusion equation [11,24]

$$u(t,x) = \mathbf{E}\left[\phi_0\left(B^x(\tau_0(t))\right)\mathbf{1}_{\{\tau_0(t)<\tau_\Omega(x)\}}\right] + \mathbf{E}\left[\int_0^{\tau_0(t)\wedge\tau_\Omega(x)} f\left(-X^{t,(\rho)}(s), B^x(s)\right) ds\right].$$

Remark 4.13. The solution in Theorem 4.10 will be continuous at $t = 0$ for every $x \in \Omega$ if ϕ is continuous in $\{0\} \times \Omega$ and $\tau_0 : [0,T] \to \mathbb{R}$ is continuous. This can be proved by a stochastic continuity argument for the first term of the solution (3.3), and for the second term one can use $\mathbf{E}[\tau_0(t)] \to 0$ as $t \downarrow 0$ (which is a consequence of the continuity of τ_0). However, the solution (3.3) will in general fail to be continuous at $t = 0$ even for smooth data. This is for example the case of integrable kernels $\int_0^{\infty} \rho(r)\,dr < \infty$ (see [41, Remark A.3]).

5. Numerical results

In this section, we present some numerical results to illustrate those theoretical findings, and explain how to apply the derived Feymann–Kac formula to numerically solve the nonlocal-in-time diffusion problem. To this end, we test the one-dimensional nonlocal diffusion problem (1.1) in $\Omega = (-1,1)$, and consider the non-integrable kernel function

$$\rho_\delta(r) = (1-\alpha)\delta^{\alpha-1}r^{-\alpha-1}\mathbf{1}_{(0,\delta)}(r), \tag{5.1}$$

with $\alpha \in (0,1)$ and the following data:

(i) initial data $\phi(x,t) = e^{5t}(1+x)(1-x)^2x$ and zero source term $f \equiv 0$;
(ii) trivial initial data $\phi(x,t) = 0$ and source term $f = e^t x^2 \sin(2\pi x)$.

The kernel function is proposed in this way in order to keep that $\int_0^{\delta} r\rho_\delta(r)\,dr = 1$ and hence the nonlocal operator recovers the infinitesimal first-order derivative as the nonlocal horizon diminishes. The analytical property of the model has been extensively studied in [16].

The stochastic process generated by the spatially second-order derivative (with zero boundary conditions), which is well-known as the killed Brownian motion in Ω, can be simply approximated by the lattice random walk. Specifically, we divided the interval Ω into M small

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[1] The uniqueness argument for the initial value problem in [16, Section 3] can be easily extended to time-dependent kernels ρ satisfying (H0).

intervals, with the uniform mesh size $h = 2/M$ and grid points $x_j = jh - 1$, $j = 0, 1, \dots, M$. Then in each time level, the particle standing in x_j will randomly move to x_{j-1} or x_{j+1}. In case that the particle hits the boundary of Ω, then the time is set as $\tau_\Omega(x_j)$. Here we let $B_h^{x_j}(t)$ be the position where the particle starting at position x_j arrives at time t.

Similarly, the stochastic process generated by the operator

$$-D_\delta^{(\rho)} u(t) = -\int_0^\delta (u(t) - u(t-r))\rho_\delta(r)\, dr$$

with historical initial data could also be approximated by a one-dimensional lattice random walk, where the trajectory of the particle involves some long-distance jumps. To numerically simulate the stochastic process, we discretize $[0, T]$ into K small intervals $[t_{n-1}, t_n]$ with $n = 1, 2, \dots, K$ and let $k = T/K$. Then we consider the discretization (assume that $\delta = mk$)

$$\begin{aligned} D_\delta^{(\rho)} u(t_n) &\approx \frac{u(t_n) - u(t_{n-1})}{k} \int_0^k r\rho_\delta(r)\, ds + \sum_{j=2}^m (u(t_n) - u(t_{n-k})) \int_{(j-1)k}^{jk} \rho_\delta(r)\, dr \\ &= \frac{1}{k^\alpha}\Big(\omega_0 u(t_n) - \sum_{j=1}^m \omega_j u(t_{n-j})\Big) =: \bar{D}_\delta^{(\rho)} u(t_n). \end{aligned} \tag{5.2}$$

Here the weights $\{\omega_j\}_{j=0}^m$ are computed exactly as $\omega_0 = \delta^{\alpha-1}\Big(1 + \frac{1-\alpha}{\alpha}(1 - m^{-\alpha})\Big)$, $\quad \omega_1 = \delta^{\alpha-1}$ and $\omega_j = \delta^{\alpha-1}\frac{1-\alpha}{\alpha}((j-1)^{-\alpha} - j^{-\alpha})$, $\quad j = 2, 3, \dots, m$. At each time level, the particle standing at the grid point t_j will jump to one of the grid points t_{j-i}, for $i = 1, 2, \dots, m$, with the probability $p_i = \omega_i/\omega_0$. It is easy to verify that $\sum_{j=1}^m \omega_j = \omega_0$ and hence $\sum_{j=1}^m p_j = 1$. We let $\tau_0(t_n)$ be the time that the particle starting at t_n passes 0, and $X_k^{t_n,(\rho)}(\tau_0(t_n))$ be the position where the particle arrives below 0. Then by applying the scaling $2\alpha k^\alpha = h^2\delta^{\alpha-1}$, the solution to (1.1) can be approximated by

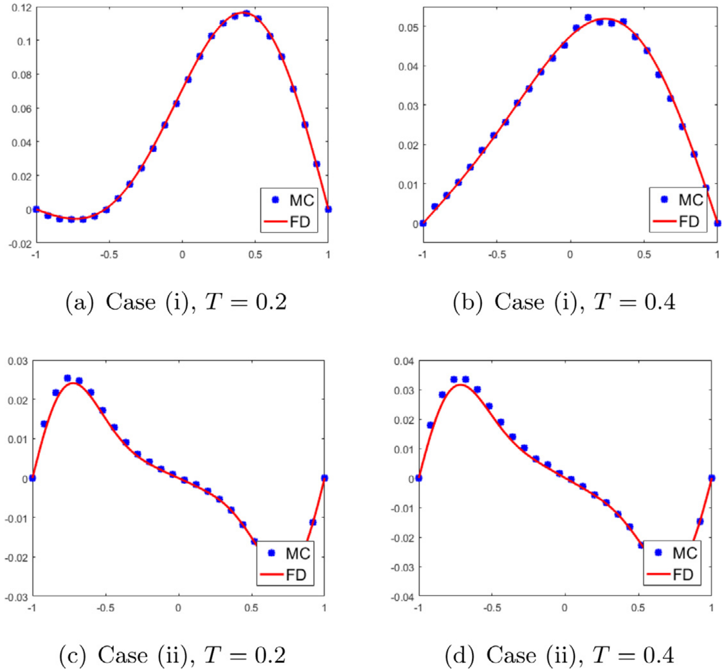
$$\begin{aligned} U_h^n =& \mathbf{E}\Big[\phi\left(-X_k^{t_n,(\rho)}(\tau_0(t)), B_h^{x_j}(\tau_0(t_n))\right)\mathbf{1}_{\{\tau_0(t_n) < \tau_\Omega(x_j)\}}\Big] \\ &+ \mathbf{E}\left[\int_0^{\tau_0(t_n)\wedge\tau_\Omega(x_j)} f\left(-X_k^{t_n,(\rho)}(s), B_h^{x_j}(s)\right) ds\right], \end{aligned} \tag{5.3}$$

using Monte Carlo method, where the integral is computed by trapezoidal rule.

In Fig. 1, we plot the numerical solution at different time levels, where the kernel function is defined in (5.1) with $\alpha = 0.75$ and $\delta = 0.2$. In the computation, we let $h = 0.02$ and $k = \sqrt[\alpha]{h^2\delta^{\alpha-1}/2\alpha}$, and use 50 000 Monte Carlo trials. Since the closed form of the analytical solution is not available, the benchmark solutions are computed by the finite difference scheme

$$\bar{D}_\delta^{(\rho)} u_h^n - \bar{\partial}_{xx}^h u_h^n = f^n$$

with a very fine mesh, say $k = 10^{-4}$ and $h = 10^{-3}$, where the discrete operator in time $\bar{D}_\delta^{(\rho)}$ is given by (5.2) and the spatial one $\bar{\partial}_{xx}^h$ is the central difference approximation to the second order derivative. We observe that the numerical solution computed by the stochastic approach is very close to the one computed by the finite difference scheme, which supports our theoretical results. Moreover, in Table 1 we present the ℓ^2-error of the Monte Carlo solution with different N (number of Monte Carlo trails) at different time level T, with fixed $\delta = 0.2$, $\alpha = 0.75$, $h = 0.02$ and $k = \sqrt[\alpha]{h^2\delta^{\alpha-1}/2\alpha}$, where we use the finite difference solutions as the benchmark solutions. The Monte Carlo solution converges with the order $O(N^{-\frac{1}{2}})$. While this is formally expected, a rigorous analysis can be a very interesting question to be addressed in a future work.



(a) Case (i), $T = 0.2$ (b) Case (i), $T = 0.4$

(c) Case (ii), $T = 0.2$ (d) Case (ii), $T = 0.4$

Fig. 1. Numerical results with $\delta = 0.2$ and $\alpha = 0.75$. (Blue dots: numerical solutions computed by (5.3) and Monte Carlo method (MC); Red curves: reference solutions computed by finite difference method (FD).)

Table 1
Case (i): ℓ^2-error of the numerical solution computed by (5.3) and Monte Carlo method (MC), with different T and N.

$T\backslash N$	1000	2000	4000	8000	16 000	Rate
0.1	6.55e−3	4.63e−3	3.39e−3	2.61e−3	1.93e−3	≈ -0.44
0.2	6.51e−3	4.58e−3	3.05e−3	2.23e−3	1.50e−3	≈ -0.53
0.4	4.65e−3	3.64e−3	2.42e−3	1.75e−3	1.31e−3	≈ -0.46

6. Concluding remarks

In this paper, we study the stochastic representation for an initial–boundary value problem of a nonlocal-in-time evolution equation (1.1), where the nonlocal operator appearing in the model is the Markovian generator of a $(-\infty, T]$-valued decreasing Lévy-type process. Under certain hypothesis, we derive the Feynman–Kac formula of the solution by reformulating the original problem into a Caputo-type nonlocal model with a specific forcing term. The case of weak data is also studied by energy arguments. The stochastic representation leads to a numerical scheme based on the Monte Carlo approach. The current theoretical results could be used to give more rigorous analysis of the stochastic algorithms for the nonlocal-in-time model. It is also an interesting topic to study some quantitative properties, such as asymptotical compatibility with shrinking nonlocal horizon parameter, of those algorithms.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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