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Fully decoupled, linear and unconditionally energy stable time discretization scheme for solving the magneto-hydrodynamic equations

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ABSTRACT

In this paper, we consider numerical approximations for solving the magneto-hydrodynamic equations, which couples the Navier–Stokes equations and Maxwell equations together. A challenging issue to solve this model numerically is the time discretization, i.e., how to develop suitable temporal discretizations for the nonlinear terms in order to preserve the energy stability at the discrete level. We solve this issue in this paper by developing a linear, fully decoupled first order time marching scheme, by combining the projection method for Navier–Stokes equations and some subtle implicit–explicit treatments for nonlinear coupling terms. We further prove that the scheme is unconditional energy stable and derive the optimal error estimates of the semi-discretization rigorously. Various numerical simulations are implemented to demonstrate the stability and the accuracy.

1. Introduction

The magneto-hydrodynamical (MHD) system models the behaviors of conducting fluids, such as plasmas, liquid metals, salt water and electrolytes, under external electromagnetic field. It has wide applications in geophysics, astrophysics and confinement for controlled thermonuclear fusion, see [1–3]. The fundamental concept behind the MHD system is that magnetic fields can induce currents in a moving conductive fluid, which in turn polarizes the fluid and reciprocally changes the magnetic field itself. Thus the governing equations that describe the MHD system are a nonlinear system to couple the Navier–Stokes equations for hydrodynamics and Maxwell's equations for electromagnetism. About the extensive theoretical modeling/numerical analysis for the MHD system, we refer to [4–23] and the references therein.

To solve the MHD system numerically, the main challenging issue is to develop proper temporal discretizations for those coupling terms, including (i) the coupling of the velocity and pressure in the fluid momentum equation, and (ii) the nonlinear coupling between the magnetic field and the velocity field through convection and Lorentz forces. It is

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well-known that simple discretizations, like fully explicit or implicit type schemes, can lead to considerable instabilities or suffer from costly time expense. Therefore, people are particularly interested in designing energy stable schemes, in the sense that the discrete energy dissipation laws hold. Meanwhile, while keeping the energy stable feature, it is also desirable to develop schemes that are easy-to-implement. Here the term "easy-to-implement" is referred to "linear" and "decoupled" in comparison with its counter parts: "nonlinear" and "coupled".

It is remarkable that many attempts have been made in this direction recently. In [8], the authors developed two implicit–explicit type methods where the first order method is shown to be unconditionally stable and the second order method is shown to be conditionally stable. However, the model considered in [8] is the reduced version, namely, the magnetic field is assumed to be a fixed function. In [24,25], the authors developed a decoupled type scheme for the full MHD system, but it is conditionally energy stable with a time step constraint similar to [8]. In [9], the authors developed a totally decoupled scheme where the computations of Navier–Stokes equations are based on the commutator of Laplacian and Leray projection, and all nonlinear and coupling terms are treated explicitly. However, the scheme is still conditionally stable. In [26], the authors had developed a "partially" decoupled scheme where the computations of magnetic field is totally decoupled from the velocity field since all nonlinear terms are treated explicitly, but the velocity is coupled with the pressure in the Navier–Stokes equations. Furthermore, a severe time step constraint ($\delta t \lesssim h^3$ where h is the grid size of space), which can be very costly in large-scale computations, has to be used to ensure stability. In [27], the authors developed some unconditionally energy stable schemes based on the projection type methods for the Navier–Stokes equations. However, the velocity field and the magnetic field are still coupled together.

Therefore, the aim of this paper is to develop a time marching scheme that is not only easy-to-implement (*linear and decoupled*), but also unconditionally energy stable. We achieve such a goal by combining several effective approaches, including, (i) an auxiliary intermediate velocity variable to decouple the computation of the magnetic field from the velocity; (ii) the projection method to decouple the pressure from the velocity; and (iii) some subtle implicit–explicit treatments to discretize the nonlinear convection and Lorentz force terms. We adopt the first order backward Euler scheme for time discretization, first order implicit–explicit treatments for nonlinear terms, and first order projection method for fluid equations. Thus, our final scheme is a first order time marching scheme. We rigorously prove the energy stability and derive the optimal error estimates for the developed scheme. Furthermore, we implement various numerical simulations, including the convergence test, energy stability test and a physical benchmark problem, the Kelvin–Helmholtz shear instability, to demonstrate the stability and accuracy of the scheme. Our decoupled idea is somewhat similar to the matrix splitting algorithms in [16]. The key ingredients are to introduce a new auxiliary velocity to decouple the computations of velocity and magnetic field. In [16], one also needs to solve an intermediate magnetic field $\tilde{b}^{n+1}$, then update the b^{n+1} through one step pseudo magnetic pressure correction. Furthermore, the error estimates and numerical studies of unconditional stabilities are not presented in [16].

The rest of paper is organized as follows. In Section 2, we present the model and derive the associated energy dissipation law. In Section 3, we develop the fully decoupled scheme and prove its associated energy stability. In Section 4, we derive its optimal error estimates. In Section 5, various numerical experiments are presented to demonstrate the stability and accuracy of the scheme. Finally, some concluding remarks are given in Section 6.

2. The MHD model and its energy law

Here and after, for two vector functions $\boldsymbol{x}, \boldsymbol{y}$, we denote the L^2 inner product as $(\boldsymbol{x}, \boldsymbol{y}) = \int_\Omega \boldsymbol{x} \cdot \boldsymbol{y} dx$ and L^2 norm $\|\boldsymbol{x}\|^2 = (\boldsymbol{x}, \boldsymbol{x})$. We use $H^1(\Omega)$ and $H^2(\Omega)$ to denote the usual Sobolev spaces, and use $\|\cdot\|_1$ for the norm in $H^1(\Omega)$ and $\|\cdot\|_2$ for the norm in $H^2(\Omega)$. We also define $H_0^1(\Omega) = \left\{\phi \in H^1(\Omega) : \phi|_{\partial\Omega} = 0\right\}$, $L_0^2(\Omega) = \left\{\phi \in L^2(\Omega) : \int_\Omega \phi dx = 0\right\}$, $H_\tau^1(\Omega) = \{\boldsymbol{w} \in H^1(\Omega)^d : \boldsymbol{n} \times \boldsymbol{w}|_\Omega = 0\}$ and $H = \{\boldsymbol{u} \in L^2(\Omega)^d, \nabla \cdot \boldsymbol{u} = 0, \boldsymbol{u} \cdot \boldsymbol{n}|_{\partial\Omega} = 0\}$, $d = 2, 3$. The following Poincare inequalities and embedding inequalities are well known [28,29]:

$$\|u\| \le c\|\nabla u\|, \quad u \in H_0^1(\Omega), \tag{2.1}$$

$$\|\boldsymbol{w}\|_1 \le c\|\nabla \cdot \boldsymbol{w}\| + c\|\nabla \times \boldsymbol{w}\| \quad \boldsymbol{w} \in H_\tau^1(\Omega), \tag{2.2}$$

$$\|u\|_{L^p} \le c\|u\|_1, \quad 2 \le p \le 6, \quad u \in H^1(\Omega), \tag{2.3}$$

$$\|u\|_{L^3} \le c\|u\|^{\frac{1}{2}}\|u\|_1^{\frac{1}{2}} \quad u \in H^1(\Omega). \tag{2.4}$$

Let X be a Banach space. The space $L^p(0, T; X)$, $1 \le p \le \infty$, is the space of classes of L^p functions from $(0, T)$ into X, which is a Banach space with the norm

$$\left(\int_0^T \|u(t)\|_X^p dt\right)^{1/p} \text{ if } 1 \le p < \infty, \quad \text{ess sup}_{t\in[0,T]}\|u(t)\|_X \text{ if } p = \infty.$$

For simplifying our notations, we write $\nabla \times \boldsymbol{a} \times \boldsymbol{b}$ to denote $(\nabla \times \boldsymbol{a}) \times \boldsymbol{b}$.

We consider the following incompressible MHD equations:

$$\boldsymbol{u}_t - \nu\Delta\boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p + s\boldsymbol{B} \times \nabla \times \boldsymbol{B} = 0, \tag{2.5}$$

$$\boldsymbol{B}_t + \eta\nabla \times \nabla \times \boldsymbol{B} - \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = 0, \tag{2.6}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{2.7}$$
$$\nabla \cdot \boldsymbol{B} = 0, \tag{2.8}$$

for $(\boldsymbol{x}, t) \in \Omega \times [0, T)$ with $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, where $\boldsymbol{u}$ denotes the velocity field, p is the pressure, and $\boldsymbol{B}$ is the magnetic field. For the physical parameters, $\nu^{-1} = R_e$ (fluid Reynolds number), $\eta^{-1} = R_m$ (magnetic Reynolds number), and s is the coupling coefficient, which are given by

$$R_e = \frac{UL}{\mu_f}, \quad R_m = \mu_m \sigma UL, \quad s = \frac{B^2}{\rho \mu_m U^2},$$

where U is the characteristic velocity, L is the characteristic length, μ_f is the kinematic viscosity, μ_m is the magnetic permeability, σ is the electric conductivity, B is the characteristic magnetic field, and ρ is the fluid density. The system is equipped with the following boundary conditions

$$\boldsymbol{u}|_{\partial\Omega} = 0, \quad \boldsymbol{B} \times \boldsymbol{n}|_{\partial\Omega} = 0, \tag{2.9}$$

and initial conditions

$$\boldsymbol{u}|_{(t=0)} = \boldsymbol{u}_0(\boldsymbol{x}), \quad \boldsymbol{B}|_{(t=0)} = \boldsymbol{B}_0(\boldsymbol{x}), \tag{2.10}$$

with $\nabla \cdot \boldsymbol{u}_0 = 0$, $\nabla \cdot \boldsymbol{B}_0 = 0$, where $\boldsymbol{n}$ denotes the outward unit normal of $\partial\Omega$. We also assume Ω is a bounded connected regular domain such that the H^2 regularity of elliptic problems holds.

The model (2.5)–(2.8) follows the energy dissipation law. By taking the L^2 inner product of (2.5) with $\boldsymbol{u}$, and of (2.6) with $s\boldsymbol{B}$, using (2.7)–(2.9) and integration by parts, we have

$$\begin{aligned}
&(\boldsymbol{u}_t, \boldsymbol{u}) + \nu\|\nabla \boldsymbol{u}\|^2 + s(\boldsymbol{B} \times \nabla \times \boldsymbol{B}, \boldsymbol{u}) = 0, \\
&s(\boldsymbol{B}_t, \boldsymbol{B}) + s\eta\|\nabla \times \boldsymbol{B}\|^2 - s(\boldsymbol{u} \times \boldsymbol{B}, \nabla \times \boldsymbol{B}) = 0.
\end{aligned}$$

By taking the summation of the two equalities, we obtain

$$\frac{d}{dt}E(\boldsymbol{u}, \boldsymbol{B}) = -\nu\|\nabla \boldsymbol{u}\|^2 - s\eta\|\nabla \times \boldsymbol{B}\|^2,$$

where

$$E(\boldsymbol{u}, \boldsymbol{B}) = \frac{1}{2}\|\boldsymbol{u}\|^2 + \frac{s}{2}\|\boldsymbol{B}\|^2$$

represents the total energy of the system (2.5)–(2.8).

3. Numerical scheme

We now construct a semi-discrete time marching numerical scheme for solving the model system (2.5)–(2.8) and prove the corresponding energy stability. It will be clear that the energy stabilities of the semi-discrete schemes are also valid in the fully discrete formulation, for instance by finite element or spectral spatial discretizations.

Let $\delta t > 0$ denote the time step size and set $t_n = n\delta t$ for $0 \le n \le [\frac{T}{\delta t}]$ with the final time T. Our numerical scheme reads as follows.

Given the initial conditions $(\boldsymbol{u}^0, \boldsymbol{B}^0, p^0)$, where $p^0 = p(0)$ is obtained from $\boldsymbol{u}_0, \boldsymbol{B}_0$ and Eqs. (2.5), having computed $(\boldsymbol{u}^n, \boldsymbol{B}^n, p^n)$ for $n > 0$, we compute $(\boldsymbol{u}^{n+1}, \boldsymbol{B}^{n+1}, p^{n+1})$ by the following steps.

Step 1.

$$\frac{\boldsymbol{B}^{n+1} - \boldsymbol{B}^n}{\delta t} + \eta\nabla \times \nabla \times \boldsymbol{B}^{n+1} - \nabla \times (\boldsymbol{u}_\star^n \times \boldsymbol{B}^n) = 0, \tag{3.1}$$
$$\frac{\boldsymbol{u}_\star^n - \boldsymbol{u}^n}{\delta t} + s\boldsymbol{B}^n \times \nabla \times \boldsymbol{B}^{n+1} = 0, \tag{3.2}$$
$$\boldsymbol{B}^{n+1} \times \boldsymbol{n}|_{\partial\Omega} = 0. \tag{3.3}$$

Step 2.

$$\frac{\tilde{\boldsymbol{u}}^{n+1} - \boldsymbol{u}_\star^n}{\delta t} + (\boldsymbol{u}^n \cdot \nabla)\tilde{\boldsymbol{u}}^{n+1} - \nu\Delta\tilde{\boldsymbol{u}}^{n+1} + \nabla p^n = 0, \tag{3.4}$$
$$\tilde{\boldsymbol{u}}^{n+1}|_{\partial\Omega} = 0. \tag{3.5}$$

Step 3.

$$\frac{\boldsymbol{u}^{n+1} - \tilde{\boldsymbol{u}}^{n+1}}{\delta t} + \nabla(p^{n+1} - p^n) = 0, \tag{3.6}$$
$$\nabla \cdot \boldsymbol{u}^{n+1} = 0, \tag{3.7}$$

$$\boldsymbol{u}^{n+1}\cdot\boldsymbol{n}|_{\partial\Omega}=0. \tag{3.8}$$

Several remarks are in order.

Remark 3.1. For the semi-discrete form, we notice the divergence free condition holds for $\boldsymbol{u}^n$. In practice, $\boldsymbol{u}^n$ may not be divergence free pointwise. For this case, a common practice is to define a skew-symmetric form

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})=\frac{1}{2}((\boldsymbol{u}\cdot\nabla)\boldsymbol{v},\boldsymbol{w})-\frac{1}{2}((\boldsymbol{u}\cdot\nabla)\boldsymbol{w},\boldsymbol{v}) \tag{3.9}$$

which was first introduced by Temam [30]. If the velocity $\boldsymbol{u}\in H$, then $b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v})=((\boldsymbol{u}\cdot\nabla)\boldsymbol{u},\boldsymbol{v})$ which is consistent with original equation.

Remark 3.2. To decouple the computation of the pressure from that of the velocity, we use the first order pressure-correction scheme [31–38]. To further decouple the computations of $\boldsymbol{B}$ from the velocity field $\boldsymbol{u}$, inspired by [33,37,39–45], we introduce a new, explicit, convective velocity $\boldsymbol{u}^n_\star$, that can be computed directly from (3.2), i.e.,

$$\boldsymbol{u}^n_\star=\boldsymbol{u}^n+\delta ts\nabla\times\boldsymbol{B}^{n+1}\times\boldsymbol{B}^n. \tag{3.10}$$

Indeed, if plugging (3.10) into (3.1), one obtains a linear equation for $\boldsymbol{B}^{n+1}$ as

$$\frac{\boldsymbol{B}^{n+1}}{\delta t}+\eta\nabla\times\nabla\times\boldsymbol{B}^{n+1}+\delta ts\nabla\times\Big(\boldsymbol{B}^n\times(\nabla\times\boldsymbol{B}^{n+1}\times\boldsymbol{B}^n)\Big)=g^n, \tag{3.11}$$

where $g^n=-\nabla\times(\boldsymbol{B}^n\times\boldsymbol{u}^n)+\frac{1}{\delta t}\boldsymbol{B}^n$. Thus, in Step 1, we only need to solve the above fully decoupled, linear problem (3.11) with the boundary condition (3.3).

Remark 3.3. For the pressure equation, indeed, by taking the divergence for (3.6), we get

$$-\Delta p^{n+1}=-\frac{1}{\delta t}\nabla\cdot\tilde{\boldsymbol{u}}^{n+1}-\Delta p^n, \tag{3.12}$$

associated with the Neumann boundary conditions $\partial_{\boldsymbol{n}}(p^{n+1}-p^n)|_{\partial\Omega}=0$. Once p^{n+1} is obtained, we update $\boldsymbol{u}^{n+1}$ from $\boldsymbol{u}^{n+1}=\tilde{\boldsymbol{u}}^{n+1}-\delta t\nabla(p^{n+1}-p^n)$.

Proposition 1. *The problem (3.11) with (3.3) of Step 1 is well-posedness.*

Proof. The associated weak form of (3.11) can be written as: Find $\boldsymbol{B}\in H^1_\tau(\Omega)$ such that

$$a(\boldsymbol{B},\boldsymbol{C})=(g^n,\boldsymbol{C}),\quad \forall\boldsymbol{C}\in H^1_\tau(\Omega)$$

where

$$a(\boldsymbol{B},\boldsymbol{C})=\frac{1}{\delta t}(\boldsymbol{B},\boldsymbol{C})+\eta(\nabla\times\boldsymbol{B},\nabla\times\boldsymbol{C})+\delta ts(\boldsymbol{B}^n\times\nabla\times\boldsymbol{B},\boldsymbol{B}^n\times\nabla\times\boldsymbol{C}).$$

It can be seen the bilinear form $a(\cdot,\cdot)$ is symmetric, namely,

$$a(\boldsymbol{B},\boldsymbol{C})=a(\boldsymbol{C},\boldsymbol{B}).$$

In addition, using (2.2) and $\nabla\cdot\boldsymbol{B}=0$ ($\nabla\cdot\boldsymbol{B}^n=0$ can be deduced by taking divergence of (3.11)) we have

$$a(\boldsymbol{B},\boldsymbol{B})\geq C(\Omega,\eta)\|\boldsymbol{B}\|_1^2,$$

and

$$a(\boldsymbol{B},\boldsymbol{C})\leq C(\delta t,\eta,s,\|\boldsymbol{B}^n\|_{L^\infty})\|\boldsymbol{B}\|_1\|\boldsymbol{C}\|_1.$$

Thus, it is well-posedness using the Lax–Milgram Theorem. □

In the below, we prove the energy stability of the scheme (3.1)–(3.8) as follows.

Theorem 3.1. *The scheme (3.1)–(3.8) is unconditionally energy stable in the sense that*

$$\begin{aligned}&s\|\boldsymbol{B}^{n+1}\|^2+\|\boldsymbol{u}^{n+1}\|^2+\delta t^2\|\nabla p^{n+1}\|^2+2\delta t\Big(s\eta\|\nabla\times\boldsymbol{B}^{n+1}\|^2+\nu\|\nabla\tilde{\boldsymbol{u}}^{n+1}\|^2\Big)\\&\quad\leq s\|\boldsymbol{B}^n\|^2+\|\boldsymbol{u}^n\|^2+\delta t^2\|\nabla p^n\|^2.\end{aligned} \tag{3.13}$$

Moreover, $\nabla\cdot\boldsymbol{B}^{n+1}=\cdots=\nabla\cdot\boldsymbol{B}^0=0$ *for* $1\leq n\leq N=[\frac{T}{\delta t}]-1$.

Proof. By taking the L^2 inner product of (3.1) with $s\boldsymbol{B}^{n+1}$ and of (3.2) with $\boldsymbol{u}_\star^n$, we obtain

$$\frac{s}{2\delta t}\left(\|\boldsymbol{B}^{n+1}-\boldsymbol{B}^n\|^2+\|\boldsymbol{B}^{n+1}\|^2-\|\boldsymbol{B}^n\|^2\right)+s\eta\|\nabla\times\boldsymbol{B}^{n+1}\|^2+s(\boldsymbol{B}^n\times\boldsymbol{u}_\star^n,\nabla\times\boldsymbol{B}^{n+1})=0, \tag{3.14}$$

and

$$\frac{1}{2\delta t}\left(\|\boldsymbol{u}_\star^n-\boldsymbol{u}^n\|^2+\|\boldsymbol{u}_\star^n\|^2-\|\boldsymbol{u}^n\|^2\right)-s(\nabla\times\boldsymbol{B}^{n+1}\times\boldsymbol{B}^n,\boldsymbol{u}_\star^n)=0. \tag{3.15}$$

By taking the L^2 inner product of (3.4) with $\tilde{\boldsymbol{u}}^{n+1}$ and using the well known property of

$$(\boldsymbol{u}\cdot\nabla\boldsymbol{v},\boldsymbol{v})=0,\forall\,\boldsymbol{u}\in H,\,\boldsymbol{v}\in H_0^1(\Omega)^2, \tag{3.16}$$

thus we derive

$$\frac{1}{2\delta t}\left(\|\tilde{\boldsymbol{u}}^{n+1}-\boldsymbol{u}_\star^n\|^2+\|\tilde{\boldsymbol{u}}^{n+1}\|^2-\|\boldsymbol{u}_\star^n\|^2\right)+\nu\|\nabla\tilde{\boldsymbol{u}}^{n+1}\|^2+(\nabla p^n,\tilde{\boldsymbol{u}}^{n+1})=0. \tag{3.17}$$

We rewrite (3.6) as

$$\frac{1}{\delta t}\boldsymbol{u}^{n+1}+\nabla p^{n+1}=\frac{1}{\delta t}\tilde{\boldsymbol{u}}^{n+1}+\nabla p^n, \tag{3.18}$$

and take the L^2 inner product of the above with itself on both sides, we obtain

$$\frac{1}{2\delta t}\|\boldsymbol{u}^{n+1}\|^2-\frac{1}{2\delta t}\|\tilde{\boldsymbol{u}}^{n+1}\|^2+\frac{\delta t}{2}\|\nabla p^{n+1}\|^2-\frac{\delta t}{2}\|\nabla p^n\|^2=(\tilde{\boldsymbol{u}}^{n+1},\nabla p^n). \tag{3.19}$$

Then, by taking the summations (3.14), (3.15), (3.17) and (3.19), we obtain

$$\begin{aligned}&\frac{s}{2\delta t}(\|\boldsymbol{B}^{n+1}\|^2-\|\boldsymbol{B}^n\|^2+\|\boldsymbol{B}^{n+1}-\boldsymbol{B}^n\|^2)+s\eta\|\nabla\times\boldsymbol{B}^{n+1}\|^2\\&\quad+\frac{1}{2\delta t}\left(\|\boldsymbol{u}^{n+1}\|^2-\|\boldsymbol{u}^n\|^2+\|\boldsymbol{u}_\star^n-\boldsymbol{u}^n\|^2\right)+\frac{1}{2\delta t}\|\tilde{\boldsymbol{u}}^{n+1}-\boldsymbol{u}_\star^n\|^2\\&\quad+\nu\|\nabla\tilde{\boldsymbol{u}}^{n+1}\|^2+\frac{\delta t}{2}\|\nabla p^{n+1}\|^2-\frac{\delta t}{2}\|\nabla p^n\|^2=0.\end{aligned} \tag{3.20}$$

After multiplying with $2\delta t$ and dropping some positive terms, we obtain (3.13). Finally, by taking the divergence for (3.1), we get $\nabla\cdot\boldsymbol{B}^{n+1}=\nabla\cdot\boldsymbol{B}^n=\cdots=\nabla\cdot\boldsymbol{B}^0=0$. □

Remark 3.4. By Theorem 3.1, summing up the inequality (3.13) from $n=0$ to m $(\leq[\frac{T}{\delta t}]-1)$, we get the stable bound

$$\begin{aligned}s\|\boldsymbol{B}^{m+1}\|^2+\|\boldsymbol{u}^{m+1}\|^2+\delta t^2\|\nabla p^{m+1}\|^2+2\delta t\sum_{n=0}^{m}\left(s\eta\|\nabla\times\boldsymbol{B}^{n+1}\|^2+\nu\|\nabla\tilde{\boldsymbol{u}}^{n+1}\|^2\right)\\\leq s\|\boldsymbol{B}^0\|^2+\|\boldsymbol{u}^0\|^2+\delta t^2\|\nabla p^0\|^2.\end{aligned}$$

4. Error analysis

In this section, we first prove the error estimates for velocity and magnetic field in Section 4.1. Then we improve the convergence order for pressure in Section 4.2. We denote by C a generic constant that is independent of δt but possibly depends on the data and the solution, and use $f\lesssim g$ to say that there is a generic constant C such that $f\leq Cg$.

We shall use repeatedly the following discrete Gronwall inequality [46].

Lemma 4.1. *Let g_0, a_n, b_n, c_n and γ_n be a sequence of nonnegative numbers for integers $n\geq 0$ such that*

$$a_n+\delta t\sum_{j=0}^{n}b_j\leq\delta t\sum_{j=0}^{n}\gamma_j a_j+\delta t\sum_{j=0}^{n}c_j+g_0.$$

Assume that $\gamma_j\delta t\leq 1$ for all j, and set $\sigma_j=(1-\gamma_j\delta t)^{-1}$. Then, for all $n\geq 0$,

$$a_n+\delta t\sum_{j=0}^{n}b_j\leq\exp\left(\delta t\sum_{j=0}^{n}\sigma_j\gamma_j\right)\left(\delta t\sum_{j=0}^{n}c_j+g_0\right).$$

The following lemma will be used in Lemma 4.4.

Lemma 4.2. *Let c_1, c_2, c_3 be nonnegative numbers, a_n be a sequence of nonnegative numbers for $n\geq 0$ such that*

$$a_{n+1}\leq c_1+c_2\delta t a_n+c_3\delta t^2 a_n^2.$$

If $\max\{c_2, \sqrt{c_3}\}D\delta t \leq 1$, *then, for* $n \geq 0$

$$a_n \leq D,$$

where $D = \max\{a_0, c_1\} + 2$.

This lemma can be proved by the mathematical induction method. We skip it.

4.1. Error estimates for velocity and magnetic field

We rewrite (2.5)–(2.6) as follows.

$$\frac{\boldsymbol{B}(t_{n+1}) - \boldsymbol{B}(t_n)}{\delta t} + \eta\nabla\times\nabla\times\boldsymbol{B}(t_{n+1}) - \nabla\times(\boldsymbol{u}(t_n)\times\boldsymbol{B}(t_n)) = R_b^{n+1}, \tag{4.1}$$

$$\begin{aligned}\frac{\boldsymbol{u}(t_{n+1}) - \boldsymbol{u}(t_n)}{\delta t} &- \nu\Delta\boldsymbol{u}(t_{n+1}) + (\boldsymbol{u}(t_n)\cdot\nabla)\boldsymbol{u}(t_{n+1}) + \nabla p(t_n)\\ &+s\boldsymbol{B}(t_n)\times\nabla\times\boldsymbol{B}(t_{n+1}) = R_u^{n+1},\end{aligned} \tag{4.2}$$

$$\frac{\boldsymbol{u}(t_{n+1}) - \boldsymbol{u}(t_{n+1})}{\delta t} + \nabla(p(t_{n+1}) - p(t_n)) = R_p^{n+1}, \tag{4.3}$$

where

$$\begin{aligned}R_b^{n+1} &= \frac{\boldsymbol{B}(t_{n+1}) - \boldsymbol{B}(t_n)}{\delta t} - \boldsymbol{B}_t(t_{n+1}) + \nabla\times(\boldsymbol{u}(t_{n+1})\times\boldsymbol{B}(t_{n+1})) - \nabla\times(\boldsymbol{u}(t_n)\times\boldsymbol{B}(t_n)),\\ R_u^{n+1} &= \frac{\boldsymbol{u}(t_{n+1}) - \boldsymbol{u}(t_n)}{\delta t} - \boldsymbol{u}_t(t_{n+1}) + (\boldsymbol{u}(t_n) - \boldsymbol{u}(t_{n+1}))\nabla\boldsymbol{u}(t_{n+1}) + \nabla p(t_n) - \nabla p(t_{n+1})\\ &\quad + s(\boldsymbol{B}(t_n) - \boldsymbol{B}(t_{n+1}))\times\nabla\times\boldsymbol{B}(t_{n+1}),\\ R_p^{n+1} &= \nabla p(t_{n+1}) - \nabla p(t_n)\end{aligned}$$

are truncation errors. The existence and uniqueness of solution to the MHD system (2.5)–(2.8) have been studied in [4]. Here we make some regularity assumptions about the solution $(\boldsymbol{u}, \boldsymbol{B}, p)$ of the system (2.5)–(2.8),

$$(\mathbf{A}): \quad \begin{cases}\boldsymbol{u}, \boldsymbol{B} \in L^\infty(0, T; H^2(\Omega)), p \in L^\infty(0, T; H^1(\Omega)),\\ \boldsymbol{u}_t, \boldsymbol{B}_t \in L^\infty(0, T; H^1(\Omega)\cap L^\infty(\Omega)), p_t \in L^\infty(0, T; H^1(\Omega)),\\ \boldsymbol{u}_{tt}, \boldsymbol{B}_{tt} \in L^\infty(0, T; L^2(\Omega))\cap L^2(0, T; H^1(\Omega)), p_{tt} \in L^2(0, T; H^1(\Omega)),\\ \boldsymbol{u}_{ttt}, \boldsymbol{B}_{ttt} \in L^2(0, T; L^2(\Omega)).\end{cases} \tag{4.4}$$

One can easily establish the following estimates for the truncation errors, provided that the exact solutions are sufficiently smooth or satisfy the assumption (A).

Lemma 4.3. *Under the Assumption (A), the truncation errors satisfy*

$$\|R_u^n\| + \|R_b^n\| + \|R_p^n\| \lesssim \delta t, 0 \leq n \leq [\frac{T}{\delta t}].$$

Proof. Since the proof is rather standard and similar to the proof in Lemma 4.5, due to the page limit, we leave it to the interested readers. □

To derive the error estimates, we denote the error functions as

$$\begin{cases} e_b^n = \boldsymbol{B}(t_n) - \boldsymbol{B}^n, \quad \tilde{e}_u^n = \boldsymbol{u}(t_n) - \tilde{\boldsymbol{u}}^n,\\ e_u^n = \boldsymbol{u}(t_n) - \boldsymbol{u}^n, \quad e_p^n = p(t_n) - p^n.\end{cases}$$

By subtracting (3.1) from (4.1), (3.4) from (4.2) and applying (3.2), and (3.6) from (4.3), we obtain the following error equations,

$$\frac{e_b^{n+1} - e_b^n}{\delta t} + \eta\nabla\times\nabla\times e_b^{n+1} + \nabla\times(\boldsymbol{B}(t_n)\times\boldsymbol{u}(t_n)) - \nabla\times(\boldsymbol{B}^n\times\boldsymbol{u}_\star^n) = R_b^{n+1}, \tag{4.5}$$

$$\begin{aligned}\frac{\tilde{e}_u^{n+1} - e_u^n}{\delta t} &- \nu\Delta\tilde{e}_u^{n+1} + (\boldsymbol{u}(t_n)\cdot\nabla)\boldsymbol{u}(t_{n+1}) - (\boldsymbol{u}^n\cdot\nabla)\tilde{\boldsymbol{u}}^{n+1} + \nabla e_p^n\\ &+s\boldsymbol{B}(t_n)\times\nabla\times\boldsymbol{B}(t_{n+1}) - s\boldsymbol{B}^n\times\nabla\times\boldsymbol{B}^{n+1} = R_u^{n+1},\end{aligned} \tag{4.6}$$

$$\frac{e_u^{n+1} - \tilde{e}_u^{n+1}}{\delta t} + \nabla(e_p^{n+1} - e_p^n) = R_p^{n+1}. \tag{4.7}$$

We first show the L^∞ stability of $\boldsymbol{B}^n$, that plays a key role in the error estimates. Let

$$\kappa = \max_{0\le t\le T} \|\boldsymbol{B}(t)\|_{L^\infty} + 1,$$

the preliminary result is given in the following lemma.

Lemma 4.4. *Assuming that the solution to (2.5)–(2.8) satisfies Assumption (A), there exists a constant C such that if $\delta t \le C$ the solution $\boldsymbol{B}^n$ of scheme (3.1)–(3.8) satisfies*

$$\|\boldsymbol{B}^n\|_{L^\infty} \le \kappa, \quad n = 0, 1, \dots, [\frac{T}{\delta t}]. \tag{4.8}$$

Proof. We use the mathematical induction method to prove this lemma.

When $n = 0$, we have $\|\boldsymbol{B}^0\|_{L^\infty} \le \kappa$.

Assuming that $\|\boldsymbol{B}^n\|_{L^\infty} \le \kappa$ is valid for $n = 0, 1, \dots, N$, we will show $\|\boldsymbol{B}^{N+1}\|_{L^\infty} \le \kappa$ is also valid through the following three steps. In *Step i*, using the induction assumptions, we first give a convergence result. Then in *Step ii*, using the convergence result obtained in *Step i*, we prove the H^2 stability for $\boldsymbol{B}^{N+1}$. Finally, in *Step iii*, by the convergence result and H^2 stability proved in *Steps i and ii* and Sobolev inequalities, we bound the L^∞ norm of $\boldsymbol{B}^{N+1}$.

(*Step i*). By taking the L^2 inner product of (4.5) with e_b^{n+1}, using integration by parts and the identity

$$(a-b, 2a) = |a|^2 - |b|^2 + |a-b|^2,$$

we obtain

$$\begin{aligned}&\frac{1}{2\delta t}(\|e_b^{n+1} - e_b^n\|^2 + \|e_b^{n+1}\|^2 - \|e_b^n\|^2) + \eta\|\nabla\times e_b^{n+1}\|^2\\ &\quad + (\boldsymbol{B}(t_n)\times\boldsymbol{u}(t_n) - \boldsymbol{B}^n\times\boldsymbol{u}_\star^n, \nabla\times e_b^{n+1}) = (R_b^{n+1}, e_b^{n+1}).\end{aligned} \tag{4.9}$$

By taking the L^2 inner product of (4.6) with $\tilde{e}_u^{n+1}$, we derive

$$\begin{aligned}&\frac{1}{2\delta t}(\|\tilde{e}_u^{n+1} - e_u^n\|^2 + \|\tilde{e}_u^{n+1}\|^2 - \|e_u^n\|^2) + \nu\|\nabla\tilde{e}_u^{n+1}\|^2 + (\nabla e_p^n, \tilde{e}_u^{n+1})\\ &\quad + \left((\boldsymbol{u}(t_n)\cdot\nabla)\boldsymbol{u}(t_{n+1}) - (\boldsymbol{u}^n\cdot\nabla)\tilde{\boldsymbol{u}}^{n+1}, \tilde{e}_u^{n+1}\right)\\ &\quad + s\left(\boldsymbol{B}(t_n)\times\nabla\times\boldsymbol{B}(t_{n+1}) - \boldsymbol{B}^n\times\nabla\times\boldsymbol{B}^{n+1}, \tilde{e}_u^{n+1}\right) = (R_u^{n+1}, \tilde{e}_u^{n+1}).\end{aligned} \tag{4.10}$$

We rewrite (4.7) to obtain

$$\frac{1}{\delta t}e_u^{n+1} + \nabla e_p^{n+1} = \frac{1}{\delta t}\tilde{e}_u^{n+1} + \nabla e_p^n + R_p^{n+1}. \tag{4.11}$$

By taking the L^2 inner product of (4.11) with itself on both sides, we obtain

$$\begin{aligned}(\tilde{e}_u^{n+1}, \nabla e_p^n) =&\frac{1}{2\delta t}(\|e_u^{n+1}\|^2 - \|\tilde{e}_u^{n+1}\|^2) + \frac{\delta t}{2}(\|\nabla e_p^{n+1}\|^2 - \|\nabla e_p^n\|^2)\\ &- (\tilde{e}_u^{n+1}, R_p^{n+1}) - \delta t(\nabla e_p^n, R_p^{n+1}) - \frac{\delta t}{2}\|R_p^{n+1}\|^2.\end{aligned} \tag{4.12}$$

We combine (4.9)–(4.12) to obtain

$$\begin{aligned}&\frac{1}{2\delta t}(\|e_b^{n+1} - e_b^n\|^2 + \|e_b^{n+1}\|^2 - \|e_b^n\|^2) + \eta\|\nabla\times e_b^{n+1}\|^2 + \frac{\delta t}{2}(\|\nabla e_p^{n+1}\|^2 - \|\nabla e_p^n\|^2)\\ &+ \frac{1}{2\delta t}(\|\tilde{e}_u^{n+1} - e_u^n\|^2 + \|e_u^{n+1}\|^2 - \|e_u^n\|^2) + \nu\|\nabla\tilde{e}_u^{n+1}\|^2\\ &= -(\boldsymbol{B}(t_n)\times\boldsymbol{u}(t_n) - \boldsymbol{B}^n\times\boldsymbol{u}_\star^n, \nabla\times e_b^{n+1}) && (:\text{ term A})\\ &\quad - \left((\boldsymbol{u}(t_n)\cdot\nabla)\boldsymbol{u}(t_{n+1}) - (\boldsymbol{u}^n\cdot\nabla)\tilde{\boldsymbol{u}}^{n+1}, \tilde{e}_u^{n+1}\right) && (:\text{ term B})\\ &\quad - s\left(\boldsymbol{B}(t_n)\times\nabla\times\boldsymbol{B}(t_{n+1}) - \boldsymbol{B}^n\times\nabla\times\boldsymbol{B}^{n+1}, \tilde{e}_u^{n+1}\right) && (:\text{ term C})\\ &\quad + (\tilde{e}_u^{n+1}, R_p^{n+1}) + \delta t(\nabla e_p^n, R_p^{n+1}) && (:\text{ term D})\\ &\quad + (R_b^{n+1}, e_b^{n+1}) + (R_u^{n+1}, \tilde{e}_u^{n+1}) && (:\text{ term E})\\ &\quad + \frac{\delta t}{2}\|R_p^{n+1}\|^2.\end{aligned} \tag{4.13}$$

For $n \leq N$, for term A, using the definition of $u_\star^n$ in (3.10), induction assumptions, assumption (A) and Young inequality $ab \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2$, we derive

$$\begin{aligned}
(\text{term A}) &\leq \left|(\boldsymbol{B}(t_n) \times \boldsymbol{u}(t_n) - \boldsymbol{B}^n \times \boldsymbol{u}_\star^n, \nabla \times e_b^{n+1})\right| \\
&= \left|(e_b^n \times \boldsymbol{u}(t_n) + \boldsymbol{B}^n \times (\boldsymbol{u}(t_n) - \boldsymbol{u}_\star^n), \nabla \times e_b^{n+1})\right| \\
&= \left|(e_b^n \times \boldsymbol{u}(t_n), \nabla \times e_b^{n+1}) + (\boldsymbol{B}^n \times e_u^n, \nabla \times e_b^{n+1}) - s\delta t(\boldsymbol{B}^n \times (\nabla \times \boldsymbol{B}^{n+1} \times \boldsymbol{B}^n), \nabla \times e_b^{n+1})\right| \\
&\leq \left(\|e_b^n\|\|\boldsymbol{u}(t_n)\|_{L^\infty} + \|\boldsymbol{B}^n\|_{L^\infty}\|e_u^n\| + s\delta t\|\boldsymbol{B}^n\|_{L^\infty}^2\|\nabla \times \boldsymbol{B}^{n+1}\|\right)\|\nabla \times e_b^{n+1}\| \\
&\lesssim \frac{\eta}{6}\|\nabla \times e_b^{n+1}\|^2 + \|e_b^n\|^2 + \|e_u^n\|^2 + \delta t^2\|\nabla \times \boldsymbol{B}^{n+1}\|^2.
\end{aligned}$$

For term B, using (3.16), we derive

$$\begin{aligned}
(\text{term B}) &\leq \left|((\boldsymbol{u}(t_n) \cdot \nabla)\boldsymbol{u}(t_{n+1}) - (\boldsymbol{u}^n \cdot \nabla)\tilde{\boldsymbol{u}}^{n+1}, \tilde{e}_u^{n+1})\right| \\
&= \left|((e_u^n \cdot \nabla)\boldsymbol{u}(t_{n+1}), \tilde{e}_u^{n+1}) + ((\boldsymbol{u}^n \cdot \nabla)\tilde{e}_u^{n+1}, \tilde{e}_u^{n+1})\right| \\
&= \left|((e_u^n \cdot \nabla)\boldsymbol{u}(t_{n+1}), \tilde{e}_u^{n+1})\right| = \left|((e_u^n \cdot \nabla)\tilde{e}_u^{n+1}, \boldsymbol{u}(t_{n+1}))\right| \\
&\leq \|e_u^n\|\;\|\nabla\tilde{e}_u^{n+1}\|\|\boldsymbol{u}(t_{n+1})\|_{L^\infty} \lesssim \frac{\nu}{8}\|\nabla\tilde{e}_u^{n+1}\|^2 + \|e_u^n\|^2.
\end{aligned}$$

For term C, using $\tilde{e}_u^{n+1} = e_u^{n+1} + \delta t(\nabla e_p^{n+1} - \nabla e_p^n) - \delta t R_p^{n+1}$, Assumption (A) and (2.3), we derive

$$\begin{aligned}
(\text{term C}) &\leq s\left|(\nabla \times \boldsymbol{B}(t_{n+1}) \times \boldsymbol{B}(t_n) - \nabla \times \boldsymbol{B}^{n+1} \times \boldsymbol{B}^n, \tilde{e}_u^{n+1})\right| \\
&= s\left|(\nabla \times \boldsymbol{B}(t_{n+1}) \times e_b^n, \tilde{e}_u^{n+1}) + (\nabla \times e_b^{n+1} \times \boldsymbol{B}^n, \tilde{e}_u^{n+1})\right| \\
&\leq s\|\nabla \times \boldsymbol{B}(t_{n+1})\|_{L^4}\|e_b^n\|\;\|\tilde{e}_u^{n+1}\|_{L^4} + s\|\nabla \times e_b^{n+1}\|\;\|\boldsymbol{B}^n\|_{L^\infty}\|\tilde{e}_u^{n+1}\| \\
&\lesssim \|e_b^n\|\|\nabla\tilde{e}_u^{n+1}\| + \|\nabla \times e_b^{n+1}\|(\|e_u^{n+1}\| + \delta t\|\nabla e_p^{n+1} - \nabla e_p^n\| + \delta t\|R_p^{n+1}\|) \\
&\lesssim \frac{\nu}{8}\|\nabla\tilde{e}_u^{n+1}\|^2 + \frac{\eta}{6}\|\nabla \times e_b^{n+1}\|^2 + \|e_b^n\|^2 + \|e_u^{n+1}\|^2 + \delta t^2\|\nabla e_p^{n+1} - \nabla e_p^n\|^2 + \delta t^2\|R_p^{n+1}\|^2.
\end{aligned}$$

For term D and term E, we derive

$$\begin{aligned}
(\text{term D}) &\leq |(\tilde{e}_u^{n+1}, R_p^{n+1})| + \delta t|(\nabla e_p^n, R_p^{n+1})| \\
&\leq \|\tilde{e}_u^{n+1}\|\|R_p^{n+1}\| + \delta t\|\nabla e_p^n\|\|R_p^{n+1}\| \\
&\lesssim \frac{\nu}{8}\|\nabla\tilde{e}_u^{n+1}\|^2 + \|R_p^{n+1}\|^2 + \delta t^2\|\nabla e_p^n\|^2 + \|R_p^{n+1}\|^2,
\end{aligned}$$

and

$$\begin{aligned}
(\text{term E}) &\leq |(R_b^{n+1}, e_b^{n+1})| + |(R_u^{n+1}, \tilde{e}_u^{n+1})| \\
&\leq \|R_b^{n+1}\|\;\|e_b^{n+1}\| + \|R_u^{n+1}\|\;\|\tilde{e}_u^{n+1}\| \\
&\lesssim \|R_b^{n+1}\|\;\|\nabla \times e_b^{n+1}\| + \|R_u^{n+1}\|\;\|\nabla\tilde{e}_u^{n+1}\| \\
&\lesssim \frac{\eta}{6}\|\nabla \times e_b^{n+1}\|^2 + \frac{\nu}{8}\|\nabla\tilde{e}_u^{n+1}\|^2 + \|R_b^{n+1}\|^2 + \|R_u^{n+1}\|^2.
\end{aligned}$$

By combining the above estimates with (4.13), we obtain

$$\begin{aligned}
\|e_b^{n+1}\|^2 - \|e_b^n\|^2 &+ \delta t\eta\|\nabla \times e_b^{n+1}\|^2 + \|e_u^{n+1}\|^2 - \|e_u^n\|^2 + \delta t\nu\|\nabla\tilde{e}_u^{n+1}\|^2 + \delta t^2(\|\nabla e_p^{n+1}\|^2 - \|\nabla e_p^n\|^2) \\
&\lesssim \delta t\|e_b^n\|^2 + \delta t\|e_u^n\|^2 + \delta t\|e_u^{n+1}\|^2 + \delta t^3\|\nabla e_p^{n+1} - \nabla e_p^n\|^2 + \delta t^3\|\nabla e_p^n\|^2 \\
&\quad + \delta t^3\|\nabla \times \boldsymbol{B}^{n+1}\|^2 + \delta t^3\|R_p^{n+1}\|^2 + \delta t\|R_p^{n+1}\|^2 \\
&\quad + \delta t\|R_b^{n+1}\|^2 + \delta t\|R_u^{n+1}\|^2 + \delta t^2\|R_p^{n+1}\|^2.
\end{aligned} \tag{4.14}$$

Summing up the above inequality from $n = 0$ to $m (m \leq N)$ and using $e_u^0 = e_b^0 = e_p^0 = 0$, $\|\nabla e_p^{n+1} - \nabla e_p^n\|^2 \lesssim \|\nabla e_p^{n+1}\|^2 + \|\nabla e_p^n\|^2$ and Lemma 4.3, we obtain

$$\begin{aligned}
\|e_b^{m+1}\|^2 + \|e_u^{m+1}\|^2 + \delta t^2\|\nabla e_p^{m+1}\|^2 + \delta t\sum_{n=0}^{m}\left(\eta\|\nabla \times e_b^{n+1}\|^2 + \nu\|\nabla\tilde{e}_u^{n+1}\|^2\right) \\
\lesssim \delta t\sum_{n=0}^{m}\left(\|e_b^{n+1}\|^2 + \|e_u^{n+1}\|^2 + \delta t^2\|\nabla e_p^{n+1}\|^2\right) + \delta t^2.
\end{aligned} \tag{4.15}$$

Here we also use $\delta t^3 \sum_{n=0}^{m} \|\nabla \times \boldsymbol{B}^{n+1}\|^2 \lesssim \delta t^2$ that is obtained from Remark 3.4. Therefore, applying the discrete Gronwall inequality in Lemma 4.1 to (4.15), there exist positive constants C_0 and C_1, such that

$$\|e_b^{m+1}\|^2 + \|e_u^{m+1}\|^2 + \delta t^2 \|\nabla e_p^{m+1}\|^2 + \delta t \sum_{n=0}^{m} \left(\eta \|\nabla \times e_b^{n+1}\|^2 + \nu \|\nabla \tilde{e}_u^{n+1}\|^2\right) \leq C_1 \delta t^2, \tag{4.16}$$

for $\delta t \leq C_0$ and $m \leq N$.

(*Step ii*). For $n \leq N$, by taking the divergence for (3.6), we obtain

$$-\delta t \Delta (p^n - p^{n-1}) = -\nabla \cdot \tilde{\boldsymbol{u}}^n.$$

From (4.16), we find

$$\delta t \|p^n - p^{n-1}\|_2 \lesssim \|\nabla \cdot \tilde{\boldsymbol{u}}^n\| = \|\nabla \cdot \tilde{e}_u^n\| \lesssim \delta t^{\frac{1}{2}}. \tag{4.17}$$

Due to the identity $\nabla \times \nabla \times \boldsymbol{w} = -\Delta \boldsymbol{w} + \nabla \nabla \cdot \boldsymbol{w}$, Eqs. (3.1) can be transformed as

$$-\eta \Delta \boldsymbol{B}^{n+1} = \nabla \times (\boldsymbol{u}_\star^n \times \boldsymbol{B}^n) - \frac{\boldsymbol{B}^{n+1} - \boldsymbol{B}^n}{\delta t}.$$

By the H^2 regularity of the elliptic problem, there holds

$$\begin{aligned} \|\boldsymbol{B}^{n+1}\|_2 &\lesssim \left\| \frac{\boldsymbol{B}^{n+1} - \boldsymbol{B}^n}{\delta t} \right\| + \|\nabla \times (\boldsymbol{B}^n \times \boldsymbol{u}_\star^n)\| \\ &\lesssim \left\| \frac{\boldsymbol{B}^{n+1} - \boldsymbol{B}^n}{\delta t} \right\| + \|\boldsymbol{u}_\star^n \nabla \boldsymbol{B}^n\| + \|\boldsymbol{B}^n \nabla \boldsymbol{u}_\star^n\| + \|\boldsymbol{B}^n \nabla \cdot \boldsymbol{u}_\star^n\|, \end{aligned} \tag{4.18}$$

where we use $\nabla \cdot \boldsymbol{B}^n = 0$ and the following identity

$$\nabla \times (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{b} \cdot \nabla \boldsymbol{a} - \boldsymbol{a} \cdot \nabla \boldsymbol{b} + \boldsymbol{a} \nabla \cdot \boldsymbol{b} - \boldsymbol{b} \nabla \cdot \boldsymbol{a}. \tag{4.19}$$

For the first term on the right hand side of (4.18), from (4.16) and assumption (A), there exists a constant C_2 such that

$$\begin{aligned} \left\| \frac{\boldsymbol{B}^{n+1} - \boldsymbol{B}^n}{\delta t} \right\| &= \left\| \frac{-e_b^{n+1} + e_b^n}{\delta t} + \frac{\boldsymbol{B}(t_{n+1}) - \boldsymbol{B}(t_n)}{\delta t} \right\| \\ &\leq \left\| \frac{e_b^{n+1} - e_b^n}{\delta t} \right\| + \left\| \frac{\boldsymbol{B}(t_{n+1}) - \boldsymbol{B}(t_n)}{\delta t} \right\| \leq C_2. \end{aligned} \tag{4.20}$$

For other terms of (4.18) on the right hand side, by combining (3.2) and (3.6), we obtain

$$\boldsymbol{u}_\star^n = \tilde{\boldsymbol{u}}^n - \delta t \nabla (p^n - p^{n-1}) + \delta t s \nabla \times \boldsymbol{B}^{n+1} \times \boldsymbol{B}^n,$$

thus by (2.1), (2.3), (2.4), (4.17) and induction assumptions, we derive

$$\begin{aligned} \|\boldsymbol{u}_\star^n \nabla \boldsymbol{B}^n\| &\leq \|\tilde{\boldsymbol{u}}^n \nabla \boldsymbol{B}^n\| + \delta t \|\nabla (p^n - p^{n-1}) \nabla \boldsymbol{B}^n\| + \delta t s \|(\nabla \times \boldsymbol{B}^{n+1} \times \boldsymbol{B}^n) \nabla \boldsymbol{B}^n\| \\ &\leq \|\tilde{e}_u^n \nabla \boldsymbol{B}^n\| + \|\boldsymbol{u}(t_n) \nabla \boldsymbol{B}^n\| + \delta t \|\nabla (p^n - p^{n-1}) \nabla \boldsymbol{B}^n\| + \delta t s \|(\nabla \times \boldsymbol{B}^{n+1} \times \boldsymbol{B}^n) \nabla \boldsymbol{B}^n\| \\ &\lesssim \|\tilde{e}_u^n\|_{L^6} \|\nabla \boldsymbol{B}^n\|_{L^3} + \|\boldsymbol{u}(t_n)\|_{L^\infty} \|\nabla \boldsymbol{B}^n\| + \delta t \|\nabla (p^n - p^{n-1})\|_{L^6} \|\nabla \boldsymbol{B}^n\|_{L^3} \\ &\quad + \delta t \|\nabla \times \boldsymbol{B}^{n+1}\|_{L^3} \|\boldsymbol{B}^n\|_{L^\infty} \|\nabla \boldsymbol{B}^n\|_{L^6} \\ &\lesssim \|\nabla \tilde{e}_u^n\| \|\nabla \boldsymbol{B}^n\|^{\frac{1}{2}} \|\boldsymbol{B}^n\|_2^{\frac{1}{2}} + \|\nabla \boldsymbol{B}^n\| + \delta t \|p^n - p^{n-1}\|_2 \|\nabla \boldsymbol{B}^n\|^{\frac{1}{2}} \|\boldsymbol{B}^n\|_2^{\frac{1}{2}} \\ &\quad + \delta t \|\nabla \times \boldsymbol{B}^{n+1}\|^{\frac{1}{2}} \|\boldsymbol{B}^{n+1}\|_2^{\frac{1}{2}} \|\boldsymbol{B}^n\|_2 \\ &\lesssim \|\nabla \boldsymbol{B}^n\| + \|\nabla \tilde{e}_u^n\|^2 \|\boldsymbol{B}^n\|_2 + \delta t^2 \|p^n - p^{n-1}\|_2^2 \|\boldsymbol{B}^n\|_2 \\ &\quad + \frac{1}{6} \|\boldsymbol{B}^{n+1}\|_2 + \delta t^2 \|\nabla \times \boldsymbol{B}^{n+1}\| \|\boldsymbol{B}^n\|_2^2 \\ &\lesssim C_3 + \frac{1}{6} \|\boldsymbol{B}^{n+1}\|_2 + \delta t \|\boldsymbol{B}^n\|_2 + \delta t^2 \|\boldsymbol{B}^n\|_2^2, \end{aligned} \tag{4.21}$$

where we actually use $\|\nabla \boldsymbol{B}^n\|$ and $\|\nabla \times \boldsymbol{B}^{n+1}\|$ are all bounded (this can be simply proved from $\|\nabla \boldsymbol{B}^n\| \leq \|\nabla e_b^n\| + \|\nabla \boldsymbol{B}(t_n)\| \lesssim \|\nabla \times e_b^n\| + \|\nabla \boldsymbol{B}(t_n)\| \leq C$ by using (2.2), (4.16) and assumption (A)), and use $\|\nabla \tilde{e}_u^n\| \lesssim \delta t^{\frac{1}{2}}$ that is obtained from (4.16).

Similarly, by (2.3), (2.4), (4.17) and induction assumptions, we derive

$$\begin{aligned}\|\boldsymbol{B}^n\nabla\boldsymbol{u}_\star^n\| &\le \|\boldsymbol{B}^n\nabla\tilde{\boldsymbol{u}}^n\| + \delta t\|\boldsymbol{B}^n\nabla\nabla(p^n-p^{n-1})\| + \delta ts\|\boldsymbol{B}^n\nabla(\nabla\times\boldsymbol{B}^{n+1}\times\boldsymbol{B}^n)\| \\ &\lesssim \|\boldsymbol{B}^n\|_{L^\infty}\|\nabla\tilde{\boldsymbol{u}}^n\| + \delta t\|\boldsymbol{B}^n\|_{L^\infty}\|p^n-p^{n-1}\|_2 \\ &\quad + \delta t(\|\boldsymbol{B}^n\|_{L^\infty}^2\|\boldsymbol{B}^{n+1}\|_2 + \|\boldsymbol{B}^n\|_{L^\infty}\|\nabla\boldsymbol{B}^{n+1}\|_{L^3}\|\nabla\boldsymbol{B}^n\|_{L^6}) \\ &\lesssim \|\nabla\tilde{\boldsymbol{u}}^n\| + \delta t\|p^n-p^{n-1}\|_2 + \delta t(\|\boldsymbol{B}^{n+1}\|_2 + \|\nabla\boldsymbol{B}^{n+1}\|^{\frac12}\|\boldsymbol{B}^{n+1}\|_2^{\frac12}\|\boldsymbol{B}^n\|_2) \\ &\lesssim C_4 + \delta t\|\boldsymbol{B}^{n+1}\|_2 + \frac16\|\boldsymbol{B}^{n+1}\|_2 + \delta t^2\|\boldsymbol{B}^n\|_2^2,\end{aligned} \tag{4.22}$$

where we use $\|\nabla\tilde{\boldsymbol{u}}^n\|$ and $\|\nabla\boldsymbol{B}^{n+1}\|$ are bounded by constants.

Likewise, for the last term in (4.18), we also have

$$\begin{aligned}\|\boldsymbol{B}^n\nabla\cdot\boldsymbol{u}_\star^n\| &\le \|\boldsymbol{B}^n\nabla\cdot\tilde{\boldsymbol{u}}^n\| + \delta t\|\boldsymbol{B}^n\Delta(p^n-p^{n-1})\| + \delta ts\|\boldsymbol{B}^n\nabla\cdot(\nabla\times\boldsymbol{B}^{n+1}\times\boldsymbol{B}^n)\| \\ &\lesssim C_5 + \delta t\|\boldsymbol{B}^{n+1}\|_2 + \frac16\|\boldsymbol{B}^{n+1}\|_2 + \delta t^2\|\boldsymbol{B}^n\|_2^2.\end{aligned} \tag{4.23}$$

By combining (4.18), (4.20), (4.21), (4.22) and (4.23), if $\delta t \le \widehat{C}_0$, there exist three positive constants C_6, C_7 and C_8 such that for $n \le N$

$$\|\boldsymbol{B}^{n+1}\|_2 \le C_6 + C_7\delta t\|\boldsymbol{B}^n\|_2 + C_8\delta t^2\|\boldsymbol{B}^n\|_2^2.$$

Therefore, by Lemma 4.2, if $\max\{C_7,\sqrt{C_8}\}D'\delta t \le 1$, i.e., $\delta t \le \frac{1}{\max\{C_7,\sqrt{C_8}\}D'}$, we have

$$\|\boldsymbol{B}^{N+1}\|_2 \le D' \quad (D' = \max\{\|\boldsymbol{B}_0\|_2, C_6\} + 2). \tag{4.24}$$

(*Step iii*). From (4.24) and the assumption (A), there exists a positive constant C_9 such that

$$\|e_b^{N+1}\|_2 \le \|\boldsymbol{B}^{N+1}\|_2 + \|\boldsymbol{B}(t_{N+1})\|_2 \le C_9.$$

Finally, from (4.16), we have

$$\begin{aligned}\|\boldsymbol{B}^{N+1}\|_{L^\infty} &\le \|e_b^{N+1}\|_{L^\infty} + \|\boldsymbol{B}(t_{N+1})\|_{L^\infty} \\ &\le C_{10}\|e_b^{N+1}\|_2^{\frac34}\|e_b^{N+1}\|^{\frac14} + \|\boldsymbol{B}(t_{N+1})\|_{L^\infty} \\ &\le C_{10}C_9^{\frac34}C_1^{\frac18}\delta t^{\frac14} + \|\boldsymbol{B}(t_{N+1})\|_{L^\infty}.\end{aligned}$$

Thus, if $C_{10}^4C_9^3C_1^{\frac12}\delta t \le 1$, i.e., $\delta t \le \frac{1}{C_{10}^4C_9^3C_1^{\frac12}}$, we have

$$\|\boldsymbol{B}^{N+1}\|_{L^\infty} \le 1 + \|\boldsymbol{B}(t_{N+1})\|_{L^\infty} \le \kappa.$$

Then we obtain (4.8) by induction for $\delta t \le C$, $C = \min\left\{C_0, \widehat{C}_0, \frac{1}{\max\{C_7,\sqrt{C_8}\}D'}, \frac{1}{C_{10}^4C_9^3C_1^{\frac12}}\right\}$. □

Now, based on the above lemma, we can easily derive the following error estimate.

Theorem 4.1. *Suppose the solution to (2.5)–(2.8) satisfies Assumption (A). Then, the scheme (3.1)–(3.8) is unconditionally convergent and has the following error estimate: for* $0 \le m \le [\frac{T}{\delta t}] - 1$,

$$\|e_b^{m+1}\|^2 + \|e_u^{m+1}\|^2 + \delta t^2\|\nabla e_p^{m+1}\|^2 + \delta t\sum_{n=0}^m\left(\eta\|\nabla\times e_b^{n+1}\|^2 + \nu\|\nabla\tilde{e}_u^{n+1}\|^2\right) \lesssim \delta t^2.$$

Proof. Since $\|\boldsymbol{B}^n\|_{L^\infty} \le \kappa$ is established for any $0 \le n \le [\frac{T}{\delta t}]$, by following the proof of *Step i* of Lemma 4.4, we obtain that (4.16) is valid for any $0 \le m \le [\frac{T}{\delta t}] - 1$ provided $\delta t \le C$.

On the other hand, if $\delta t \ge C$, using Remark 3.4 and assumption (A), we deduce that there exists a constant C_{11} such that

$$\begin{aligned}&\|e_b^{m+1}\|^2 + \|e_u^{m+1}\|^2 + \delta t^2\|\nabla e_p^{m+1}\|^2 + \delta t\sum_{n=0}^m\left(\eta\|\nabla\times e_b^{n+1}\|^2 + \nu\|\nabla\tilde{e}_u^{n+1}\|^2\right) \\ &\quad \le C_{11} = \frac{C_{11}}{C^2}C^2 \le \frac{C_{11}}{C^2}(\delta t)^2 \lesssim \delta t^2.\end{aligned}$$

Therefore, the proof is finished by combining the two cases. □

Remark 4.1. Once the bound $\|\boldsymbol{B}^n\|_{L^\infty} \le \kappa$ and the error estimate in Theorem 4.1 are obtained, by following the Step ii of the proof for Lemma 4.4, we can establish the H^2 stability of $\boldsymbol{B}^n$, namely, there exists a constant $\widehat{\kappa}$, such that

$$\max_{0\le n\le[\frac{T}{\delta t}]} \|\boldsymbol{B}^n\|_2 \le \widehat{\kappa}.$$

Remark 4.2. Define P_H as the L^2 orthogonal projector from $L^2(\Omega)^d$ to H, i.e.,

$$(\boldsymbol{u} - P_H\boldsymbol{u}, \boldsymbol{v}) = 0, \quad \forall \boldsymbol{u} \in L^2(\Omega)^d, \quad \boldsymbol{v} \in H.$$

By the H^1 stability of P_H [47], and (2.1), we have

$$\|e_u^n\|_1 = \|P_H\tilde{e}_u^n\|_1 \le \|\tilde{e}_u^n\|_1 \lesssim \|\nabla\tilde{e}_u^n\|, \tag{4.25}$$

which together with Theorem 4.1 imply

$$\delta t\sum_{n=0}^{[\frac{T}{\delta t}]} \|e_u^n\|_1^2 \lesssim \delta t^2. \tag{4.26}$$

4.2. Error estimate for pressure

Noting Theorem 4.1, the order of the pressure is not optimal, therefore we need to improve it by the following process. We denote $d_t w^n = \frac{w^n - w^{n-1}}{\delta t}$, $d_t w(t_n) = \frac{w(t_n)-w(t_{n-1})}{\delta t}$ for any variable w, $w(t)$. By applying d_t to (4.5)–(4.7), we obtain

$$\begin{aligned}&\frac{d_t\tilde{e}_u^{n+1} - d_t e_u^n}{\delta t} - \nu\Delta d_t\tilde{e}_u^{n+1} + \boldsymbol{u}(t_n)\nabla d_t\boldsymbol{u}(t_{n+1}) + d_t\boldsymbol{u}(t_n)\nabla\boldsymbol{u}(t_n) - \boldsymbol{u}^n\nabla d_t\tilde{\boldsymbol{u}}^{n+1}\\ &\quad -d_t\boldsymbol{u}^n\nabla\tilde{\boldsymbol{u}}^n + \nabla d_t e_p^n + s\boldsymbol{B}(t_n)\times\nabla\times d_t\boldsymbol{B}(t_{n+1}) + sd_t\boldsymbol{B}(t_n)\times\nabla\times\boldsymbol{B}(t_n)\\ &\quad -s\boldsymbol{B}^n\times\nabla\times d_t\boldsymbol{B}^{n+1} - sd_t\boldsymbol{B}^n\times\nabla\times\boldsymbol{B}^n = d_t R_u^{n+1},\end{aligned} \tag{4.27}$$

$$\frac{d_t e_u^{n+1} - d_t\tilde{e}_u^{n+1}}{\delta t} + \nabla(d_t e_p^{n+1} - d_t e_p^n) = d_t R_p^{n+1}, \tag{4.28}$$

$$\begin{aligned}&\frac{d_t e_b^{n+1} - d_t e_b^n}{\delta t} + \eta\nabla\times\nabla\times d_t e_b^{n+1} + \nabla\times(d_t\boldsymbol{B}(t_n)\times\boldsymbol{u}(t_n)) + \nabla\times(\boldsymbol{B}(t_{n-1})\times d_t\boldsymbol{u}(t_n))\\ &\quad -\nabla\times(d_t\boldsymbol{B}^n\times\boldsymbol{u}_\star^n) - \nabla\times(\boldsymbol{B}^{n-1}\times d_t\boldsymbol{u}_\star^n) = d_t R_b^{n+1}.\end{aligned} \tag{4.29}$$

The truncation terms $d_t R_u^{n+1}$, $d_t R_p^{n+1}$, $d_t R_b^{n+1}$ in (4.27)–(4.29) have the following property.

Lemma 4.5. *Under the Assumption (A), the truncation errors satisfy*

$$\delta t\sum_{n=1}^{[\frac{T}{\delta t}]}(\|d_t R_u^{n+1}\|^2 + \|d_t R_b^{n+1}\|^2 + \|d_t R_p^{n+1}\|^2) \lesssim \delta t^2.$$

Proof. By the definition of $d_t R_u^{n+1} = \frac{R_u^{n+1} - R_u^n}{\delta t}$, we get

$$\begin{aligned}d_t R_u^{n+1} = &\frac{1}{\delta t}(d_t\boldsymbol{u}(t_{n+1}) - \boldsymbol{u}_t(t_{n+1}) - d_t\boldsymbol{u}(t_n) + \boldsymbol{u}_t(t_n)) - d_t\boldsymbol{u}(t_{n+1})\nabla\boldsymbol{u}(t_{n+1}) + d_t\boldsymbol{u}(t_n)\nabla\boldsymbol{u}(t_n)\\ &+ \nabla d_t p(t_n) - \nabla d_t p(t_{n+1}) - sd_t\boldsymbol{B}(t_{n+1})\times\nabla\times\boldsymbol{B}(t_{n+1}) + sd_t\boldsymbol{B}(t_n)\times\nabla\times\boldsymbol{B}(t_n).\end{aligned}$$

We estimate above equation term by term as follows. By some basic calculations and (2.1), (2.3), Assumption (A), we get

$$\begin{aligned}&\frac{1}{\delta t^2}\|d_t\boldsymbol{u}(t_{n+1}) - \boldsymbol{u}_t(t_{n+1}) - d_t\boldsymbol{u}(t_n) + \boldsymbol{u}_t(t_n)\|^2\\ &= \frac{1}{4\delta t^4}\left\|\int_{t_n}^{t_{n+1}}(t-t_n)^2\boldsymbol{u}_{ttt}dt - \int_{t_{n-1}}^{t_n}(t-t_{n-1})^2\boldsymbol{u}_{ttt}dt - \delta t^2\int_{t_n}^{t_{n+1}}\boldsymbol{u}_{ttt}dt\right\|^2\\ &\lesssim \delta t\int_{t_n}^{t_{n+1}}\|\boldsymbol{u}_{ttt}\|^2dt + \delta t\int_{t_{n-1}}^{t_n}\|\boldsymbol{u}_{ttt}\|^2dt,\end{aligned}$$

$$
\begin{aligned}
&\|d_t\boldsymbol{u}(t_{n+1})\nabla\boldsymbol{u}(t_{n+1}) - d_t\boldsymbol{u}(t_n)\nabla\boldsymbol{u}(t_n)\|^2\\
&= \|(d_t\boldsymbol{u}(t_{n+1}) - d_t\boldsymbol{u}(t_n))\nabla\boldsymbol{u}(t_{n+1}) + d_t\boldsymbol{u}(t_n)\nabla(\boldsymbol{u}(t_{n+1}) - \boldsymbol{u}(t_n))\|^2\\
&\le \|d_t\boldsymbol{u}(t_{n+1}) - d_t\boldsymbol{u}(t_n)\|_{L^4}^2\|\nabla\boldsymbol{u}(t_{n+1})\|_{L^4}^2 + \|d_t\boldsymbol{u}(t_n)\|_{L^\infty}^2\|\nabla(\boldsymbol{u}(t_{n+1}) - \boldsymbol{u}(t_n))\|^2\\
&\lesssim \|\nabla(d_t\boldsymbol{u}(t_{n+1}) - d_t\boldsymbol{u}(t_n))\|^2 + \|\nabla(\boldsymbol{u}(t_{n+1}) - \boldsymbol{u}(t_n))\|^2\\
&= \left\|\frac{1}{\delta t}\int_{t_n}^{t_{n+1}}(t_{n+1}-t)\nabla\boldsymbol{u}_{tt}dt + \frac{1}{\delta t}\int_{t_{n-1}}^{t_n}(t-t_{n-1})\nabla\boldsymbol{u}_{tt}dt\right\|^2 + \left\|\int_{t_n}^{t_{n+1}}\nabla\boldsymbol{u}_t dt\right\|^2\\
&\lesssim \delta t\int_{t_n}^{t_{n+1}}\|\nabla\boldsymbol{u}_{tt}\|^2dt + \delta t\int_{t_{n-1}}^{t_n}\|\nabla\boldsymbol{u}_{tt}\|^2dt + \delta t\int_{t_n}^{t_{n+1}}\|\nabla\boldsymbol{u}_t\|^2dt,
\end{aligned}
$$

$$
\begin{aligned}
\|\nabla d_t p(t_n) - \nabla d_t p(t_{n+1})\|^2 &= \left\|\frac{1}{\delta t}\int_{t_n}^{t_{n+1}}(t_{n+1}-t)\nabla p_{tt}dt + \frac{1}{\delta t}\int_{t_{n-1}}^{t_n}(t-t_{n-1})\nabla p_{tt}dt\right\|^2\\
&\lesssim \delta t\int_{t_n}^{t_{n+1}}\|\nabla p_{tt}\|^2dt + \delta t\int_{t_{n-1}}^{t_n}\|\nabla p_{tt}\|^2dt,
\end{aligned}
$$

and

$$
\begin{aligned}
&\|d_t\boldsymbol{B}(t_{n+1})\times\nabla\times\boldsymbol{B}(t_{n+1}) - d_t\boldsymbol{B}(t_n)\times\nabla\times\boldsymbol{B}(t_n)\|^2\\
&\lesssim \left\|\frac{1}{\delta t}\int_{t_n}^{t_{n+1}}(t_{n+1}-t)\nabla\times\boldsymbol{B}_{tt}dt + \frac{1}{\delta t}\int_{t_{n-1}}^{t_n}(t-t_{n-1})\nabla\times\boldsymbol{B}_{tt}dt\right\|^2 + \left\|\int_{t_n}^{t_{n+1}}\nabla\times\boldsymbol{B}_t dt\right\|^2\\
&\lesssim \delta t\int_{t_n}^{t_{n+1}}\|\nabla\boldsymbol{B}_{tt}\|^2dt + \delta t\int_{t_{n-1}}^{t_n}\|\nabla\boldsymbol{B}_{tt}\|^2dt + \delta t\int_{t_n}^{t_{n+1}}\|\nabla\boldsymbol{B}_t\|^2dt.
\end{aligned}
$$

Thus, we combine the above estimates, using assumption (A), to find

$$
\delta t\sum_{n=1}^{[\frac{T}{\delta t}]}\|d_t R_u^{n+1}\|^2 \lesssim \delta t^2\int_0^T \|\boldsymbol{u}_{ttt}\|^2 + \|\nabla\boldsymbol{u}_{tt}\|^2 + \|\nabla\boldsymbol{u}_t\|^2 + \|\nabla p_{tt}\|^2 + \|\nabla\boldsymbol{B}_{tt}\|^2 + \|\nabla\boldsymbol{B}_t\|^2dt \lesssim \delta t^2.
$$

Using the very similar procedures, we can also get

$$
\delta t\sum_{n=1}^{[\frac{T}{\delta t}]}(\|d_t R_p^{n+1}\|^2 + \|d_t R_b^{n+1}\|^2) \lesssim \delta t^2.
$$

Therefore, the proof is finished. □

To derive the optimal convergence order of pressure, we also need the following first step error bound.

Lemma 4.6. *Under the assumption (A), there holds*

$$
\|d_t e_u^1\|^2 + \|d_t e_b^1\|^2 + \delta t^2\|\nabla d_t e_p^1\|^2 \lesssim \delta t^2. \tag{4.30}
$$

Proof. (i). By taking $n = 0$ in (4.5) and from $e_b^0 = 0$, we obtain

$$
\frac{e_b^1}{\delta t} + \eta\nabla\times\nabla\times e_b^1 + \nabla\times(\boldsymbol{B}_0\times\boldsymbol{u}_0) - \nabla\times(\boldsymbol{B}_0\times\boldsymbol{u}_\star^0) = R_b^1.
$$

By taking the L^2 inner product of above equation with $\frac{1}{\delta t}e_b^1$, we have

$$
\|\frac{e_b^1}{\delta t}\|^2 + \frac{\eta}{\delta t}\|\nabla\times e_b^1\|^2 + \left(\nabla\times(\boldsymbol{B}_0\times\boldsymbol{u}_0) - \nabla\times(\boldsymbol{B}_0\times\boldsymbol{u}_\star^0), \frac{e_b^1}{\delta t}\right) = \left(R_b^1, \frac{e_b^1}{\delta t}\right). \tag{4.31}
$$

From (3.10), using Remark 4.1, we obtain

$$\begin{aligned}
&\left|\left(\nabla\times(\boldsymbol{B}_0\times\boldsymbol{u}_0)-\nabla\times(\boldsymbol{B}_0\times\boldsymbol{u}_\star^0),\frac{e_b^1}{\delta t}\right)\right|\\
&=\delta ts\left|\left(\nabla\times(\boldsymbol{B}_0\times(\boldsymbol{B}_0\times\nabla\times\boldsymbol{B}^1)),\frac{e_b^1}{\delta t}\right)\right|\\
&=\delta ts\left|\left((\boldsymbol{B}_0\times\nabla\times\boldsymbol{B}^1)\nabla\boldsymbol{B}_0-\boldsymbol{B}_0\nabla(\boldsymbol{B}_0\times\nabla\times\boldsymbol{B}^1)+\boldsymbol{B}_0\nabla\cdot(\boldsymbol{B}_0\times\nabla\times\boldsymbol{B}^1),\frac{e_b^1}{\delta t}\right)\right|\\
&\lesssim\delta t\|\boldsymbol{B}_0\|_{L^\infty}\|\boldsymbol{B}_0\|_2\|\boldsymbol{B}^1\|_2\left\|\frac{e_b^1}{\delta t}\right\|+\delta t\|\boldsymbol{B}_0\|_{L^\infty}\|\boldsymbol{B}_0\|_2\|\boldsymbol{B}^1\|_2\left\|\frac{e_b^1}{\delta t}\right\|+\delta t\|\boldsymbol{B}_0\|_{L^\infty}^2\|\boldsymbol{B}^1\|_2\left\|\frac{e_b^1}{\delta t}\right\|\\
&\lesssim\frac{1}{4}\|\frac{e_b^1}{\delta t}\|^2+\delta t^2,
\end{aligned}$$

and

$$\left|\left(R_b^1,\frac{e_b^1}{\delta t}\right)\right|\leq\|R_b^1\|\left\|\frac{e_b^1}{\delta t}\right\|\lesssim\frac{1}{4}\left\|\frac{e_b^1}{\delta t}\right\|^2+\|R_b^1\|^2\lesssim\frac{1}{4}\left\|\frac{e_b^1}{\delta t}\right\|^2+\delta t^2.$$

Therefore, since $e_b^0=0$, (4.31) implies

$$\|\frac{e_b^1-e_b^0}{\delta t}\|^2+\frac{\eta}{\delta t}\|\nabla\times e_b^1\|^2\lesssim\delta t^2. \tag{4.32}$$

(ii). By taking $n=0$ in (4.6) and using $e_u^0=e_p^0=0$, we obtain

$$\frac{\tilde{e}_u^1}{\delta t}-\nu\Delta\tilde{e}_u^1+(\boldsymbol{u}_0\cdot\nabla)\tilde{e}_u^1+s\boldsymbol{B}_0\times\nabla\times e_b^1=R_u^1.$$

By taking the L^2 inner product of the above equation with $\frac{1}{\delta t}\tilde{e}_u^1$, we have

$$\left\|\frac{\tilde{e}_u^1}{\delta t}\right\|^2+\frac{1}{\delta t}\nu\|\nabla\tilde{e}_u^1\|^2+s\left(\boldsymbol{B}_0\times\nabla\times e_b^1,\frac{\tilde{e}_u^1}{\delta t}\right)=\left(R_u^1,\frac{\tilde{e}_u^1}{\delta t}\right). \tag{4.33}$$

By taking $n=0$ in (4.7), it gives

$$\frac{e_u^1}{\delta t}+\nabla e_p^1=\frac{\tilde{e}_u^1}{\delta t}+R_p^1.$$

By taking the L^2 inner products of the above equation with itself on both sides, we obtain

$$\frac{1}{2}(\left\|\frac{e_u^1}{\delta t}\right\|^2-\left\|\frac{\tilde{e}_u^1}{\delta t}\right\|^2)+\frac{1}{2}\|\nabla e_p^1\|^2-\left(\frac{\tilde{e}_u^1}{\delta t},R_p^1\right)-\frac{1}{2}\|R_p^1\|^2=0. \tag{4.34}$$

We combine (4.33) and (4.34) to obtain

$$\begin{aligned}
&\frac{1}{2}\left\|\frac{\tilde{e}_u^1}{\delta t}\right\|^2+\frac{1}{2}\left\|\frac{e_u^1}{\delta t}\right\|^2+\frac{1}{\delta t}\nu\|\nabla\tilde{e}_u^1\|^2+\frac{1}{2}\|\nabla e_p^1\|^2\\
&\qquad=\left(\frac{\tilde{e}_u^1}{\delta t},R_p^1\right)+\left(R_u^1,\frac{\tilde{e}_u^1}{\delta t}\right)+\frac{1}{2}\|R_p^1\|^2-s\left(\boldsymbol{B}_0\times\nabla\times e_b^1,\frac{\tilde{e}_u^1}{\delta t}\right).
\end{aligned} \tag{4.35}$$

The terms on the right hand side of (4.35) can be estimated by

$$\begin{aligned}
&\left|\left(\frac{\tilde{e}_u^1}{\delta t},R_p^1\right)\right|+\left|\left(R_u^1,\frac{\tilde{e}_u^1}{\delta t}\right)\right|\lesssim\frac{1}{8}\left\|\frac{\tilde{e}_u^1}{\delta t}\right\|^2+\|R_p^1\|^2+\|R_u^1\|^2,\\
&s\left|\left(\boldsymbol{B}_0\times\nabla\times e_b^1,\frac{\tilde{e}_u^1}{\delta t}\right)\right|\lesssim\frac{1}{8}\left\|\frac{\tilde{e}_u^1}{\delta t}\right\|^2+\|\nabla\times e_b^1\|^2.
\end{aligned}$$

From $e_u^0=e_p^0=0$, thus (4.35) implies

$$\begin{aligned}
&\frac{1}{4}\left\|\frac{\tilde{e}_u^1}{\delta t}\right\|^2+\frac{1}{2}\left\|\frac{e_u^1-e_u^0}{\delta t}\right\|^2+\frac{1}{\delta t}\nu\|\nabla\tilde{e}_u^1\|^2+\frac{\delta t^2}{2}\left\|\frac{\nabla e_p^1-\nabla e_p^0}{\delta t}\right\|^2\\
&\qquad\lesssim\|R_p^1\|^2+\|R_u^1\|^2+\|\nabla\times e_b^1\|^2\lesssim\delta t^2,
\end{aligned} \tag{4.36}$$

in which we use (4.32) and Lemma 4.3.

Finally, we obtain (4.30) by combining (4.32) and (4.36) together. □

Based on the above lemmas, we prove the following result which will lead to the optimal error order of pressure.

Lemma 4.7. *Under assumption (A), there exists a constant* $\widehat{C}$ *such that, when* $\delta t \leq \widehat{C}$, *the following estimate holds for* $1 \leq m \leq [\frac{T}{\delta t}] - 1$,

$$\|d_t e_u^{m+1}\|^2 + \|d_t e_b^{m+1}\|^2 + \delta t^2 \|\nabla d_t e_p^{m+1}\|^2 + \delta t \sum_{n=1}^{m} (\nu \|\nabla d_t \tilde{e}_u^{n+1}\|^2 + \eta \|\nabla \times d_t e_b^{n+1}\|^2) \lesssim \delta t^2.$$

Proof. By taking the L^2 inner product of (4.27) with $d_t \tilde{e}_u^{n+1}$, we have

$$\begin{aligned}
&\frac{1}{2\delta t}(\|d_t \tilde{e}_u^{n+1} - d_t e_u^n\|^2 + \|d_t \tilde{e}_u^{n+1}\|^2 - \|d_t e_u^n\|^2) + \nu \|\nabla d_t \tilde{e}_u^{n+1}\|^2 \\
&\quad + (\boldsymbol{u}(t_n) \nabla d_t \boldsymbol{u}(t_{n+1}), d_t \tilde{e}_u^{n+1}) - (\boldsymbol{u}^n \nabla d_t \tilde{\boldsymbol{u}}^{n+1}, d_t \tilde{e}_u^{n+1}) \\
&\quad + (d_t \boldsymbol{u}(t_n) \nabla \boldsymbol{u}(t_n), d_t \tilde{e}_u^{n+1}) - (d_t \boldsymbol{u}^n \nabla \tilde{\boldsymbol{u}}^n, d_t \tilde{e}_u^{n+1}) \\
&\quad + s(\boldsymbol{B}(t_n) \times \nabla \times d_t \boldsymbol{B}(t_{n+1}), d_t \tilde{e}_u^{n+1}) - s(\boldsymbol{B}^n \times \nabla \times d_t \boldsymbol{B}^{n+1}, d_t \tilde{e}_u^{n+1}) \\
&\quad + s(d_t \boldsymbol{B}(t_n) \times \nabla \times \boldsymbol{B}(t_n), d_t \tilde{e}_u^{n+1}) - s(d_t \boldsymbol{B}^n \times \nabla \times \boldsymbol{B}^n, d_t \tilde{e}_u^{n+1}) \\
&= (d_t R_u^{n+1}, d_t \tilde{e}_u^{n+1}) - (\nabla d_t e_p^n, d_t \tilde{e}_u^{n+1}).
\end{aligned} \tag{4.37}$$

From (4.28), we derive

$$\frac{d_t e_u^{n+1}}{\delta t} + \nabla d_t e_p^{n+1} = \frac{d_t \tilde{e}_u^{n+1}}{\delta t} + \nabla d_t e_p^n + d_t R_p^{n+1}. \tag{4.38}$$

By taking the L^2 inner product of (4.38) with itself on both sides, we obtain

$$\begin{aligned}
&\frac{1}{2\delta t}(\|d_t e_u^{n+1}\|^2 - \|d_t \tilde{e}_u^{n+1}\|^2) + \frac{\delta t}{2}(\|\nabla d_t e_p^{n+1}\|^2 - \|\nabla d_t e_p^n\|^2) \\
&\quad = (d_t \tilde{e}_u^{n+1}, d_t R_p^{n+1}) + \delta t (\nabla d_t e_p^n, d_t R_p^{n+1}) + \frac{\delta t}{2}\|d_t R_p^{n+1}\|^2 + (d_t \tilde{e}_u^{n+1}, \nabla d_t e_p^n).
\end{aligned} \tag{4.39}$$

By taking the L^2 inner product of (4.29) with $d_t e_b^{n+1}$, we have

$$\begin{aligned}
&\frac{1}{2\delta t}(\|d_t e_b^{n+1} - d_t e_b^n\|^2 + \|d_t e_b^{n+1}\|^2 - \|d_t e_b^n\|^2) + \eta \|\nabla \times d_t e_b^{n+1}\|^2 \\
&\quad + (d_t \boldsymbol{B}(t_n) \times \boldsymbol{u}(t_n), \nabla \times d_t e_b^{n+1}) - (d_t \boldsymbol{B}^n \times \boldsymbol{u}_\star^n, \nabla \times d_t e_b^{n+1}) \\
&\quad + (\boldsymbol{B}(t_{n-1}) \times d_t \boldsymbol{u}(t_n), \nabla \times d_t e_b^{n+1}) - (\boldsymbol{B}^{n-1} \times d_t \boldsymbol{u}_\star^n, \nabla \times d_t e_b^{n+1}) \\
&= (d_t R_b^{n+1}, d_t e_b^{n+1}).
\end{aligned} \tag{4.40}$$

Combining (4.37), (4.39) and (4.40) together, we obtain

$$\begin{aligned}
&\frac{1}{2\delta t}(\|d_t \tilde{e}_u^{n+1} - d_t e_u^n\|^2 + \|d_t e_u^{n+1}\|^2 - \|d_t e_u^n\|^2) + \nu \|\nabla d_t \tilde{e}_u^{n+1}\|^2 \\
&\quad + \frac{\delta t}{2}(\|\nabla d_t e_p^{n+1}\|^2 - \|\nabla d_t e_p^n\|^2) + \eta \|\nabla \times d_t e_b^{n+1}\|^2 \\
&\quad + \frac{1}{2\delta t}(\|d_t e_b^{n+1} - d_t e_b^n\|^2 + \|d_t e_b^{n+1}\|^2 - \|d_t e_b^n\|^2) \\
&\quad + (\boldsymbol{u}(t_n) \nabla d_t \boldsymbol{u}(t_{n+1}), d_t \tilde{e}_u^{n+1}) - (\boldsymbol{u}^n \nabla d_t \tilde{\boldsymbol{u}}^{n+1}, d_t \tilde{e}_u^{n+1}) && (\text{: term I}) \\
&\quad + (d_t \boldsymbol{u}(t_n) \nabla \boldsymbol{u}(t_n), d_t \tilde{e}_u^{n+1}) - (d_t \boldsymbol{u}^n \nabla \tilde{\boldsymbol{u}}^n, d_t \tilde{e}_u^{n+1}) && (\text{: term II}) \\
&\quad + s(\boldsymbol{B}(t_n) \times \nabla \times d_t \boldsymbol{B}(t_{n+1}), d_t \tilde{e}_u^{n+1}) - s(\boldsymbol{B}^n \times \nabla \times d_t \boldsymbol{B}^{n+1}, d_t \tilde{e}_u^{n+1}) && (\text{: term III}) \\
&\quad + s(d_t \boldsymbol{B}(t_n) \times \nabla \times \boldsymbol{B}(t_n), d_t \tilde{e}_u^{n+1}) - s(d_t \boldsymbol{B}^n \times \nabla \times \boldsymbol{B}^n, d_t \tilde{e}_u^{n+1}) && (\text{: term IV}) \\
&\quad + (d_t \boldsymbol{B}(t_n) \times \boldsymbol{u}(t_n), \nabla \times d_t e_b^{n+1}) - (d_t \boldsymbol{B}^n \times \boldsymbol{u}_\star^n, \nabla \times d_t e_b^{n+1}) && (\text{: term V}) \\
&\quad + (\boldsymbol{B}(t_{n-1}) \times d_t \boldsymbol{u}(t_n), \nabla \times d_t e_b^{n+1}) - (\boldsymbol{B}^{n-1} \times d_t \boldsymbol{u}_\star^n, \nabla \times d_t e_b^{n+1}) && (\text{: term VI}) \\
&\quad = (d_t R_u^{n+1}, d_t \tilde{e}_u^{n+1}) + (d_t R_b^{n+1}, d_t e_b^{n+1}) && (\text{: term VII}) \\
&\quad + (d_t \tilde{e}_u^{n+1}, d_t R_p^{n+1}) + \delta t (\nabla d_t e_p^n, d_t R_p^{n+1}) && (\text{: term VIII}) \\
&\quad + \frac{\delta t}{2}\|d_t R_p^{n+1}\|^2.
\end{aligned} \tag{4.41}$$

For term I, we estimate as

$$\begin{aligned}(\text{term I}) &\le |(\boldsymbol{u}(t_n)\nabla d_t\boldsymbol{u}(t_{n+1}), d_t\tilde{e}_u^{n+1}) - (\boldsymbol{u}^n\nabla d_t\tilde{\boldsymbol{u}}^{n+1}, d_t\tilde{e}_u^{n+1})|\\ &= |(e_u^n\nabla d_t\boldsymbol{u}(t_{n+1}), d_t\tilde{e}_u^{n+1}) + (\boldsymbol{u}^n\nabla d_t\tilde{e}_u^{n+1}, d_t\tilde{e}_u^{n+1})|\\ &= |(e_u^n\nabla d_t\tilde{e}_u^{n+1}, d_t\boldsymbol{u}(t_{n+1}))|\\ &\lesssim \|\nabla e_u^n\|\|d_t\boldsymbol{u}(t_{n+1})\|_{L^4}\|\nabla d_t\tilde{e}_u^{n+1}\| \lesssim \frac{\nu}{12}\|\nabla d_t\tilde{e}_u^{n+1}\|^2 + \|\nabla e_u^n\|^2.\end{aligned} \tag{4.42}$$

where we use $\boldsymbol{u}_t \in L^\infty(0,T;L^4)$ and (3.16).

For term II, using (2.1), (2.3) and (2.4), we estimate as

$$\begin{aligned}(\text{term II}) &\le |(d_t\boldsymbol{u}(t_n)\nabla\boldsymbol{u}(t_n), d_t\tilde{e}_u^{n+1}) - (d_t\boldsymbol{u}^n\nabla\tilde{\boldsymbol{u}}^n, d_t\tilde{e}_u^{n+1})|\\ &= |(d_te_u^n\nabla\boldsymbol{u}(t_n), d_t\tilde{e}_u^{n+1}) + (d_t\boldsymbol{u}^n\nabla\tilde{e}_u^n, d_t\tilde{e}_u^{n+1})|\\ &= |(d_te_u^n\nabla\boldsymbol{u}(t_n), d_t\tilde{e}_u^{n+1}) - (d_te_u^n\nabla\tilde{e}_u^n, d_t\tilde{e}_u^{n+1}) + (d_t\boldsymbol{u}(t_n)\nabla\tilde{e}_u^n, d_t\tilde{e}_u^{n+1})|\\ &\lesssim \|d_te_u^n\|\|\nabla\boldsymbol{u}(t_n)\|_{L^4}\|\nabla d_t\tilde{e}_u^{n+1}\| + (\|d_te_u^n\|_{L^3} + \|d_t\boldsymbol{u}(t_n)\|_{L^4})\|\nabla\tilde{e}_u^n\|\|\nabla d_t\tilde{e}_u^{n+1}\|\\ &\lesssim \frac{\nu}{24}\|\nabla d_t\tilde{e}_u^{n+1}\|^2 + \|d_te_u^n\|^2 + \|d_te_u^n\|^{\frac12}\|d_te_u^n\|_1^{\frac12}\|\nabla\tilde{e}_u^n\|\|\nabla d_t\tilde{e}_u^{n+1}\| + \|d_t\boldsymbol{u}(t_n)\|_{L^4}^2\|\nabla\tilde{e}_u^n\|^2\\ &\lesssim \frac{\nu}{12}\|\nabla d_t\tilde{e}_u^{n+1}\|^2 + \|d_te_u^n\|^2 + \|d_te_u^n\|\|d_te_u^n\|_1\|\nabla\tilde{e}_u^n\|\|\nabla\tilde{e}_u^n\| + \|\nabla\tilde{e}_u^n\|^2\\ &\lesssim \frac{\nu}{12}\|\nabla d_t\tilde{e}_u^{n+1}\|^2 + \|d_te_u^n\|^2 + \|d_te_u^n\|\|\nabla\tilde{e}_u^n\| + \|\nabla\tilde{e}_u^n\|^2\\ &\lesssim \frac{\nu}{12}\|\nabla d_t\tilde{e}_u^{n+1}\|^2 + \|d_te_u^n\|^2 + \|\nabla\tilde{e}_u^n\|^2,\end{aligned} \tag{4.43}$$

where we actually use the fact that $\boldsymbol{u}_t \in L^\infty(0,T;L^4)$, and $\|d_te_u^n\|_1\|\nabla\tilde{e}_u^n\|$ is bounded since, from (4.25) and Theorem 4.1, we have

$$\|d_te_u^n\|_1\|\nabla\tilde{e}_u^n\| \le \frac{1}{\delta t}(\|e_u^n\|_1 + \|e_u^{n-1}\|_1)\|\nabla\tilde{e}_u^n\| \lesssim \frac{1}{\delta t}(\|\nabla\tilde{e}_u^n\|^2 + \|\nabla\tilde{e}_u^{n-1}\|^2) \le C.$$

For term III, using (2.1), (2.2), (2.3), Remark 4.1, we estimate as

$$\begin{aligned}(\text{term III}) &\le |(\boldsymbol{B}(t_n)\times\nabla\times d_t\boldsymbol{B}(t_{n+1}), d_t\tilde{e}_u^{n+1}) - (\boldsymbol{B}^n\times\nabla\times d_t\boldsymbol{B}^{n+1}, d_t\tilde{e}_u^{n+1})|\\ &= |(e_b^n\times\nabla\times d_t\boldsymbol{B}(t_{n+1}), d_t\tilde{e}_u^{n+1}) + (\boldsymbol{B}^n\times\nabla\times d_te_b^{n+1}, d_t\tilde{e}_u^{n+1})|\\ &= |(\nabla\times(d_t\tilde{e}_u^{n+1}\times e_b^n), d_t\boldsymbol{B}(t_{n+1})) - (\nabla\times(\boldsymbol{B}^n\times d_t\tilde{e}_u^{n+1}), d_te_b^{n+1})|\\ &= |(e_b^n\nabla d_t\tilde{e}_u^{n+1} - d_t\tilde{e}_u^{n+1}\nabla e_b^n - e_b^n\nabla\cdot d_t\tilde{e}_u^{n+1}, d_t\boldsymbol{B}(t_{n+1}))\\ &\quad - (d_t\tilde{e}_u^{n+1}\nabla\boldsymbol{B}^n - \boldsymbol{B}^n\nabla d_t\tilde{e}_u^{n+1} + \boldsymbol{B}^n\nabla\cdot d_t\tilde{e}_u^{n+1}, d_te_b^{n+1})|\\ &\lesssim \|\nabla e_b^n\|\|\nabla d_t\tilde{e}_u^{n+1}\|\|d_t\boldsymbol{B}(t_{n+1})\|_{L^4} + (\|\nabla\boldsymbol{B}^n\|_{L^4} + \|\boldsymbol{B}^n\|_{L^\infty})\|\nabla d_t\tilde{e}_u^{n+1}\|\|d_te_b^{n+1}\|\\ &\lesssim \frac{\nu}{12}\|\nabla d_t\tilde{e}_u^{n+1}\|^2 + \|\nabla e_b^n\|^2 + \|\nabla\boldsymbol{B}^n\|_{L^4}^2\|d_te_b^{n+1}\|^2 + \|\boldsymbol{B}^n\|_{L^\infty}^2\|d_te_b^{n+1}\|^2\\ &\lesssim \frac{\nu}{12}\|\nabla d_t\tilde{e}_u^{n+1}\|^2 + \|\nabla\times e_b^n\|^2 + \|d_te_b^{n+1}\|^2,\end{aligned} \tag{4.44}$$

where we use $\boldsymbol{B}_t \in L^\infty(0,T;L^4)$ and $\|\nabla e_b^n\|^2 \lesssim \|\nabla\cdot e_b^n\|^2 + \|\nabla\times e_b^n\|^2$ and $\nabla\cdot e_b^n = 0$.

For term IV, using (2.1)–(2.4), we estimate as

$$\begin{aligned}(\text{term IV}) &\le |(d_t\boldsymbol{B}(t_n)\times\nabla\times\boldsymbol{B}(t_n), d_t\tilde{e}_u^{n+1}) - (d_t\boldsymbol{B}^n\times\nabla\times\boldsymbol{B}^n, d_t\tilde{e}_u^{n+1})|\\ &= |(d_te_b^n\times\nabla\times\boldsymbol{B}(t_n), d_t\tilde{e}_u^{n+1}) + (d_t\boldsymbol{B}^n\times\nabla\times e_b^n, d_t\tilde{e}_u^{n+1})|\\ &\lesssim \|d_te_b^n\|\|\boldsymbol{B}(t_n)\|_2\|\nabla d_t\tilde{e}_u^{n+1}\| + \|d_t\boldsymbol{B}^n\|_{L^3}\|\nabla\times e_b^n\|\|d_t\tilde{e}_u^{n+1}\|_{L^6}\\ &\lesssim \|d_te_b^n\|\|\nabla d_t\tilde{e}_u^{n+1}\| + (\|d_te_b^n\|_{L^3} + \|d_t\boldsymbol{B}(t_n)\|_{L^3})\|\nabla\times e_b^n\|\|\nabla d_t\tilde{e}_u^{n+1}\|\\ &\lesssim \frac{\nu}{12}\|\nabla d_t\tilde{e}_u^{n+1}\|^2 + \|d_te_b^n\|^2 + \|d_te_b^n\|_{L^3}^2\|\nabla\times e_b^n\|^2 + \|d_t\boldsymbol{B}(t_n)\|_{L^3}^2\|\nabla\times e_b^n\|^2\\ &\lesssim \frac{\nu}{12}\|\nabla d_t\tilde{e}_u^{n+1}\|^2 + \|d_te_b^n\|^2 + \|d_te_b^n\|\|\nabla\times d_te_b^n\|\|\nabla\times e_b^n\|^2 + \|\nabla\times e_b^n\|^2\\ &\lesssim \frac{\nu}{12}\|\nabla d_t\tilde{e}_u^{n+1}\|^2 + \|d_te_b^n\|^2 + \|d_te_b^n\|\|\nabla\times e_b^n\| + \|\nabla\times e_b^n\|^2\\ &\lesssim \frac{\nu}{12}\|\nabla d_t\tilde{e}_u^{n+1}\|^2 + \|d_te_b^n\|^2 + \|\nabla\times e_b^n\|^2,\end{aligned} \tag{4.45}$$

where we use $\boldsymbol{B}_t \in L^\infty(0,T;L^3)$, $\boldsymbol{B} \in L^\infty(0,T;H^2)$ and a fact that $\|\nabla \times d_t e_b^n\|\|\nabla \times e_b^n\|$ is bounded since

$$\|\nabla \times d_t e_b^n\|\|\nabla \times e_b^n\| \le \frac{1}{\delta t}(\|\nabla \times e_b^n\| + \|\nabla \times e_b^{n-1}\|)\|\nabla \times e_b^n\| \lesssim \frac{1}{\delta t}(\|\nabla \times e_b^n\|^2 + \|\nabla \times e_b^{n-1}\|^2),$$

that is bounded from Theorem 4.1.

For term V, using (2.2)–(2.4), we estimate as

$$\begin{aligned}
(\text{term V}) &\le |(d_t\boldsymbol{B}(t_n)\times \boldsymbol{u}(t_n), \nabla \times d_t e_b^{n+1}) - (d_t\boldsymbol{B}^n \times \boldsymbol{u}_\star^n, \nabla \times d_t e_b^{n+1})| \\
&= |(d_t e_b^n \times \boldsymbol{u}(t_n), \nabla \times d_t e_b^{n+1}) + (d_t\boldsymbol{B}^n \times (\boldsymbol{u}(t_n) - \boldsymbol{u}_\star^n), \nabla \times d_t e_b^{n+1})| \\
&= |(d_t e_b^n \times \boldsymbol{u}(t_n), \nabla \times d_t e_b^{n+1}) + ((d_t\boldsymbol{B}(t_n) - d_t e_b^n) \times (\boldsymbol{u}(t_n) - \boldsymbol{u}_\star^n), \nabla \times d_t e_b^{n+1})| \\
&\lesssim \|d_t e_b^n\|\|\boldsymbol{u}(t_n)\|_{L^\infty}\|\nabla \times d_t e_b^{n+1}\| + \|d_t e_b^n\|_{L^3}\|\boldsymbol{u}(t_n) - \boldsymbol{u}_\star^n\|_{L^6}\|\nabla \times d_t e_b^{n+1}\| \\
&\quad + \|d_t\boldsymbol{B}(t_n)\|_{L^3}\|\boldsymbol{u}(t_n) - \boldsymbol{u}_\star^n\|_{L^6}\|\nabla \times d_t e_b^{n+1}\| \\
&\lesssim \|d_t e_b^n\|\|\nabla \times d_t e_b^{n+1}\| + \|d_t e_b^n\|^{\frac12}\|\nabla \times d_t e_b^n\|^{\frac12}\|\boldsymbol{u}(t_n) - \boldsymbol{u}_\star^n\|_1\|\nabla \times d_t e_b^{n+1}\| \\
&\quad + \|\boldsymbol{u}(t_n) - \boldsymbol{u}_\star^n\|_1\|\nabla \times d_t e_b^{n+1}\| \\
&\le \frac{\eta}{6}\|\nabla \times d_t e_b^{n+1}\|^2 + \|d_t e_b^n\|^2 + \|d_t e_b^n\|\|\nabla \times d_t e_b^n\|\|\boldsymbol{u}(t_n) - \boldsymbol{u}_\star^n\|_1^2 + \|\boldsymbol{u}(t_n) - \boldsymbol{u}_\star^n\|_1^2 \\
&\le \frac{\eta}{6}\|\nabla \times d_t e_b^{n+1}\|^2 + \|d_t e_b^n\|^2 + \|d_t e_b^n\|^2\|\nabla \times d_t e_b^n\|^2\|\boldsymbol{u}(t_n) - \boldsymbol{u}_\star^n\|_1^2 + \|\boldsymbol{u}(t_n) - \boldsymbol{u}_\star^n\|_1^2,
\end{aligned}$$

where we use $\boldsymbol{u} \in L^\infty(0,T;H^2)$, $\boldsymbol{B}_t \in L^\infty(0,T,H^1)$. In fact, by using (3.10), Remark 4.1 and (4.26), we get

$$\begin{aligned}
\|\boldsymbol{u}(t_n) - \boldsymbol{u}_\star^n\| &\le \|e_u^n\| + s\delta t\|\boldsymbol{B}^n \times \nabla \times \boldsymbol{B}^{n+1}\| \\
&\lesssim \|e_u^n\| + \delta t\|\boldsymbol{B}^n\|_{L^\infty}\|\nabla \times \boldsymbol{B}^{n+1}\| \lesssim \delta t, \\
\|\nabla(\boldsymbol{u}(t_n) - \boldsymbol{u}_\star^n)\| &\le \|\nabla e_u^n\| + s\delta t\|\nabla(\boldsymbol{B}^n \times \nabla \times \boldsymbol{B}^{n+1})\| \\
&\lesssim \|\nabla e_u^n\| + \delta t(\|\boldsymbol{B}^n\|_2\|\boldsymbol{B}^{n+1}\|_2 + \|\boldsymbol{B}^n\|_{L^\infty}\|\boldsymbol{B}^{n+1}\|_2) \\
&\lesssim \|\nabla e_u^n\| + \delta t,
\end{aligned}$$

which also imply

$$\delta t\sum_{n=1}^{N}\|\boldsymbol{u}(t_n) - \boldsymbol{u}_\star^n\|_1^2 \lesssim \delta t^2. \tag{4.46}$$

Therefore, from (4.46) and Theorem 4.1, we easily know that $\|\nabla\times d_t e_b^n\|\|\boldsymbol{u}(t_n) - \boldsymbol{u}_\star^n\|_1 \lesssim \frac{1}{\delta t}(\|\nabla\times e_b^n\| + \|\nabla\times e_b^{n-1}\|)\|\boldsymbol{u}(t_n) - \boldsymbol{u}_\star^n\|_1$ is bounded by a constant, thus term V can be further estimated as

$$(\text{term V}) \lesssim \frac{\eta}{6}\|\nabla \times d_t e_b^{n+1}\|^2 + \|d_t e_b^n\|^2 + \|\boldsymbol{u}(t_n) - \boldsymbol{u}_\star^n\|_1^2. \tag{4.47}$$

For term VI, we estimate as

$$\begin{aligned}
(\text{term VI}) &\le |(\boldsymbol{B}(t_{n-1}) \times d_t\boldsymbol{u}(t_n), \nabla \times d_t e_b^{n+1}) - (\boldsymbol{B}^{n-1} \times d_t\boldsymbol{u}_\star^n, \nabla \times d_t e_b^{n+1})| \\
&= |(e_b^{n-1} \times d_t\boldsymbol{u}(t_n), \nabla \times d_t e_b^{n+1}) + (\boldsymbol{B}^{n-1} \times d_t(\boldsymbol{u}(t_n) - \boldsymbol{u}_\star^n), \nabla \times d_t e_b^{n+1})| \\
&= |(e_b^{n-1} \times d_t\boldsymbol{u}(t_n), \nabla \times d_t e_b^{n+1}) + \left(\boldsymbol{B}^{n-1} \times d_t(e_u^n + s\delta t\boldsymbol{B}^n \times \nabla \times \boldsymbol{B}^{n+1}), \nabla \times d_t e_b^{n+1}\right)| \\
&\lesssim \|\nabla \times e_b^{n-1}\|\|d_t\boldsymbol{u}(t_n)\|_{L^4}\|\nabla \times d_t e_b^{n+1}\| + \|\boldsymbol{B}^{n-1}\|_{L^\infty}\|d_t e_u^n\|\|\nabla \times d_t e_b^{n+1}\| \\
&\quad + \delta t|(\boldsymbol{B}^{n-1} \times d_t(\boldsymbol{B}^n \times \nabla \times \boldsymbol{B}^{n+1}), \nabla \times d_t e_b^{n+1})| \\
&\lesssim \|\nabla \times e_b^{n-1}\|\|\nabla \times d_t e_b^{n+1}\| + \|d_t e_u^n\|\|\nabla \times d_t e_b^{n+1}\| \\
&\quad + \delta t|(\boldsymbol{B}^{n-1} \times d_t(\boldsymbol{B}^n \times \nabla \times \boldsymbol{B}^{n+1}), \nabla \times d_t e_b^{n+1})| \\
&\lesssim \frac{\eta}{12}\|\nabla \times d_t e_b^{n+1}\|^2 + \|\nabla \times e_b^{n-1}\|^2 + \|d_t e_u^n\|^2 + \delta t|(\boldsymbol{B}^{n-1} \times d_t(\boldsymbol{B}^n \times \nabla \times \boldsymbol{B}^{n+1}), \nabla \times d_t e_b^{n+1})|,
\end{aligned} \tag{4.48}$$

where we use $\boldsymbol{u}_t \in L^\infty(0,T;H^1(\Omega))$. The last term on the right hand side of (4.48) can be estimated as

$$\begin{aligned}
&\delta t|(\boldsymbol{B}^{n-1} \times d_t(\boldsymbol{B}^n \times \nabla \times \boldsymbol{B}^{n+1}), \nabla \times d_t e_b^{n+1})| \\
&\quad \lesssim \|\boldsymbol{B}^{n-1}\|_{L^\infty}\|\boldsymbol{B}^n \times \nabla \times \boldsymbol{B}^{n+1} - \boldsymbol{B}^{n-1} \times \nabla \times \boldsymbol{B}^n\|\|\nabla \times d_t e_b^{n+1}\| \\
&\quad \lesssim \|\boldsymbol{B}^n \times \nabla \times (\boldsymbol{B}^{n+1} - \boldsymbol{B}^n) + (\boldsymbol{B}^n - \boldsymbol{B}^{n-1}) \times \nabla \times \boldsymbol{B}^n\|\|\nabla \times d_t e_b^{n+1}\| \\
&\quad \lesssim \left(\|\boldsymbol{B}^n \times \nabla \times (\boldsymbol{B}^{n+1} - \boldsymbol{B}^n)\| + \|(\boldsymbol{B}^n - \boldsymbol{B}^{n-1}) \times \nabla \times \boldsymbol{B}^n\|\right)\|\nabla \times d_t e_b^{n+1}\| \\
&\quad \lesssim \|\boldsymbol{B}^n \times \nabla \times (e_b^n - e_b^{n+1} + \boldsymbol{B}(t_{n+1}) - \boldsymbol{B}(t_n))\|\|\nabla \times d_t e_b^{n+1}\| \\
&\qquad + \|(e_b^{n-1} - e_b^n + \boldsymbol{B}(t_n) - \boldsymbol{B}(t_{n-1})) \times \nabla \times \boldsymbol{B}^n\|\|\nabla \times d_t e_b^{n+1}\|
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \left(\|\nabla\times(e_b^n-e_b^{n+1})\|+\|\nabla\times(\boldsymbol{B}(t_{n+1})-\boldsymbol{B}(t_n))\|\right)\|\nabla\times d_te_b^{n+1}\| \\
&\quad+\left(\|e_b^{n-1}-e_b^n\|_{L^4}+\|\boldsymbol{B}(t_n)-\boldsymbol{B}(t_{n-1})\|_1\right)\|\boldsymbol{B}^n\|_2\|\nabla\times d_te_b^{n+1}\| \\
&\lesssim \frac{\eta}{12}\|\nabla\times d_te_b^{n+1}\|^2+\|\nabla\times(e_b^n-e_b^{n+1})\|^2+\|\boldsymbol{B}(t_{n+1})-\boldsymbol{B}(t_n)\|_1^2 \\
&\quad+\|\nabla\times(e_b^{n-1}-e_b^n)\|^2+\|\boldsymbol{B}(t_n)-\boldsymbol{B}(t_{n-1})\|_1^2,
\end{aligned}
$$

where we use (2.2), (2.3), Remark 4.1 and Lemma 4.4. Therefore, for term VI, we obtain

$$
\begin{aligned}
(\text{term VI}) &\lesssim \frac{\eta}{6}\|\nabla\times d_te_b^{n+1}\|^2+\|\nabla\times e_b^{n-1}\|^2+\|d_te_u^n\|^2+\|\nabla\times(e_b^n-e_b^{n+1})\|^2 \\
&\quad+\|\nabla\times(e_b^{n-1}-e_b^n)\|^2+\|\boldsymbol{B}(t_{n+1})-\boldsymbol{B}(t_n)\|_1^2+\|\boldsymbol{B}(t_n)-\boldsymbol{B}(t_{n-1})\|_1^2.
\end{aligned}
\tag{4.49}
$$

For term VII, using (2.1)–(2.3), we have

$$
\begin{aligned}
(\text{term VII}) &\le |(d_tR_u^{n+1},d_t\tilde{e}_u^{n+1})+(d_tR_b^{n+1},d_te_b^{n+1})| \\
&\lesssim \frac{\nu}{12}\|\nabla d_t\tilde{e}_u^{n+1}\|^2+\frac{\eta}{6}\|\nabla\times d_te_b^{n+1}\|^2+\|d_tR_u^{n+1}\|^2+\|d_tR_b^{n+1}\|^2.
\end{aligned}
\tag{4.50}
$$

For term VIII, we have

$$
\begin{aligned}
(\text{term VIII}) &\le |(d_t\tilde{e}_u^{n+1},d_tR_p^{n+1})-\delta t(\nabla d_te_p^n,d_tR_p^{n+1})| \\
&\lesssim \frac{\nu}{12}\|\nabla d_t\tilde{e}_u^{n+1}\|^2+\|d_tR_p^{n+1}\|^2+\delta t^2\|\nabla d_te_p^n\|^2.
\end{aligned}
\tag{4.51}
$$

By combining (4.41), (4.42), (4.43), (4.44), (4.45), (4.47), (4.49), (4.50) and (4.51), we have

$$
\begin{aligned}
&\|d_te_u^{n+1}\|^2-\|d_te_u^n\|^2+\delta t\nu\|\nabla d_t\tilde{e}_u^{n+1}\|^2+\delta t^2(\|\nabla d_te_p^{n+1}\|^2-\|\nabla d_te_p^n\|^2) \\
&\quad+\|d_te_b^{n+1}\|^2-\|d_te_b^n\|^2+\delta t\eta\|\nabla\times d_te_b^{n+1}\|^2 \\
&\quad\lesssim \delta t(\|\nabla e_u^n\|^2+\|\nabla\tilde{e}_u^n\|^2+\|\nabla\times e_b^n\|^2+\|\nabla\times e_b^{n-1}\|^2+\|\nabla\times e_b^{n+1}\|^2) \\
&\quad+\delta t(\|d_te_u^n\|^2+\|d_te_b^n\|^2+\|d_te_b^{n+1}\|^2+\delta t^2\|\nabla d_te_p^n\|^2) \\
&\quad+\delta t(\|d_tR_p^{n+1}\|^2+\|d_tR_u^{n+1}\|^2+\|d_tR_b^{n+1}\|^2) \\
&\quad+\delta t\left(\|\boldsymbol{B}(t_{n+1})-\boldsymbol{B}(t_n)\|_1^2+\|\boldsymbol{B}(t_n)-\boldsymbol{B}(t_{n-1})\|_1^2\right) \\
&\quad+\delta t\|\boldsymbol{u}(t_n)-\boldsymbol{u}_\star^n\|_1^2.
\end{aligned}
\tag{4.52}
$$

By taking the summation of (4.52) from $n=1$ to m, using Theorem 4.1, Lemma 4.5, Lemma 4.6, (4.26), (4.46) and the assumption (A) we obtain

$$
\begin{aligned}
&\|d_te_u^{m+1}\|^2+\|d_te_b^{m+1}\|^2+\delta t^2\|\nabla d_te_p^{m+1}\|^2+\delta t\sum_{n=1}^m(\nu\|\nabla d_t\tilde{e}_u^{n+1}\|^2+\eta\|\nabla\times d_te_b^{n+1}\|^2) \\
&\quad\lesssim \|d_te_u^1\|^2+\|d_te_b^1\|^2+\delta t^2\|\nabla d_te_p^1\|^2+\delta t\sum_{n=1}^m(\|d_te_u^n\|^2+\|d_te_b^n\|^2+\|d_te_b^{n+1}\|^2+\delta t^2\|\nabla d_te_p^n\|^2) \\
&\quad+\delta t\sum_{n=1}^m(\|d_tR_p^{n+1}\|^2+\|d_tR_u^{n+1}\|^2+\|d_tR_b^{n+1}\|^2)+\delta t^2 \\
&\quad\lesssim \delta t\sum_{n=1}^m(\|d_te_u^n\|^2+\|d_te_b^n\|^2+\|d_te_b^{n+1}\|^2+\delta t^2\|\nabla d_te_p^n\|^2)+\delta t^2.
\end{aligned}
$$

From the Gronwall's inequality in Lemma 4.1, there exists a constant $\widehat{C}$, such that

$$
\|d_te_u^{m+1}\|^2+\|d_te_b^{m+1}\|^2+\delta t^2\|\nabla d_te_p^{m+1}\|^2+\delta t\sum_{n=1}^m(\nu\|\nabla d_t\tilde{e}_u^{n+1}\|^2+\eta\|\nabla\times d_te_b^{n+1}\|^2)\lesssim\delta t^2,
$$

holds for $\delta t\le\widehat{C}$ and $1\le m\le[\frac{T}{\delta t}]-1$, that concludes this lemma. □

Now, we can prove the optimal error estimate for pressure. Meantime, the H^1 error estimates for velocity and magnetic field are also obtained.

Theorem 4.2. *Under the assumptions of Lemma 4.7, we have*

$$
\|\nabla e_u^n\|+\|\nabla\tilde{e}_u^n\|+\|\nabla\times e_b^n\|+\|e_p^n\|\lesssim\delta t,\ \ 1\le n\le[\frac{T}{\delta t}].
\tag{4.53}
$$

Proof. By taking the summation of (4.6) and (4.7), we obtain the error equations

$$\begin{aligned}\frac{e_u^{n+1}-e_u^n}{\delta t} &- \nu\Delta\tilde{e}_u^{n+1} + (\boldsymbol{u}(t_n)\cdot\nabla)\boldsymbol{u}(t_{n+1}) - (\boldsymbol{u}^n\cdot\nabla)\tilde{\boldsymbol{u}}^{n+1} + \nabla e_p^{n+1} \\ &+ s\boldsymbol{B}(t_n)\times\nabla\times\boldsymbol{B}(t_{n+1}) - s\boldsymbol{B}^n\times\nabla\times\boldsymbol{B}^{n+1} = R_u^{n+1} + R_p^{n+1}.\end{aligned} \tag{4.54}$$

By taking the L^2 inner product of (4.54) with any test function $\boldsymbol{v}$ $(\in H_0^1(\Omega)^d)$, from assumption (A), we obtain

$$\begin{aligned}(e_p^{n+1},\nabla\cdot\boldsymbol{v}) &= \left(\frac{e_u^{n+1}-e_u^n}{\delta t},\boldsymbol{v}\right) + \nu(\nabla\tilde{e}_u^{n+1},\nabla\boldsymbol{v}) + \left((e_u^n\cdot\nabla)\boldsymbol{u}(t_{n+1}) + (\boldsymbol{u}^n\cdot\nabla)\tilde{e}_u^{n+1},\boldsymbol{v}\right) \\ &\quad + s\left(e_b^n\times\nabla\times\boldsymbol{B}(t_{n+1}) + \boldsymbol{B}^n\times\nabla\times e_b^{n+1},\boldsymbol{v}\right) - (R_u^{n+1}+R_p^{n+1},\boldsymbol{v}) \\ &\lesssim \|d_t e_u^{n+1}\|\|\nabla\boldsymbol{v}\| + \|\nabla\tilde{e}_u^{n+1}\|\|\nabla\boldsymbol{v}\| + \|e_u^n\|\|\nabla\boldsymbol{v}\| + \|\nabla\tilde{e}_u^{n+1}\|\|\nabla\boldsymbol{v}\| \\ &\quad + \|e_b^n\|\|\nabla\boldsymbol{v}\| + \|\nabla\times e_b^{n+1}\|\|\nabla\boldsymbol{v}\| + \|R_u^{n+1}\|\|\nabla\boldsymbol{v}\| + \|R_p^{n+1}\|\|\nabla\boldsymbol{v}\|.\end{aligned}$$

Using the inf–sup condition, there exists a positive constant β such that

$$\beta\|q\| \le \sup_{\boldsymbol{w}\in H_0^1(\Omega)^d}\frac{(\nabla\cdot\boldsymbol{w},q)}{\|\nabla\boldsymbol{w}\|},\quad \forall q\in L_0^2(\Omega),$$

we obtain

$$\begin{aligned}\beta\|e_p^{n+1}\| &\lesssim \|d_t e_u^{n+1}\| + \|\nabla\tilde{e}_u^{n+1}\| + \|e_u^n\| + \|e_b^n\| + \|\nabla\times e_b^{n+1}\| + \|R_u^{n+1}\| + \|R_p^{n+1}\| \\ &\lesssim \delta t + \|\nabla\tilde{e}_u^{n+1}\| + \|\nabla\times e_b^{n+1}\|,\end{aligned} \tag{4.55}$$

where we also use Lemma 4.7, Theorem 4.1 and Lemma 4.3.

From Lemma 4.7 and Hölder's inequality, for $1\le m\le[\frac{T}{\delta t}]-1$, we have

$$\begin{aligned}\|\nabla\tilde{e}_u^{m+1}\| - \|\nabla\tilde{e}_u^1\| &= \sum_{n=1}^m(\|\nabla\tilde{e}_u^{n+1}\| - \|\nabla\tilde{e}_u^n\|) \le \sum_{n=1}^m\|\nabla\tilde{e}_u^{n+1}-\nabla\tilde{e}_u^n\| \\ &\le \left(\delta t^2\sum_{n=1}^m\left\|\frac{\nabla\tilde{e}_u^{n+1}-\nabla\tilde{e}_u^n}{\delta t}\right\|^2\right)^{\frac12}(\sum_{n=1}^m 1^2)^{\frac12} \lesssim (\delta t^3)^{\frac12}(\frac{T}{\delta t})^{\frac12} \lesssim \delta t.\end{aligned} \tag{4.56}$$

Thus, from (4.36), we obtain

$$\|\nabla\tilde{e}_u^{m+1}\| \lesssim \delta t + \|\nabla\tilde{e}_u^1\| \lesssim \delta t,\ 0\le m\le[\frac{T}{\delta t}]-1. \tag{4.57}$$

In addition, due to (4.25), we also have

$$\|\nabla e_u^{m+1}\| \le \|e_u^{m+1}\|_1 \lesssim \|\nabla\tilde{e}_u^{m+1}\| \lesssim \delta t. \tag{4.58}$$

Similarly, we can also deduce

$$\|\nabla\times e_b^{m+1}\| \lesssim \delta t. \tag{4.59}$$

Finally, from (4.55), (4.57), (4.58), (4.59), we conclude (4.53). □

Remark 4.3. In this work we have proved the optimal error estimates in Theorem 4.1 and Theorem 4.2 for the time-marching scheme (3.1)–(3.8). The error estimates for the fully discrete scheme is much more complicated, we omit the details here due to the page limits. About the convergence estimates for the fully discrete schemes in the context of finite element method/or spectral method related to the Navier–Stokes equations, we refer to [48–61].

5. Numerical examples

We now implement some numerical experiments to validate the stability and accuracy of the scheme. We use the inf–sup stable P2/P1 element [62] for the velocity and pressure, and linear element for the magnetic field. The fully discrete finite element scheme reads as

Step 1. Find $\boldsymbol{B}_h^{n+1}\in C_h\subset H_\tau^1(\Omega)$ such that for all $\boldsymbol{C}_h\in C_h$

$$\begin{aligned}\left(\frac{\boldsymbol{B}_h^{n+1}-\boldsymbol{B}_h^n}{\delta t},\boldsymbol{C}_h\right) &+ \eta(\nabla\times\boldsymbol{B}_h^{n+1},\nabla\times\boldsymbol{C}_h) + \eta(\nabla\cdot\boldsymbol{B}_h^{n+1},\nabla\cdot\boldsymbol{C}_h) + (\boldsymbol{B}_h^n\times\boldsymbol{u}_h^n,\nabla\times\boldsymbol{C}_h) \\ &+ \delta ts(\boldsymbol{B}_h^n\times\nabla\times\boldsymbol{B}_h^{n+1},\boldsymbol{B}_h^n\times\nabla\times\boldsymbol{C}_h) = 0.\end{aligned}$$

Table 1
The numerical errors and convergence order for $\|e_u\|_{L^2}, \|e_u\|_{H^1}, \|e_p\|_{L^2}, \|e_b\|_{L^2}, \|e_b\|_{H^1}$ at $t = 1$ that are computed using various temporal resolutions with the exact solutions of (5.1). The physical parameters are $\nu = \eta = s = 1$.

δt	$\|e_u\|_{L^2}$	Order	$\|e_u\|_{H^1}$	Order	$\|e_p\|_{L^2}$	Order	$\|e_b\|_{L^2}$	Order	$\|e_b\|_{H^1}$	Order
1/8	2.44e−4	–	2.94e−3	–	1.34e−2	–	2.49e−3	–	1.20e−2	–
1/16	1.07e−4	1.18	1.15e−3	1.34	6.65e−3	1.01	1.30e−3	0.94	6.25e−3	0.94
1/32	4.57e−5	1.23	3.76e−4	1.62	3.01e−3	1.14	6.59e−4	0.97	3.18e−3	0.98
1/64	2.25e−5	1.02	1.74e−4	1.10	1.44e−3	1.05	3.32e−4	0.99	1.60e−3	0.99
1/128	1.13e−5	1.00	8.67e−5	1.00	7.12e−4	1.01	1.66e−4	1.00	8.02e−4	1.00
1/256	5.63e−6	1.00	4.33e−5	1.00	3.54e−4	1.00	8.35e−5	1.00	4.02e−4	1.00

Step 2. Find $\tilde{\boldsymbol{u}}_h^{n+1} \in V_h \subset H_0^1(\Omega)^d$ such that for all $\boldsymbol{v}_h \in V_h$

$$\left(\frac{\tilde{\boldsymbol{u}}_h^{n+1} - \boldsymbol{u}_h^n}{\delta t}, \boldsymbol{v}_h\right) + \nu(\nabla \tilde{\boldsymbol{u}}_h^{n+1}, \nabla \boldsymbol{v}_h) + b(\boldsymbol{u}_h^n, \tilde{\boldsymbol{u}}_h^{n+1}, \boldsymbol{v}_h) - (p_h^n, \nabla \cdot \boldsymbol{v}_h) + s(\boldsymbol{B}_h^n \times \nabla \times \boldsymbol{B}_h^{n+1}, \boldsymbol{v}_h) = 0.$$

Step 3. Find $p_h^{n+1} \in M_h \subset L_0^2(\Omega)$ from

$$(\nabla p_h^{n+1}, \nabla q_h) = -\frac{1}{\delta t}(\nabla \cdot \tilde{\boldsymbol{u}}_h^{n+1}, q_h) + (\nabla p_h^n, \nabla q_h) \quad q_h \in M_h.$$

Step 4. Update $\boldsymbol{u}_h^{n+1}$ from

$$\boldsymbol{u}_h^{n+1} = \tilde{\boldsymbol{u}}_h^{n+1} - \delta t \nabla p_h^{n+1} + \delta t \nabla p_h^n.$$

Similar to Theorem 3.1, we can prove the fully discrete scheme is also unconditionally energy stable in the sense that

$$\begin{aligned} s\|\boldsymbol{B}_h^{n+1}\|^2 + \|\boldsymbol{u}_h^{n+1}\|^2 + \delta t^2\|\nabla p_h^{n+1}\|^2 + 2\delta t(s\eta\|\nabla \times \boldsymbol{B}_h^{n+1}\|^2 + s\eta\|\nabla \cdot \boldsymbol{B}_h^{n+1}\|^2 + \nu\|\nabla \tilde{\boldsymbol{u}}_h^{n+1}\|^2) \\ \leq s\|\boldsymbol{B}_h^n\|^2 + \|\boldsymbol{u}_h^n\|^2 + \delta t^2\|\nabla p_h^n\|^2. \end{aligned}$$

5.1. Accuracy test

We first perform numerical simulations to test the convergence rates of the proposed scheme. The computational domain is $\Omega = [0, 1] \times [0, 1]$. We assume the following functions

$$\begin{cases} \boldsymbol{u} = (ye^{-t}, x\cos(t)), \\ p = 0, \\ \boldsymbol{B} = (y\cos(t), xe^{-t}) \end{cases} \tag{5.1}$$

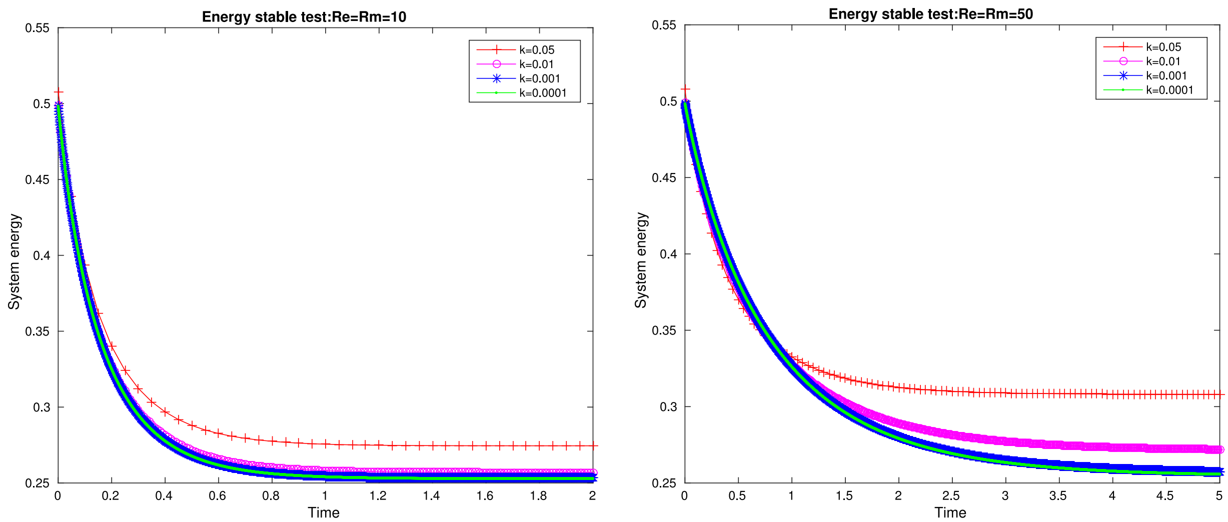
to be the exact solution, and impose some suitable force fields such that the given solution can satisfy the system. For simplicity, the physical parameters are set as $\nu = \eta = s = 1$. Note that the exact solution are linear functions in space. The approximate errors mainly come from the time discretization. We fix space mesh size $h = \frac{1}{8}$ and refine the time step size δt to test the convergence orders about the time discretization. In Table 1, we list the numerical errors between the numerical solution and the exact solution at $T = 1$ with different time step sizes. We observe the first order accuracy asymptotically for $\|e_u\|_{L^2}, \|e_u\|_{H^1}, \|e_p\|_{L^2}, \|e_b\|_{L^2}$ and $\|e_b\|_{H^1}$, as predicted theoretically.

5.2. Stability test

We show the evolution of the total free energy in this example. We set the computed domain to be $\Omega = [0, 1]^2$, and the initial conditions for $\boldsymbol{u}, p, \boldsymbol{B}$ are

$$\begin{cases} \boldsymbol{u}^0 = (x^2(x-1)^2y(y-1)(2y-1),\ -y^2(y-1)^2x(x-1)(2x-1)), \\ p^0 = 0, \\ \boldsymbol{B}^0 = (\sin(\pi x)\cos(\pi y),\ -sin(\pi y)\cos(\pi x)). \end{cases} \tag{5.2}$$

We test the energy stability over matching time of the proposed scheme under variant physical parameters of $R_e = R_m = 10$ and 50. The coupling parameter is fixed as $s = 1$, and mesh size is $h = 1/64$. In Fig. 1, we present the time evolution of the total free energy for four different time steps of $k = 0.05, 0.01, 0.001, 0.0001$ until $T = 5$. We observe that all four energy curves show decays monotonically for all time step sizes, which numerically confirms that our algorithm is unconditionally energy stable.



(a) $Re = Rm = 10$.　(b) $Re = Rm = 50$.

Fig. 1. Time evolution of the free energy functional till $T = 5$ for four different time steps and two sets of order parameters (a) $R_e = R_m = 10$ and (b) $R_e = R_m = 50$. The energy curves show the decays for all time steps $\delta t = 0.05, 0.01, 0.001, 0.0001$, which confirms that our algorithm is unconditionally stable.

5.3. Hydromagnetic Kelvin–Helmholtz instability

The Kelvin–Helmholtz (K–H) instability in sheared flow configurations is an efficient mechanism to initiate mixing of fluids, transport of momentum and energy, and the development of turbulence. Such a problem is of interest in investigating a variety of space, astrophysical, and geophysical situations involving sheared plasma flows. Configurations where it is relevant include the interface between the solar wind and the magnetosphere, coronal streamers moving through the solar wind, etc. Since most astrophysical environments are electrically conducting and relevant fluids are likely to be magnetized, it is thus of prime importance to understand the role of magnetic fields in the K-H instability. About the theoretical and numerical study of Hydromagnetic K-H instability, we refer to [2,15,63–67] and the references therein.

We revisit the occurring of the K–H instability in a single shear flow configuration that is embedded in a uniform flow-aligned magnetic field. The simulation is performed in the computed domain of $[0, 2] \times [0, 1]$. The initial velocity field is $\boldsymbol{u}_0 = (1.5, 0)$ in the top half domain, and $\boldsymbol{u}_0 = (-1.5, 0)$ in the bottom half domain. The sheared initial magnetic field is $\boldsymbol{B}_0 = (\tanh(y/\epsilon), 0)$ where $\epsilon = 0.07957747154595$ (cf. [64]). The velocity $\boldsymbol{u}$, magnetic field $\boldsymbol{B}$ and pressure p are periodic boundary conditions on left and right boundaries. On the top and bottom boundary, the second component v of the velocity field $\boldsymbol{u} = (u, v)$ is imposed. The boundary conditions for $\boldsymbol{B}$ are $\boldsymbol{B} \times \boldsymbol{n} = \boldsymbol{B}_0 \times \boldsymbol{n}$ for the top boundary and $-\boldsymbol{B}_0 \times \boldsymbol{n}$ for the bottom. The order parameters are $R_e = R_m = 1000$, $s = 0.2$. We use the time step $\delta t = 0.01$ and grid size $h = \frac{1}{40}$.

In Fig. 2, we show snapshots of the magnitude of $\boldsymbol{B}_1$ that is the first component of $\boldsymbol{B} = (\boldsymbol{B}_1, \boldsymbol{B}_2)$ together with the velocity field $\boldsymbol{u}$ at $t = 0.2, 1, 2.5, 3, 3.2, 3.5$. When time evolves, we can observe the vortexes start to form around $t = 1$. After $t = 2.5$, the profiles of vortexes and the magnetic field show the typical structure of K-H instability, and soon it deforms and rotates along with the flow. The obtained numerical results coincide well with the numerical/experimental results discussed in [63,65–71], qualitatively.

6. Concluding remarks

In this paper, we develop an efficient numerical scheme for solving the MHD system. The scheme is (a) fully decoupled, (b) unconditionally energy stable, (c) linear and easy-to-implement. Moreover, we theoretically establish the unconditional energy stability and provide rigorous error estimates for the scheme. A series of numerical simulations, including the convergence test, energy stability test and a physical benchmark problem, are presented to validate the stability and accuracy of the scheme.

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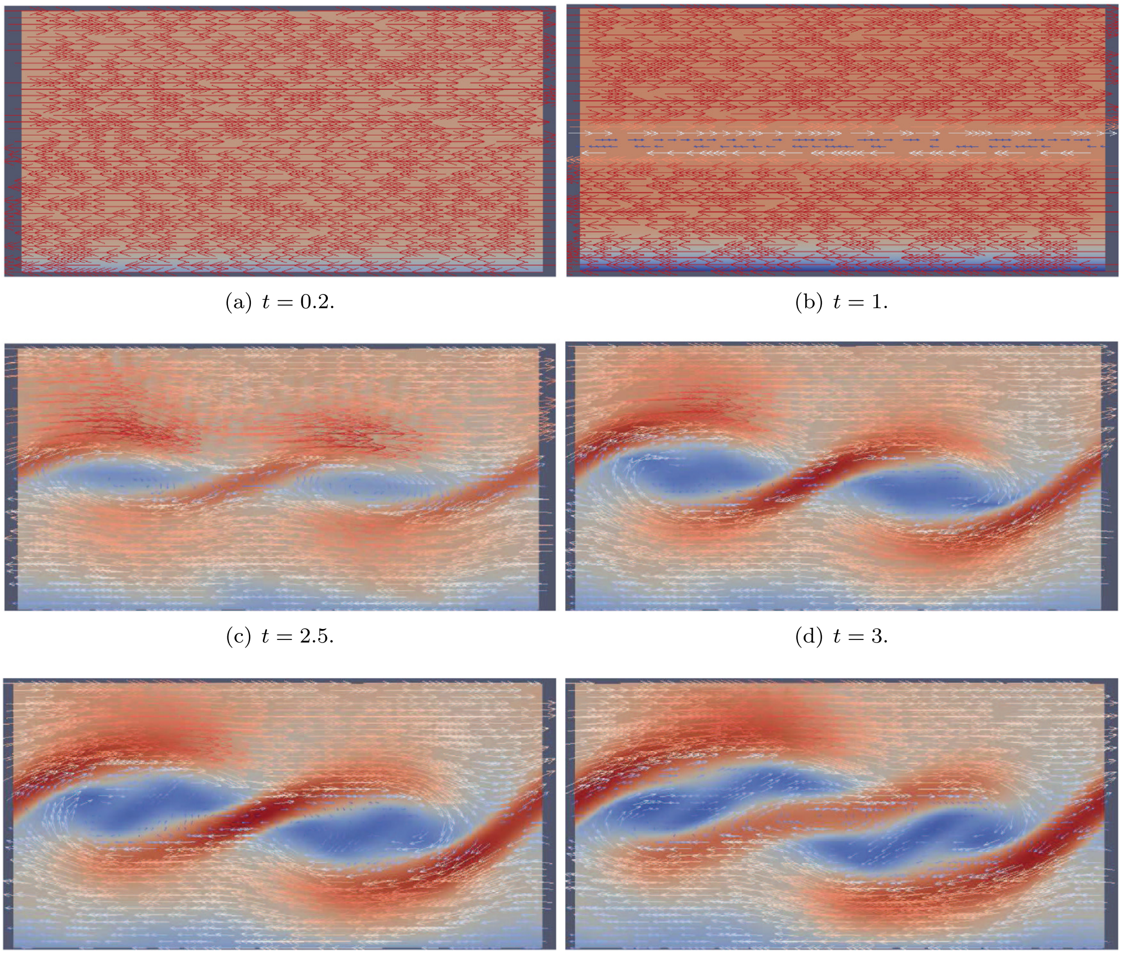

(a) $t = 0.2$. (b) $t = 1$.

(c) $t = 2.5$. (d) $t = 3$.

(e) $t = 3.2$. (f) $t = 3.5$.

Fig. 2. The dynamical behaviors of the magnetic field together with the velocity field that shows the hydromagnetic K-H instability. Snapshots of the numerical approximation are taken at $t = 0.2, 1, 2.5, 3, 3.2, 3.5$.

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