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EXTERNAL OPTIMAL CONTROL OF FRACTIONAL PARABOLIC PDES*

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Abstract. In [Antil *et al.* *Inverse Probl.* **35** (2019) 084003.] we introduced a new notion of optimal control and source identification (inverse) problems where we allow the control/source to be outside the domain where the fractional elliptic PDE is fulfilled. The current work extends this previous work to the parabolic case. Several new mathematical tools have been developed to handle the parabolic problem. We tackle the Dirichlet, Neumann and Robin cases. The need for these novel optimal control concepts stems from the fact that the classical PDE models only allow placing the control/source either on the boundary or in the interior where the PDE is satisfied. However, the nonlocal behavior of the fractional operator now allows placing the control/source in the exterior. We introduce the notions of weak and very-weak solutions to the fractional parabolic Dirichlet problem. We present an approach on how to approximate the fractional parabolic Dirichlet solutions by the fractional parabolic Robin solutions (with convergence rates). A complete analysis for the Dirichlet and Robin optimal control problems has been discussed. The numerical examples confirm our theoretical findings and further illustrate the potential benefits of nonlocal models over the local ones.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded open set with boundary $\partial\Omega$. Consider the Banach spaces (Z_D, U_D) and (Z_R, U_R), where the subscripts D and R denote Dirichlet and Robin, respectively. The goal of this paper is to study the following parabolic external optimal control (or source identification) problems:

– **Fractional parabolic Dirichlet exterior control (source identification) problem:** Given $\xi \geq 0$ a constant penalty parameter we consider the following minimization problem:

$$\min_{(u,z)\in(U_D,Z_D)} \Big(J(u) + \frac{\xi}{2}\|z\|^2_{Z_D}\Big), \tag{1.1a}$$

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subject to the fractional parabolic Dirichlet exterior value problem: Find $u \in U_D$ solving

$$\begin{cases} \partial_t u + (-\Delta)^s u = 0 & \text{in } Q := (0,T)\times\Omega, \\ u = z & \text{in } \Sigma := [0,T)\times(\mathbb{R}^N\setminus\Omega), \\ u(0,\cdot) = 0 & \text{in } \Omega, \end{cases} \tag{1.1b}$$

and the control constraints

$$z \in Z_{ad,D}, \tag{1.1c}$$

with $Z_{ad,D} \subset Z_D$ being a closed and convex subset. Here, $Z_D := L^2((0,T);L^2(\mathbb{R}^N\setminus\Omega))$, $U_D := L^2((0,T);L^2(\Omega))$ and the functional J is assumed to be weakly lower-semicontinuous and satisfies suitable conditions. We refer to Section 4.1 for more details.

– **Fractional parabolic Robin exterior control (source identification) problem:** Given $\xi \geq 0$ a constant penalty parameter we consider the minimization problem

$$\min_{(u,z)\in(U_R,Z_R)} \Big(J(u) + \frac{\xi}{2}\|z\|^2_{Z_R} \Big), \tag{1.2a}$$

subject to the fractional parabolic Robin exterior value problem: Find $u \in U_R$ solving

$$\begin{cases} \partial_t u + (-\Delta)^s u = 0 & \text{in } Q, \\ \mathcal{N}_s u + \kappa u = \kappa z & \text{in } \Sigma, \\ u(0,\cdot) = 0 & \text{in } \Omega, \end{cases} \tag{1.2b}$$

and the control constraints

$$z \in Z_{ad,R}, \tag{1.2c}$$

with $Z_{ad,R} \subset Z_R$ being a closed and convex subset. In (1.2b), $\mathcal{N}_s$ denotes the interaction operator and is given in (2.8) below, $\kappa \in L^1(\mathbb{R}^N\setminus\Omega)\cap L^\infty(\mathbb{R}^N\setminus\Omega)$ and is non-negative. The Banach spaces $Z_R := L^2((0,T);L^2(\mathbb{R}^N\setminus\Omega,\mu))$, $U_R := L^2((0,T);W^{s,2}_{\Omega,\kappa})\cap H^1((0,T);(W^{s,2}_{\Omega,\kappa})^\star)$ and the functional J is also weakly lower-semicontinuous and satisfies suitable conditions. We refer to Section 2 for the definition of the spaces involved and to Section 4.2 for further details on the functional J.

A widely used example of a functional J is as follows. Let $u_d \in L^2((0,T);L^2(\Omega))$ be given and consider the functional J defined by

$$J(u) := \frac{1}{2}\|u-u_d\|^2_{L^2((0,T);L^2(\Omega))}.$$

A typical example of a control constraint set, for instance, in the case of the Robin problem is as follows: given z_a, z_b with $z_a \leq z_b$, we can take

$$Z_{ad,R} := \{z \in Z_R \ : \ z_a(t,x) \leq z(t,x) \leq z_b(t,x), \ \text{a.e. in } \Sigma\}.$$

Nevertheless, our approach is not limited to these choices.

We would like to extend our previous work [11] on the elliptic (stationary) case to the parabolic (non-stationary) case. We have to develop several new tools to study the parabolic case. More details on the similarities and differences between the two cases will be discussed shortly. Notice that (1.2b) is a generalized exterior value problem and all the details (with minor modifications) transfer to the case when instead of $\mathcal{N}_s u + \kappa u = \kappa z$ in Σ we consider $\mathcal{N}_s u = \kappa z$ in Σ, where z denotes the control/source. The resulting optimal control problem is the parabolic Neumann exterior control problem. We mention that we can also deal with the following more general system:

$$\begin{cases} \partial_t u + (-\Delta)^s u = f & \text{in } Q, \\ \mathcal{N}_s u + \kappa u = \kappa z & \text{in } \Sigma, \\ u(0,\cdot) = u_0 & \text{in } \Omega. \end{cases}$$

In fact, one has to decompose the solution u of the above system as $u = u_1 + u_2$, where u_1 satisfies (1.2b) and u_2 solves the system

$$\begin{cases} \partial_t u_2 + (-\Delta)^s u_2 = f & \text{in } Q, \\ \mathcal{N}_s u_2 + \kappa u_2 = 0 & \text{in } \Sigma, \\ u_2(0,\cdot) = u_0 & \text{in } \Omega, \end{cases}$$

and uses semigroups method.

The classical parabolic models, such as diffusion equations (with $s = 1$), are too restrictive. They only allow a source or a control placement either inside the domain Ω or on the boundary $\partial\Omega$. Notice that in both (1.1b) and (1.2b) the source/control z is placed in the exterior domain $\mathbb{R}^N \setminus \Omega$, disjoint from Ω. This is not possible for the classical models. The authors in [11] have recently introduced the notion of exterior optimal control with elliptic fractional PDEs as constraints. The current paper develops a complete theoretical framework for the parabolic case. The paper [11] has been inspired by the work of M. Warma [48] where the author has shown that the classical notion of controllability from the boundary does not make sense for fractional PDEs involving the fractional Laplace operator, and therefore, it must be replaced by a control that is localized outside the open set where the PDE is solved. For completeness, we would like to mention that the authors have recently considered the case where the source/control is located in the interior [6], see also [5] for the case when the source/control is the diffusion coefficient. We also mention the works on the interior control in the case of the so-called spectral fractional Laplacian [6, 7] and for boundary controls, see [10]. Some interesting (but not directly related) works on fractional Calderón type inverse problems have been investigated in [30, 36, 43] and the references therein. Notice that fractional order operators further provide flexibility to approximate arbitrary functions (see *e.g.* [22, 25, 32, 35]).

The key difficulties and novelties of this paper are as follows:

(i) **Nonlocal diffusion operator and exterior conditions.** The fractional Laplacian $(-\Delta)^s$ is a nonlocal operator and its evaluation at a point requires information over the entire $\mathbb{R}^N$. In addition, $(-\Delta)^s u$ may be nonsmooth even if u is smooth (see *e.g.* [42], Rem. 7.2). Moreover, we do not have the notion of boundary conditions, but the exterior conditions in $\mathbb{R}^N \setminus \Omega$.

(ii) **Nonlocal normal derivative.** $\mathcal{N}_s u$ is the nonlocal normal derivative of u. This can be thought of as a restricted fractional Laplacian in $\mathbb{R}^N \setminus \Omega$. It is a very difficult object to handle both at the continuous and discrete levels. Indeed, the best known regularity result for $\mathcal{N}_s$ is given in Lemma 2.3 which says that $\mathcal{N}_s u \in W^{s,2}_{\text{loc}}(\mathbb{R}^N \setminus \Omega)$ whenever $u \in W^{s,2}(\mathbb{R}^N)$. Higher regularity results are currently unknown.

(iii) **Approximation of the Dirichlet problem by a Robin problem.** In the case of the parabolic Dirichlet problem (1.1), it is imperative to deal with $\mathcal{N}_s$. Indeed, we need to approximate the very-weak solution to the parabolic Dirichlet problem (1.1b) which requires computing $\mathcal{N}_s$ of the test functions (see (3.13)). Moreover, the optimality system for the parabolic Dirichlet control problem (1.1) requires an approximation of the $\mathcal{N}_s$ of the adjoint variable (see (4.4)). We circumvent the first difficulty by approximating the parabolic Dirichlet problem (1.1b) by a parabolic Robin problem. We also prove a rate of convergence for this approximation. Under this new setup, the first order optimality conditions do not require an approximation of the $\mathcal{N}_s$ of the adjoint variable.

(iv) **Weak and very-weak solutions.** We study the notion of weak-solutions to the parabolic Dirichlet problem (1.1b) which requires a higher regularity on the datum $z \in H^1((0,T); W^{s,2}(\mathbb{R}^N \setminus \Omega))$. Since for the control problem (1.1) we only assume that $Z_D := L^2((0,T); L^2(\mathbb{R}^N \setminus \Omega))$, we have to introduce an even weaker notion of solutions to (1.1b). We call it very-weak solutions. We also develop the notion of weak-solutions to the Robin problem (1.2b) and prove their existence and uniqueness.

(v) **Optimal control problems.** We establish the well-posedness of solutions to both parabolic Dirichlet and Robin control problems.

Even though the outline of this paper is similar to our previous paper on fractional elliptic (stationary) control/inverse problems [11], however, the elliptic and parabolic cases are fundamentally different. The results presented here are novel in the context of parabolic (non-stationary) problems and in many situations the techniques used in the elliptic case either cannot be directly used or they must be carefully adapted to the parabolic case. In Definition 3.4 we introduce the notion of weak-solution to the non-homogeneous fractional parabolic Dirichlet problem. Notice that we need an additional regularity (H^1-in time) to establish the notion of weak solution which is different to the elliptic case. In Definition 3.7 we introduce the notion of very-weak solutions to the fractional parabolic problem which requires an integration-by-parts in both space and time. The duality argument to show the existence and uniqueness of solutions in the parabolic case is more involved (*cf.* Theorem 3.8) when compared to the elliptic case. In Definition 3.9 we introduce the notion of weak solutions to the Robin problem whose existence and uniqueness is shown in Theorem 3.10 by using the notion of integrated semigroups. Notice that the concept of integrated semigroups is not needed in the elliptic case. Section 4 deals with the Dirichlet and Robin fractional parabolic optimal control problems. The proofs in this section use standard calculus of variations techniques, this is similar to the elliptic case, however one has to deal with both the space and time variables and the notion of solutions to the parabolic problems. Notice that in addition, now one has to solve the adjoint equation backward in time. In Theorem 5.3 we approximate the Dirichlet solutions by the Robin solutions. A similar result in the elliptic case was considered in our recent work. The proof in the parabolic case to a some extent is motivated by the elliptic case but one has to deal with both the space and time variables which requires a careful analysis and several changes in the previous arguments, for instance, the duality arguments are different. In Theorem 5.4 we present the approximation of the Dirichlet control/inverse problem by the Robin control/source problem. The arguments in this case are similar to the elliptic case after adapting it to the parabolic case (the proof has been omitted). Finally, in Section 6 we present a numerical scheme to approximate the fractional parabolic state equation and the control/inverse problems. All the results presented here are completely new as such parabolic problems have not been considered before in the literature. Notice that the parabolic control/inverse problems are computationally much more involved than the elliptic problem since one has to solve the state equation forward in time and the adjoint equation backward in time.

Models with fractional derivatives are becoming increasingly popular which can be attributed to their role in many applications. These models appear in (but not limited to) image denoising, image segmentation and phase field modeling [2, 4, 12]; and magnetotellurics (geophysics) [49].

In many realistic applications, the source/control is placed outside the domain where a PDE is fulfilled. Some examples of problems where this *may be of relevance* are: (a) magnetic drug delivery: the drug with ferromagnetic particles is injected in the body and an external magnetic field is used to steer it to a desired location [8, 9, 40]; (b) acoustic testing: the aerospace structures are subjected to the sound from the loudspeakers [37].

The rest of the paper is organized as follows. We begin with Section 2 which introduces the notations and some preliminary results. The content of this section is well-known. Our main work starts from Section 3 where we first study the notion of weak and very weak solutions to the parabolic Dirichlet problem in Section 3.1. This is followed by the notion of weak solution to the Robin problem in Section 3.2. The emphasis of Section 4 is on the parabolic Dirichlet and the parabolic Robin optimal control problems. In Section 5, we discuss the approximation of the parabolic Dirichlet problem and the parabolic Dirichlet control problem by the parabolic

Robin ones. Finally, in Section 6 we discuss the numerical approximations of all these problems. The numerical experiments confirm our theoretical estimates. The experiments on the control/source identification problem illustrate the strength of the nonlocal approach over the local ones.

2. Notation and preliminaries

The purpose of this section is to introduce the notations and some preliminary results. The results of this section are well-known. We follow the notations from [11, 48]. Unless otherwise stated, $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded open set and $0 < s < 1$. Let

$$W^{s,2}(\Omega) := \left\{ u \in L^2(\Omega) : \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \, \mathrm{d}x\mathrm{d}y < \infty \right\}, \tag{2.1}$$

and we endow it with the norm defined by

$$\|u\|_{W^{s,2}(\Omega)} := \left(\int_\Omega |u|^2 \, \mathrm{d}x + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \, \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}}.$$

If $s = 1$, then we shall denote $W^{1,2}(\Omega) := \{u \in L^2(\Omega) : \ |\nabla u| \in L^2(\Omega)\}$ and $W_0^{1,2}(\Omega) := \overline{\mathcal{D}(\Omega)}^{W^{1,2}(\Omega)}$ by $H^1(\Omega)$ and $H_0^1(\Omega)$, respectively.

In order to study the Dirichlet problem (1.1b) we also shall need to define

$$W_0^{s,2}(\overline{\Omega}) := \left\{ u \in W^{s,2}(\mathbb{R}^N) : \ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \right\},$$

where $W^{s,2}(\mathbb{R}^N)$ is defined as in (2.1) with Ω replaced by $\mathbb{R}^N$. In this case

$$\|u\|_{W_0^{s,2}(\overline{\Omega})} := \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \, \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}}$$

defines an equivalent norm on $W_0^{s,2}(\overline{\Omega})$.

The dual spaces of $W^{s,2}(\mathbb{R}^N)$ and $W_0^{s,2}(\overline{\Omega})$ are denoted by $W^{-s,2}(\mathbb{R}^N)$ and $W^{-s,2}(\overline{\Omega})$, respectively. Moreover, $\langle \cdot, \cdot \rangle$ shall denote their duality pairing whenever it is clear from the context.

The local fractional order Sobolev space is defined as

$$W_{\text{loc}}^{s,2}(\mathbb{R}^N \setminus \Omega) := \left\{ u \in L_{\text{loc}}^2(\mathbb{R}^N \setminus \Omega) : \ u\varphi \in W^{s,2}(\mathbb{R}^N \setminus \Omega), \ \forall \ \varphi \in \mathcal{D}(\mathbb{R}^N \setminus \Omega) \right\}. \tag{2.2}$$

To study the Robin problem we shall need the following Sobolev space introduced in [26]. For $\kappa \in L^1(\mathbb{R}^N \setminus \Omega)$ fixed, we let

$$W_{\Omega,\kappa}^{s,2} := \left\{ u : \mathbb{R}^N \to \mathbb{R} \ \text{ measurable and } \ \|u\|_{W_{\Omega,\kappa}^{s,2}} < \infty \right\},$$

where

$$\|u\|_{W_{\Omega,\kappa}^{s,2}} := \left(\|u\|_{L^2(\Omega)}^2 + \||\kappa|^{\frac{1}{2}} u\|_{L^2(\mathbb{R}^N \setminus \Omega)}^2 + \int \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}}. \tag{2.3}$$

Let μ be the measure on $\mathbb{R}^N \setminus \Omega$ given by $\mathrm{d}\mu = |\kappa|\mathrm{d}x$. With this setting, the norm in (2.3) can be rewritten as

$$\|u\|_{W^{s,2}_{\Omega,\kappa}} := \left(\|u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\mathbb{R}^N\setminus\Omega,\mu)} + \int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}}. \tag{2.4}$$

If $\kappa = 0$, then we shall let $W^{s,2}_{\Omega,0} = W^{s,2}_{\Omega}$. The following result has been proved in ([26], Prop. 3.1).

Proposition 2.1. *Let $\kappa \in L^1(\mathbb{R}^N \setminus \Omega)$. Then $W^{s,2}_{\Omega,\kappa}$ is a Hilbert space.*

Next, for a Banach $\mathbb{X}$, we shall denote the vector-valued Banach spaces

$$H^1_{0,0}((0,T);\mathbb{X}) := \Big\{u \in H^1((0,T);\mathbb{X}) :\ u(0,\cdot) = 0\Big\},$$

and

$$H^1_{0,T}((0,T);\mathbb{X}) := \Big\{u \in H^1((0,T);\mathbb{X}) :\ u(T,\cdot) = 0\Big\},$$

and

$$H^1_{0}((0,T);\mathbb{X}) := \Big\{u \in H^1((0,T);\mathbb{X}) :\ u(0,\cdot) = u(T,\cdot) = 0\Big\}.$$

We notice that the continuous embedding $H^1((0,T);\mathbb{X}) \hookrightarrow C([0,T];\mathbb{X})$ holds, so that, for $u \in H^1((0,T);\mathbb{X})$, the values $u(0,\cdot)$ and $u(T,\cdot)$ make sense.

Finally, we are ready to introduce the fractional Laplace operator. We set

$$\mathbb{L}^1_s(\mathbb{R}^N) := \left\{u : \mathbb{R}^N \to \mathbb{R} \ \text{ measurable and } \ \int_{\mathbb{R}^N} \frac{|u(x)|}{(1+|x|)^{N+2s}}\,\mathrm{d}x < \infty\right\}.$$

For $u \in \mathbb{L}^1_s(\mathbb{R}^N)$ and $\varepsilon > 0$ we let

$$(-\Delta)^s_\varepsilon u(x) := C_{N,s}\int_{\{y\in\mathbb{R}^N, |y-x|>\varepsilon\}} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\mathrm{d}y, \ \ x \in \mathbb{R}^N,$$

where the normalization constant $C_{N,s}$ is given by

$$C_{N,s} := \frac{s2^{2s}\Gamma\left(\frac{2s+N}{2}\right)}{\pi^{\frac{N}{2}}\Gamma(1-s)}, \tag{2.5}$$

and Γ is the usual Euler Gamma function (see *e.g.* [17, 19–21, 24, 46, 47]). The **fractional Laplacian** $(-\Delta)^s$ is defined for $u \in \mathbb{L}^1_s(\mathbb{R}^N)$ by the formula

$$(-\Delta)^s u(x) = C_{N,s}\mathrm{P.V.}\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\mathrm{d}y = \lim_{\varepsilon\downarrow 0}(-\Delta)^s_\varepsilon u(x), \ \ x \in \mathbb{R}^N, \tag{2.6}$$

provided that the limit exists for a.e. $x \in \mathbb{R}^N$. It has been shown in ([18], Prop. 2.2) that for $u \in \mathcal{D}(\Omega)$, we have

$$\lim_{s\uparrow 1^-}\int_{\mathbb{R}^N} u(-\Delta)^s u\ \mathrm{d}x = \int_{\mathbb{R}^N} |\nabla u|^2\mathrm{d}x = -\int_{\mathbb{R}^N} u\Delta u\ \mathrm{d}x = -\int_{\Omega} u\Delta u\ \mathrm{d}x.$$

This is where the constant $C_{N,s}$ plays a crucial role.

Next, we define the operator $(-\Delta)_D^s$ in $L^2(\Omega)$ as follows.

$$D((-\Delta)_D^s) := \Big\{u|_\Omega :\ u \in W_0^{s,2}(\overline{\Omega}) \text{ and } (-\Delta)^s u \in L^2(\Omega)\Big\},\ \ (-\Delta)_D^s(u|_\Omega) := ((-\Delta)^s u)|_\Omega. \tag{2.7}$$

Then $(-\Delta)_D^s$ is the realization in $L^2(\Omega)$ of $(-\Delta)^s$ with the Dirichlet exterior condition $u = 0$ in $\mathbb{R}^N \setminus \Omega$. The following result is well-known (see *e.g.* [16, 44]).

Proposition 2.2. *The operator $(-\Delta)_D^s$ has a compact resolvent and $-(-\Delta)_D^s$ generates a strongly continuous submarkovian semigroup $(e^{-t(-\Delta)_D^s})_{t\geq 0}$ on $L^2(\Omega)$. The operator $(-\Delta)_D^s$ can be also viewed as a bounded operator from $W_0^{s,2}(\overline{\Omega})$ into $W^{-s,2}(\overline{\Omega})$. In this case $-(-\Delta)_D^s$ also generates a strongly continuous semigroup $(e^{-t(-\Delta)_D^s})_{t\geq 0}$ on $W^{-s,2}(\overline{\Omega})$.*

Next, for $u \in W_\Omega^{s,2}$ we define the nonlocal normal derivative $\mathcal{N}_s$ as follows:

$$\mathcal{N}_s u(x) := C_{N,s} \int_\Omega \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy, \quad x \in \mathbb{R}^N \setminus \overline{\Omega}. \tag{2.8}$$

We shall call $\mathcal{N}_s$ the *interaction operator*. Notice that the origin of the term "interaction" goes back to [27]. Clearly $\mathcal{N}_s$ is a nonlocal operator and it is well defined as shown the following result (see *e.g.* [29], Lem. 3.2).

Lemma 2.3. *The interaction operator $\mathcal{N}_s$ maps $W^{s,2}(\mathbb{R}^N)$ into $W^{s,2}_{\mathrm{loc}}(\mathbb{R}^N \setminus \Omega)$.*

Despite the fact that $\mathcal{N}_s$ is defined in $\mathbb{R}^N \setminus \Omega$, it is still known as the "normal" derivative. This is due to its similarity with the classical normal derivative (see *e.g.* [11], Prop. 2.2).

We conclude this section by stating the integration by parts formula for the fractional Laplacian.

Proposition 2.4 (The integration by parts formula for $(-\Delta)^s$). *Assume that Ω has a Lipschitz continuous boundary. Let $u \in W_\Omega^{s,2}$ be such that $(-\Delta)^s u \in L^2(\Omega)$ and $\mathcal{N}_s u \in L^2(\mathbb{R}^N \setminus \Omega)$. Then, for every $v \in W_\Omega^{s,2} \cap L^2(\mathbb{R}^N \setminus \Omega)$ we have*

$$\frac{C_{N,s}}{2} \int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dxdy = \int_\Omega v(-\Delta)^s u\, dx + \int_{\mathbb{R}^N\setminus\Omega} v\mathcal{N}_s u\, dx, \tag{2.9}$$

where $\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2 = (\Omega \times \Omega) \cup (\Omega \times (\mathbb{R}^N \setminus \Omega)) \cup ((\mathbb{R}^N \setminus \Omega) \times \Omega)$.

The proof of the preceding proposition is included in ([26], Lem. 3.2) for smooth functions. The version given here is obtained by using an approximation argument (see *e.g.* [39], Prop. 3.7).

3. The parabolic state equations

Before analyzing the optimal control problems (1.1) and (1.2), for a given function z, we shall focus on the Dirichlet (1.1b) and Robin (1.2b) exterior value problems. Throughout the remainder of the paper, we shall assume that Ω is a bounded domain with a Lipschitz continuous boundary. This regularity assumption is needed, firstly, to be able to extend functions in $W^{s,2}(\mathbb{R}^N \setminus \Omega)$ to functions in $W^{s,2}(\mathbb{R}^N)$ and secondly, to apply the integration by parts formula given in (2.9).

3.1. The parabolic Dirichlet problem for the fractional Laplacian

Let us consider first the following auxiliary problem:

$$\begin{cases} \partial_t w + (-\Delta)^s w &= f \quad \text{in } Q, \\ w &= 0 \quad \text{in } \Sigma, \\ w(0,\cdot) &= 0 \quad \text{in } \Omega, \end{cases} \tag{3.1}$$

that is, a fractional parabolic equation with a nonzero right-hand-side but a zero exterior condition. Notice that (3.1) can be rewritten as the following Cauchy problem:

$$\begin{cases} \partial_t w + (-\Delta)_D^s w = f & \text{in } Q, \\ w(0,\cdot) = 0 & \text{in } \Omega, \end{cases} \tag{3.2}$$

where we recall that $(-\Delta)_D^s$ is the operator defined in (2.7). Throughout this subsection $\langle\cdot,\cdot\rangle$ shall denote the duality pairing between $W^{-s,2}(\overline{\Omega})$ and $W_0^{s,2}(\overline{\Omega})$.

We next introduce our notion of weak solutions to (3.1).

Definition 3.1 (**Weak solutions: the homogeneous Dirichlet case).** Let $f \in L^2((0,T);W^{-s,2}(\overline{\Omega}))$. A function $w \in \mathbb{U}_0 := L^2((0,T);W_0^{s,2}(\overline{\Omega})) \cap H^1_{0,0}((0,T);W^{-s,2}(\overline{\Omega}))$ is said to be a weak solution to (3.1) if

$$\langle \partial_t w(t,\cdot), v\rangle + \frac{C_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(w(t,x)-w(t,y))(v(x)-v(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y = \langle f(t,\cdot), v\rangle,$$

for every $v \in W_0^{s,2}(\overline{\Omega})$ and almost every $t \in (0,T)$.

Remark 3.2. We mention the following facts.

(a) A weak solution to (3.1) belongs to $C([0,T],L^2(\Omega))$ (see *e.g.* [38], Rem. 9).
(b) If $f \in L^2((0,T);L^2(\Omega))$, then it has been shown in [16] (by using semigroup theory) that a weak solution to (3.1) enjoys the following regularity:

$$u \in C([0,T];D((-\Delta)_D^s)) \cap H^1_{0,0}((0,T);L^2(\Omega)).$$

The existence and uniqueness of weak solutions to (3.1) was shown in ([38], Thm. 26).

Proposition 3.3 (**Weak solutions to (3.1)).** *Let $f \in L^2((0,T);W^{-s,2}(\overline{\Omega}))$. Then there exists a unique weak solution $w \in \mathbb{U}_0$ to (3.1) in the sense of Definition 3.1 and is given by*

$$w(t,x) = \int_0^t e^{-(t-\tau)(-\Delta)_D^s} f(\tau,x)\,\mathrm{d}\tau,$$

where $(e^{-t(-\Delta)_D^s})_{t\geq 0}$ is the semigroup mentioned in Proposition 2.2. In addition, there is a constant $C>0$ such that

$$\|w\|_{\mathbb{U}_0} \leq C\|f\|_{L^2((0,T);W^{-s,2}(\overline{\Omega}))}. \tag{3.3}$$

We next introduce our notion of weak solutions to the nonhomogeneous problem (1.1b). Notice the higher regularity requirement on the datum z.

Definition 3.4 (**Weak solutions: the nonhomogenous Dirichlet case**)**.** Let $z \in H^1_{0,0}((0,T); W^{s,2}(\mathbb{R}^N \setminus \Omega))$ and $\tilde{z} \in H^1_{0,0}((0,T); W^{s,2}(\mathbb{R}^N))$ be such that $\tilde{z}|_\Sigma = z$. Then a function $u \in \mathbb{U} := L^2((0,T); W^{s,2}(\mathbb{R}^N)) \cap H^1_{0,0}((0,T); W^{-s,2}(\overline{\Omega}))$ is said to be a weak solution to (1.1b) if $u - \tilde{z} \in \mathbb{U}_0$ and

$$\langle \partial_t u(t,\cdot), v\rangle + \frac{C_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(t,x)-u(t,y))(v(x)-v(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y = 0,$$

for every $v \in W^{s,2}_0(\overline{\Omega})$ and almost every $t \in (0,T)$.

Throughout the following, without any specific mention we shall let

$$\mathbb{U}_0 := L^2((0,T); W^{s,2}_0(\overline{\Omega})) \cap H^1_{0,0}((0,T); W^{-s,2}(\overline{\Omega}))$$

and

$$\mathbb{U} := L^2((0,T); W^{s,2}(\mathbb{R}^N)) \cap H^1_{0,0}((0,T); W^{-s,2}(\overline{\Omega})).$$

Next, we show the well-posedness of (1.1b).

Theorem 3.5 (**Weak solutions to (1.1b)**)**.** *Let $z \in H^1_{0,0}((0,T); W^{s,2}(\mathbb{R}^N \setminus \Omega))$. Then there exists a unique weak solution $u \in \mathbb{U}$ to (1.1b). In addition, there is a constant $C > 0$ such that*

$$\|u\|_{\mathbb{U}} \le C\|z\|_{H^1((0,T);W^{s,2}(\mathbb{R}^N\setminus\Omega))}. \tag{3.4}$$

Proof. Before we proceed with the proof, we need some preparation. Let us first assume that z depends only on the spatial variable x and consider the s-Harmonic extension $\tilde{z} \in W^{s,2}(\mathbb{R}^N)$ of $z \in W^{s,2}(\mathbb{R}^N \setminus \Omega)$ that solves the following Dirichlet problem:

$$\begin{cases} (-\Delta)^s \tilde{z} = 0 & \text{in } \Omega,\\ \tilde{z} = z & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.5}$$

in a weak sense. That is, given $z \in W^{s,2}(\mathbb{R}^N \setminus \Omega)$, there exists a unique $\tilde{z} \in W^{s,2}(\mathbb{R}^N)$ such that $\tilde{z}|_{\mathbb{R}^N\setminus\Omega} = z$ and $\tilde{z}$ solves (3.5) in the sense that

$$\frac{C_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\tilde{z}(x)-\tilde{z}(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y = 0 \quad \text{for all } v \in W^{s,2}_0(\overline{\Omega}),$$

and there is a constant $C > 0$ such that

$$\|\tilde{z}\|_{W^{s,2}(\mathbb{R}^N)} \le C\|z\|_{W^{s,2}(\mathbb{R}^N\setminus\Omega)}. \tag{3.6}$$

The existence of a weak solution to (3.5) and the continuous dependence on the datum z have been shown in [33] (see also [29, 45]), under the assumption that Ω has a Lipschitz continuous boundary. If z is a function of (x,t) and belongs to $H^1_{0,0}((0,T); W^{s,2}(\mathbb{R}^N \setminus \Omega))$, then it follows from the above arguments that $\tilde{z} \in H^1_{0,0}((0,T); W^{s,2}(\mathbb{R}^N))$.

Next, we show the existence of a unique solution to (1.1b) by using a lifting argument. We define $w := u - \tilde{z}$. Then $w|_\Sigma = 0$. Moreover, a simple calculation shows that w fulfills

$$\begin{cases} \partial_t w + (-\Delta)^s w = -\partial_t \tilde{z} & \text{in } Q,\\ w = 0 & \text{in } \Sigma,\\ w(0,\cdot) = 0 & \text{in } \Omega. \end{cases} \tag{3.7}$$

Since $\partial_t z \in L^2((0,T);W^{s,2}(\mathbb{R}^N \setminus \Omega))$, it follows from the above discussion that $\partial_t \tilde{z} \in L^2((0,T);W^{s,2}(\mathbb{R}^N))$. Hence, using Proposition 3.3, we get that there exists a unique $w \in \mathbb{U}_0$ solving (3.7). Thus, the unique solution $u \in \mathbb{U}$ is given by $u = w + \tilde{z}$. It remains to show the estimate (3.4). Firstly, since $w = 0$ in Σ, it follows from (3.3) that there is a constant $C > 0$ such that

$$\|w\|_{\mathbb{U}} = \|w\|_{\mathbb{U}_0} \le C\|\partial_t \tilde{z}\|_{L^2((0,T);W^{-s,2}(\overline{\Omega}))}. \tag{3.8}$$

Secondly, it follows from (3.6) that there is a constant $C > 0$ such that

$$\|\tilde{z}\|_{L^2((0,T);W^{s,2}(\mathbb{R}^N))} \le C\|z\|_{L^2((0,T);W^{s,2}(\mathbb{R}^N\setminus\Omega))}. \tag{3.9}$$

Thirdly, using (3.8) and (3.9) we get that there is a constant $C > 0$ such that

$$\begin{aligned}
\|u\|_{\mathbb{U}} &= \|w + \tilde{z}\|_{\mathbb{U}} \le \|w\|_{\mathbb{U}} + \|\tilde{z}\|_{\mathbb{U}} \\
&\le C\left(\|\partial_t \tilde{z}\|_{L^2((0,T);W^{-s,2}(\overline{\Omega}))} + \|z\|_{L^2((0,T);W^{s,2}(\mathbb{R}^N\setminus\Omega))} + \|\tilde{z}\|_{H^1((0,T);W^{-s,2}(\overline{\Omega}))}\right) \\
&\le C\left(\|\partial_t \tilde{z}\|_{L^2((0,T);W^{-s,2}(\overline{\Omega}))} + \|z\|_{L^2((0,T);W^{s,2}(\mathbb{R}^N\setminus\Omega))} + \|\tilde{z}\|_{L^2((0,T);W^{-s,2}(\overline{\Omega}))}\right),
\end{aligned} \tag{3.10}$$

where in the last estimate we have used the fact that

$$\|\tilde{z}\|^2_{H^1((0,T);W^{-s,2}(\overline{\Omega}))} = \|\tilde{z}\|^2_{L^2((0,T);W^{-s,2}(\overline{\Omega}))} + \|\partial_t \tilde{z}\|^2_{L^2((0,T);W^{-s,2}(\overline{\Omega}))}.$$

Since $\tilde{z} \in L^2((0,T);W^{s,2}(\mathbb{R}^N))$, it follows from (3.6) that

$$\begin{aligned}
\|\tilde{z}\|_{L^2((0,T);W^{-s,2}(\overline{\Omega}))} &\le C\|\tilde{z}\|_{L^2((0,T);W^{-s,2}(\mathbb{R}^N))} \le C\|\tilde{z}\|_{L^2((0,T);W^{s,2}(\mathbb{R}^N))} \\
&\le C\|z\|_{L^2((0,T);W^{s,2}(\mathbb{R}^N\setminus\Omega))}.
\end{aligned} \tag{3.11}$$

Note that $\partial_t \tilde{z}$ is a solution of the Dirichlet problem (3.5) with z replaced with $\partial_t z$. This shows that $\partial_t \tilde{z} \in L^2((0,T);W^{s,2}(\mathbb{R}^N))$. Hence, using (3.6) again, we obtain that

$$\begin{aligned}
\|\partial_t \tilde{z}\|_{L^2((0,T);W^{-s,2}(\overline{\Omega}))} &\le C\|\partial_t \tilde{z}\|_{L^2((0,T);W^{-s,2}(\mathbb{R}^N))} \le C\|\partial_t \tilde{z}\|_{L^2((0,T);W^{s,2}(\mathbb{R}^N))} \\
&\le C\|\partial_t z\|_{L^2((0,T);W^{s,2}(\mathbb{R}^N\setminus\Omega))}.
\end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12) we get from (3.10) that

$$\|u\|_{\mathbb{U}} \le C\left(\|z\|_{L^2((0,T);W^{s,2}(\mathbb{R}^N\setminus\Omega))} + \|\partial_t z\|_{L^2((0,T);W^{s,2}(\mathbb{R}^N\setminus\Omega))}\right).$$

We have shown (3.4) and the proof is finished. □

Remark 3.6. *Let $(\varphi_n)_{n\in\mathbb{N}}$ be the orthonormal basis of eigenfunctions of $(-\Delta)^s_D$ associated with the eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$. If in Theorem 3.5 one assumes that $z \in H^1_0((0,T);W^{s,2}(\mathbb{R}^N \setminus \Omega))$, then it has been shown in ([48], Thm. 18) that the unique weak solution u of (1.1b) is given by*

$$u(t,x) = -\sum_{n=1}^{\infty}\left(\int_0^t (z(t-\tau,\cdot), \mathcal{N}_s\varphi_n)_{L^2(\mathbb{R}^N\setminus\Omega)}\, e^{-\lambda_n\tau}\, d\tau\right)\varphi_n(x).$$

Our next goal is to reduce the regularity requirements on the datum z in both space and time. We shall call the resulting solution u a very-weak solution.

Definition 3.7 (Very-weak solution: the nonhomogenous Dirichlet case). Let the function $z \in L^2((0,T);L^2(\mathbb{R}^N\setminus\Omega))$. A function $u \in L^2((0,T);L^2(\mathbb{R}^N))$ is said to be a very-weak solution to (1.1b) if the identity

$$\int_Q u\left(-\partial_t v + (-\Delta)^s v\right)\, dxdt = -\int_\Sigma z\mathcal{N}_s v \, dxdt, \tag{3.13}$$

holds for every $v \in L^2((0,T);V)\cap H^1_{0,T}((0,T);L^2(\Omega))$, where $V := \{v \in W_0^{s,2}(\overline{\Omega}) : \ (-\Delta)^s v \in L^2(\Omega)\}$.

Throughout the remainder of the article, without any mention we shall let

$$V := \{v \in W_0^{s,2}(\overline{\Omega}) : \ (-\Delta)^s v \in L^2(\Omega)\}.$$

The following result shows the existence and uniqueness of a very-weak solution to (1.1b) in the sense of Definition 3.7. We will prove this result by using a duality argument (see *e.g.* [31] for the case $s = 1$).

Theorem 3.8. *Let $z \in L^2((0,T);L^2(\mathbb{R}^N\setminus\Omega))$. Then there exists a unique very-weak solution u to (1.1b) according to Definition 3.7 that fulfills*

$$\|u\|_{L^2((0,T);L^2(\Omega))} \le C\|z\|_{L^2((0,T);L^2(\mathbb{R}^N\setminus\Omega))}, \tag{3.14}$$

for a constant $C > 0$. In addition, if $z \in H^1_{0,0}((0,T);W^{s,2}(\mathbb{R}^N\setminus\Omega))$, then the following assertions hold.

(a) Every weak solution of (1.1b) is also a very-weak solution.
(b) Every very-weak solution of (1.1b) that belongs to $\mathbb{U}$ is also a weak solution.

Proof. For a given $\zeta \in L^2((0,T);L^2(\Omega))$, we begin by considering the following "dual" problem:

$$\begin{cases} -\partial_t v + (-\Delta)^s v &= \zeta \quad \text{in } Q,\\ v &= 0 \quad \text{in } \Sigma,\\ v(T,\cdot) &= 0 \quad \text{in } \Omega. \end{cases} \tag{3.15}$$

We notice that in (3.15), it is not required that $\zeta(T,\cdot) = 0$ in Ω. Using semigroups theory as in Proposition 3.3 (see also Rem. 3.2), we have that the problem (3.15) has a unique weak solution $v \in L^2((0,T);V)\cap H^1_{0,T}((0,T);L^2(\Omega))$. Hence, $\partial_t v \in L^2(Q)$ and $(-\Delta)^s v \in L^2(Q)$.

Since $v \in L^2((0,T);V)\cap H^1_{0,T}((0,T);L^2(\Omega))$, we have that $\mathcal{N}_s v \in L^2((0,T);L^2(\mathbb{R}^N\setminus\Omega))$. We define the mapping

$$\mathcal{M} : L^2((0,T);L^2(\Omega)) \to L^2((0,T);L^2(\mathbb{R}^N\setminus\Omega)), \qquad \zeta \mapsto \mathcal{M}\zeta := -\mathcal{N}_s v.$$

We notice that $\mathcal{M}$ is linear and continuous because there is a constant $C > 0$ such that

$$\|\mathcal{M}\zeta\|_{L^2((0,T);L^2(\mathbb{R}^N\setminus\Omega))} = \|\mathcal{N}_s v\|_{L^2((0,T);L^2(\mathbb{R}^N\setminus\Omega))} \le C\|v\|_{L^2((0,T);W_0^{s,2}(\overline{\Omega}))} \le C\|\zeta\|_{L^2((0,T);L^2(\Omega))}.$$

Let $u := \mathcal{M}^* z$. Then we have

$$\int_Q u\zeta \, dxdt = \int_Q u\left(-\partial_t v + (-\Delta)^s v\right)\, dxdt = \int_Q (\mathcal{M}^* z)\zeta\, dxdt = -\int_\Sigma z\mathcal{N}_s v\, dxdt.$$

We have constructed a function $u \in L^2((0,T);L^2(\mathbb{R}^N))$ that solves (3.13). Next, we show the uniqueness of very-weak solutions. Assume that the system (1.1b) has two very weak-solutions u_1 and u_2 with the same exterior value z. Then, it follows from (3.13) that

$$\int_Q (u_1 - u_2)\left(-\partial_t v + (-\Delta)^s v\right)\, \mathrm{d}x\mathrm{d}t = 0,$$

for every $v \in L^2((0,T);V) \cap H^1_{0,T}((0,T);L^2(\Omega))$. Using the fundamental lemma of the calculus of variation, we can deduce from the preceding identity that $u_1 = u_2$ a.e. in Q. Since $u_1 = u_2$ a.e in Σ, we can conclude that $u_1 = u_2$ a.e. in $(0,T) \times \mathbb{R}^N$. We have shown the uniqueness of solutions.

Finally, we notice that there is a constant $C > 0$ such that

$$\left|\int_Q u\zeta \,\mathrm{d}x\mathrm{d}t\right| \le \|z\|_{L^2((0,T);L^2(\mathbb{R}^N\setminus\Omega))}\|\mathcal{N}_s v\|_{L^2((0,T);L^2(\mathbb{R}^N\setminus\Omega))} \le C\|z\|_{L^2((0,T);L^2(\mathbb{R}^N\setminus\Omega))}\|\zeta\|_{L^2((0,T);L^2(\Omega))}.$$

Dividing both sides of the preceding estimate by $\|\zeta\|_{L^2((0,T);L^2(\Omega))}$ and taking the supremum over $\zeta \in L^2((0,T);L^2(\Omega))$, we obtain (3.14).

Next, we prove the last two assertions of the theorem. Assume that $z \in H^1_{0,0}((0,T);W^{s,2}(\mathbb{R}^N \setminus \Omega))$.

(a) Let $u \in \mathbb{U} \hookrightarrow L^2((0,T);L^2(\mathbb{R}^N))$ be a weak solution to (1.1b). It follows from the definition that $u = z$ on Σ and in particular, we have that

$$\langle \partial_t u(t,\cdot), v\rangle + \frac{C_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(t,x)-u(t,y))(v(t,x)-v(t,y))}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y = 0, \tag{3.16}$$

for every $v \in L^2((0,T);V) \cap H^1_{0,T}((0,T);L^2(\Omega))$ and almost every $t \in (0,T)$. Since $v(t,\cdot) = 0$ in $\mathbb{R}^N \setminus \Omega$, we have that

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} &\frac{(u(t,x)-u(y,y))(v(t,x)-v(t,y))}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y \\ &= \int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2} \frac{(u(t,x)-u(t,y))(v(t,x)-v(t,y))}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y. \end{aligned}\tag{3.17}$$

Using (3.16), (3.17), the integration by parts formula (2.9) together with the fact that $u = z$ in $\mathbb{R}^N \setminus \Omega$, we get that

$$\begin{aligned}\langle \partial_t u(t,\cdot), v\rangle + \frac{C_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} &\frac{(u(t,x)-u(t,y))(v(t,x)-v(t,y))}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y \\ &= 0 \\ &= \langle \partial_t u(t,\cdot), v(t,\cdot)\rangle + \int_\Omega u(t,x)(-\Delta)^s v(t,x)\,\mathrm{d}x + \int_{\mathbb{R}^N\setminus\Omega} u(t,x)\mathcal{N}_s v(t,x)\,\mathrm{d}x \\ &= \langle \partial_t u(t,\cdot), v(t,\cdot)\rangle + \int_\Omega u(t,x)(-\Delta)^s v(t,x)\,\mathrm{d}x + \int_{\mathbb{R}^N\setminus\Omega} z(t,x)\mathcal{N}_s v(t,x)\,\mathrm{d}x. \end{aligned}$$

Integrating the previous identity by parts over $(0,T)$, we get that

$$-\int_0^T \langle u(t,\cdot), \partial_t v(t,\cdot)\rangle\,\mathrm{d}t + \int_Q u(-\Delta)^s v\,\mathrm{d}x\mathrm{d}t + \int_\Sigma z\mathcal{N}_s v\,\mathrm{d}x\mathrm{d}t = 0.$$

Since $u(t,\cdot)$, $\partial_t v(t\cdot)\in L^2(\Omega)$, it follows from the preceding identity that

$$\int_Q u\Big(-\partial_t v+(-\Delta)^s v\Big)\;dxdt=-\int_\Sigma z\mathcal{N}_s v\;dxdt$$

for every $v\in L^2((0,T);V)\cap H^1_{0,T}((0,T);L^2(\Omega))$. Thus, u is a very-weak solution of (1.1b).

(b) Let u be a very-weak solution to (1.1b) and assume that $u\in\mathbb{U}$. Then $u=z$ in Σ. Moreover, $z\in H^1_{0,0}((0,T);W^{s,2}(\mathbb{R}^N\setminus\Omega))$ and if $\tilde z\in H^1_{0,0}((0,T);W^{s,2}(\mathbb{R}^N))$ is such that $\tilde z|_\Sigma=z$, then clearly $u-\tilde z\in\mathbb{U}_0$. Since u is a very-weak solution to (1.1b), it follows from Definition 3.7 that for every $v\in L^2((0,T);V)\cap H^1_{0,T}((0,T);L^2(\Omega))$, we have

$$\int_Q u(-\partial_t v+(-\Delta)^s v)\;dx=-\int_\Sigma z\mathcal{N}_s v\;dx. \tag{3.18}$$

Since $u\in\mathbb{U}$, $v=0$ on Σ, using the integration by parts formula (2.9), we get that

$$\begin{aligned}
&\int_0^T\langle\partial_t u(t,\cdot),(t,\cdot)v\rangle\;dt+\int_0^T\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(t,x)-u(t,y))(v(t,x)-v(t,y))}{|x-y|^{N+2s}}\;dxdydt\\
&=\int_0^T\langle\partial_t u(t,\cdot),v(t,\cdot)\rangle\;dt+\int_0^T\int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2}\frac{(u(t,x)-u(t,y))(v(t,x)-v(t,y))}{|x-y|^{N+2s}}\;dxdydt\\
&=\int_Q u\Big(\partial_t v+(-\Delta)^s v\Big)\;dxdt+\int_\Sigma u\mathcal{N}_s v\;dxdt\\
&=\int_Q u\Big(\partial_t v+(-\Delta)^s v\Big)\;dxdt+\int_\Sigma z\mathcal{N}_s v\;dxdt.
\end{aligned}\tag{3.19}$$

It follows form (3.18) and (3.19) that for every $v\in L^2((0,T);V)\cap H^1_{0,T}((0,T);L^2(\Omega))$ we have the identity

$$\int_0^T\langle\partial_t u(t,\cdot),v(t,\cdot)\rangle\;dt+\int_0^T\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(t,x)-u(t,y))(v(t,x)-v(t,y))}{|x-y|^{N+2s}}\;dxdydt=0. \tag{3.20}$$

Since V is dense in $W^{s,2}_0(\overline{\Omega})$ and $L^2(\Omega)$ is dense in $W^{-s,2}(\overline{\Omega})$, it follows that (3.20) remains true for every $v\in L^2((0,T);W^{s,2}_0(\overline{\Omega}))\cap H^1_{0,T}((0,T);W^{-s,2}(\overline{\Omega}))$. Notice that $v(t,\cdot)\in W^{s,2}_0(\overline{\Omega})$ for a.e. $t\in(0,T]$. As a result, we have that the following pointwise formulation

$$\langle\partial_t u(t,\cdot),v\rangle\;+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(t,x)-u(t,y))(v(x)-v(y))}{|x-y|^{N+2s}}\;dxdy=0, \tag{3.21}$$

holds for every $v\in W^{s,2}_0(\overline{\Omega})$ and a.e. $t\in(0,T)$. We have shown that u is the unique weak solution to (1.1b) according to Definition 3.4 and the proof is complete. □

3.2. The parabolic Robin problem for the fractional Laplacian

In this section, we consider the Robin problem (1.2b). Let $W^{s,2}_{\Omega,\kappa}$ be the Banach space introduced in (2.3) and the measure μ on $\mathbb{R}^N\setminus\Omega$ given by $d\mu=|\kappa|dx=\kappa dx$, since we have assumed that κ is non-negative. In this subsection $\langle\cdot,\cdot\rangle$ shall denote the duality pairing between $(W^{s,2}_{\Omega,\kappa})^\star$ and $W^{s,2}_{\Omega,\kappa}$. Next, we introduce our notion of weak solutions to the Robin problem.

Definition 3.9. Let $z \in L^2((0,T);L^2(\mathbb{R}^N \setminus \Omega,\mu))$. A function $u \in L^2((0,T);W^{s,2}_{\Omega,\kappa}) \cap H^1_{0,0}((0,T);(W^{s,2}_{\Omega,\kappa})^\star)$ is said to be a weak solution of (1.2b) if the identity

$$\begin{aligned}\langle \partial_t u(t,\cdot), v\rangle + \int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2} \frac{(u(t,x)-u(t,y))(v(x)-v(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N\setminus\Omega} \kappa(x)u(t,x)v(x)\,\mathrm{d}x \\ = \int_{\mathbb{R}^N\setminus\Omega} \kappa(x)z(t,x)v(x)\,\mathrm{d}x, \end{aligned} \tag{3.22}$$

holds for every $v \in W^{s,2}_{\Omega,\kappa}$ and almost every $t \in (0,T)$.

Throughout the following, for $u, v \in W^{s,2}_{\Omega,\kappa}$ we shall denote

$$\mathcal{E}(u,v) := \frac{C_{N,s}}{2}\int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N\setminus\Omega} \kappa uv\,\mathrm{d}x.$$

Next, we show the existence and uniqueness of solutions.

Theorem 3.10. *Let $\kappa \in L^1(\mathbb{R}^N \setminus \Omega) \cap L^\infty(\mathbb{R}^N \setminus \Omega)$ be non-negative. Then for every $z \in L^2((0,T);L^2(\mathbb{R}^N \setminus \Omega,\mu))$, there exists a unique weak solution $u \in L^2((0,T);W^{s,2}_{\Omega,\kappa}) \cap H^1_{0,0}((0,T);(W^{s,2}_{\Omega,\kappa})^\star)$ of* (1.2b).

Proof. We prove the result in several steps.
Step 1. Define the operator A in $L^2(\Omega) \times L^2(\mathbb{R}^N \setminus \Omega,\mu)$ as follows:

$$\begin{cases} D(A) := \Big\{(u,0):\ u \in W^{s,2}_{\Omega,\kappa},\ (-\Delta)^s u \in L^2(\Omega),\ \mathcal{N}_s u \in L^2(\mathbb{R}^N \setminus \Omega,\mu)\Big\}, \\ A(u,0) = (-(-\Delta)^s u, -\mathcal{N}_s u - \kappa u)\,. \end{cases}$$

Let $(f,g) \in L^2(\Omega) \times L^2(\mathbb{R}^N \setminus \Omega,\mu)$. We claim that $(u,0) \in D(A)$ with $-A(u,0) = (f,g)$ if and only if

$$\mathcal{E}(u,v) = \int_\Omega fv\,\mathrm{d}x + \int_{\mathbb{R}^N\setminus\Omega} gv\,\mathrm{d}\mu, \tag{3.23}$$

for all $v \in W^{s,2}_{\Omega,\kappa}$. Indeed, we have that $(u,0) \in D(A)$ with $-A(u,0) = (f,g)$ if and only if u is a weak solution of the following elliptic problem:

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ \mathcal{N}_s u + \kappa u = \kappa g & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{3.24}$$

It has been shown in [11] (see also [39]) that u solves (3.24) if and only if (3.23) holds and the claim is proved.

Step 2. Firstly, let $\lambda > 0$ be a real number. We show that the operator $\lambda - A : D(A) \to L^2(\Omega) \times L^2(\mathbb{R}^N \setminus \Omega,\mu)$ is invertible. It is clear that for every $\lambda > 0$ there is a constant $\alpha > 0$ such that

$$\lambda \int_\Omega |u|^2\,\mathrm{d}x + \mathcal{E}(u,u) \geq \alpha \|u\|^2_{W^{s,2}_{\Omega,\kappa}} \tag{3.25}$$

for all $u \in W^{s,2}_{\Omega,\kappa}$. Hence, by Lax-Milgram's Theorem, for every $(f,g) \in L^2(\Omega) \times L^2(\mathbb{R}^N \setminus \Omega, \mu)$ there exists a unique $u \in W^{s,2}_{\Omega,\kappa}$ such that

$$\lambda \int_\Omega uv \, \mathrm{d}x + \mathcal{E}(u,v) = \int_\Omega fv \, \mathrm{d}x + \int_{\mathbb{R}^N \setminus \Omega} gv \, \mathrm{d}\mu, \tag{3.26}$$

for all $v \in W^{s,2}_{\Omega,\kappa}$. By Step 1, this means that there is a unique $u \in W^{s,2}_{\Omega,\kappa}$ with $(u,0) \in D(A)$ and

$$(\lambda - A)(u,0) = (\lambda u, 0) - A(u,0) = (f,g).$$

We have shown that $\lambda - A : D(A) \to L^2(\Omega) \times L^2(\mathbb{R}^N \setminus \Omega, \mu)$ is a bijection for every $\lambda > 0$.

Secondly, assume now that $f \leq 0$ a.e. in Ω and $g \leq 0$ μ-a.e. in $\mathbb{R}^N \setminus \Omega$. Let the function $(u,0) := (\lambda - A)^{-1}(f,g)$ and set $v := u^+ := \max\{u, 0\}$. It follows from [47] that $u^+ \in W^{s,2}_{\Omega,\kappa}$. Let $u^- := \max\{-u, 0\}$. Since

$$\begin{aligned}\Big(u^-(x) - u^-(y)\Big)\Big(u^+(x) - u^+(y)\Big) =& u^-(x)u^+(x) - u^-(x)u^+(y) - u^-(y)u^+(x) + u^-(y)u^+(y)\\ =& -u^-(x)u^+(y) - u^-(y)u^+(x) \leq 0,\end{aligned}$$

we have that $\mathcal{E}(u^-, u^+) \leq 0$. Hence,

$$\mathcal{E}(u,v) = \mathcal{E}(u^+ - u^-, u^+) = \mathcal{E}(u^+, u^+) - \mathcal{E}(u^-, u^+) \geq 0.$$

Then by (3.26), we have that

$$0 \leq \lambda \int_\Omega |u|^2 \, \mathrm{d}x + \mathcal{E}(u, u^+) = \int_\Omega fu^+ \, \mathrm{d}x + \int_{\mathbb{R}^N \setminus \Omega} gu^+ \, \mathrm{d}\mu \leq 0.$$

By (3.25) this implies that $u^+ = 0$, that is, $u \leq 0$ almost everywhere. We have shown that the resolvent $(\lambda - A)^{-1}$ is a positive operator. Since every positive linear operator is continuous (see *e.g.*, [13]), we can deduce that $(\lambda - A)$ is in fact invertible.

Thirdly, we have in particular shown that the operator A is closed since $-A$ is the operator associated with the closed form $\mathcal{E}$. Hence, $D(A)$ endowed with the graph norm is a Banach space and by definition of A, we have that $D(A) \subset W^{s,2}_{\Omega,\kappa} \times \{0\}$. Since both of these spaces are continuously embedded into $L^2(\Omega) \times L^2(\mathbb{R}^N \setminus \Omega, \mu)$, we can deduce from the closed graph theorem that $D(A)$ is continuously embedded into $W^{s,2}_{\Omega,\kappa} \times \{0\}$.

Step 3. Now since $L^2(\Omega) \times L^2(\mathbb{R}^N \setminus \Omega, \mu)$ is a Banach lattice with an order continuous norm and by Step 2 the operator A is resolvent positive, it follows from ([14], Thm. 3.11.7) that $-A$ generates a once integrated semigroup on $L^2(\Omega) \times L^2(\mathbb{R}^N \setminus \Omega, \mu)$. Hence, using the theory of integrated semigroups and abstract Cauchy problems studied in ([14], Sect. 3.11) and proceeding as in ([41], Sect. 2), we can deduce that for every $z \in L^2((0,T); L^2(\mathbb{R}^N \setminus \Omega, \mu))$, the problem (1.2b) has a unique weak solution. The proof is finished. □

We conclude this section by showing that if z is more regular in the time variable, then the existence of weak solutions can be easily proved without using the theory of integrated semigroups as in the proof of Theorem 3.10.

Proposition 3.11. *Let $\kappa \in L^1(\mathbb{R}^N \setminus \Omega) \cap L^\infty(\mathbb{R}^N \setminus \Omega)$. Then for every $z \in H^1_{0,0}((0,T); L^2(\mathbb{R}^N \setminus \Omega, \mu))$, there exists a unique weak solution $u \in L^2((0,T); W^{s,2}_{\Omega,\kappa}) \cap H^1_{0,0}((0,T); (W^{s,2}_{\Omega,\kappa})^\star)$ of (1.2b).*

Proof. We proceed as in the proof of Theorem 3.5. Firstly, assume that $z \in L^2(\mathbb{R}^N \setminus \Omega, \mu)$ does not depend on the time variable. Let $\tilde{z}$ be the solution of the following elliptic Robin problem:

$$\begin{cases} (-\Delta)^s \tilde{z} = 0 & \text{in } \Omega, \\ \mathcal{N}_s \tilde{z} + \kappa \tilde{z} = \kappa z & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.27}$$

in the sense that $\tilde{z} \in W^{s,2}_{\Omega,\kappa}$ and

$$\int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2} \frac{(\tilde{z}(x)-\tilde{z}(y))(v(x)-v(y))}{|x-y|^{N+2s}} \,\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N\setminus\Omega} \kappa\tilde{z}v \,\mathrm{d}x = \int_{\mathbb{R}^N\setminus\Omega} \kappa z v \,\mathrm{d}x, \tag{3.28}$$

for every $v \in W^{s,2}_{\Omega,\kappa}$. Under our assumptions, it has been shown in [11] that (3.27) has a unique solution $\tilde{z}$.

Secondly, assume that $z \in H^1_{0,0}((0,T); L^2(\mathbb{R}^N \setminus \Omega, \mu))$. Since in this case $\partial_t \tilde{z}$ will be a solution of (3.27) with z replaced by $\partial_t z$, we can deduce that (3.27) has a unique solution $\tilde{z} \in H^1_{0,0}((0,T); W^{s,2}_{\Omega,\kappa})$.

Consider the following parabolic problem with $w := u - \tilde{z}$:

$$\begin{cases} \partial_t w + (-\Delta)^s w = -\partial_t \tilde{z} & \text{in } Q, \\ \mathcal{N}_s w + \kappa w = 0 & \text{in } \Sigma, \\ w(0,\cdot) = 0 & \text{in } \Omega. \end{cases} \tag{3.29}$$

Let $(-\Delta)^s_R$ be the realization in $L^2(\Omega)$ of $(-\Delta)^s$ with the zero Robin exterior condition $\mathcal{N}_s w + \kappa w = 0$ in $\mathbb{R}^N \setminus \Omega$. We refer to [23] for a precise description of this operator. Then the parabolic problem (3.29) can be rewritten as the following Cauchy problem

$$\begin{cases} \partial_t w + (-\Delta)^s_R w = -\partial_t \tilde{z} & \text{in } Q, \\ w(0,\cdot) = 0 & \text{in } \Omega. \end{cases}$$

It has been shown in [23] (see also [39]) that the operator $-(-\Delta)^s_R$ generates a strongly continuous submarkovian semigroup $(e^{-t(-\Delta)^s_R})_{t\geq 0}$ in $L^2(\Omega)$. Hence, using semigroups theory, we can deduce that (3.29) has a unique weak solution w that belongs to $L^2((0,T); W^{s,2}_{\Omega,\kappa}) \cap H^1_{0,0}((0,T); (W^{s,2}_{\Omega,\kappa})^\star)$ and is given by

$$w(t,x) = -\int_0^t e^{-(t-\tau)(-\Delta)^s_R} \partial_\tau \tilde{z}(\tau,x) \,\mathrm{d}x.$$

It is clear that $u := w + \tilde{z}$ is the unique weak solution of (1.2b). The proof is finished. □

4. Exterior optimal control problems

The purpose of this section is to study the Dirichlet and Robin optimal control problems (1.1) and (1.2), respectively. These are the subjects investigated in Sections 4.1 and 4.2, respectively.

4.1. Fractional Dirichlet exterior control problem

We recall the function spaces Z_D and U_D given by

$$Z_D := L^2((0,T); L^2(\mathbb{R}^N \setminus \Omega)), \quad U_D := L^2((0,T); L^2(\Omega)).$$

Due to Theorem 3.8, the control-to-state (solution) map

$$S : Z_D \to U_D, \quad z \mapsto Sz =: u,$$

is well-defined, linear and continuous. Furthermore, for $z \in Z_D$, we have $u := Sz \in L^2((0,T); L^2(\mathbb{R}^N))$. Let $J : U_D \to \mathbb{R}$ and consider the reduced functional

$$\mathcal{J} : Z_D \to \mathbb{R}, \;\; z \mapsto \mathcal{J}(z) := \Big(J(Sz) + \frac{\xi}{2}\|z\|^2_{Z_D} \Big).$$

Then we can write the reduced Dirichlet exterior parabolic optimal control problem as follows:

$$\min_{z \in Z_{ad,D}} \mathcal{J}(z). \tag{4.1}$$

Next, we state the well-posedness result for (1.1) and equivalently for (4.1).

Theorem 4.1. *Let $Z_{ad,D}$ be a closed and convex subset of Z_D. Let either $\xi > 0$ with $J \geq 0$ or $Z_{ad,D}$ bounded and $J : U_D \to \mathbb{R}$ weakly lower-semicontinuous. Then there exists a solution $\bar{z}$ to (4.1) and equivalently to (1.1). If either J is convex and $\xi > 0$ or J is strictly convex and $\xi \geq 0$, then $\bar{z}$ is unique.*

Proof. The proof is based on the direct method or the Weierstrass theorem ([15], Thm. 3.2.1). We sketch the proof here for completeness. For the functional $\mathcal{J} : Z_{ad,D} \to \mathbb{R}$, it is possible to construct a minimizing sequence $\{z_n\}_{n\in\mathbb{N}}$ (see [15], Thm. 3.2.1) such that $\inf_{z\in Z_{ad,D}} \mathcal{J}(z) = \lim_{n\to\infty} \mathcal{J}(z_n)$. If $\xi > 0$ with $J \geq 0$ or $Z_{ad,D} \subset Z_D$ is bounded, then $\{z_n\}_{n\in\mathbb{N}}$ is a bounded sequence in Z_D which is a Hilbert space. As a result, we have that (up to a subsequence if necessary) $z_n \rightharpoonup \bar{z}$ (weak convergence) in Z_D as $n \to \infty$. Since $Z_{ad,D}$ is closed and convex, hence, is weakly closed, we have that $\bar{z} \in Z_{ad,D}$.

It remains to show that $(S\bar{z}, \bar{z})$ fullfills the state equation according to Definition 3.7 and $\bar{z}$ is a minimizer to (4.1). In order to show that $(S\bar{z}, \bar{z})$ fulfills the state equation, we need to focus on the identity

$$\int_Q u_n \left(-\partial_t v + (-\Delta)^s v \right) \, \mathrm{d}x\mathrm{d}t = -\int_\Sigma z_n \mathcal{N}_s v \, \mathrm{d}x\mathrm{d}t \tag{4.2}$$

for all $v \in L^2((0,T); V) \cap H^1_{0,T}((0,T); L^2(\Omega))$, as $n \to \infty$. Since (passing to a subsequence if necessary) $u_n := Sz_n \rightharpoonup S\bar{z} =: \bar{u}$ in U_D as $n \to \infty$, and $z_n \rightharpoonup \bar{z}$ in Z_D as $n \to \infty$, we can immediately take the limit in (4.2) as $n \to \infty$, and conclude that $(\bar{u}, \bar{z}) \in U_D \times Z_{ad,D}$ fulfills the state equation according to Definition 3.7.

Next, that $\bar{z}$ is the minimizer of (4.1) follows from the fact that $\mathcal{J}$ is weakly lower semicontinuous. Indeed, $\mathcal{J}$ is the sum of two weakly lower semicontinuous functions (recall that the norm is continuous and convex therefore weakly lower semicontinuous).

Finally, the uniqueness of $\bar{z}$ follows from the stated assumptions on J and ξ which leads to the strict convexity of the functional $\mathcal{J}$. The proof is finished. □

In order to derive the first order necessary optimality conditions, we need an expression of the adjoint operator S^*. We discuss this next. We notice that for every measurable set $E \subset \mathbb{R}^N$, we have that $L^2((0,T); L^2(E)) = L^2((0,T) \times E)$ with equivalent norms.

Lemma 4.2. *The adjoint operator $S^* : U_D \to Z_D$ for the state equation (1.1b) is given by*

$$S^* w = -\mathcal{N}_s p \in Z_D,$$

where $w \in U_D$ and $p \in L^2((0,T); W_0^{s,2}(\overline{\Omega})) \cap H^1_{0,T}((0,T); W^{-s,2}(\overline{\Omega}))$ is the weak solution to the following adjoint problem:

$$\begin{cases} -\partial_t p + (-\Delta)^s p & = w \quad \textit{in } Q, \\ p & = 0 \quad \textit{in } \Sigma, \\ p(T,\cdot) & = 0 \quad \textit{in } \Omega. \end{cases} \tag{4.3}$$

Proof. First of all, since S is linear and bounded, it follows that S^* is well-defined. Now for every $w \in U_D$ and $z \in Z_D$, we have that

$$(w, Sz)_{L^2((0,T);L^2(\Omega))} = (S^* w, z)_{L^2((0,T);L^2(\mathbb{R}^N \setminus \Omega))}.$$

We notice that using semigroups theory (see *e.g.* Rem. 3.2 and [16]) we have that $p \in L^2((0,T); V) \cap H^1_{0,T}((0,T); L^2(\Omega))$. Thus, $\partial_t p$, $(-\Delta)^s p \in L^2(Q)$. Next, testing the equation (4.3) with Sz which solves the state equation in the very-weak sense (*cf.* Def. 3.13) we get that

$$\begin{aligned} (w, Sz)_{L^2((0,T);L^2(\Omega))} &= (-\partial_t p + (-\Delta)^s p, Sz)_{L^2((0,T);L^2(\Omega))} \\ &= -(z, \mathcal{N}_s p)_{L^2((0,T);L^2(\mathbb{R}^N\setminus\Omega))} = (z, S^* w)_{L^2((0,T);L^2(\mathbb{R}^N\setminus\Omega))}, \end{aligned}$$

and the proof is complete. □

For the remainder of this section, we will assume that $\xi > 0$.

Theorem 4.3. *Let $\mathcal{Z} \subset Z_D$ be open such that $Z_{ad,D} \subset \mathcal{Z}$ and let the assumptions of Theorem 4.1 hold. Moreover, let $u \mapsto J(u) : U_D \to \mathbb{R}$ be continuously Fréchet differentiable with $J'(u) \in U_D$. If $\bar{z}$ is a minimizer of (4.1) over $Z_{ad,D}$, then the first order necessary optimality conditions are given by*

$$(-\mathcal{N}_s \bar{p} + \xi \bar{z}, z - \bar{z})_{L^2((0,T);L^2(\mathbb{R}^N\setminus\Omega))} \geq 0, \quad \forall z \in Z_{ad,D}, \tag{4.4}$$

where $\bar{p} \in L^2((0,T); W_0^{s,2}(\overline{\Omega})) \cap H^1_{0,T}((0,T); W^{-s,2}(\overline{\Omega}))$ solves the adjoint equation

$$\begin{cases} -\partial_t \bar{p} + (-\Delta)^s \bar{p} = J'(\bar{u}) & \textit{in } Q, \\ \bar{p} = 0 & \textit{in } \Sigma, \\ \bar{p}(T,\cdot) = 0 & \textit{in } \Omega, \end{cases} \tag{4.5}$$

with $\bar{u} := S\bar{z}$. Finally, (4.4) is equivalent to

$$\bar{z} = \mathcal{P}_{Z_{ad},D}\left(\xi^{-1} \mathcal{N}_s \bar{p}\right), \tag{4.6}$$

where $\mathcal{P}_{Z_{ad},D}$ is the projection onto the set $Z_{ad,D}$. Moreover, if J is convex, then (4.4) is a sufficient condition.

Proof. The statements are a direct consequence of the differentiability properties of J and the chain rule, combined with Lemma 4.2. Notice that, we have introduced the open set $\mathcal{Z}$ to properly define the derivative of $\mathcal{J}$. Let $h \in \mathcal{Z}$ be given. Then the directional derivative of $\mathcal{J}$ is given by

$$\begin{aligned} \mathcal{J}'(\bar{z})h &= (J'(S\bar{z}), Sh)_{L^2((0,T);L^2(\Omega))} + \xi(\bar{z}, h)_{L^2((0,T);L^2(\mathbb{R}^N\setminus\Omega))} \\ &= (S^* J'(S\bar{z}) + \xi \bar{z}, h)_{L^2((0,T);L^2(\Omega))}, \end{aligned} \tag{4.7}$$

where we have used that $J'(S\bar{z}) \in \mathcal{L}(L^2((0,T);L^2(\Omega)),\mathbb{R}) = L^2((0,T);L^2(\Omega))$. Next from Lemma 4.2, we have that

$$S^* J'(S\bar{z}) = -\mathcal{N}_s\bar{p},$$

where $\bar{p}$ solves (4.5). Recall that $\bar{p} \in L^2((0,T);W_0^{s,2}(\overline{\Omega})) \cap H^1_{0,T}((0,T);W^{-s,2}(\overline{\Omega}))$ solving (4.5) also has the following regularity: $\partial_t\bar{p} \in L^2((0,T);L^2(\Omega))$ and this implies that $(-\Delta)^s\bar{p} \in L^2((0,T);L^2(\Omega))$. This implies that $\mathcal{N}_s\bar{p} \in L^2((0,T);L^2(\mathbb{R}^N \setminus \Omega))$. Substituting this expression of $S^*J'(S\bar{z})$ in (4.7), we obtain that

$$\mathcal{J}'(\bar{z})h = (-\mathcal{N}_s\bar{p} + \xi\bar{z}, h)_{L^2((0,T);L^2(\Omega))}.$$

The remainder of the steps to obtain (4.4) are standard, see for instance [3, 34].

Finally, (4.6) follows by using ([15], Thm. 3.3.5]). The proof is finished. □

4.2. Fractional Robin optimal control problem

Next, we shall focus on the Robin optimal control problem (1.2). Recall that

$$Z_R := L^2((0,T);L^2(\mathbb{R}^N \setminus \Omega,\mu)), \quad U_R := L^2((0,T);W^{s,2}_{\Omega,\kappa}) \cap H^1_{0,0}((0,T);(W^{s,2}_{\Omega,\kappa})^\star).$$

Recall also that $\mathrm{d}\mu = \kappa \mathrm{d}x$ with $\kappa \in L^1(\mathbb{R}^N \setminus \Omega) \cap L^\infty(\mathbb{R}^N \setminus \Omega)$ and is non-negative. Due to Theorem 3.10, the following control-to-state (solution) map

$$S : Z_R \to U_R, \quad z \mapsto Sz =: u,$$

is well-defined. In addition, S is linear and continuous. Owing to the continuous embedding $U_R \hookrightarrow L^2((0,T);L^2(\Omega))$, we can instead define

$$S : Z_R \to L^2((0,T);L^2(\Omega)).$$

Letting

$$\mathcal{J}(z) : Z_R \to \mathbb{R},\ z \mapsto \mathcal{J}(z) := \Big(J(Sz) + \frac{\xi}{2}\|z\|^2_{Z_R}\Big),$$

then the reduced Robin exterior parabolic optimal control problem is given by

$$\min_{z \in Z_{ad,R}} \mathcal{J}(z). \tag{4.8}$$

The following well-posedness result holds.

Theorem 4.4. *Let $Z_{ad,R}$ be a convex and closed subset of Z_R and let either $\xi > 0$ with $J \geq 0$ or $Z_{ad,R} \subset Z_R$ bounded. If $J : L^2((0,T);L^2(\Omega)) \to \mathbb{R}$ is weakly lower-semicontinuous, then there exists a solution $\bar{z}$ to (4.8) and equivalently to (1.2). If either J is convex and $\xi > 0$ or J is strictly convex and $\xi \geq 0$, then $\bar{z}$ is unique.*

Proof. The proof is similar to the proof of Theorem 4.1. We only discuss the part where $\{z_n\}_{n\in\mathbb{N}}$ is a minimizing sequence such that, passing to a subsequence if necessary, $z_n \rightharpoonup \bar{z}$ in $L^2((0,T);L^2(\mathbb{R}^N \setminus \Omega,\mu))$ as $n \to \infty$. Let (Sz_n, z_n), $n \in \mathbb{N}$, be the solution of (1.2b). We need to show that there is a subsequence which converges to $(S\bar{z}, \bar{z})$

in $L^2((0,T);W^{s,2}_{\Omega,\kappa})\cap H^1_{0,0}((0,T);(W^{s,2}_{\Omega,\kappa})^\star)$ as $n\to\infty$ and $(S\bar{z},\bar{z})$ solves (1.2b) in the weak sense (*cf.* Def. 3.9). Since $u_n := Sz_n \in L^2((0,T);W^{s,2}_{\Omega,\kappa})\cap H^1_{0,0}((0,T);(W^{s,2}_{\Omega,\kappa})^\star)$ solves (1.2b), we have that the identity

$$\langle \partial_t u_n(t,\cdot),v\rangle + \mathcal{E}(u_n(t,\cdot),v) = \int_{\mathbb{R}^N\setminus\Omega} z_n(t,x)v(x)\,\mathrm{d}\mu, \tag{4.9}$$

holds for every $v\in W^{s,2}_{\Omega,\kappa}$ and a.e. $t\in(0,T)$, where $\mathcal{E}$ is as defined in (3.28). We note that the mapping S is bounded due to Theorem 3.10. As a result, passing to a subsequence if necessary, we have that $Sz_n = u_n \rightharpoonup S\bar{z}=\bar{u}$ in $L^2((0,T);W^{s,2}_{\Omega,\kappa})\cap H^1_{0,0}((0,T);(W^{s,2}_{\Omega,\kappa})^\star)$ as $n\to\infty$. Then, taking the limit as $n\to\infty$ in (4.9) we get that

$$\langle \partial_t \bar{u}(t,\cdot),v\rangle + \mathcal{E}(\bar{u}(t,\cdot),v) = \int_{\mathbb{R}^N\setminus\Omega} \bar{z}(t,x)v(x)\,\mathrm{d}\mu.$$

That is, $(S\bar{z},\bar{z})$ solves (1.2b) in the weak sense (*cf.* Def. 3.9). The proof is finished. □

As in the previous section, before we state the first order optimality conditions, we shall derive the expression of the adjoint operator S^*.

Lemma 4.5. *The adjoint operator $S^*: L^2((0,T);L^2(\Omega))\to Z_R$ is given by*

$$(S^*w,z)_{Z_R} = \int_\Sigma pz\,\mathrm{d}\mu\mathrm{d}t \quad \forall z\in Z_R,$$

where $w\in L^2((0,T);L^2(\Omega))$ and $p\in L^2((0,T);W^{s,2}_{\Omega,\kappa})\cap H^1_{0,T}((0,T);(W^{s,2}_{\Omega,\kappa})^\star)$ is the weak solution to

$$\begin{cases} -\partial_t p + (-\Delta)^s p = w & \text{in } Q,\\ \mathcal{N}_s p + \kappa p = 0 & \text{in } \Sigma,\\ p(T,\cdot)=0 & \text{in } \Omega. \end{cases} \tag{4.10}$$

Proof. Let $w\in L^2((0,T);L^2(\Omega))$ and $z\in Z_R$. Since $Sz\in L^2((0,T);W^{s,2}_{\Omega,\kappa})\cap H^1((0,T);(W^{s,2}_{\Omega,\kappa})^\star)\subset L^2((0,T);L^2(\Omega))$, we can write

$$(w,Sz)_{L^2((0,T);L^2(\Omega))} = (S^*w,z)_{Z_R}.$$

Furthermore, testing (4.10) with $Sz=u$ we obtain that

$$\begin{aligned}(w,Sz)_{L^2((0,T);L^2(\Omega))} &= (-\partial_t p + (-\Delta)^s p, Sz)_{L^2((0,T);L^2(\Omega))}\\ &= \int_\Sigma zp\,\mathrm{d}\mu\mathrm{d}t = (S^*w,z)_{Z_R},\end{aligned}$$

where we have used the integration-by-parts in both space and time and the fact that $Sz=u$ solves the state equation according to Definition 3.9. The proof is complete. □

We conclude this section with the following first order optimality conditions result whose proof is similar to the Dirichlet case and is omitted for brevity. We shall assume that $\xi>0$.

Theorem 4.6. *Let $\mathcal{Z}\subset Z_R$ be open such that $Z_{ad,R}\subset\mathcal{Z}$ and let the assumptions of Theorem 4.4 hold. Let $u\mapsto J(u): L^2((0,T);L^2(\Omega))\to\mathbb{R}$ be continuously Fréchet differentiable with $J'(u)\in L^2((0,T);L^2(\Omega))$. If $\bar{z}$ is* *a minimizer of* (4.8)*, then the first order necessary optimality conditions are given by*

$$\int_{\Sigma} (\bar{p} + \xi \bar{z})(z - \bar{z}) \, \mathrm{d}\mu \mathrm{d}t \geq 0, \quad z \in Z_{ad,R} \tag{4.11}$$

where $\bar{p} \in L^2((0,T); W^{s,2}_{\Omega,\kappa}) \cap H^1_{0,T}((0,T); (W^{s,2}_{\Omega,\kappa})^\star)$ *solves the following adjoint equation:*

$$\begin{cases} -\partial_t \bar{p} + (-\Delta)^s \bar{p} = J'(\bar{u}) & \textit{in } Q, \\ \mathcal{N}_s \bar{p} + \kappa \bar{p} = 0 & \textit{in } \Sigma, \\ \bar{p}(T, \cdot) = 0 & \textit{in } \Omega. \end{cases} \tag{4.12}$$

Moreover, (4.11) *is equivalent to*

$$\bar{z} = \mathcal{P}_{Z_{ad,R}}(-\xi^{-1} \bar{p})$$

where $\mathcal{P}_{Z_{ad,R}}$ *is the projection onto the set* $Z_{ad,R}$*. If* J *is convex, then* (4.11) *is also sufficient.*

5. Approximation of the Dirichlet exterior value and control problems

Recall that the Dirichlet control problem requires approximations of the nonlocal normal derivative of the test functions (*cf.* (3.13)) and the solutions of the adjoint system (*cf.* (4.4)). The Nonlocal normal derivative $\mathcal{N}_s$ is a delicate object to handle both at the continuous level and at the discrete level. Indeed, the best known regularity result for $\mathcal{N}_s u$ is as given in Lemma 2.3. Moreover, a numerical approximation of this object is a daunting task. In order to circumvent the approximations of $\mathcal{N}_s u$ both in (3.13) and (4.4), in this section we propose to approximate the parabolic Dirichlet problem by the following parabolic Robin problem.

Let $n \in \mathbb{N}$. In this section we are interested in solutions u_n to the following parabolic Robin problem:

$$\begin{cases} \partial_t u_n + (-\Delta)^s u_n = 0 & \text{in } Q, \\ \mathcal{N}_s u_n + n\kappa u_n = n\kappa z & \text{in } \Sigma, \\ u_n(0, \cdot) = 0 & \text{in } \Omega, \end{cases} \tag{5.1}$$

that belong to the space $L^2((0,T); W^{s,2}_{\Omega,\kappa} \cap L^2(\mathbb{R}^N \setminus \Omega)) \cap H^1_{0,0}((0,T); (W^{s,2}_{\Omega,\kappa} \cap L^2(\mathbb{R}^N \setminus \Omega))^\star)$. Notice that $W^{s,2}_{\Omega,\kappa} \cap L^2(\mathbb{R}^N \setminus \Omega)$ is endowed with the norm

$$\|u\|_{W^{s,2}_{\Omega,\kappa} \cap L^2(\mathbb{R}^N \setminus \Omega)} := \left(\|u\|^2_{W^{s,2}_{\Omega,\kappa}} + \|u\|^2_{L^2(\mathbb{R}^N \setminus \Omega)} \right)^{\frac{1}{2}}, \quad u \in W^{s,2}_{\Omega,\kappa} \cap L^2(\mathbb{R}^N \setminus \Omega). \tag{5.2}$$

Moreover, in our application we shall take κ such that its support $\mathrm{supp}[\kappa]$ has a positive Lebesgue measure. Thus, we make the following assumption.

Assumption 5.1. We assume that $\kappa \in L^1(\mathbb{R}^N \setminus \Omega) \cap L^\infty(\mathbb{R}^N \setminus \Omega)$ and $\kappa > 0$ almost everywhere in $K := \mathrm{supp}[\kappa] \subset \mathbb{R}^N \setminus \Omega$, and the Lebesgue measure $|K| > 0$.

It follows from Assumption 5.1 that $\displaystyle\int_{\mathbb{R}^N \setminus \Omega} \kappa \, \mathrm{d}x > 0$.

We recall that a solution to (5.1) belongs to $L^2((0,T); W^{s,2}_{\Omega,\kappa}) \cap H^1_{0,0}((0,T); (W^{s,2}_{\Omega,\kappa})^\star)$ (this follows from Prop. 3.11). In order to show that this solution also belongs to $L^2((0,T); W^{s,2}_{\Omega,\kappa} \cap L^2(\mathbb{R}^N \setminus \Omega)) \cap H^1_{0,0}((0,T); (W^{s,2}_{\Omega,\kappa} \cap L^2(\mathbb{R}^N \setminus \Omega))^\star)$, we recall a result from ([11], Lem. 6.2).

Lemma 5.2 (**[11], Lem. 6.2**). *Assume that Assumption 5.1 holds. Then*

$$\|u\|_W := \left(\int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N\setminus\Omega} |u|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \tag{5.3}$$

defines an equivalent norm on $W^{s,2}_{\Omega,\kappa} \cap L^2(\mathbb{R}^N \setminus \Omega)$.

We are now ready to state the main result of this section whose proof is motivated by the elliptic case studied by the authors in [11].

Theorem 5.3 (**Approximation of weak solutions to the Dirichlet problem**). *Let Assumption 5.1 hold. Then the following assertions hold.*

(a) Let $z \in H^1_{0,0}((0,T); W^{s,2}(\mathbb{R}^N \setminus \Omega))$ *and* $u_n \in L^2((0,T); W^{s,2}_{\Omega,\kappa} \cap L^2(\mathbb{R}^N \setminus \Omega)) \cap H^1_{0,0}((0,T); (W^{s,2}_{\Omega,\kappa} \cap L^2(\mathbb{R}^N \setminus \Omega))^\star)$ *be the weak solution of (5.1). Let* $u \in \mathbb{U}$ *be the weak solution to the state equation (1.1b). Then there is a constant* $C > 0$ *(independent of* n*) such that*

$$\|u - u_n\|_{L^2((0,T);L^2(\mathbb{R}^N))} \le \frac{C}{n}\|u\|_{L^2((0,T);W^{s,2}(\mathbb{R}^N))}. \tag{5.4}$$

In particular, u_n *converges strongly to* u *in* $L^2((0,T); L^2(\Omega))$ *as* $n \to \infty$.

(b) Let $z \in L^2((0,T); L^2(\mathbb{R}^N \setminus \Omega))$ *and* $u_n \in L^2((0,T); W^{s,2}_{\Omega,\kappa} \cap L^2(\mathbb{R}^N \setminus \Omega)) \cap H^1_{0,0}((0,T); (W^{s,2}_{\Omega,\kappa} \cap L^2(\mathbb{R}^N \setminus \Omega))^\star)$ *be the weak solution of (5.1). Then there is a subsequence that we still denote by* $\{u_n\}_{n\in\mathbb{N}}$ *and a* $\tilde{u} \in L^2((0,T); L^2(\mathbb{R}^N))$ *such that* $u_n \rightharpoonup \tilde{u}$ *in* $L^2((0,T); L^2(\mathbb{R}^N))$ *as* $n \to \infty$*, and* $\tilde{u}$ *satisfies*

$$\int_Q \tilde{u}\Big(-\partial_t v + (-\Delta)^s v\Big)\,\mathrm{d}x\mathrm{d}t = -\int_\Sigma \tilde{u}\mathcal{N}_s v\,\mathrm{d}x\mathrm{d}t, \tag{5.5}$$

for all $v \in L^2((0,T); V) \cap H^1_{0,T}((0,T); L^2(\Omega))$.

Proof. (a) We begin by discussing the well-posedness of (5.1). We first notice that under our assumption, we have that $W^{s,2}(\mathbb{R}^N \setminus \Omega) \hookrightarrow L^2(\mathbb{R}^N \setminus \Omega) \hookrightarrow L^2(\mathbb{R}^N \setminus \Omega, \mu)$. Now a weak solution $u_n \in L^2((0,T); W^{s,2}_{\Omega,\kappa} \cap L^2(\mathbb{R}^N \setminus \Omega)) \cap H^1_{0,0}((0,T); (W^{s,2}_{\Omega,\kappa} \cap L^2(\mathbb{R}^N \setminus \Omega))^\star)$ to (5.1) fulfills the identity

$$\begin{aligned}\langle \partial_t u_n(t,\cdot), v\rangle + \frac{C_{N,s}}{2}\int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2} \frac{(u_n(t,x)-u_n(t,y))(v(x)-v(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y \\ + n\int_{\mathbb{R}^N\setminus\Omega} u_n(t,x)v(x)\,\mathrm{d}\mu = n\int_{\mathbb{R}^N\setminus\Omega} z(t,x)v(x)\,\mathrm{d}\mu,\end{aligned} \tag{5.6}$$

for every $v \in W^{s,2}_{\Omega,\kappa} \cap L^2(\mathbb{R}^N \setminus \Omega)$ and almost every $t \in (0,T)$. For every $n \in \mathbb{N}$, the existence of a unique solution u_n to (5.1) follows by using the arguments of Proposition 3.11.

Next, we prove the estimate (5.4). For $v, w \in W^{s,2}_{\Omega,\kappa} \cap L^2(\mathbb{R}^N \setminus \Omega)$, we shall let

$$\mathcal{E}_n(v,w) := \frac{C_{N,s}}{2}\int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2} \frac{(v(x)-v(y))(w(x)-w(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y + n\int_{\mathbb{R}^N\setminus\Omega} vw\,\mathrm{d}\mu. \tag{5.7}$$

It is not difficult to see (*cf.* [11], Eq. (6.17)) that there is a constant $C > 0$ (independent of n) such that

$$\frac{C_{N,s}}{2}\int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2} \frac{|u_n(t,x)-u_n(t,y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y + n\int_{\mathbb{R}^N\setminus\Omega} |u_n(t,x)|^2\,\mathrm{d}x \le C\mathcal{E}_n(u_n(t,\cdot),u_n(t,\cdot)). \tag{5.8}$$

Next, let $u \in \mathbb{U}$ be the weak solution to the Dirichlet problem (1.1b) according to Definition 3.4 and let $v \in W^{s,2}_{\Omega,\kappa} \cap L^2(\mathbb{R}^N \setminus \Omega)$. Using the integration by parts formula (2.9) we get that

$$
\begin{aligned}
&\langle \partial_t (u-u_n)(t,\cdot), v\rangle + \mathcal{E}_n((u-u_n)(t,\cdot), v) \\
&= \int_\Omega \Big(\partial_t (u-u_n)(t,x) + (-\Delta)^s (u-u_n)(t,x)\Big) v \, \mathrm{d}x + \int_{\mathbb{R}^N\setminus\Omega} \mathcal{N}_s (u-u_n)(t,x) v(x)\, \mathrm{d}x \\
&\quad + n \int_{\mathbb{R}^N\setminus\Omega} (u-u_n)(t,x) v(x)\, \mathrm{d}\mu \\
&= \int_\Omega \left(\partial_t (u-u_n)(t,x) + (-\Delta)^s (u-u_n)(t,x)\right) v(x) \, \mathrm{d}x + \int_{\mathbb{R}^N\setminus\Omega} v(x) \mathcal{N}_s u(t,x)\, \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N\setminus\Omega} \left(\mathcal{N}_s u_n(t,x) + n\kappa(x)(u_n - z)\right)(t,x) v(x)\, \mathrm{d}x \\
&= \int_{\mathbb{R}^N\setminus\Omega} v(x) \mathcal{N}_s u(t,x)\, \mathrm{d}x, \qquad (5.9)
\end{aligned}
$$

where we have used that

$$
\partial_t (u-u_n) + (-\Delta)^s (u-u_n) = 0 \ \text{ in } Q \ \text{ and } \ \mathcal{N}_s u_n + n\kappa(u_n - z) = 0 \text{ in } \ \Sigma,
$$

which follows from the fact that u is a solution to the Dirichlet problem (1.1b) and u_n a solution of (5.1). Letting $v := (u-u_n)(t,\cdot)$ in (5.9) and using (5.8), we can conclude that there is a constant $C > 0$ (independent of n) such that

$$
\begin{aligned}
&C\langle \partial_t (u-u_n)(t,\cdot), (u-u_n)(t,\cdot)\rangle + n\|(u-u_n)(t,\cdot)\|^2_{L^2(\mathbb{R}^N\setminus\Omega)} \\
&\quad \le C\Big(\langle \partial_t (u-u_n)(t,\cdot), (u-u_n)(t,\cdot)\rangle + \mathcal{E}_n((u-u_n)(t,\cdot), (u-u_n)(t,\cdot))\Big) \\
&\quad = C\int_{\mathbb{R}^N\setminus\Omega} (u-u_n)(t,x) \mathcal{N}_s u(t,x)\, \mathrm{d}x \\
&\quad \le C\|(u-u_n)(t,\cdot)\|_{L^2(\mathbb{R}^N\setminus\Omega)} \|\mathcal{N}_s u(t,\cdot)\|_{L^2(\mathbb{R}^N\setminus\Omega)} \\
&\quad \le C\|(u-u_n)(t,\cdot)\|_{L^2(\mathbb{R}^N\setminus\Omega)} \|u(t,\cdot)\|_{W^{s,2}(\mathbb{R}^N)} \\
&\quad \le \frac{n}{2}\|(u-u_n)(t,\cdot)\|^2_{L^2(\mathbb{R}^N\setminus\Omega)} + \frac{C^2}{2n}\|u(t,\cdot)\|^2_{W^{s,2}(\mathbb{R}^N)}.
\end{aligned}
$$

Hence,

$$
C\langle \partial_t (u-u_n)(t,\cdot), (u-u_n)(t,\cdot)\rangle + \frac{n}{2}\|(u-u_n)(t,\cdot)\|^2_{L^2(\mathbb{R}^N\setminus\Omega)} \le \frac{C}{n}\|u(t,\cdot)\|^2_{W^{s,2}(\mathbb{R}^N)},
$$

where we have replaced the constant C^2 by C. Since

$$
\langle \partial_t (u-u_n)(t,\cdot), (u-u_n)(t,\cdot)\rangle = \frac{1}{2}\partial_t \|(u-u_n)(t,\cdot)\|^2_{L^2(\mathbb{R}^N\setminus\Omega)},
$$

it follows that

$$
\frac{C}{2}\partial_t \|(u-u_n)(t,\cdot)\|^2_{L^2(\mathbb{R}^N\setminus\Omega)} + \frac{n}{2}\|(u-u_n)(t,\cdot)\|^2_{L^2(\mathbb{R}^N\setminus\Omega)} \le \frac{C}{n}\|u(t,\cdot)\|^2_{W^{s,2}(\mathbb{R}^N)}.
$$

Thus,

$$\frac{C}{2}\|(u-u_n)(t,\cdot)\|^2_{L^2(\mathbb{R}^N\setminus\Omega)}+\frac{n}{2}\int_0^t\|(u-u_n)(\tau,\cdot)\|^2_{L^2(\mathbb{R}^N\setminus\Omega)}\,d\tau\le\frac{C}{n}\int_0^t\|u(\tau,\cdot)\|^2_{W^{s,2}(\mathbb{R}^N)}\,\mathrm{d}\tau$$

which implies that

$$\begin{cases}\|u-u_n\|_{L^\infty((0,T);L^2(\mathbb{R}^N\setminus\Omega))}\le\frac{C}{\sqrt{n}}\|u\|_{L^2((0,T);W^{s,2}(\mathbb{R}^N))}\\[2ex] \|u-u_n\|_{L^2((0,T);L^2(\mathbb{R}^N\setminus\Omega))}\le\frac{C}{n}\|u\|_{L^2((0,T);W^{s,2}(\mathbb{R}^N))}.\end{cases}\tag{5.10}$$

In order to obtain (5.4), it remains to estimate $\|u-u_n\|_{L^2((0,T);L^2(\Omega))}$.

We notice that $L^2((0,T);L^2(\Omega))=L^2((0,T)\times\Omega)$ with equivalent norms and

$$\|u-u_n\|_{L^2((0,T);L^2(\Omega))}=\sup_{\eta\in L^2((0,T);L^2(\Omega))}\frac{\left|\int_0^T\int_\Omega(u-u_n)\eta\,\mathrm{d}x\mathrm{d}t\right|}{\|\eta\|_{L^2((0,T);L^2(\Omega))}}.\tag{5.11}$$

For any $\eta\in L^2((0,T);L^2(\Omega))$, let $w\in L^2((0,T);W_0^{s,2}(\overline{\Omega}))\cap H^1_{0,T}((0,T);W^{-s,2}(\overline{\Omega}))$ solve the following dual problem:

$$\begin{cases}-\partial_t w+(-\Delta)^s w &=\eta\quad\text{in } Q,\\ w &=0\quad\text{in } \Sigma,\\ w(T,\cdot) &=0\quad\text{in } \Omega.\end{cases}\tag{5.12}$$

It follows from Proposition 3.3 that there is a unique solution w to (5.12) that fulfills

$$\|w\|_{L^2((0,T);W^{s,2}(\mathbb{R}^N))}\le C\|\eta\|_{L^2((0,T);L^2(\Omega))}.\tag{5.13}$$

Notice that $w\in L^2((0,T);W_0^{s,2}(\overline{\Omega}))$ and using (5.9) we obtain that

$$\begin{aligned}&\int_0^T\int_\Omega(u-u_n)(-\partial_t w+(-\Delta)^s w)\,\mathrm{d}x\mathrm{d}t\\ &=\int_0^T\langle\partial_t(u-u_n),w\rangle\,\mathrm{d}t-\int_0^T\int_{\mathbb{R}^N\setminus\Omega}(u-u_n)\mathcal{N}_s w\,\mathrm{d}x\mathrm{d}t\\ &\quad+\frac{C_{N,s}}{2}\int_0^T\int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2}\frac{((u-u_n)(t,x)-(u-u_n)(t,y))(w(t,x)-w(t,y))}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y\mathrm{d}t\\ &=\int_0^T\langle\partial_t(u-u_n),w\rangle\,\mathrm{d}t+\int_0^T\mathcal{E}_n(u-u_n,w)\,\mathrm{d}t-\int_0^T\int_{\mathbb{R}^N\setminus\Omega}(u-u_n)\mathcal{N}_s w\,\mathrm{d}x\mathrm{d}t\\ &=\int_0^T\int_{\mathbb{R}^N\setminus\Omega}w\mathcal{N}_s u\,\mathrm{d}x\mathrm{d}t-\int_0^T\int_{\mathbb{R}^N\setminus\Omega}(u-u_n)\mathcal{N}_s w\,\mathrm{d}x\mathrm{d}t\\ &=-\int_0^T\int_{\mathbb{R}^N\setminus\Omega}(u-u_n)\mathcal{N}_s w\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Using the preceding identity, (5.10) and (5.13), we obtain that

$$\begin{aligned}\left|\int_0^T\int_\Omega (u-u_n)(-\partial_t w+(-\Delta)^s w)\,\mathrm{d}x\mathrm{d}t\right| &= \left|\int_0^T\int_{\mathbb{R}^N\setminus\Omega}(u-u_n)\mathcal{N}_s w\,\mathrm{d}x\mathrm{d}t\right|\\ &\le \|u-u_n\|_{L^2((0,T);L^2(\mathbb{R}^N\setminus\Omega))}\|\mathcal{N}_s w\|_{L^2((0,T);L^2(\mathbb{R}^N\setminus\Omega))}\\ &\le \frac{C}{n}\|u\|_{L^2((0,T);W^{s,2}(\mathbb{R}^N))}\|w\|_{L^2((0,T);W^{s,2}(\mathbb{R}^N))}\\ &\le \frac{C}{n}\|u\|_{L^2((0,T);W^{s,2}(\mathbb{R}^N))}\|\eta\|_{L^2((0,T);L^2(\Omega))}. \end{aligned} \tag{5.14}$$

Using (5.11) and (5.14) we get that

$$\|u-u_n\|_{L^2((0,T);L^2(\Omega))} \le \frac{C}{n}\|u\|_{L^2((0,T);W^{s,2}(\mathbb{R}^N))}. \tag{5.15}$$

Now the estimate (5.4) follows from (5.10) and (5.15). The proof of Part (a) is complete.

(b) Let $z\in L^2((0,T);L^2(\mathbb{R}^N\setminus\Omega))$. Using our assumption, we immediately notice that we have the continuous embedding $L^2(\mathbb{R}^N\setminus\Omega)\hookrightarrow L^2(\mathbb{R}^N\setminus\Omega,\mu)$. In addition, $\{u_n\}_{n\in\mathbb{N}}$ satisfies (5.6). Then proceeding similarly as in (5.8) we can deduce that

$$\begin{aligned} C\langle\partial_t u_n(t,\cdot),u_n(t,\cdot)\rangle + n\|u_n(t,\cdot)\|^2_{L^2(\mathbb{R}^N\setminus\Omega)} &\le C\Big(\langle\partial_t u_n(t,\cdot),u_n(t,\cdot)\rangle+\mathcal{E}_n(u_n,u_n)\Big)\\ &\le nC\|\kappa\|_{L^\infty(\mathbb{R}^N\setminus\Omega)}\|z(t,\cdot)\|_{L^2(\mathbb{R}^N\setminus\Omega)}\|u_n(t,\cdot)\|_{L^2(\mathbb{R}^N\setminus\Omega)}, \end{aligned}$$

for almost every $t\in(0,T)$. Since $\langle\partial_t u_n(t,\cdot),u_n(t,\cdot)\rangle=\frac{1}{2}\partial_t\|u_n(t,\cdot)\|^2_{L^2(\mathbb{R}^N\setminus\Omega)}$, we have that

$$\|u_n\|_{L^2((0,T);L^2(\mathbb{R}^N\setminus\Omega))}\le C\|z\|_{L^2((0,T);L^2(\mathbb{R}^N\setminus\Omega))}. \tag{5.16}$$

In order to show that $\|u_n\|_{L^2((0,T);L^2(\Omega))}$ is uniformly bounded, we can proceed as in (5.15), *i.e.*, by using a duality argument. Let $\eta\in L^2((0,T);L^2(\Omega))$ and $w\in\mathbb{U}_0$ be the weak solution of (5.12). Then using (5.6) and taking $w\in L^2((0,T);V)\cap H^1_{0,T}((0,T);L^2(\Omega))$, we get that

$$\begin{aligned}\int_Q u_n\eta\,\mathrm{d}x\mathrm{d}t &= \int_Q u_n(-\partial_t w+(-\Delta)^s w)\,\mathrm{d}x\mathrm{d}t\\ &= \int_0^T\langle\partial_t u_n,w\rangle\,\mathrm{d}t-\int_0^T\int_{\mathbb{R}^N\setminus\Omega}u_n\mathcal{N}_s w\,\mathrm{d}x\mathrm{d}t\\ &\quad+\frac{C_{N,s}}{2}\int_0^T\int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2}\frac{(u_n(t,x)-u_n(t,y))(w(t,x)-w(t,y))}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y\mathrm{d}t\\ &= -\int_0^T\int_{\mathbb{R}^N\setminus\Omega}u_n\mathcal{N}_s w\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Using the above identity, (5.16) and (5.13) we obtain that

$$\left|\int_0^T\int_\Omega u_n\eta\,\mathrm{d}x\mathrm{d}t\right| = \left|\int_0^T\int_{\mathbb{R}^N\setminus\Omega}u_n\mathcal{N}_s w\,\mathrm{d}x\mathrm{d}t\right|$$

$$
\begin{aligned}
&\le \|u_n\|_{L^2((0,T);L^2(\mathbb{R}^N\setminus\Omega))}\|\mathcal{N}_s w\|_{L^2((0,T);L^2(\mathbb{R}^N\setminus\Omega))}\\
&\le C\|z\|_{L^2((0,T);L^2(\mathbb{R}^N\setminus\Omega))}\|w\|_{L^2((0,T);W^{s,2}(\mathbb{R}^N))}\\
&\le C\|z\|_{L^2((0,T);L^2(\mathbb{R}^N\setminus\Omega))}\|\eta\|_{L^2((0,T);L^2(\Omega))}.
\end{aligned} \tag{5.17}
$$

Thus,

$$
\|u_n\|_{L^2((0,T);L^2(\Omega))} \le C\|z\|_{L^2((0,T);L^2(\mathbb{R}^N\setminus\Omega))}. \tag{5.18}
$$

Combining (5.16)–(5.18) we get that

$$
\|u_n\|_{L^2((0,T);L^2(\mathbb{R}^N))} \le C\|z\|_{L^2((0,T);L^2(\mathbb{R}^N\setminus\Omega))}. \tag{5.19}
$$

Therefore, the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $L^2((0,T);L^2(\mathbb{R}^N)) = L^2((0,T)\times\mathbb{R}^N)$. Thus, passing to a subsequence if necessary, we have that u_n converges weakly to some $\tilde{u}$ in $L^2((0,T);L^2(\mathbb{R}^N))$ as $n\to\infty$.

It remains to show (5.5). Notice that $W_0^{s,2}(\overline{\Omega}) \hookrightarrow W_{\Omega,\kappa}^{s,2} \cap L^2(\mathbb{R}^N\setminus\Omega)$. Thus, by (5.6) we have that

$$
\int_0^T \langle \partial_t u_n(t,\cdot), v(t,\cdot)\rangle \,\mathrm{d}t + \frac{C_{N,s}}{2}\int_0^T \int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2} \frac{(u_n(t,x)-u_n(t,y))(v(t,x)-v(t,y))}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y\mathrm{d}t = 0, \tag{5.20}
$$

for every $v\in L^2((0,T);V)\cap H^1_{0,T}((0,T);L^2(\Omega))$. Next, applying the integration by parts formula (2.9) we can deduce that

$$
\begin{aligned}
\int_0^T \langle \partial_t u_n(t,\cdot), v(t,\cdot)\rangle \,\mathrm{d}t + \frac{C_{N,s}}{2}\int_0^T \int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2} \frac{(u_n(t,x)-u_n(t,y))(v(t,x)-v(t,y))}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y\mathrm{d}t\\
= \int_Q u_n(-\partial_t v + (-\Delta)^s v)\,\mathrm{d}x\mathrm{d}t + \int_\Sigma u_n \mathcal{N}_s v\,\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{5.21}
$$

Combining (5.20)–(5.21) we get the identity

$$
\int_Q u_n(-\partial_t v + (-\Delta)^s v)\,\mathrm{d}x\mathrm{d}t + \int_\Sigma u_n \mathcal{N}_s v\,\mathrm{d}x\mathrm{d}t = 0. \tag{5.22}
$$

Taking the limit as $n\to\infty$ in (5.22), we obtain that

$$
\int_Q \tilde{u}(-\partial_t v + (-\Delta)^s v)\,\mathrm{d}x\mathrm{d}t + \int_\Sigma \tilde{u} \mathcal{N}_s v\,\mathrm{d}x\mathrm{d}t = 0,
$$

for every $v\in L^2((0,T);V)\cap H^1_{0,T}((0,T);L^2(\Omega))$. We have shown (5.5) and the proof is finished. □

Next, we show the approximation of the parabolic Dirichlet control problem (1.1).

Let $Z_R := L^2((0,T);L^2(\mathbb{R}^N\setminus\Omega))$ and consider the following minimization problem:

$$
\min_{(u,z)\in(U_R,Z_R)} \Big(J(u) + \frac{\xi}{2}\|z\|^2_{Z_R}\Big), \tag{5.23a}
$$

subject to the fractional parabolic Robin exterior value problem: Find $u \in U_R$ solving

$$\begin{cases} \partial_t u + (-\Delta)^s u = 0 & \text{in } Q, \\ \mathcal{N}_s u + n\kappa u = n\kappa z & \text{in } \Sigma, \\ u(0,\cdot) = 0 & \text{in } \Omega, \end{cases} \tag{5.23b}$$

and the control constraints

$$z \in Z_{ad,R}. \tag{5.23c}$$

Theorem 5.4 (**Approximation of the parabolic Dirichlet control problem).** *The problem (5.23) admits a minimizer $(z_n, u(z_n)) \in Z_{ad,R} \times L^2((0,T); W^{s,2}_{\Omega,\kappa} \cap L^2(\mathbb{R}^N \setminus \Omega)) \cap H^1_{0,0}((0,T); (W^{s,2}_{\Omega,\kappa} \cap L^2(\mathbb{R}^N \setminus \Omega))^\star)$. If $Z_R = H^1_{0,0}((0,T); W^{s,2}(\mathbb{R}^N \setminus \Omega))$ and $Z_{ad,R} \subset Z_R$ is bounded, then for any sequence $\{n_\ell\}_{\ell=1}^\infty$ with $n_\ell \to \infty$, there exists a subsequence still denoted by $\{n_\ell\}_{\ell=1}^\infty$, such that $z_{n_\ell} \rightharpoonup \tilde{z}$ in $H^1_{0,0}((0,T); W^{s,2}(\mathbb{R}^N \setminus \Omega))$ and $u(z_{n_\ell}) \to u(\tilde{z})$ in $L^2((0,T); L^2(\mathbb{R}^N))$ as $n_\ell \to \infty$, with $(\tilde{z}, u(\tilde{z}))$ solving the parabolic Dirichlet control problem (1.1) with $Z_{ad,D}$ replaced by $Z_{ad,R}$.*

Proof. The proof is similar to the elliptic case studied in [11] with the obvious modifications and is omitted for brevity. ☐

We conclude this section by writing the stationarity system corresponding to (5.23): Find $(z,u,p) \in Z_{ad,R} \times \Big(L^2((0,T); W^{s,2}_{\Omega,\kappa} \cap L^2(\mathbb{R}^N \setminus \Omega)) \cap H^1((0,T); (W^{s,2}_{\Omega,\kappa} \cap L^2(\mathbb{R}^N \setminus \Omega))^\star)\Big)^2$ with $u(0,\cdot) = p(T,\cdot) = 0$ in Ω such that

$$\begin{cases} \langle \partial_t u(t,\cdot), v\rangle + \mathcal{E}_n(u(t,\cdot), v) & = \int_{\mathbb{R}^N \setminus \Omega} n\kappa(x) z(t,x) v(x)\,\mathrm{d}x, \quad \text{a.e. } t \in (0,T), \\ \langle -\partial_t p(t,\cdot), w\rangle + \mathcal{E}_n(w, p(t,\cdot)) & = \int_\Omega J'(u(t,x)) w(x)\,\mathrm{d}x, \quad \text{a.e. } t \in (0,T), \\ \int_\Sigma (n\kappa(x) p(t,x) + \xi z(t,x))(\widetilde{z} - z)(t,x)\,\mathrm{d}x & \geq 0, \end{cases} \tag{5.24}$$

for all $(\widetilde{z}, v, w) \in Z_{ad,R} \times (W^{s,2}_{\Omega,\kappa} \cap L^2(\mathbb{R}^N \setminus \Omega)) \times (W^{s,2}_{\Omega,\kappa} \cap L^2(\mathbb{R}^N \setminus \Omega))$. Here $\mathcal{E}_n$ is as in (5.7).

6. Numerical approximations

In this section, we shall introduce the numerical approximation of all the problems we have considered so far. We remark that solving parabolic fractional PDEs is a delicate issue. One has to assemble the integrals with singular kernels and the resulting system matrices are dense. On the top of that, the optimal control problem requires solving the state equation forward in time and adjoint equation backward in time. This can be prohibitively expensive. The purpose of this section is simply to illustrate that the numerical results are in agreement with the theory and to show the benefits of the fractional optimal control problem.

The rest of the section is organized as follows: in Section 6.1 we first focus on the approximations of the Robin problem (5.1). With the help of a numerical example, we illustrate the sharpness of Theorem 5.3. This is followed by a source identification problem in Section 6.2. The numerical example presented in Section 6.2 clearly indicates the strength and flexibility of nonlocal problems over the local ones.

6.1. Approximation of parabolic Dirichlet problems by parabolic Robin problems

We begin by introducing a discrete scheme for the parabolic Robin problem (5.1) and recall that we can approximate the parabolic Dirichlet problem by the parabolic Robin problem. Let $\widetilde{\Omega}$ be an open bounded set that contains Ω, the support of z, and the support of κ. We consider a conforming simplicial triangulation of Ω

and $\widetilde{\Omega} \setminus \Omega$ such that the resulting partition remains admissible. Throughout the following, we will assume that the support of z and κ are contained in $\widetilde{\Omega} \setminus \Omega$. Let $\mathbb{V}_h$ (on $\widetilde{\Omega}$) be the finite element space of continuous piecewise linear functions. We use the backward-Euler to carry out the time discretization: Let K denote the number of time intervals, we set the time-step to be $\tau = T/K$. Then for $k = 1, \dots, K$, the fully discrete approximation of (5.1) with nonzero right-hand-side f and initial datum $u^{(0)} = u(0, \cdot)$ is given by: find $u_h^{(k)} \in \mathbb{V}_h$ such that

$$\begin{aligned} \int_\Omega u_h^{(k)} v \ \mathrm{d}x + \tau \mathcal{E}_n(u_h^{(k)}, v) = \tau \langle f^{(k)}, v \rangle + \tau \int_{\widetilde{\Omega}\setminus\Omega} n\kappa z^{(k)} v \ \mathrm{d}x \\ + \int_\Omega u_h^{(k-1)} v \ \mathrm{d}x \quad \forall v \in \mathbb{V}_h, \end{aligned} \tag{6.1}$$

where $\mathcal{E}_n$ is as in (5.7). The approximation of the double integral over $\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2$ is carried out using the approach of [1]. The remaining integrals are computed using quadrature which is accurate for polynomials of degree less than and equal to 4. All the implementations are carried in Matlab and we use the direct solver to solve the linear systems. We emphasize that all our spatial meshes are generated using Gmsh [28].

We next consider an example of a parabolic Dirichlet problem with nonzero exterior conditions. Let $\Omega = B_0(1/2) \subset \mathbb{R}^2$ and $T = 1$. We aim to find u solving

$$\begin{cases} \partial_t u + (-\Delta)^s u &= u_{\text{exact}} + e^t \quad \text{in } Q, \\ u(t, \cdot) &= u_{\text{exact}}(t, \cdot) \quad \text{in } \Sigma, \\ u(0, \cdot) &= u_{\text{exact}}(0, \cdot) \quad \text{in } \Omega. \end{cases} \tag{6.2}$$

The exact solution for this problem is given by

$$u_{\text{exact}}(t, x) = \frac{2^{-2s} e^t}{\Gamma(1+s)^2} \left(1 - |x|^2\right)_+^s .$$

We set $\widetilde{\Omega} = B_0(1.5)$ and approximate (6.2) by using (6.1). Moreover, we set $\kappa = 1$ on its support. We divide the time interval $(0, 1)$ into 1800 subintervals. For a fixed $s = 0.6$ and spatial Degrees of Freedom (DoFs) = 6017, we study the $L^2((0, T); L^2(\Omega))$ error $\|u_{\text{exact}} - u_h\|_{L^2((0,T);L^2(\Omega))}$ with respect to n in Figure 1 (left). We obtain a convergence rate of $1/n$, as predicted by Theorem 5.3a.

In the right panel, in Figure 1, we have shown the error $\|u_{\text{exact}} - u_h\|_{L^2(0,T;L^2(\Omega))}$ for a fixed $s = 0.6$, but $n = 1e4, 1e5, 1e6, 1e7$, as a function of DoFs. We observe that the error remains stable with respect to n as we refine the spatial mesh. Moreover, the observed rate of convergence is $(\text{DoFs})^{-\frac{1}{2}}$.

For the same example, next we study the behavior of $\|u_{\text{exact}} - u_h\|_{L^2((0,T);L^2(\Omega))}$ as $s \to 1$ in Figure 2. We observe that the error remains stable.

We conclude this section, with another example where f and z are less regular than in the above example. We set

$$f(t, x) := \left(|0.1 - x_2|^{0.01} + |-0.1 - x_2|^{0.01}\right) e^t \quad \text{and} \quad z(t, x) := \left(|0.6 - x_2|^{0.01} + |-0.6 - x_2|^{0.01}\right) e^t. \tag{6.3}$$

Notice that in this case we do not have access to u_{exact}. Instead, we set u_{exact} to be the solution with $n = 10^8$. The error $\|u_{\text{exact}} - u_h\|_{L^2((0,T);L^2(\Omega))}$ with respect to n is shown in Figure 3. The example seems to give a convergence rate of $n^{-0.7473}$ which is lower than the rate we predicted in Theorem 5.3a. This appears to indicate that the result of Theorem 5.3a are *sharp*, as in this example we expect $u \notin L^2((0, T); W^{s,2}(\mathbb{R}^N))$.

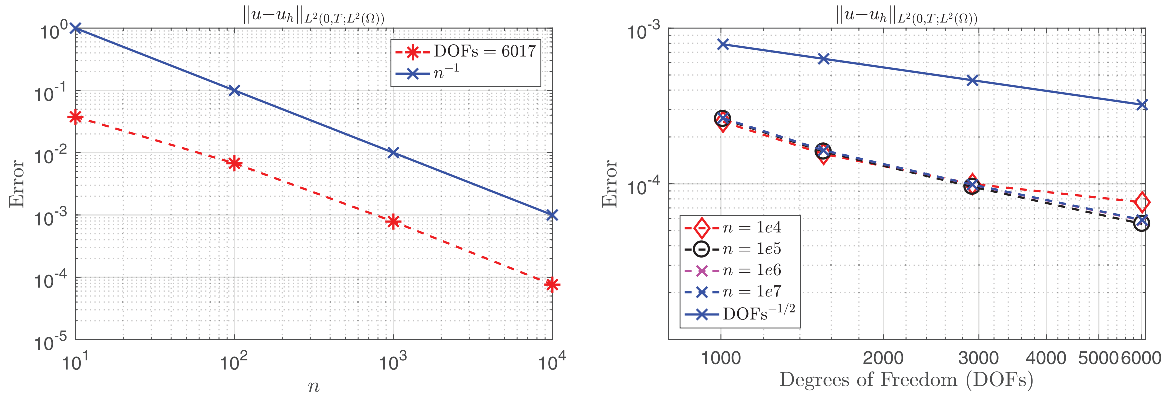

FIGURE 1. *Left panel*: fix $s = 0.6$, Degrees of Freedom (DoFs) = 6017. The number of time intervals is 1800. The solid line denotes the reference line and the dottle line is the actual error. We observe that the error $\|u_{\text{exact}} - u_h\|_{L^2((0,T);L^2(\Omega))}$ with respect to n decays at the rate of $1/n$ as predicted by the estimate (5.4) in Theorem 5.3a. *Right panel*: let $s = 0.6$ and number of time intervals = 1800, be fixed. We have shown that the error with respect to spatial DoFs, for $n = 10^4, n = 10^5, n = 10^6$, and $n = 10^7$, behaves as $(\text{DoFs})^{-\frac{1}{2}}$.

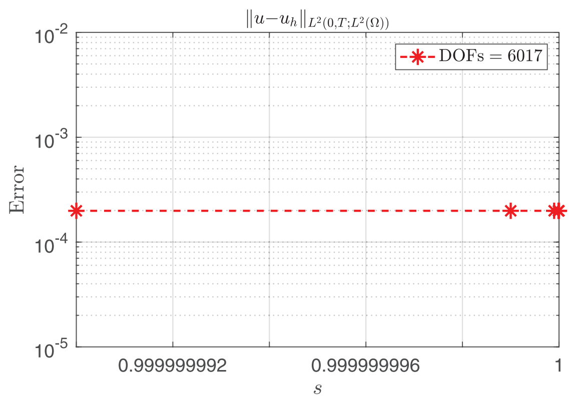

FIGURE 2. Behavior of $\|u_{\text{exact}} - u_h\|_{L^2(0,T;L^2(\Omega))}$ as $s \to 1$. We notice that the error remains stable.

6.2. Parabolic source/control identification problems

After the validation in the previous example, we are now ready to consider a source/control identification problem where the source/control is located outside the domain Ω. The optimality system is as given in (5.24). The spatial discretization of all the optimization variables (u, z, p) is carried out using continuous piecewise linear finite elements and time discretization using backward-Euler. We set the objective function to be

$$j(u, z) := J(u) + \frac{\xi}{2}\|z\|^2_{L^2((0,T);L^2(\mathbb{R}^N\setminus\Omega))}, \quad \text{with} \quad J(u) := \frac{1}{2}\|u - u_d\|^2_{L^2((0,T);L^2(\Omega))},$$

where $u_d : L^2((0,T);L^2(\Omega)) \to \mathbb{R}$ is the given data (observations). Moreover, we let $Z_{ad,R} := \{z \in L^2((0,T);L^2(\mathbb{R}^N \setminus \Omega)) : z \geq 0, \text{ a.e. in } (0,T) \times \widehat{\Omega}\}$ where $\widehat{\Omega}$ is the support set of the control z that is contained in $\widetilde{\Omega} \setminus \Omega$. We solve the optimization problem using the projected-BFGS method with Armijo line search.

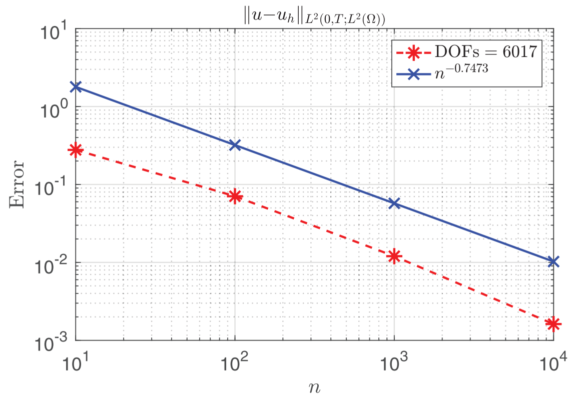

FIGURE 3. We use the non-smooth data given in (6.3) and show the error $\|u_{\text{exact}} - u_h\|_{L^2((0,T);L^2(\Omega))}$ with respect to n. We observe a rate of convergence less than predicted in Theorem 5.3a. This seems to indicate that the result of Theorem 5.3a is sharp, since in this example we expect $u \notin L^2((0,T);W^{s,2}(\mathbb{R}^N))$.

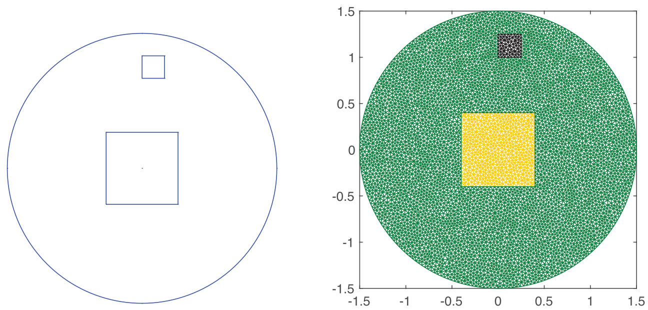

FIGURE 4. *Left panel*: the circle denotes $\widetilde{\Omega}$ and the larger square denotes the domain Ω. Moreover, the outer square inside $\widetilde{\Omega} \setminus \Omega$ is $\widehat{\Omega}$, *i.e.*, the region where the source/control is supported. *Right panel*: a finite element mesh.

We consider the domain as given in Figure 4. The circle denotes $\widetilde{\Omega} = B_0(3/2)$ and the larger square denotes the domain $\Omega = [-0.4, 0.4]^2$. The smaller square, inside $\widetilde{\Omega} \setminus \Omega$, denoted by $\widehat{\Omega}$, is where the source/control is supported. The right panel shows a finite element mesh with DoFs = 6103.

We generate the data u_d as follows: for $z = 1$, we solve the state equation (first equation in (5.24)). We then add a normally distributed noise with mean zero and standard deviation 0.005. We call the resulting expression u_d. In addition, we set $\kappa = 1$ on its support and $n = 1e7$.

Next, we identify the source $\bar{z}_h$ by solving the optimality system (5.24) using the aforementioned optimization algorithm. For $\xi = 1e$-8, our results are shown in Figure 5. In the first two rows, we have plotted $\bar{z}_h$ (as a by-product of our optimization algorithm) for $s = 0.1$ at 4 time instances $t = 0.25, 0.3, 0.43, 0.58$. The third row shows $\bar{z}_h$ for $s = 0.8$ at only one of these four time instances since $\bar{z}_h$ is zero at the remaining three time instances. The zero $\bar{z}_h$ for all these time instances can be explained as follows: we know that when s approaches 1, the fractional Laplacian approaches the standard Laplacian $-\Delta$. The latter operator only imposes boundary conditions on $\partial\Omega$, but not exterior conditions as in the case of the fractional Laplacian.

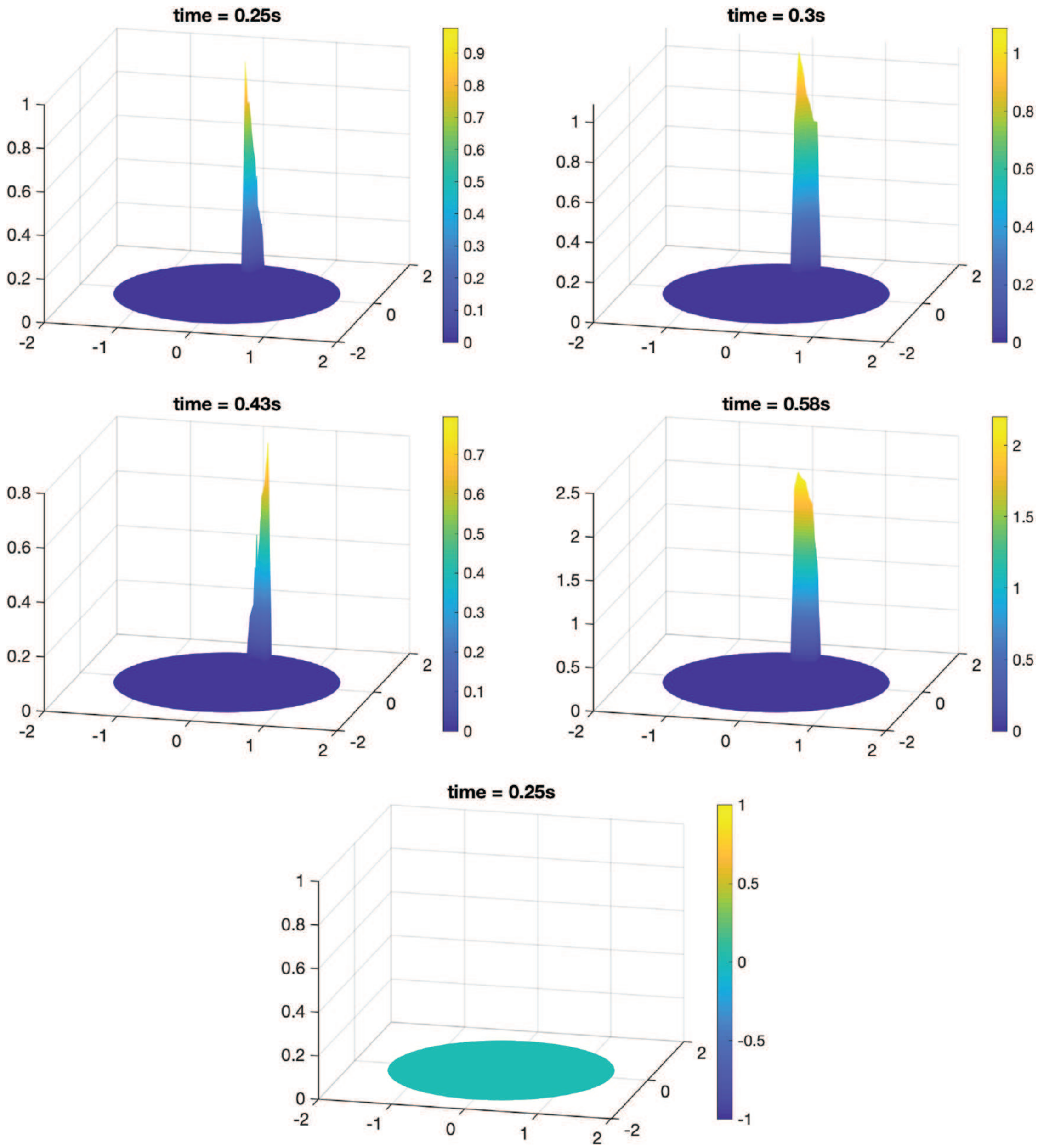

FIGURE 5. The first and second row show the source $\bar{z}_h$ for exponent $s = 0.1$ at 4 different time instances, $t = 0.25, 0.3, 0.43, 0.58$. The last row shows $\bar{z}_h$ for exponent $s = 0.8$ at $t = 0.25$. Notice that $\bar{z}_h \equiv 0$ at $t = 0.25$. For $s = 0.8$, we also obtain that $\bar{z}_h \equiv 0$ at $t = 0.3, 0.43, 0.58$ therefore we have omitted those plots. This comparison between $\bar{z}_h$ for $s = 0.1$ and $s = 0.8$ clearly indicates that we can recover the sources for smaller values of s but when s approaches 1, since the fractional Laplacian approaches the standard Laplacian, we cannot see the external source at all times, *i.e.*, we obtain $\bar{z}_h \equiv 0$. Recall that, the standard Laplacian does not allow imposing exterior conditions.

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