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Optimal Control of Fractional Elliptic PDEs with State Constraints and Characterization of the Dual of Fractional-Order Sobolev Spaces

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Abstract

This paper introduces the notion of state constraints for optimal control problems governed by fractional elliptic partial differential equations. Several mathematical tools are developed during the process to study these problems, for instance, the characterization of the dual of fractional-order Sobolev spaces and the well-posedness of fractional elliptic equations with measure-valued data. These tools are widely applicable. We show well-posedness of the optimal control problem and derive the first-order optimality conditions. Notice that the adjoint equation is a fractional partial differential equation with a measure as the right-hand-side datum. We use the characterization of the fractional- order dual spaces to study the regularity of solutions of the state and adjoint equations. We emphasize that the classical case was considered by E. Casas, but almost none of the existing results are applicable to our fractional case. As an application of the regularity result of the adjoint equation, we establish the Sobolev regularity of the optimal control. In addition, under this setup, even weaker controls can be used.

Keywords Optimal control with PDE constraint · State and control constraints · Fractional Laplacian · Measure valued data · Characterization of fractional dual spaces · Regularity of solutions to state and adjoint equations · Regularity of optimal control

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1 Introduction

The main goal of this paper is to introduce and study an optimal control problem with fractional partial differential equation (PDE) as constraints. We shall denote the fractional exponent by $s \in]0, 1[$. The main novelty of the paper is due to additional constraints on the state u and the control z. Recently, article [1] has derived a fractional Helmholtz equation using the first principle arguments. The authors have also shown a direct qualitative match between the numerical simulations and real data for a problem in geophysical electromagnetics. This fractional Helmholtz equation is a generalization of our state equation in (2a). We further notice that the fractional operators are also starting to play a pivotal role in other applications: imaging science, phase field models, diffusion of biological species, and data science, see [2] and the references therein. In view of these applications, control or source identification problems for fractional equations are natural and have also motivated the current study.

Optimal control of fractional PDEs with control constraints has recently received a lot of attention. We refer to [3] for the optimal control of fractional semilinear PDEs with both spectral and integral fractional Laplacians with distributed control, see also [4] for such a control of an integral operator. We refer to [5] for the boundary control with the spectral fractional Laplacian and [6,7] for the exterior optimal control of fractional PDEs. See [8] for the optimal control of quasi-linear fractional PDEs where the control is in the coefficients.

We remark that the classical case ($s = 1$), is well known, see, for instance, [9–13] and the references therein. Nevertheless, none of these existing works are directly applicable to the case of fractional state constraints as stated in (2b). To be more precise, for the classical case $s = 1$, the result on boundedness of solutions assumes that $p > \frac{N}{2}$ (cf. (2c)) and the classical notion of very weak solutions (cf. Definition 4.2) does not extend to the fractional case. Moreover, the characterization of the dual of classical integer-order Sobolev spaces, which is needed to establish the regularity of solutions to the adjoint equation, was not known for fractional-order Sobolev spaces (cf. Corollary 7.3). This additional adjoint regularity is then used to establish a higher regularity result for the optimal control.

The rest of the paper is organized as follows. In Sect. 2, we introduce the problem under consideration and state the main difficulties and novelties. In Sect. 3, we introduce the underlying notations and state some preliminary results. Our main work starts from Sect. 4, where we establish the continuity of solutions to the state equation and the well-posedness of the fractional PDEs with measured valued data. In Sect. 5, we show the well-posedness of the optimal control problem and derive the optimality conditions. In Sect. 6, we derive the characterization of the dual spaces of the fractional-order Sobolev spaces. We conclude the paper by giving higher regularity results for the associated adjoint and control variables in Sect. 7. In addition, we also discuss the case where we have weaker than L^p-controls.

2 Problem Formulation

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) be a bounded open set with boundary $\partial\Omega$. The main goal of this paper is to introduce and study an optimal control problem with both control and state constraints:

$$\min_{(u,z)\in(U,Z)} J(u,z) \tag{1}$$

subject to the fractional elliptic PDE: find $u \in U$ solving

$$(-\Delta)^s u = z \quad \text{in } \Omega, \qquad u = 0 \quad \text{in } \mathbb{R}^N \backslash \Omega\,, \tag{2a}$$

as constraints and additional state constraints

$$u|_\Omega \in \mathcal{K} := \left\{ w \in C_0(\Omega) \ : \ w(x) \leq u_b(x), \quad \forall x \in \overline{\Omega} \right\}, \tag{2b}$$

where $C_0(\Omega)$ is the space of all continuous functions in $\overline{\Omega}$ that vanish on $\partial\Omega$ and $u_b \in C(\overline{\Omega})$. Moreover, we also consider the control constraints

$$z \in Z_{ad} \subset L^p(\Omega) \tag{2c}$$

with Z_{ad} being a nonempty, closed, and convex set. In (2c), the real number p satisfies

$$p > \frac{N}{2s} \quad \text{if } N > 2s, \quad p > 1 \quad \text{if } N = 2s, \quad p = 1 \quad \text{if } N < 2s. \tag{3}$$

Notice that for $z \in L^p(\Omega)$, with p as in (3), we have that u solving (2a) belong to $L^\infty(\Omega)$ (see, e.g., [3]). We refer to Sect. 5 for more details and the precise assumptions on the functional J.

We remark that the case $s = 1$ is classical, see, for instance, [9–13] and the references therein. Nevertheless, none of these existing works are directly applicable to the case of fractional state constraints as stated in (2b). To be more precise, for the classical case $s = 1$, the result on boundedness of solutions assumes that $p > \frac{N}{2}$ and the notion of very weak solutions does not extend to the case $0 < s < 1$. Moreover, the characterization of the dual of integer-order Sobolev spaces, which is needed to establish the regularity of solutions to the adjoint equation, was not known for fractional-order Sobolev spaces. This adjoint regularity is then used to establish a higher regularity result for the optimal control.

Main Difficulties and Novelties of the Paper

(a) *Nonlocal equation.* Equation (2a) is nonlocal. In order to realize $(-\Delta)^s u$ in Ω, it is necessary to know u in all of $\mathbb{R}^N$. In addition, in order for the system (2a) to be well-posed, the condition $u = 0$ must be prescribed in $\mathbb{R}^N \setminus \Omega$ and not on $\partial\Omega$, as in the classical case of the Laplace operator.

(b) *Continuity of the state solution with L^p-data.* Similarly to the classical case, we need to show that the solution u of (2a) is continuous whenever $z \in L^p(\Omega)$. We shall show that if $z \in L^p(\Omega)$ with p satisfying (3), then every weak solution to (2a) belongs to $C_0(\Omega)$. This continuity result, in a sense, weakens the regularity requirements on z in comparison with [14, Proposition 1.1] where they have assumed that $z \in L^\infty(\Omega)$, and it also allows us to solve our control problem by taking $Z_{ad} \subset L^p(\Omega)$.

(c) *Equation with measure valued data.* The adjoint equation (see, Eq. (11)) associated with (2a) is a fractional PDE with a measure-valued datum. Firstly, we shall show in Theorem 4.2 the well-posedness of such PDEs in $L^{\frac{p}{p-1}}(\Omega)$ where p is as in (3), and secondly, we shall prove in Corollary 7.2 that solutions belong to $\widetilde{W}_0^{t,\frac{p}{p-1}}(\Omega)$ under suitable assumptions on p and $0 < t < 1$.

(d) *Characterization of the dual space* $\widetilde{W}^{-s,p'}(\Omega)$. Let $1 \leq p < \infty$, $p' := \frac{p}{p-1}$ and let $\widetilde{W}^{-s,p'}(\Omega)$ denote the dual of $\widetilde{W}_0^{s,p}(\Omega)$ (see, Sect. 3). In Theorem 6.1, we shall show that if $1 \leq p < \infty$ and $f \in \widetilde{W}^{-s,p'}(\Omega)$, then there is pair of functions $(f^0, f^1) \in L^{p'}(\Omega) \times L^{p'}(\mathbb{R}^N \times \mathbb{R}^N)$ such that for every $v \in \widetilde{W}_0^{s,p}$, we have: $f(v) = \langle f, v\rangle = \int_\Omega f^0 v \, dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f^1(x,y)\frac{v(x)-v(y)}{|x-y|^{\frac{N}{p}+s}} \, dxdy$. This characterization is one of the main novelties of the current paper.

(e) *Higher regularity of solutions to the Dirichlet problem* (2a). Using the above characterization of the dual spaces, we shall show in Corollary 7.1 that if $0 < t < s < 1$ and $2 < \frac{N}{s} < p \leq \infty$, or $1 \leq p' < 2$ and $\frac{1}{p'} < t < 1$, and $z \in \widetilde{W}^{-t,p}$, then weak solutions of the Dirichlet problem (2a) are also continuous up to the boundary of Ω. This is the first time that such a regularity result has been proved (with very weak right-hand-side data) for the fractional Laplace operator.

(f) *Higher regularity of the optimal control.* Using the above higher regularity of the adjoint variable, we shall establish the $W^{t,\frac{p}{p-1}}(\Omega)$-regularity of the optimal control. This higher regularity of the optimal control is crucial to establish some rates of convergence of the numerical methods.

3 Notations and Preliminaries

We follow the notations from our previous work [6]. Unless otherwise stated, $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) is an arbitrary bounded open set, $0 < s < 1$ and $1 \leq p < \infty$. For a function u defined in $\mathbb{R}^N$ (or in Ω), we shall denote by $D_{s,p}u$, the function defined in $\mathbb{R}^N \times \mathbb{R}^N$ (or in $\Omega \times \Omega$) by $D_{s,p}u[x,y] := \dfrac{u(x)-u(y)}{|x-y|^{\frac{N}{p}+s}}$.

Then, we define the Sobolev space

$$W^{s,p}(\Omega) := \Big\{u \in L^p(\Omega) : \; D_{s,p}u \in L^p(\Omega \times \Omega)\Big\}$$

and we endow it with the norm $\|u\|_{W^{s,p}(\Omega)} := \left(\int_\Omega |u|^p \, dx + \|D_{s,p}u\|^p_{L^p(\Omega\times\Omega)}\right)^{\frac{1}{p}}$. We let $W_0^{s,p}(\Omega) := \overline{\mathcal{D}(\Omega)}^{W^{s,p}(\Omega)}$, where $\mathcal{D}(\Omega)$ is the space of test functions.

Since Ω is assumed to be bounded, we have the following continuous embeddings:

$$W_0^{s,2}(\Omega) \hookrightarrow \begin{cases} L^p(\Omega), & \text{if } N \geq 2s, \\ C^{0,s-\frac{N}{2}}(\overline{\Omega}), & \text{if } N < 2s, \end{cases} \tag{4}$$

with $p = 2^\star := \dfrac{2N}{N-2s}$ if $N > 2s$, and $p \in [1, \infty[$ arbitrary if $N = 2s$.

A complete characterization of $W_0^{s,p}(\Omega)$, for arbitrary bounded open sets, is given in [15]. By [16, Theorem 1.4.2.4, p. 25] (see also, [15,17]) if Ω has a Lipschitz continuous boundary and $\frac{1}{p} < s < 1$, then

$$\|u\|_{W_0^{s,p}(\Omega)} = \|D_{s,p}u\|_{L^p(\Omega\times\Omega)} \tag{5}$$

defines an equivalent norm on $W_0^{s,p}(\Omega)$. In that case, we shall use this norm for $W_0^{s,p}(\Omega)$.

In order to study the problem (2a), we need to consider the following function space:

$$\widetilde{W}_0^{s,p}(\Omega) := \Big\{u \in W^{s,p}(\mathbb{R}^N) : \; u = 0 \; \text{ on } \; \mathbb{R}^N \setminus \Omega\Big\}.$$

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with a Lipschitz continuous boundary. By [18, Theorem 6], $\mathcal{D}(\Omega)$ is dense in $\widetilde{W}_0^{s,p}(\Omega)$. Moreover, for every $0 < s < 1$ a simple calculation gives

$$\begin{aligned} \|u\|^p_{\widetilde{W}_0^{s,p}(\Omega)} &:= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\mathrm{d}y \\ &= \|D_{s,p}u\|^p_{L^p(\Omega\times\Omega)} + \int_\Omega |u|^p \kappa(x)\,\mathrm{d}x, \end{aligned} \tag{6}$$

where $\kappa(x) := 2\displaystyle\int_{\mathbb{R}^N\setminus\Omega} \frac{1}{|x-y|^{N+sp}}\,\mathrm{d}y, \quad x \in \Omega.$

Remark 3.1 (a) The embeddings (4) hold with $W_0^{s,2}(\Omega)$ replaced by $\widetilde{W}_0^{s,2}(\Omega)$.
(b) Let p satisfy (3) and $p' := \frac{p}{p-1}$. Then, it is easy to see that $\widetilde{W}_0^{s,2}(\Omega) \hookrightarrow L^{p'}(\Omega)$.

We next state an important result for $\widetilde{W}_0^{s,p}(\Omega)$ taken from [16, Corollary 1.4.4.5].

Theorem 3.1 *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with a Lipschitz continuous boundary and $1 < p < \infty$. If $\frac{1}{p} < s < 1$, then $\widetilde{W}_0^{s,p}(\Omega) = W_0^{s,p}(\Omega)$ with equivalent norms.*

Under the hypotheses of Theorem 3.1, we have that if $\frac{1}{p} < s < 1$, then

$$\|u\|_{\widetilde{W}_0^{s,p}(\Omega)} = \|D_{s,p}u\|_{L^p(\Omega\times\Omega)}. \tag{7}$$

In other words, the integral involving the function $\kappa(x)$ in (6) is not relevant.

If $0 < s < 1$, $p \in]1, \infty[$ and $p' := \frac{p}{p-1}$, then the space $\widetilde{W}^{-s,p'}(\Omega)$ is defined as the dual of $\widetilde{W}_0^{s,p}(\Omega)$.

We are now ready to define the fractional Laplacian. We set

$$\mathbb{L}_s^1(\mathbb{R}^N) := \left\{u : \mathbb{R}^N \to \mathbb{R} \text{ measurable and } \int_{\mathbb{R}^N} \frac{|u(x)|}{(1+|x|)^{N+2s}}\,\mathrm{d}x < \infty\right\}.$$

For $u \in \mathbb{L}_s^1(\mathbb{R}^N)$ and $\varepsilon > 0$, we let $(-\Delta)_\varepsilon^s u(x) := C_{N,s} \int_{\{y\in\mathbb{R}^N, |y-x|>\varepsilon\}} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\mathrm{d}y$, $x \in \mathbb{R}^N$, where $C_{N,s}$ is a normalization constant and is given by $C_{N,s} := \frac{s2^{2s}\Gamma\left(\frac{2s+N}{2}\right)}{\pi^{\frac{N}{2}}\Gamma(1-s)}$, with Γ being the standard Euler Gamma function (see, e.g., [19,20]). The fractional Laplacian for $u \in \mathbb{L}_s^1(\mathbb{R}^N)$ is defined by

$$(-\Delta)^s u(x) := C_{N,s}\text{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\mathrm{d}y = \lim_{\varepsilon\downarrow 0}(-\Delta)_\varepsilon^s u(x), \quad x \in \mathbb{R}^N, \tag{8}$$

provided that the limit exists.

Next, we define the operator $(-\Delta)_D^s$ in $L^2(\Omega)$ as follows:

$$\begin{aligned} D((-\Delta)_D^s) &:= \left\{u|_\Omega : u \in \widetilde{W}_0^{s,2}(\Omega) \text{ and } (-\Delta)^s u \in L^2(\Omega)\right\}, \\ (-\Delta)_D^s(u|_\Omega) &:= (-\Delta)^s u \text{ in } \Omega. \end{aligned} \tag{9}$$

Then, $(-\Delta)_D^s$ is the realization in $L^2(\Omega)$ of $(-\Delta)^s$ with the Dirichlet exterior condition $u = 0$ in $\mathbb{R}^N \setminus \Omega$.

Finally, we close this section by recalling the integration-by-parts formula for $(-\Delta)^s$ (see, e.g., [21]).

Proposition 3.1 *Let $u \in \widetilde{W}_0^{s,2}(\Omega)$ with $(-\Delta)^s u \in L^2(\Omega)$. Then, for every $v \in \widetilde{W}_0^{s,2}(\Omega)$ we have*

$$\frac{C_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dxdy = \int_\Omega v(-\Delta)^s u\,dx. \tag{10}$$

4 The State and Adjoint Equations

Throughout the remainder of the paper, given a Banach space X and its dual $X^\star$, we shall denote by $\langle\cdot,\cdot\rangle_{X^\star,X}$ their duality pairing.

The purpose of this section is to show that weak solutions of (2a) are continuous up to the boundary of Ω, and to study the existence and uniqueness of very weak solutions to the system

$$(-\Delta)^s u = \mu \quad \text{in } \Omega, \qquad u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \tag{11}$$

where $\mu \in \mathcal{M}(\Omega)$, the space of all Radon measures on Ω. More precisely, $\mathcal{M}(\Omega) = (C_0(\Omega))^*$, i.e., $\mathcal{M}(\Omega)$ is the dual of $C_0(\Omega)$ such that

$$\langle \mu, v\rangle_{(C_0(\Omega))^\star, C_0(\Omega)} = \int_\Omega v \, \mathrm{d}\mu, \quad \mu \in \mathcal{M}(\Omega), \quad v \in C_0(\Omega).$$

In addition, we have the following norm on this space: $\|\mu\|_{\mathcal{M}(\Omega)} = \sup_{v \in C_0(\Omega), |v| \le 1} \int_\Omega v \, \mathrm{d}\mu$.

We will first show the continuity of weak solutions to (2a). We recall that the paper [14] proves the optimal Hölder C^s-regularity of u under the condition that the datum $z \in L^\infty(\Omega)$. However, in our setting, we have only assumed that $z \in L^p(\Omega)$. Therefore, the result of [14] does not apply. We state the notion of weak solutions to (2a).

Definition 4.1 (*Weak solutions to the Dirichlet problem*) Given $z \in \widetilde{W}^{-s,2}(\Omega)$. A function $u \in \widetilde{W}_0^{s,2}(\Omega)$ is said to be a weak solution of (2a) if for every $v \in \widetilde{W}_0^{s,2}(\Omega)$, we have

$$\frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, \mathrm{d}x\mathrm{d}y = \langle z, v\rangle.$$

The following theorem is the first main result of this section.

Theorem 4.1 *Let Ω be a bounded Lipschitz domain satisfying the exterior cone condition. Assume that $z \in L^p(\Omega)$ with p as in (3). Then, every weak solution u of (2a) belongs to $C_0(\Omega)$ and there is a constant $C = C(N, s, p, \Omega) > 0$ such that*

$$\|u\|_{C_0(\Omega)} \le C\|z\|_{L^p(\Omega)}. \tag{12}$$

Proof Let $z \in L^p(\Omega)$ and let $\{z_n\}_{n\in\mathbb{N}} \subset L^\infty(\Omega)$ be a sequence such that $\|z_n - z\|_{L^p(\Omega)} \to 0$ as $n \to \infty$.
For each $n \in \mathbb{N}$, let u_n solve the following Dirichlet problem:

$$(-\Delta)^s u_n = z_n \quad \text{in } \Omega, \qquad u_n = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \tag{13}$$

By [14, Proposition 1.1], $u_n \in C^s(\mathbb{R}^N)$. Next, subtracting (2a) from (13), we deduce that

$$(-\Delta)^s (u_n - u) = z_n - z \quad \text{in } \Omega, \qquad (u_n - u) = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega.$$

Since $(z_n - z) \in L^p(\Omega)$, applying [3, Theorem 3.7], we get that $\|u_n - u\|_{L^\infty(\Omega)} \le C\|z_n - z\|_{L^p(\Omega)} \to 0$ as $n \to \infty$. Thus, $\|u_n - u\|_{L^\infty(\Omega)} \to 0$ as $n \to \infty$. Since $u_n \in C_0(\Omega)$, it follows that $u \in C_0(\Omega)$. □

In Corollary 7.1, we shall reduce the assumed $L^p(\Omega)$-regularity requirement on z in Theorem 4.1.

Next, we introduce the notion of very weak solutions to the problem (11).

Definition 4.2 (*Very weak solutions to the Dirichlet problem with measure data*) Let p be as in (3) and $\frac{1}{p}+\frac{1}{p'}=1$. Let $\mu \in \mathcal{M}(\Omega)$. A function $u \in L^{p'}(\Omega)$ is said to be a very weak solution to (11), if for every $v \in V := \{v \in C_0(\Omega) \cap \widetilde{W}_0^{s,2}(\Omega) : \ (-\Delta)^s v \in L^p(\Omega)\}$ we have

$$\int_\Omega u(-\Delta)^s v \,\mathrm{d}x = \int_\Omega v \,\mathrm{d}\mu. \tag{14}$$

The following theorem is the second main result of this section.

Theorem 4.2 *Let Ω be a bounded Lipschitz domain satisfying the exterior cone condition. Let $\mu \in \mathcal{M}(\Omega)$, p as in (3) and $p' := \frac{p}{p-1}$. Then, there exists a unique $u \in L^{p'}(\Omega)$ that solves (11) according to Definition 4.2, and there is a constant $C = C(N, s, p, \Omega) > 0$ such that*

$$\|u\|_{L^{p'}(\Omega)} \le C\|\mu\|_{\mathcal{M}(\Omega)}. \tag{15}$$

Proof For a given $\xi \in L^p(\Omega)$, we begin by considering the following auxiliary problem:

$$(-\Delta)^s v = \xi \quad \text{in } \Omega, \qquad v = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \tag{16}$$

Since $L^p(\Omega) \hookrightarrow L^{p'}(\Omega) \hookrightarrow \widetilde{W}^{-s,2}(\Omega)$ (by Remark 3.1), it follows that there is a unique $v \in \widetilde{W}_0^{s,2}(\Omega)$ satisfying (16). By Theorem 4.1, $v \in C_0(\Omega)$.

Consider a mapping $\Xi : L^p(\Omega) \to C_0(\Omega), \quad \xi \mapsto \Xi\xi := v$. Notice that, Ξ is linear and continuous (by Theorem 4.1). Let us define $u := \Xi^*\mu$. Then, $u \in L^{p'}(\Omega)$. We show that u solves (11). Notice that,

$$\int_\Omega u\xi \,\mathrm{d}x = \int_\Omega u(-\Delta)^s v \,\mathrm{d}x = \int_\Omega (\Xi^*\mu)\xi \,\mathrm{d}x = \int_\Omega v \,\mathrm{d}\mu, \tag{17}$$

for every $v \in V$. Thus, we have constructed a function $u \in L^{p'}(\Omega)$ that solves (11), according to Definition 4.2. Next, we show uniqueness of solutions. Assume that (11) has two solutions u_1 and u_2 with the same right-hand-side datum μ. Then, it follows from (14) that $\int_\Omega (u_1 - u_2)(-\Delta)^s v \,\mathrm{d}x = 0$, for every $v \in V$. It follows from the fundamental lemma of the calculus of variations that $u_1 = u_2$ a.e. in Ω and we have shown the uniqueness of solutions. It remains to prove the required estimate (15). From (17), we have that

$$\left|\int_\Omega u\xi \,\mathrm{d}x\right| \le \|\mu\|_{\mathcal{M}(\Omega)}\|v\|_{C_0(\Omega)} \le C\|\mu\|_{\mathcal{M}(\Omega)}\|\xi\|_{L^p(\Omega)}, \tag{18}$$

where in the last estimate we have used Theorem 4.1. Then, dividing both sides of (18) by $\|\xi\|_{L^p(\Omega)}$ and taking the supremum over $\xi \in L^p(\Omega)$, we obtain (15). The proof is finished. $\square$

The regularity of solutions to (11), given in Theorem 4.2, will be improved in Corollary 7.2.

5 The Optimal Control Problem

The purpose of this section is to study the existence of solutions to the optimal control problem (2) and establish the first-order optimality conditions. Throughout this section, we shall assume that Ω is a bounded Lipschitz domain satisfying the exterior cone condition. Moreover, p is as in (3).

We begin by rewriting the optimal control problem (2). Let $(-\Delta)^s_D$ be the operator defined in (9). Then, the problem (2) can be rewritten as follows:

$$\min_{(u,z)\in(U,Z)} J(u,z)$$

subject to the constraints: (19)

$$(-\Delta)^s_D u = z \quad \text{in } \Omega, \qquad u|_\Omega \in \mathcal{K} \quad \text{and} \quad z \in Z_{ad}.$$

Next, we introduce the relevant function spaces. We let

$$Z := L^p(\Omega) \text{ and } U := \left\{u \in \widetilde{W}_0^{s,2}(\Omega) \cap C_0(\Omega) : ((-\Delta)^s_D)u|_\Omega \in L^p(\Omega)\right\}.$$

Then, U is a Banach space with the graph norm $\|u\|_U := \|u\|_{\widetilde{W}_0^{s,2}(\Omega)} + \|u\|_{C_0(\Omega)} + \|(-\Delta)^s_D u\|_{L^p(\Omega)}$. We let $Z_{ad} \subset Z$ a nonempty, closed, and convex set and $\mathcal{K}$ as in (2b), i.e.,

$$\mathcal{K} := \left\{w \in C_0(\Omega) \ : \ w(x) \le u_b(x), \quad \forall x \in \overline{\Omega}\right\}. \tag{20}$$

Notice that for every $z \in Z$, due to Theorem 4.1, there is a unique $u \in U$ that solves the state equation (2a).Thus, the control-to-state (solution) map, $S : Z \to U, \quad z \mapsto Sz =: u$, is well-defined, linear, and continuous. Since $U \hookrightarrow C_0(\Omega)$, we can consider the control-to-state map as $E \circ S : Z \to C_0(\Omega)$.

Next, we define the admissible control set as $\widehat{Z}_{ad} := \{z \in Z : z \in Z_{ad} \text{ and } (E \circ S)z \in \mathcal{K}\}$, and as a result, the reduced minimization problem is given by

$$\min_{z \in \widehat{Z}_{ad}} \mathcal{J}(z) := J((E \circ S)z, z). \tag{21}$$

Next, we state the well-posedness result for (2) and equivalently for (21).

Theorem 5.1 *Let Z_{ad} be a bounded, closed, and convex subset of Z and $\mathcal{K}$ a convex and closed subset of $C_0(\Omega)$ such that $\widehat{Z}_{ad} \neq \emptyset$. If $J : L^2(\Omega) \times L^p(\Omega) \to \mathbb{R}$, with p as in (3), is weakly lower semicontinuous, then there is a solution to (21).*

Proof The proof is based on the so-called direct method or the Weierstrass theorem [22, Theorem 3.2.1]. We provide some details for completeness. We can always construct a minimizing sequence $\{z_n\}_{n=1}^{\infty} \subset Z$ such that $\inf_{z \in Z_{ad}} \mathcal{J}(z) = \lim_{n\to\infty} \mathcal{J}(z_n)$. Since Z_{ad} is bounded, it follows that $\{z_n\}_{n=1}^{\infty}$ is a bounded sequence. Since Z is reflexive, we have that there exists a weakly convergent subsequence $\{z_n\}_{n=1}^{\infty}$ (not relabeled) such that $z_n \rightharpoonup \bar{z}$ in Z as $n \to \infty$. Next, since Z_{ad} is closed and convex, thus weakly closed, we have that $\bar{z} \in Z_{ad}$.

Next, we notice that $C_0(\Omega)$ is non-reflexive. However, we have that $u_n = Sz_n \in U \hookrightarrow C_0(\Omega)$ and $S \in \mathcal{L}(Z, C_0(\Omega))$. Thus, there is a subsequence $\{u_n\}$ (not-relabeled) that converges weakly* to $\bar{u}$ in $C_0(\Omega)$ as $n \to \infty$. Since $\mathcal{K}$ is also weakly closed, we have that $\bar{u} \in \mathcal{K}$.

Owing to the uniqueness of the limit and the assumption that $\widehat{Z}_{ad}$ is nonempty, we can deduce that $\bar{z} \in \widehat{Z}_{ad}$. Finally, it remains to show that $\bar{z}$ is a solution to (21). This follows from the weak lower-semicontinuity assumption on J. The proof is finished. □

Before deriving the first-order necessary optimality conditions, we make the following assumption.

Assumption 5.2 (*Compatibility condition between $\mathcal{K}$ and Z_{ad}*) There is a pair $(\hat{u}, \hat{z}) \in U \times Z$ that fulfills

$$(-\Delta)^s_D \hat{u} = \hat{z} \quad \text{in } \Omega, \qquad \hat{z} \in Z_{ad}, \qquad \hat{u}(x) < u_b(x) \quad \forall x \in \overline{\Omega}. \tag{22}$$

Notice that the last condition in Assumption 5.2 says that the state constraints in $\mathcal{K}$ are satisfied strictly. Assumption 5.2 is a compatibility condition between Z_{ad} and $\mathcal{K}$. For instance, in the absence of state constraints, it is immediately fulfilled. In addition, if $Z_{ad} = Z$, then again Assumption 5.2 is satisfied, see [12, p. 87] for the classical case. But having both control and state constraints requires a compatibility condition between $\mathcal{K}$ and Z_{ad} as otherwise the solution set might be empty. We need the state constraints to be strictly satisfied for the existence of Lagrange multipliers, see [13, p. 340] for the classical case.

Using the definition of U, we have that $(-\Delta)^s_D : U \mapsto Z$ is a bounded operator and from Theorem 4.1, it is also surjective. We have the following first-order necessary optimality conditions.

Theorem 5.3 *Let $J : L^2(\Omega) \times L^p(\Omega) \to \mathbb{R}$, with p as in (3), be continuously Fréchet differentiable and assume that (22) holds. Let $(\bar{u}, \bar{z})$ be a solution to the optimization problem (2). Then, there exist a Lagrange multiplier $\bar{\mu} \in (C_0(\Omega))^\star$ and an adjoint variable $\bar{\xi} \in L^{p'}(\Omega)$ such that*

$$(-\Delta)^s_D \bar{u} = \bar{z} \quad \text{in } \Omega, \tag{23a}$$

$$\langle \bar{\xi}, (-\Delta)^s_D v \rangle_{L^{p'}(\Omega), L^p(\Omega)} = (J_u(\bar{u}, \bar{z}), v)_{L^2(\Omega)} + \int_\Omega v \, d\bar{\mu}, \quad \forall\, v \in U \tag{23b}$$

$$\langle \bar{\xi} + J_z(\bar{u}, \bar{z}), z - \bar{z} \rangle_{L^{p'}(\Omega), L^p(\Omega)} \geq 0, \qquad \forall\, z \in Z_{ad} \tag{23c}$$

$$\bar{\mu} \geq 0, \quad \bar{u}(x) \leq u_b(x) \text{ in } \Omega, \quad \text{and} \quad \int_\Omega (u_b - \bar{u})\, d\mu = 0. \tag{23d}$$

Proof We begin by checking the requirements for [12, Lemma 1.14]. We notice that $(-\Delta)^s_D : U \mapsto Z$ is bounded and surjective. Moreover, the condition (22) implies that the interior of the set $\mathcal{K}$ is nonempty. It remains to show the existence of a pair $(\hat{u}, \hat{z}) \in U \times Z_{ad}$ such that

$$(-\Delta)^s_D(\hat{u} - \bar{u}) - (\hat{z} - \bar{z}) = 0 \quad \text{in} \;\; \Omega. \tag{24}$$

Since $(\bar{u}, \bar{z})$ solves the state equation, it follows from (24) that

$$(-\Delta)^s_D \hat{u} = \hat{z} \quad \text{in} \;\; \Omega. \tag{25}$$

Notice that for every $\hat{z} \in Z_{ad}$, there is a unique $\hat{u}$ that solves (25) and, in particular, $(\hat{u}, \hat{z})$ works. Thus, the conditions of [12, Lemma 1.14] hold. Then, [12, Theorem 1.56] immediately implies that (23a)–(23c) hold. Instead of (23d), we obtain that

$$\bar{\mu} \in \mathcal{K}^\circ, \quad u(x) \le u_b(x), \quad x \in \Omega, \quad \text{and} \quad \langle \bar{\mu}, \bar{u} \rangle_{C_0(\Omega)^*, C_0(\Omega)} = 0, \tag{26}$$

where $\mathcal{K}^\circ$ denotes the polar cone. Then, the equivalence between (26) and (23d) follows from a classical result in functional analysis (see, e.g., [12, p. 88] for more details). The proof is finished. □

6 Characterization of the Dual of Fractional-Order Sobolev Spaces

Given $0 < s < 1$, $1 \le p < \infty$ and $p' := \frac{p}{p-1}$, the aim of this section is to give a complete characterization of the space $\widetilde{W}^{-s,p'}(\Omega)$. Recall that, $\widetilde{W}^{-s,p'}(\Omega)$ is defined as the dual of the space $\widetilde{W}^{s,p}_0(\Omega)$. Some of the arguments here are motivated by the classical case $s = 1$.

We start by stating this abstract result taken from [23, p. 194].

Lemma 6.1 *If X and W are two Banach spaces, then $X \times W$ endowed with the norm $\|(x, y)\|_{X\times W} := \|x\|_X + \|y\|_W$ is also a Banach space and the dual space $(X \times W)^\star$ is isometrically isomorphic to $X^\star \times W^\star$.*

Let $1 \le p < \infty$ and let $Y := L^p(\Omega) \times L^p(\mathbb{R}^N \times \mathbb{R}^N)$ be endowed with the norm $\|(v_1, v_2)\|_Y := \left(\|v_1\|^p_{L^p} + \|v_2\|^p_{L^p(\mathbb{R}^N\times\mathbb{R}^N)}\right)^{\frac{1}{p}}$. For $v \in \widetilde{W}^{s,p}_0(\Omega)$, we associate the vector $Pv \in Y$ given by

$$Pv := (v, D_{s,p}v). \tag{27}$$

Since $\|Pv\|_Y = \|(v, D_{s,p}v)\|_Y = \|v\|_{\widetilde{W}^{s,p}_0(\Omega)}$, we have that P is an isometry and hence injective. Therefore, $P : \widetilde{W}^{s,p}_0(\Omega) \mapsto Y$ is an isometric isomorphism of $\widetilde{W}^{s,p}_0(\Omega)$ onto its image $Z \subset Y$. Also, Z is a closed subspace of Y, because $\widetilde{W}^{s,p}_0(\Omega)$ is complete (isometries preserve completion).

Throughout this section without any mention, we shall let $Y := L^p(\Omega) \times L^p(\mathbb{R}^N \times \mathbb{R}^N)$.

Lemma 6.2 *Let $1 \leq p < \infty$. Then, for every $f \in Y^\star$, there exists a unique $u = (u_1, u_2) \in L^{p'}(\Omega) \times L^{p'}(\mathbb{R}^N \times \mathbb{R}^N)$ such that for every $v = (v_1, v_2) \in Y$, we have*

$$f(v) = \int_\Omega u_1 v_1 \, dx + \int_{\mathbb{R}^N \times \mathbb{R}^N} u_2 v_2 \, dx \quad \textit{and}$$
$$\|f\|_{Y^*} = \|u\|_{L^{p'}(\Omega)\times L^{p'}(\mathbb{R}^N\times\mathbb{R}^N)} = \|u_1\|_{L^{p'}(\Omega)} + \|u_2\|_{L^{p'}(\mathbb{R}^N\times\mathbb{R}^N)}.$$

Proof Let $w \in L^p(\Omega)$. Then, $(w, 0) \in Y$. We define $f_1(w) := f(w, 0)$. Then, $f_1 \in (L^p(\Omega))^\star$. For arbitrary $w_1, w_2, w \in L^p(\Omega)$ and scalars α, β, we have

$$\begin{aligned} f_1(\alpha w_1 + \beta w_2) &= f(\alpha w_1 + \beta w_2, 0) = f(\alpha(w_1, 0) + \beta(w_2, 0)) \\ &= \alpha f((w_1, 0)) + \beta f((w_2, 0)) = \alpha f_1(w_1) + \beta f_1(w_2), \end{aligned}$$

and $|f_1(w)| = |f((w, 0))| \leq \|f\|_{Y^\star}\|(w, 0)\|_Y = \|f\|_{Y^\star}\|w\|_{L^p(\Omega)}$. Thus, $f_1 \in (L^p(\Omega))^\star = L^{p'}(\Omega)$.

Similarly, let $w \in L^p(\mathbb{R}^N \times \mathbb{R}^N)$. Then, $(0, w) \in Y$ and if we define $f_2(w) := f(0, w)$, we have $f_2 \in (L^p(\mathbb{R}^N \times \mathbb{R}^N))^\star = L^{p'}(\mathbb{R}^N \times \mathbb{R}^N)$.

Therefore, by the Riesz Representation theorem there exist a unique $u_1 \in L^{p'}(\Omega)$ and a unique $u_2 \in L^{p'}(\mathbb{R}^N \times \mathbb{R}^N)$ such that $f(v_1, 0) = f_1(v_1) = \langle u_1, v_1\rangle_{L^{p'}(\Omega), L^p(\Omega)}$ for every $v_1 \in L^p(\Omega)$ and $f(0, v_2) = f_2(v_2) = \langle u_2, v_2\rangle_{L^{p'}(\mathbb{R}^N\times\mathbb{R}^N), L^p(\mathbb{R}^N\times\mathbb{R}^N)}$ for every $v_2 \in L^p(\mathbb{R}^N \times \mathbb{R}^N)$.

Now let $v := (v_1, v_2) \in Y$. We can write $v = (v_1, v_2) = (v_1, 0) + (0, v_2)$. Hence,

$$f(v) = f(v_1, 0) + f(0, v_2) = f_1(v_1) + f_2(v_2) = \int_\Omega u_1 v_1 \, \mathrm{d}x + \int_{\mathbb{R}^N\times\mathbb{R}^N} u_2 v_2 \, \mathrm{d}x.$$

Moreover,

$$\begin{aligned} |f(v)| &\leq \|u_1\|_{L^{p'}(\Omega)}\|v_1\|_{L^p(\Omega)} + \|u_2\|_{L^{p'}(\mathbb{R}^N\times\mathbb{R}^N)}\|v_2\|_{L^p(\mathbb{R}^N\times\mathbb{R}^N)} \\ &\leq \|u\|_{L^{p'}(\Omega)\times L^{p'}(\mathbb{R}^N\times\mathbb{R}^N)}\|v\|_Y. \end{aligned}$$

Therefore,

$$\|f\|_{Y^\star} \leq \|u\|_{L^{p'}(\Omega)\times L^{p'}(\mathbb{R}^N\times\mathbb{R}^N)}. \tag{28}$$

The proof of the first part is complete. It then remains to show that the norms in (28) are equal.

Let us first consider the case $1 < p < \infty$. Define

$$v_1(x) := \begin{cases} |u_1(x)|^{p'-2}\overline{u_1(x)}, & \text{if } u_1(x) \neq 0, \\ 0, & \text{if } u_1(x) = 0, \end{cases} \quad \text{and}$$
$$v_2(x, y) := \begin{cases} |u_2(x, y)|^{p'-2}\overline{u_2(x, y)}, & \text{if } u_2(x, y) \neq 0, \\ 0, & \text{if } u_2(x, y) = 0. \end{cases}$$

Then, for $v = (v_1, v_2)$ we have

$$
\begin{aligned}
|f(v)| &= |f(v_1, v_2)| = |f((v_1, 0) + (0, v_2))| = |f_1(v_1) + f_2(v_2)| \\
&= \Big| \langle u_1, v_1 \rangle_{L^{p'}(\Omega), L^p(\Omega)} + \langle u_2, v_2 \rangle_{L^{p'}(\mathbb{R}^N \times \mathbb{R}^N), L^p(\mathbb{R}^N \times \mathbb{R}^N)} \Big| \\
&= \|u_1\|^{p'}_{L^{p'}(\Omega)} + \|u_2\|^{p'}_{L^{p'}(\mathbb{R}^N \times \mathbb{R}^N)} = \|u\|^{p'}_{L^{p'}(\Omega) \times L^{p'}(\mathbb{R}^N \times \mathbb{R}^N)} \\
&= |\langle u, v \rangle_{Y^\star, Y}| = \|v\|_Y \|u\|_{Y^\star} = \|v\|_Y \|u\|_{L^{p'}(\Omega) \times L^{p'}(\mathbb{R}^N \times \mathbb{R}^N)},
\end{aligned}
$$

where we have used the equality in Hölder's inequality and the equality holds because $|v_i|^p = |u_i|^{p'}$. Moreover, we have used the fact that $Y^\star \cong L^{p'}(\Omega) \times L^{p'}(\mathbb{R}^N \times \mathbb{R}^N)$ due to Lemma 6.1.

Let us consider the case $p = 1$. Then, $Y = L^1(\Omega) \times L^1(\mathbb{R}^N \times \mathbb{R}^N)$ and we can set (due to Lemma 6.1) $Y^\star = L^\infty(\Omega) \times L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$. Notice that $\|u\|_{Y^\star} := \max \left\{ \|u_1\|_{L^\infty(\Omega)}, \|u_2\|_{L^\infty(\mathbb{R}^N \times \mathbb{R}^N)} \right\}$. To get the desired result, it is sufficient to show that $\|f\|_{Y^\star} \geq \|u\|_{Y^\star}$. Now, for any $\epsilon > 0$ and $k = 1$ there exists a measurable set $A \subset \Omega$ (or $\subset \mathbb{R}^N \times \mathbb{R}^N$ when $k = 2$) with finite, nonzero measure such that $|u_k(x)| \geq \|u\|_{Y^\star} - \epsilon$, for almost every $x \in A$.

Next, we define $v_k(x) := \begin{cases} \frac{\overline{u_k(x)}}{|u_k(x)|}, & \text{for } x \in A \text{ and } u_k(x) \neq 0, \\ 0, & \text{elsewhere .} \end{cases}$

Set $v := (v_k, 0)$ if $k = 1$, otherwise set $v := (0, v_k)$. Then,

$$
\begin{aligned}
|f(v)| &= \Big| \langle u_k, v_k \rangle_{Y^\star, Y} \Big| = \int_A |u_k(x)| \, \mathrm{d}x \geq \Big(\|u\|_{Y^\star} - \epsilon \Big) \|v\|_Y \\
&= \left(\|u\|_{L^\infty(\Omega) \times L^\infty \mathbb{R}^N \times \mathbb{R}^N)} - \epsilon \right) \|v\|_Y .
\end{aligned}
$$

Since ϵ is chosen arbitrarily, we have that the result follows from the definition of the operator norm. The proof is finished. □

Theorem 6.1 *Let $1 \leq p < \infty$ and $f \in \widetilde{W}^{-s,p'}(\Omega)$. Then, there exists $(f^0, f^1) \in L^{p'}(\Omega) \times L^{p'}(\mathbb{R}^N \times \mathbb{R}^N)$ such that for all $v \in \widetilde{W}^{s,p}_0(\Omega)$,*

$$
\langle f, v \rangle_{\widetilde{W}^{-s,p'}(\Omega), \widetilde{W}^{s,p}_0(\Omega)} = \int_\Omega f^0 v \, dx + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f^1(x, y) D_{s,p} v[x, y] \, dx \, dy, \quad (29)
$$

$$
\|f\|_{\widetilde{W}^{-s,p'}(\Omega)} = \inf \left\{ \|(f^0, f^1)\|_{L^{p'}(\Omega) \times L^{p'}(\mathbb{R}^N \times \mathbb{R}^N)} \right\}, \quad (30)
$$

where the infimum is taken over all $(f^0, f^1) \in L^{p'}(\Omega) \times L^{p'}(\mathbb{R}^N \times \mathbb{R}^N)$ for which (29) holds. Moreover, if $1 < p < \infty$, then (f^0, f^1) is unique.

Proof Define the linear functional $\widehat{L} : Z \to \mathbb{R}$, where $Z \subset Y$ is the range of P given in (27), by

$$
\widehat{L}(Pv) = f(v), \quad v \in \widetilde{W}^{s,p}_0(\Omega)
$$

$$
\begin{tikzcd}
\widetilde{W}_0^{s,p}(\Omega) \arrow[r, "P"] \arrow[dr, "f"'] & Z \subset Y \arrow[d, "\widehat{L}"] \\
& \mathbb{R}.
\end{tikzcd}
$$

Since P is an isometric isomorphism onto Z, it follows that $\widehat{L} \in Z^\star$ and

$$
\|\widehat{L}\|_{Z^\star} = \sup_{\|Pv\|_Y=1} |\langle \widehat{L}, Pv\rangle_{Y^\star,Y}| = \sup_{\|v\|_{\widetilde{W}_0^{s,p}}=1} |\langle f, v\rangle_{\widetilde{W}^{-s,p'}(\Omega),\widetilde{W}_0^{s,p}(\Omega)}| = \|f\|_{W^{-s,p'}(\Omega)}.
$$

By the Hahn–Banach theorem, there exists an $L \in Y^\star = L^{p'}(\Omega)\times L^{p'}(\mathbb{R}^N\times\mathbb{R}^N)$ with $\|L\|_{Y^\star} = \|\widehat{L}\|_{Z^\star}$. Since $L \in Y^\star$, using Lemma 6.2, we have that there exists $(f^0, f^1) \in L^{p'}(\Omega) \times L^{p'}(\mathbb{R}^N \times \mathbb{R}^N)$ such that $L(v) = \int_\Omega f^0 v_1 \,\mathrm{d}x + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f^1 v_2 \,\mathrm{d}x\mathrm{d}y$ for every $v = (v_1, v_2) \in Y$. Notice that when $1 < p < \infty$, (f^0, f^1) is unique due to the uniform convexity of the Banach space $L^p(\Omega) \times L^p(\mathbb{R}^N \times \mathbb{R}^N)$.

Thus, for $v \in \widetilde{W}_0^{s,p}(\Omega)$ we have $Pv \in Y$. Using the definition of $\widehat{L}$, we get

$$
f(v) = \widehat{L}(Pv) = L(Pv) = L(v, D_{s,p}v) = \int_\Omega f^0 v \,\mathrm{d}x + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f^1 D_{s,p}v \,\mathrm{d}x\mathrm{d}y,
$$

which is (29), after noticing that $\langle f, v\rangle_{\widetilde{W}^{-s,p'}(\Omega),\widetilde{W}_0^{s,p}(\Omega)} = f(v)$. Moreover, we have $\|f\|_{\widetilde{W}^{-s,p'}(\Omega)} = \|\widehat{L}\|_{Z^\star} = \|L\|_{Y^\star} = \|(f^0, f^1)\|_{L^{p'}(\Omega)\times L^{p'}(\mathbb{R}^N\times\mathbb{R}^N)}$.

Now, for arbitrary $(g^0, g^1) \in L^{p'}(\Omega) \times L^{p'}(\mathbb{R}^N \times \mathbb{R}^N)$, for which (29) holds for all $v \in \widetilde{W}_0^{s,p}(\Omega)$, we can define L_g as $L_g(u) = \langle g^0, u_1\rangle_{L^{p'}(\Omega),L^p(\Omega)} + \langle g^1, u_2\rangle_{L^{p'}(\mathbb{R}^N\times\mathbb{R}^N),L^p(\mathbb{R}^N\times\mathbb{R}^N)}, \quad \forall\, u \in Y.$

Then, $L_g \in Y^\star$ and $L_g|_Z = \widehat{L}$ (due to (29)). As a result, $\|\widehat{L}\|_{Z^\star} \le \|L_g\|_{Y^\star}$. Thus, $\|f\|_{\widetilde{W}^{-s,p'}(\Omega)} \le \|g\|_{L^{p'}(\Omega)\times L^{p'}(\mathbb{R}^N\times\mathbb{R}^N)}$.
The proof is complete. □

In view of Theorem 3.1(b), we have the following result.

Corollary 6.1 *Let $1 < p < \infty$, $\frac{1}{p} < s < 1$ and $f \in \widetilde{W}^{-s,p'}(\Omega)$. Then, there exists a unique $(f^0, f^1) \in L^{p'}(\Omega) \times L^{p'}(\Omega \times \Omega)$ such that for every $v \in \widetilde{W}_0^{s,p}(\Omega)$,*

$$
\langle f, v\rangle_{\widetilde{W}^{-s,p'}(\Omega),\widetilde{W}_0^{s,p}(\Omega)} = \int_\Omega f^0 v\, dx + \int_\Omega\int_\Omega f^1(x,y) D_{s,p}v[x,y]\, dx\, dy, \tag{31}
$$

$$
\|f\|_{\widetilde{W}^{-s,p'}(\Omega)} = \inf\left\{\|(f^0, f^1)\|_{L^{p'}(\Omega)\times L^{p'}(\Omega\times\Omega)}\right\}, \tag{32}
$$

where the infimum is taken over all $(f^0, f^1) \in L^{p'}(\Omega) \times L^{p'}(\Omega \times \Omega)$ for which (31) holds.

7 Improved Regularity of State and Higher Regularity of Adjoint

In this section, we study the higher regularity properties of solutions to the Dirichlet problem (2a), with a right-hand side $z \in \widetilde{W}^{-t,p}(\Omega)$, for suitable values of $p \in]1, \infty[$ and $0 < t < 1$.

7.1 Regularity of the State

Throughout the remainder of this section, for $u, v \in \widetilde{W}_0^{s,2}(\Omega)$, we shall let

$$\mathcal{E}(u, v) := \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, \mathrm{d}x\mathrm{d}y. \tag{33}$$

We start with the following theorem which can be viewed as the first main result of this section.

Theorem 7.1 *Let $f_0 \in L^p(\Omega)$ with $p > \frac{N}{2s}$ and $f_1 \in L^q(\Omega \times \Omega)$ with $q > \frac{N}{s}$ if $N \geq 2s$ and $q \geq 2$ if $N < 2s$. Then, there exists a unique function $u \in \widetilde{W}_0^{s,2}(\Omega)$ satisfying*

$$\mathcal{E}(u, v) = \int_\Omega f_0 v \, dx + \int_\Omega \int_\Omega f_1(x, y) D_{s,2} v[x, y] \, dxdy, \tag{34}$$

for every $v \in \widetilde{W}_0^{s,2}(\Omega)$. In addition, $u \in L^\infty(\Omega)$ and there is a constant $C > 0$ such that

$$\|u\|_{L^\infty(\Omega)} \leq C \left(\|f_0\|_{L^p(\Omega)} + \|f_1\|_{L^q(\Omega\times\Omega)} \right). \tag{35}$$

To prove the theorem, we need the following lemma taken from [24, Lemma B.1].

Lemma 7.1 *Let $\Phi = \Phi(t)$ be a nonnegative, non-increasing function on a half line $t \geq k_0 \geq 0$ such that there are positive constants c, α and δ ($\delta > 1$) with*

$$\Phi(h) \leq c(h - k)^{-\alpha} \Phi(k)^\delta \text{ for } h > k \geq k_0.$$

Then, $\Phi(k_0 + d) = 0 \quad$ with $\quad d^\alpha = c\Phi(k_0)^{\delta-1} 2^{\alpha\delta/(\delta-1)}$.

Proof of Theorem 7.1 We prove the result in several steps.

Step 1. Firstly, we show that there is a unique $u \in \widetilde{W}_0^{s,2}(\Omega)$ satisfying (34). Indeed, recall that $\widetilde{W}_0^{s,2}(\Omega) \hookrightarrow L^{p'}(\Omega)$, by Remark 3.1. Notice also that if $N \geq 2s$, then $q \geq 2$. Since Ω is bounded, we have that, in all the cases, the continuous embedding $L^q(\Omega \times \Omega) \hookrightarrow L^2(\Omega \times \Omega)$ holds. Hence, using the classical Hölder inequality, we get that there is a constant $C > 0$ such that

$$\begin{aligned} &\left| \int_\Omega f_0 v \, \mathrm{d}x + \int_\Omega \int_\Omega f_1(x, y) D_{s,2} v[x, y] \, \mathrm{d}x\mathrm{d}y \right| \\ &\quad \leq C \left(\|f_0\|_{L^p(\Omega)} + \|f_1\|_{L^2(\Omega\times\Omega)} \right) \|v\|_{\widetilde{W}_0^{s,2}(\Omega)}. \end{aligned}$$

Since the bilinear form $\mathcal{E}$ is continuous and coercive, it follows from the classical Lax–Milgram lemma that there is unique $u \in \widetilde{W}_0^{s,2}(\Omega)$ satisfying (34).

Step 2. Notice that, if $N < 2s$, then it follows from the embedding (4) that $u \in L^\infty(\Omega)$. We give the proof for the case $N > 2s$. The case $N = 2s$ follows with a simple modification of the case $N > 2s$.

Step 3. Let $u \in \widetilde{W}_0^{s,2}(\Omega)$ be the unique function satisfying (34). Let $k \geq 0$ be a real number and set $u_k := (|u| - k)^+\text{sgn}(u)$. By [15, Lemma 2.7], we have that $u_k \in \widetilde{W}_0^{s,2}(\Omega)$ for every $k \geq 0$. Proceeding as in the proof of [25, Theorem 2.9] (see also [3, Proposition 3.10 and Section 3.3]), we get that for every $k \geq 0$,

$$\mathcal{E}(u_k, u_k) \leq \mathcal{E}(u_k, u) = \int_\Omega f_0 u_k \, dx + \int_\Omega \int_\Omega f_1(x, y) D_{s,2} u_k[x, y] \, dxdy. \tag{36}$$

Let $A_k := \{x \in \Omega : \ |u(x)| \geq k\}$. Then, it is clear that

$$u_k = \Big[(|u| - k)\text{sign}(u)\Big]\chi_{A_k}. \tag{37}$$

Let $p_1 \in [1, \infty]$ be such that $\dfrac{1}{p} + \dfrac{1}{2^\star} + \dfrac{1}{p_1} = 1$, where we recall that $2^\star := \frac{2N}{N-2s}$. Since by assumption $p > \frac{N}{2s} = \frac{2^\star}{2^\star-2}$, we have that

$$\frac{1}{p_1} = 1 - \frac{1}{2^\star} - \frac{1}{p} = \frac{2^\star}{2^\star} - \frac{1}{2^\star} - \frac{1}{p} > \frac{2^\star}{2^\star} - \frac{1}{2^\star} - \frac{2^\star - 2}{2^\star} = \frac{1}{2^\star} \Longrightarrow p_1 < 2^\star. \tag{38}$$

Using (37), the continuous embedding $\widetilde{W}_0^{s,2}(\Omega) \hookrightarrow L^{2^\star}(\Omega)$, and the Hölder inequality, we get that there is a constant $C > 0$ such that for every $k \geq 0$,

$$\int_\Omega f_0 u_k \, dx = \int_{A_k} f_0 u_k \, dx \leq \|f_0\|_{L^p(\Omega)} \|u_k\|_{\widetilde{W}_0^{s,2}(\Omega)} \|\chi_{A_k}\|_{L^{p_1}(\Omega)}. \tag{39}$$

Let $\delta_1 := \frac{2^\star}{p_1}$. Then, $\delta_1 > 1$ by (38), but this not needed here. We have that for every $k \geq 0$,

$$\|\chi_{A_k}\|_{L^{p_1}(\Omega)} = |A_k|^{\frac{1}{p_1}} = \left(|A_k|^{\frac{1}{2^\star}}\right)^{\frac{2^\star}{p_1}} = \|\chi_{A_k}\|_{L^{2^\star}(\Omega)}^{\frac{2^\star}{p_1}} = \|\chi_{A_k}\|_{L^{2^\star}(\Omega)}^{\delta_1}. \tag{40}$$

Step 4. Next, let $q_1 \in [1, \infty]$ be such that $\dfrac{1}{q} + \dfrac{1}{2} + \dfrac{1}{q_1} = 1$. Since $q > \frac{N}{s} = 2\frac{N}{2s} = 2\frac{2^\star}{2^\star-2}$, we have

$$\frac{1}{q_1} = 1 - \frac{1}{2} - \frac{1}{q} = 2\frac{2^\star}{2 \cdot 2^\star} - \frac{1}{2} - \frac{1}{q} > \frac{2 \cdot 2^\star}{2 \cdot 2^\star} - \frac{1}{2} - \frac{2^\star - 2}{2 \cdot 2^\star} = \frac{1}{2^\star} \Longrightarrow q_1 < 2^\star. \tag{41}$$

Using (37), the continuous embedding $\widetilde{W}_0^{s,2}(\Omega) \hookrightarrow L^{2^\star}(\Omega)$, and the Hölder inequality again, we can deduce that there is a constant $C > 0$ such that for every $k \geq 0$,

$$\begin{aligned}\int_\Omega \int_\Omega f_1(x,y) D_{s,2}u_k[x,y]\,\mathrm{d}x\mathrm{d}y &= \int_{A_k}\int_{A_k} f_1(x,y) D_{s,2}u_k[x,y]\,\mathrm{d}x\mathrm{d}y\\ &+ \int_{A_k}\int_{\Omega\setminus A_k} f_1(x,y) D_{s,2}u_k[x,y]\,\mathrm{d}x\mathrm{d}y + \int_{\Omega\setminus A_k}\int_{A_k} f_1(x,y) D_{s,2}u_k[x,y]\,\mathrm{d}x\mathrm{d}y\\ &\leq C\|f_1\|_{L^q(\Omega\times\Omega)}\|u_k\|_{\widetilde{W}_0^{s,2}(\Omega)}\|\chi_{A_k}\|_{L^{q_1}(\Omega)}. \end{aligned} \tag{42}$$

Let $\delta_2 := \frac{2^\star}{q_1}$. Then, $\delta_2 > 1$ by (41), which is also not needed here. As in (40), for every $k \geq 0$,

$$\|\chi_{A_k}\|_{L^{q_1}(\Omega)} = \|\chi_{A_k}\|^{\delta_2}_{L^{2^\star}(\Omega)}. \tag{43}$$

Step 5. Let $\delta := \min\{\delta_1, \delta_2\} > 1$. It follows from (40) that $\|\chi_{A_k}\|_{L^{p_1}(\Omega)} = \|\chi_{A_k}\|^{\delta_1}_{L^{2^\star}(\Omega)} = \|\chi_{A_k}\|^{\delta}_{L^{2^\star}(\Omega)}\|\chi_{A_k}\|^{\delta_1-\delta}_{L^{2^\star}(\Omega)} \leq \|\chi_\Omega\|^{\delta_1-\delta}_{L^{2^\star}(\Omega)}\|\chi_{A_k}\|^{\delta}_{L^{2^\star}(\Omega)}$ for every $k \geq 0$.

Similarly, it follows from (43) that $\|\chi_{A_k}\|_{L^{q_1}(\Omega)} \leq \|\chi_\Omega\|^{\delta_2-\delta}_{L^{2^\star}(\Omega)}\|\chi_{A_k}\|^{\delta}_{L^{2^\star}(\Omega)}$ for every $k \geq 0$.

We have shown that there is a constant $C > 0$ such that for every $k \geq 0$,

$$\max\{\|\chi_{A_k}\|_{L^{p_1}(\Omega)}, \|\chi_{A_k}\|_{L^{q_1}(\Omega)}\} \leq C\|\chi_{A_k}\|^{\delta}_{L^{2^\star}(\Omega)}. \tag{44}$$

Using (36), (39), (42), (44), and the fact that there is a constant $C > 0$ such that $C\|u_k\|_{\widetilde{W}_0^{s,2}(\Omega)} \leq \mathcal{E}(u_k, u_k)$, we get that there is a constant $C > 0$ such that for every $k \geq 0$,

$$\|u_k\|_{\widetilde{W}_0^{s,2}(\Omega)} \leq C\left(\|f_0\|_{L^p(\Omega)} + \|f_1\|_{L^q(\Omega\times\Omega)}\right)\|\chi_{A_k}\|^{\delta}_{L^{2^\star}(\Omega)}. \tag{45}$$

Using the continuous embedding $\widetilde{W}_0^{s,2}(\Omega) \hookrightarrow L^{2^\star}(\Omega)$ and (45), we get that there is a constant $C > 0$ such that for every $k \geq 0$,

$$\|u_k\|_{L^{2^\star}(\Omega)} \leq C\left(\|f_0\|_{L^p(\Omega)} + \|f_1\|_{L^q(\Omega\times\Omega)}\right)\|\chi_{A_k}\|^{\delta}_{L^{2^\star}(\Omega)}. \tag{46}$$

Step 6. Now let $h > k \geq 0$. Then, $A_h \subset A_k$ and in A_h, we have that $|u_k| \geq (h-k)$. Thus, it follows from (46) that there is a constant $C > 0$ such that for every $h > k \geq 0$,

$$\|\chi_{A_h}\|_{L^{2^\star}(\Omega)} \leq C(h-k)^{-1}\left(\|f_0\|_{L^p(\Omega)} + \|f_1\|_{L^q(\Omega\times\Omega)}\right)\|\chi_{A_k}\|^{\delta}_{L^{2^\star}(\Omega)}. \tag{47}$$

Let $\Phi(k) := \|\chi_{A_k}\|_{L^{2^\star}(\Omega)}$. It follows from (47) that for all $h > k \geq 0$, we have

$$\Phi(h) \leq C(h-k)^{-1}\left(\|f_0\|_{L^p(\Omega)} + \|f_1\|_{L^q(\Omega\times\Omega)}\right)\Phi(k)^{\delta}.$$

Applying Lemma 7.1 to the function Φ, we can deduce that there is a constant $C_1 > 0$ such that $\Phi(K) = 0$ with $K := C_1 C \left(\|f_0\|_{L^p(\Omega)} + \|f_1\|_{L^q(\Omega\times\Omega)}\right)$. We have shown (35) as needed. □

The following theorem is the second main result of this section. Here, we reduce the regularity of the datum z, if one compares with [3, Theorem 3.7].

Theorem 7.2 *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with a Lipschitz continuous boundary. Let $2 < \frac{N}{s} < p \le \infty$ and $0 < \frac{p-1}{p} = \frac{1}{p'} < t < s < 1$. Then, for every $z \in \widetilde{W}^{-t,p}(\Omega)$, there is a unique solution $u \in \widetilde{W}_0^{s,2}(\Omega)$ of (2a). In addition, $u \in L^\infty(\Omega)$ and there is a constant $C > 0$ such that*

$$\|u\|_{L^\infty(\Omega)} \le C\|z\|_{\widetilde{W}^{-t,p}(\Omega)}. \tag{48}$$

Proof We prove the result in several steps.

Step 1. Firstly, for $z \in \widetilde{W}^{-t,p}(\Omega)$, by a solution to (2a), we mean a function $u \in \widetilde{W}_0^{s,2}(\overline{\Omega})$ satisfying

$$\mathcal{E}(u, v) = \langle z, v\rangle_{\widetilde{W}^{-t,p}(\Omega),\widetilde{W}^{t,p'}(\Omega)}, \quad \forall\, v \in \widetilde{W}_0^{t,p'}(\Omega), \tag{49}$$

provided that the left- and right-hand-side expressions make sense.

Step 2. Secondly, since $\frac{1}{p'} < t < 1$ and $z \in \widetilde{W}^{-t,p}(\Omega)$, it follows from Corollary 6.1 that there exists a pair of functions $(f^0, f^1) \in L^p(\Omega) \times L^p(\Omega \times \Omega)$ such that, for every $v \in \widetilde{W}_0^{t,p'}(\Omega)$, we have

$$\langle z, v\rangle_{\widetilde{W}^{-t,p}(\Omega),\widetilde{W}^{t,p'}(\Omega)} = \int_\Omega f^0 v\,\mathrm{d}x + \int_\Omega\int_\Omega f^1(x, y) D_{t,p'}v[x, y]\,\mathrm{d}x\,\mathrm{d}y. \tag{50}$$

Choose $(f^0, f^1) \in L^p(\Omega) \times L^p(\Omega \times \Omega)$ satisfying (50) and are such that

$$\|z\|_{\widetilde{W}^{-t,p}(\Omega)} = \|f^0\|_{L^p(\Omega)} + \|f^1\|_{L^p(\Omega\times\Omega)}. \tag{51}$$

Since $0 < t < s < 1$ and $2 > p'$, it follows that the continuous embedding $\widetilde{W}_0^{s,2}(\Omega) \hookrightarrow \widetilde{W}_0^{t,p'}(\Omega)$ holds. More precisely, there is a constant $C > 0$ such that

$$\begin{aligned}\left|D_{t,p'}v[x, y]\right| &= \frac{|v(x) - v(y)|}{|x - y|^{\frac{p'}{N}+t}} = \frac{|v(x) - v(y)|}{|x - y|^{\frac{2}{N}+s}}|x - y|^{s-t+\frac{2-p'}{N}}\\ &\le C\left|D_{s,2}v[x, y]\right|,\end{aligned}$$

where we used that $s-t+\frac{2-p'}{N} > 0$. Thus, $\|D_{t,p'}v\|_{L^{p'}(\Omega\times\Omega)} \le C\|D_{s,2}v\|_{L^2(\Omega\times\Omega)}$ for every $v \in \widetilde{W}_0^{s,2}(\Omega)$. Hence, (50) also holds for every $v \in \widetilde{W}_0^{s,2}(\Omega)$ and the expressions in (49) make sense.

Step 3. We claim that there is a unique $u \in \widetilde{W}_0^{s,2}(\Omega)$ satisfying (49). Indeed, let $v \in \widetilde{W}_0^{s,2}(\Omega)$. Using Step 2 and Remark 3.1, we get that there is a constant $C > 0$ such that for every $v \in \widetilde{W}_0^{t,p'}(\Omega)$,

$$
\begin{aligned}
\left|\langle z, v\rangle_{\widetilde{W}^{-t,p}(\Omega),\widetilde{W}^{t,p'}(\Omega)}\right| =& \left|\int_\Omega f^0 v\,\mathrm{d}x + \int_\Omega\int_\Omega f^1(x,y)D_{t,p'}v[x,y]\,\mathrm{d}x\,\mathrm{d}y\right| \\
\le& \|f^0\|_{L^p(\Omega)}\|v\|_{L^{p'}(\Omega)} + \|f^1\|_{L^p(\Omega\times\Omega)}\|D_{t,p'}v\|_{L^{p'}(\Omega\times\Omega)} \\
\le& C\left(\|f^0\|_{L^p(\Omega)} + \|f^1\|_{L^p(\Omega\times\Omega)}\right)\|v\|_{\widetilde{W}_0^{s,2}(\Omega)}.
\end{aligned}
$$

We have shown that the right-hand side of (50) defines a linear continuous functional on $\widetilde{W}_0^{s,2}(\Omega)$. Thus, the claim follows by applying the Lax–Milgram lemma.

Step 4. It follows from Step 3 that the unique function $u \in \widetilde{W}_0^{s,2}(\Omega)$ satisfying (49) is such that for every $v \in \widetilde{W_0}^{t,p'}(\Omega)$, we have

$$
\begin{aligned}
\mathcal{E}(u,v) =& \langle z, v\rangle_{\widetilde{W}^{-t,p}(\Omega),\widetilde{W}_0^{t,p'}(\Omega)} \\
&\le C\int_\Omega |f^0 v|\,\mathrm{d}x + \int_\Omega\int_\Omega \left|f^1(x,y)D_{s,2}v[x,y]\right|\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}
$$

Therefore, proceeding exactly as in the proof of Theorem 7.1, we get that $u \in L^\infty(\Omega)$ and there is a constant $C>0$ such that $\|u\|_{L^\infty(\Omega)} \le C\left(\|f^0\|_{L^p(\Omega)} + \|f^1\|_{L^p(\Omega\times\Omega)}\right) = C\|z\|_{\widetilde{W}^{-t,p}(\Omega)}$, where we have used (51). We have shown (48) and the proof is finished. □

We have the following regularity result as a corollary of Theorems 4.1 and 7.2.

Corollary 7.1 *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain satisfying the exterior cone condition. Let $2 < \frac{N}{s} < p \le \infty$ and $0 < \frac{p-1}{p} = \frac{1}{p'} < t < s < 1$. Let $z \in \widetilde{W}^{-t,p}(\Omega)$ and let $u \in \widetilde{W}_0^{s,2}(\Omega)$ be the unique weak solution of (2a). Then, $u \in C_0(\Omega)$.*

Proof Let $z \in \widetilde{W}^{-t,p}(\Omega)$ and $\{z_n\}_{n\ge1} \subset L^\infty(\Omega)$ a sequence such that $z_n \to z$ in $\widetilde{W}^{-t,p}(\Omega)$ as $n \to \infty$. Let $u_n \in \widetilde{W}_0^{s,2}(\Omega)$ satisfy $\mathcal{E}(u_n, v) = \langle z_n, v\rangle_{\widetilde{W}^{-t,p}(\Omega),\widetilde{W}^{t,p'}(\Omega)} = \int_\Omega z_n v\,\mathrm{d}x$ for every $v \in \widetilde{W}_0^{s,2}(\Omega)$. It follows, from Theorem 4.1, that $u_n \in C_0(\Omega)$. Since $u_n - u \in \widetilde{W}_0^{s,2}(\overline{\Omega})$ and satisfies $\mathcal{E}(u_n - u, v) = \langle z_n - z, v\rangle_{\widetilde{W}^{-t,p}(\Omega),\widetilde{W}^{t,p'}(\Omega)}$ for every $v \in \widetilde{W}_0^{s,2}(\Omega)$, it follows, from Theorem 7.2, that $(u_n - u) \in L^\infty(\Omega)$ and there is a constant $C > 0$ (independent of n) such that $\|u_n - u\|_{L^\infty(\Omega)} \le C\|z_n - z\|_{\widetilde{W}^{-t,p}(\Omega)}$. Since $u_n \in C_0(\Omega)$ and $z_n \to z$ in $\widetilde{W}^{-t,p}(\Omega)$ as $n \to \infty$, it follows from the preceding estimate that $u_n \to u$ in $L^\infty(\Omega)$ as $n \to \infty$. Thus, $u \in C_0(\Omega)$ and the proof is finished. □

Next, we improve the regularity of u solving (11) with a measure μ as the right-hand-side datum. Notice that such a result will immediately improve the regularity of the adjoint variable $\bar{\xi}$ solving (23b). Recall that the best result so far proved for solutions of (11) is given in Theorem 4.2.

Corollary 7.2 *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain satisfying the exterior cone condition. Let $2 < \frac{N}{s} < p \leq \infty$ and $0 < \frac{p-1}{p} = \frac{1}{p'} < t < s < 1$. Let $\mu \in \mathcal{M}(\Omega)$. Then, there is a unique solution $u \in \widetilde{W}_0^{t,p'}(\Omega)$ to (11) and there is a constant $C > 0$ such that $\|u\|_{\widetilde{W}_0^{t,p'}(\Omega)} \leq C\|\mu\|_{\mathcal{M}(\Omega)}$.*

Proof The proof follows exactly as the proof of Theorem 4.2 with the exception that, for the inequality (18), we use Corollary 7.1 to get $\left|\int_\Omega u\xi \, \mathrm{d}x\right| \leq \|\mu\|_{\mathcal{M}(\Omega)}\|v\|_{C_0(\Omega)} \leq C\|\mu\|_{\mathcal{M}(\Omega)}\|\xi\|_{\widetilde{W}^{-t,p}(\Omega)}$. The preceding estimate implies that $u \in (\widetilde{W}^{-t,p}(\Omega))^\star = \widetilde{W}_0^{t,p'}(\Omega)$. The proof is finished. □

Recall that the "strong form" of the adjoint equation (23b) is given by

$$(-\Delta)^s\bar{\xi} = J_u(\bar{u}, \bar{z}) + \bar{\mu} \quad \text{in } \Omega, \qquad \bar{\xi} = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \tag{52}$$

Using Corollary 7.2 and the fact that $J_u(\bar{u}, \bar{z}) \in L^2(\Omega)$, we obtain the following regularity result.

Corollary 7.3 (Regularity of the adjoint variable) *Let $\bar{\mu} \in \mathcal{M}(\Omega)$ and let $\bar{\xi}$ be the Lagrange multiplier given in Theorem 5.3. Then, under the conditions of Corollary 7.2, we have that $\bar{\xi} \in \widetilde{W}_0^{t,p'}(\Omega)$.*

7.2 Regularity of Control

In this section, we apply the results obtained in the previous section to the optimal control problem. In the literature, a typical cost functional J is given by (cf. the monograph [26])

$$J(u, z) := \frac{1}{2}\|u - u_d\|^2_{L^2(\Omega)} + g(z)\,, \tag{53}$$

where $u_d \in L^2(\Omega)$ is given. When $g(z) := \frac{\alpha}{2}\|z\|^2_{L^2(\Omega)}$, with given parameter $\alpha > 0$, then (23c) becomes

$$\bar{z} = \mathbb{P}_{Z_{ad}}\left(-\alpha^{-1}\bar{\xi}\right), \tag{54}$$

where $\mathbb{P}_{Z_{ad}}$ denotes the projection onto the set Z_{ad}. Recall that, $\bar{\xi}$ is the adjoint variable solving (23b). We emphasize that Z_{ad} is still the same as before, and with a choice of $J(\cdot, \cdot)$, all the assumptions in the previous results hold. Recall that the boundedness of Z_{ad} enforces $L^p(\Omega)$ regularity on the control z, in addition with the choice of $g(z) = \frac{\alpha}{2}\|z\|^2_{L^2(\Omega)}$ in (53), we are enforcing $L^2(\Omega)$ regularity on the control z. As a result, p is always greater than equal to 2. Finally, (54) can improve the regularity of the optimal control from $L^p(\Omega)$ to $W^{t,p'}(\Omega)$.

Theorem 7.3 (Regularity of control) *Let the conditions of Corollary 7.3 hold and J be as in (53) with $g(z) := \frac{\alpha}{2}\|z\|^2_{L^2(\Omega)}$ with $\alpha > 0$. Given $a, b \in W^{t,p'}(\Omega)$ with $a < b$ a.e.*

in Ω, *let* $Z_{ad} := \{z \in L^p(\Omega)\ :\ a(x) \le z(x) \le b(x),\ a.e.\ in\ \Omega\}$. *Then, the optimal control* $\bar{z} \in W^{t,p'}(\Omega)$.

Proof Under the assumption on Z_{ad}, the projection in (54) becomes $\mathbb{P}_{Z_{ad}}(\bar{\xi}) := \max\{a, \min\{b, \bar{\xi}\}\}$. Since $\bar{\xi}$ in $\widetilde{W}_0^{t,p'}(\Omega)$, in particular, we have $\bar{\xi}|_\Omega \in W^{t,p'}(\Omega)$. Next since $b \in W^{t,p'}(\Omega)$, using [15, Lemma 2.7], we have that $v := \min\{b, \bar{\xi}\} \in W^{t,p'}(\Omega)$. Similarly, $\max\{a, v\} \in W^{t,p'}(\Omega)$. Thus, from (54), we obtain that $\bar{z} \in W^{t,p'}(\Omega)$ and the proof is complete. □

Remark 7.1 We notice that since $\bar{z} \in W^{t,p'}(\Omega)$ (by Theorem 7.3), we have that the regularity of the corresponding solution $\bar{u}$ to the state equation (2a) can also be improved. More precisely, by [27, Theorem 7.1 and p524], we have that $\bar{u} \in H_{p'}^{s(t+2s)}(\overline{\Omega}) \cap \widetilde{W}_0^{s,2}(\Omega)$. We refer to [28, Equation (2.9)] for the precise definition of the space $H_{p'}^{s(t+2s)}(\overline{\Omega})$.

7.3 Control in $\widetilde{W}^{-t,p}(\Omega)$ Instead of $L^p(\Omega)$

Let Ω, s, t and p be as in Corollary 7.1. Then, all the results obtained in Sect. 5 for the optimal control problem hold, with obvious modification of the proofs, if one considers the spaces

$$Z := \widetilde{W}^{-t,p}(\Omega) \quad \text{and} \quad U := \Big\{u \in \widetilde{W}_0^{s,2}(\Omega) \cap C_0(\Omega) : ((-\Delta)_D^s u)|_\Omega \in \widetilde{W}^{-t,p}(\Omega)\Big\}.$$

Notice that, in this case, Z_{ad} is a closed and convex subset of $\widetilde{W}^{-t,p}(\Omega)$ instead of $L^p(\Omega)$. We further emphasize that even in this case, the adjoint variable still enjoys the higher regularity as given in Corollary 7.3. Furthermore, the result given in Theorem 7.3 remains valid if we replace $L^p(\Omega)$ by $\widetilde{W}^{-t,p}(\Omega)$.

8 Conclusions and Future Work

Summary. We have introduced a novel characterization of fractional-order Sobolev spaces. For domains with exterior cone condition, we have shown continuity of solutions to fractional PDEs and we have established well-posedness of fractional PDEs with measure-valued data. These results are crucial to study the state (and control) constrained optimal control problems discussed in the paper. They have helped to establish the well-posedness of the control problem, deriving the optimality conditions and the regularity of the optimal control.

Perspectives/Open Problems. There has already been a follow-up to the current work in [29], which describes a numerical method to solve the optimal control problem. It will be of interest to the community to relax the aforementioned exterior cone condition to show the continuity of solutions to fractional PDEs. In Corollary 6.1, where under the assumption $\frac{1}{p} < s < 1$, we are able to replace the integral over $\mathbb{R}^N$ (cf. Theorem 6.1) by an integral over Ω, it will be interesting to extend this result to the full range of s, i.e., also when $0 < s \le \frac{1}{p}$.

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