

PAPER

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External optimal control of nonlocal PDEs

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Abstract

Very recently Warma (2019 *SIAM J. Control Optim.* to appear) has shown that for nonlocal PDEs associated with the fractional Laplacian, the classical notion of controllability from the boundary does not make sense and therefore it must be replaced by a control that is localized outside the open set where the PDE is solved. Having learned from the above mentioned result, in this paper we introduce a new class of source identification and optimal control problems where the source/control is located outside the observation domain where the PDE is satisfied. The classical diffusion models lack this flexibility as they assume that the source/control is located either inside or on the boundary. This is essentially due to the locality property of the underlying operators. We use the nonlocality of the fractional operator to create a framework that now allows placing a source/control outside the observation domain. We consider the Dirichlet, Robin and Neumann source identification or optimal control problems. These problems require dealing with the nonlocal normal derivative (that we shall call interaction operator). We create a functional analytic framework and show well-posedness and derive the first order optimality conditions for these problems. We introduce a new approach to approximate, with convergence rate, the Dirichlet problem with nonzero exterior condition. The numerical examples confirm our theoretical findings and illustrate the practicality of our approach.

Keywords: fractional Laplacian, interaction operator, weak and very-weak solutions, Dirichlet control problem, Robin control problem, external control

(Some figures may appear in colour only in the online journal)

1. Introduction and motivation

In many real life applications a source or a control is placed outside (disjoint from) the observation domain Ω where the PDE is satisfied. Some examples of inverse and optimal control problems where this situation may arise are: (i) acoustic testing, when the loudspeakers are placed far from the aerospace structures [38]; (ii) magnetotellurics (MT), which is a technique to infer earth's subsurface electrical conductivity from surface measurements [48, 55]; (iii) magnetic drug targeting (MDT), where drugs with ferromagnetic particles in suspension are injected into the body and the external magnetic field is then used to steer the drug to relevant areas, for example, solid tumors [7, 8, 41]; (iv) electroencephalography (EEG) is used to record electrical activities in brain [42, 56], in case one accounts for the neurons disjoint from the brain, one will obtain an external source problem.

This is different from the traditional approaches where the source/control is placed either inside the domain Ω or on the boundary $\partial\Omega$ of Ω . This is not surprising since in many cases we do not have a direct access to $\partial\Omega$. See for instance, the setup in figure 1. In such applications the existing models can be ineffective due to their strict requirements. Indeed think of the source identification problem for the most basic Poisson equation:

$$-\Delta u = f \quad \text{in } \Omega, \quad u = z \quad \text{on } \partial\Omega, \quad (1.1)$$

where the source is either f (force or load) or z (boundary control) see [6, 39, 47]. In (1.1) there is no provision to place the source in $\hat{\Omega} \subset \mathbb{R}^N \setminus \Omega$, i.e. a domain that is disjoint from Ω , see figure 1 for two examples of Ω and $\hat{\Omega}$. The issue is that the operator Δ has 'lesser reach', in other words, it is a local operator. On the other hand the fractional Laplacian $(-\Delta)^s$ with $0 < s < 1$ (that we shall define below) is a nonlocal operator. This difference in behavior can be easily seen in our numerical examples in section 7.2 where we observe that we cannot see the external source as s approaches 1.

Recently, nonlocal diffusion operators such as the fractional Laplacian $(-\Delta)^s$ have emerged as an excellent alternative to model diffusion. Under a probabilistic framework this operator can be derived as the limit of the so-called *long jump* random walk [49]. Recall that Δ is the limit of the classical random walk or the Brownian motion. More applications of these models appear in (but not limited to) image denoising and phase field modeling [4, 10]; fractional diffusion maps (data analysis) [5]; magnetotellurics (geophysics) [55].

Coming back to the question of source/control placement, we next state the exterior value problem corresponding to $(-\Delta)^s$. Find u in an appropriate function space satisfying

$$(-\Delta)^s u = f \quad \text{in } \Omega, \quad u = z \quad \text{on } \mathbb{R}^N \setminus \Omega. \quad (1.2)$$

As in the case of (1.1), besides f being the source/control in Ω , we can also place the source/control z in the exterior domain $\mathbb{R}^N \setminus \Omega$. However, the action of z in (1.2) is significantly different from (1.1). Indeed, the source/control in (1.1) is placed on the boundary $\partial\Omega$, but the source/control z in (1.2) is placed in $\mathbb{R}^N \setminus \Omega$ which is what we wanted to achieve in figure 1. For completeness, we refer to [12] for the optimal control problem, with f being the source/control and [11, 13] for another inverse problem to identify the coefficients in the fractional p -Laplacian.

The purpose of this paper is to introduce and study a new class of the Dirichlet, Robin and Neumann source identification problems or the optimal control problems. We shall use these terms interchangeably but we will make a distinction in our numerical experiments. We emphasize that yet another class of identification where the unknown is the fractional exponent s for the spectral fractional Laplacian (which is different from the operator under

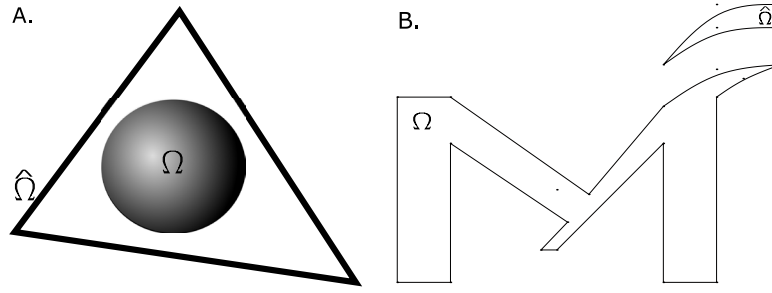


Figure 1. Let a diffusion process occurs inside a domain Ω which is the sphere in Case. (A) (left) and the letter M in Case. (B) (right). We are interested in the source identification or controlling this diffusion process by placing the source/control in a set $\hat{\Omega}$ which is disjoint from Ω . Case (A): $\hat{\Omega}$ is the triangular pipe. Case (B): $\hat{\Omega}$ is the structure on the top of the letter M.

consideration) was recently considered in [46]. We refer to [10] for the case when s is a function of $x \in \Omega$.

Now we describe our problems.

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded open set with boundary $\partial\Omega$. Let (Z_D, U_D) and (Z_R, U_R) , where the subscripts D and R indicate Dirichlet and Robin, be Banach spaces. The goal of this paper is to consider the following two external control or source identification problems. The source/control in our case is denoted by z . Our objective function consists of two parts and we shall denote by J the part that depends on the state u . The precise assumptions on J will be given in section 4.

- **Fractional Dirichlet exterior control problem:** Given $\xi \geq 0$ a constant penalty parameter, we consider the minimization problem:

$$\min_{(u,z) \in (U_D, Z_D)} J(u) + \frac{\xi}{2} \|z\|_{Z_D}^2, \quad (1.3a)$$

subject to the fractional Dirichlet exterior value problem: find $u \in U_D$ solving

$$\begin{cases} (-\Delta)^s u = 0 & \text{in } \Omega, \\ u = z & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (1.3b)$$

and the control constraints

$$z \in Z_{ad,D}, \quad (1.3c)$$

with $Z_{ad,D} \subset Z_D$ being a closed and convex subset.

- **Fractional Robin exterior control problem:** Given $\xi \geq 0$ a constant penalty parameter, we consider the minimization problem:

$$\min_{(u,z) \in (U_R, Z_R)} J(u) + \frac{\xi}{2} \|z\|_{Z_R}^2, \quad (1.4a)$$

subject to the fractional Robin exterior value problem: find $u \in U_R$ solving

$$\begin{cases} (-\Delta)^s u = 0 & \text{in } \Omega, \\ \mathcal{N}_s u + \kappa u = \kappa z & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (1.4b)$$

and the control constraints

$$z \in Z_{ad,R}, \quad (1.4c)$$

with $Z_{ad,R} \subset Z_R$ being a closed and convex subset. In (1.4b), $\mathcal{N}_s u$ is the nonlocal normal derivative of u that will be defined in section 2, $\kappa \in L^1(\mathbb{R}^N \setminus \Omega) \cap L^\infty(\mathbb{R}^N \setminus \Omega)$ and is non-negative. We notice that the latter assumption is not a restriction since otherwise we can replace κ throughout by $|\kappa|$.

The precise conditions on Ω and the Banach spaces involved will be given in the subsequent sections. Notice that both exterior value problems (1.3b) and (1.4b) are ill-posed if the conditions are enforced on $\partial\Omega$. The main difficulties in (1.3) and (1.4) stem from the following facts.

- **Nonlocal diffusion operator.** The fractional Laplacian $(-\Delta)^s$ is a nonlocal operator. This can be easily seen from its definition.
- **Nonlocal normal derivative.** The first order optimality conditions for (1.3), the very-weak solution to the Dirichlet problem (1.3b) and the Robin exterior value problem (1.4b) require to study $\mathcal{N}_s u$ which is the so-called nonlocal-normal derivative of u . Thus we not only have the nonlocal operator $(-\Delta)^s$ in the domain but also in the exterior $\mathbb{R}^N \setminus \Omega$, i.e. a double nonlocality. An approximation of $\mathcal{N}_s u$, especially numerically, is extremely challenging.
- **Exterior conditions in $\mathbb{R}^N \setminus \Omega$ and not boundary conditions on $\partial\Omega$.** The conditions in (1.3b) and (1.4b) need to be specified in $\mathbb{R}^N \setminus \Omega$ instead on $\partial\Omega$ as otherwise the problems (1.3) and (1.4) are ill-posed as we have already mentioned above.
- **Very-weak solutions of nonlocal exterior value problems.** A typical choice for Z_D is $L^2(\mathbb{R}^N \setminus \Omega)$. As a result, the Dirichlet exterior value problem (1.3b) can only have very-weak solutions (see [14, 15, 17] for the case $s = 1$). To the best of our knowledge this is the first work that considers the notion of very-weak solutions for nonlocal (fractional) exterior value problems associated with the fractional Laplace operator.
- **Regularity of the optimization variables.** The standard shift-theorem which holds for local operators such as Δ does not always hold for nonlocal operators such as $(-\Delta)^s$ (see for example [34]).

In view of all these aforementioned challenges it is clear that the standard techniques which are now well established for local problems do not directly extend to the nonlocal problems investigated in the present paper.

The purpose of this paper is to discuss our approach to deal with these nontrivial issues. We emphasize that to the best of our knowledge this is the first work that considers the optimal control problems (source identification problems) (1.3b) and (1.4b) where the control/source is applied from the outside. Let us also mention that this notion of controllability of PDEs from the exterior has been introduced by Warma in [53] for the nonlocal heat equation associated with the fractional Laplacian and in [40] for the wave type equation with the fractional Laplace operator to study their controllability properties. The case of the strong damping wave equation is included in [54] where some controllability results have been obtained. In case of problems with the spectral fractional Laplacian the boundary control has been established in [9]. For completeness, we also mention some interesting works on fractional Calderón type inverse problems [31, 37, 44]. Notice that fractional operators further provide flexibility to approximate arbitrary functions [25, 28, 34, 36].

We mention that we can also deal with the fractional Neumann exterior control problem. That is, given $\xi \geq 0$ a constant penalty parameter,

$$\min_{(u,z) \in (U_N, Z_N)} J(u) + \frac{\xi}{2} \|z\|_{Z_N}^2,$$

subject to the fractional Neumann exterior value problem: find $u \in U_N$ solving

$$\begin{cases} (-\Delta)^s u + u &= 0 & \text{in } \Omega, \\ \mathcal{N}_s u &= z & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (1.5)$$

and the control constraints

$$z \in Z_{ad,N}.$$

The term u is added in (1.4b) just to ensure the uniqueness of solutions. The proofs follow similarly as the two cases we consider in the present paper with very minor changes. Since the paper is already long, we shall not give any details on this case.

Below we mention the novelties of the present paper.

- (i) **Weak and very-weak solutions.** For the first time, we introduce and study the notion of very-weak solutions to the Dirichlet exterior value problem (1.3b) which is suitable for optimal control problems. We also study weak solutions of the Robin exterior value problem (1.4b).
- (ii) **Approximation of the Dirichlet weak and very-weak solutions by the Robin weak solutions.** We approximate the weak and very-weak solutions of the nonhomogeneous Dirichlet exterior value problem by using a suitable Robin exterior value problem. This allows us to circumvent approximating the nonlocal normal derivative and it is one of the *key contribution of this paper*. Recall that for the very-weak solution of the Dirichlet problem we need to evaluate the nonlocal normal derivative of the test functions (see definition 3.3) and for the Dirichlet control problem we need to evaluate the nonlocal normal derivative of the adjoint variable (see theorem 4.3). This is a new approach to impose non-zero exterior conditions for the fractional Dirichlet exterior value problem. We refer to an alternative approach [3] where the authors use the Lagrange multipliers to impose nonzero Dirichlet exterior conditions.
- (iii) We study both Dirichlet and Robin exterior control problems.
- (iv) We approximate (with rate) the Dirichlet exterior control problem by a suitable Robin exterior control problem.

The rest of the paper is organized as follows. We begin with section 2 where we introduce the relevant notations and the function spaces needed. The material in this section is well-known. Our main work starts from section 3 where we study first the weak and very-weak solutions for the Dirichlet exterior value problem in section 3.1. This is followed by the well-posedness of the Robin exterior value problem in section 3.2. The Dirichlet exterior control problem is considered in section 4 and Robin in section 5. We show how to approximate the weak solutions to the Dirichlet problem and the solutions to the Dirichlet exterior control problem in section 6. Section 7.1 is devoted to the experimental rate of convergence to approximate the Dirichlet exterior value problem using the Robin problem. In section 7.2 we consider a source identification problem in the classical sense, however our source is located outside the observation domain where the PDE is satisfied. Section 7.3 is devoted to two optimal control problems.

Remark 1.1 (Practical aspects). From a practical point of view, having the source/control over the entire $\mathbb{R}^N \setminus \Omega$ can be very expensive. But this can be easily fixed by appropriately describing Z_{ad} . Indeed in case of figure 1 we can set the support of functions in Z_{ad} to be in $\tilde{\Omega}$.

2. Notations and preliminaries

Unless otherwise stated, $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) is a bounded open set and $0 < s < 1$. We let

$$W^{s,2}(\Omega) := \left\{ u \in L^2(\Omega) : \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy < \infty \right\},$$

and we endow it with the norm defined by

$$\|u\|_{W^{s,2}(\Omega)} := \left(\int_{\Omega} |u|^2 dx + \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}}.$$

In order to study (1.3b) we also need to define

$$W_0^{s,2}(\overline{\Omega}) := \{u \in W^{s,2}(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega\}.$$

Then

$$\|u\|_{W_0^{s,2}(\overline{\Omega})} := \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}}$$

defines an equivalent norm on $W_0^{s,2}(\overline{\Omega})$.

We shall use $W^{-s,2}(\mathbb{R}^N)$ and $W^{-s,2}(\overline{\Omega})$ to denote the dual spaces of $W^{s,2}(\mathbb{R}^N)$ and $W_0^{s,2}(\overline{\Omega})$, respectively, and $\langle \cdot, \cdot \rangle$ to denote their duality pairing whenever it is clear from the context.

We also define the local fractional order Sobolev space

$$W_{\text{loc}}^{s,2}(\mathbb{R}^N \setminus \Omega) := \{u \in L^2(\mathbb{R}^N \setminus \Omega) : u\varphi \in W^{s,2}(\mathbb{R}^N \setminus \Omega), \forall \varphi \in \mathcal{D}(\mathbb{R}^N \setminus \Omega)\}. \quad (2.1)$$

To introduce the fractional Laplace operator, we set

$$\mathbb{L}_s^1(\mathbb{R}^N) := \left\{ u : \mathbb{R}^N \rightarrow \mathbb{R} \text{ measurable, } \int_{\mathbb{R}^N} \frac{|u(x)|}{(1 + |x|)^{N+2s}} dx < \infty \right\}.$$

For $u \in \mathbb{L}_s^1(\mathbb{R}^N)$ and $\varepsilon > 0$, we let

$$(-\Delta)_{\varepsilon}^s u(x) = C_{N,s} \int_{\{y \in \mathbb{R}^N, |y-x| > \varepsilon\}} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy, \quad x \in \mathbb{R}^N,$$

where the normalized constant $C_{N,s}$ is given by

$$C_{N,s} := \frac{s 2^{2s} \Gamma\left(\frac{2s+N}{2}\right)}{\pi^{\frac{N}{2}} \Gamma(1-s)}, \quad (2.2)$$

and Γ is the usual Euler Gamma function (see, e.g. [20, 22–24, 26, 51, 52]). The fractional Laplacian $(-\Delta)^s$ is defined for $u \in \mathbb{L}_s^1(\mathbb{R}^N)$ by the formula

$$(-\Delta)^s u(x) = C_{N,s} \text{ P.V. } \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy = \lim_{\varepsilon \downarrow 0} (-\Delta)_{\varepsilon}^s u(x), \quad x \in \mathbb{R}^N, \quad (2.3)$$

provided that the limit exists. It has been shown in [21, proposition 2.2] that for $u \in \mathcal{D}(\Omega)$, we have

$$\lim_{s \uparrow 1^-} \int_{\mathbb{R}^N} u(-\Delta)^s u dx = \int_{\mathbb{R}^N} |\nabla u|^2 dx = - \int_{\mathbb{R}^N} u \Delta u dx = - \int_{\Omega} u \Delta u dx,$$

that is where the constant $C_{N,s}$ plays a crucial role.

Next, for $u \in W^{s,2}(\mathbb{R}^N)$ we define the nonlocal normal derivative $\mathcal{N}_s$ as:

$$\mathcal{N}_s u(x) := C_{N,s} \int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy, \quad x \in \mathbb{R}^N \setminus \overline{\Omega}. \quad (2.4)$$

We shall call $\mathcal{N}_s$ the *interaction operator*. Notice that the term ‘interaction’ has also been used by Du et al in [29]. Clearly $\mathcal{N}_s$ is a nonlocal operator and it is well defined on $W^{s,2}(\mathbb{R}^N)$ as we discuss next.

Lemma 2.1. *The interaction operator $\mathcal{N}_s$ maps continuously $W^{s,2}(\mathbb{R}^N)$ into $W^{s,2}_{\text{loc}}(\mathbb{R}^N \setminus \Omega)$. As a result, if $u \in W^{s,2}(\mathbb{R}^N)$, then $\mathcal{N}_s u \in L^2(\mathbb{R}^N \setminus \Omega)$.*

Proof. We refer to [32, lemma 3.2] for the proof of the first part. The second part is a direct consequence of (2.1). $\square$

Despite the fact that $\mathcal{N}_s$ is defined on $\mathbb{R}^N \setminus \Omega$, it is still known as the ‘normal’ derivative. This is due to its similarity with the classical normal derivative as we discuss next.

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with a Lipschitz continuous boundary. Then the following assertions hold.*

- (a) **The divergence theorem for $(-\Delta)^s$.** Let $u \in C_0^2(\mathbb{R}^N)$, i.e. C^2 functions on $\mathbb{R}^N$ that vanish at $\pm\infty$. Then

$$\int_{\Omega} (-\Delta)^s u \, dx = - \int_{\mathbb{R}^N \setminus \Omega} \mathcal{N}_s u \, dx.$$

- (b) **The integration by parts formula for $(-\Delta)^s$.** Let $u \in W^{s,2}(\mathbb{R}^N)$ be such that $(-\Delta)^s u \in L^2(\Omega)$. Then for every $v \in W^{s,2}(\mathbb{R}^N)$ we have that

$$\begin{aligned} \int_{\Omega} v (-\Delta)^s u \, dx &= \frac{C_{N,s}}{2} \int \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy \\ &\quad - \int_{\mathbb{R}^N \setminus \Omega} v \mathcal{N}_s u \, dx, \end{aligned} \quad (2.5)$$

where $\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2 := (\Omega \times \Omega) \cup (\Omega \times (\mathbb{R}^N \setminus \Omega)) \cup ((\mathbb{R}^N \setminus \Omega) \times \Omega)$.

- (c) **The limit as $s \uparrow 1^-$.** Let $u, v \in C_0^2(\mathbb{R}^N)$. Then

$$\lim_{s \uparrow 1^-} \int_{\mathbb{R}^N \setminus \Omega} v \mathcal{N}_s u \, dx = \int_{\partial\Omega} v \frac{\partial u}{\partial \nu} \, d\sigma.$$

Remark 2.3. Comparing the properties (a)–(c) in proposition 2.2 with the classical properties of the standard Laplacian Δ we can immediately infer that $\mathcal{N}_s$ plays the same role for $(-\Delta)^s$ that the classical normal derivative does for Δ . For this reason, we call $\mathcal{N}_s$ the nonlocal normal derivative.

Proof of proposition 2.2. The proofs of Parts (a) and (c) are contained in [27, lemma 3.2] and [27, proposition 5.1], respectively. The proof of Part (b) for smooth functions can be found in [27, lemma 3.3]. The version given here is obtained by using a density argument (see [53, proposition 3.7]). $\square$

3. The state equations

Before analyzing the optimal control problems (1.3) and (1.4) for a given function z , we shall focus on the Dirichlet (1.3b) and Robin (1.4b) exterior value problems. We shall assume that Ω is a bounded domain with a Lipschitz continuous boundary.

3.1. The Dirichlet problem for the fractional Laplacian

We begin by rewriting the system (1.3b) in a more general form. That is,

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = z & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (3.1)$$

Here is our notion of weak solution.

Definition 3.1 (Weak solution to the Dirichlet problem). Let $f \in W^{-s,2}(\overline{\Omega})$, $z \in W^{s,2}(\mathbb{R}^N \setminus \Omega)$ and $\tilde{z} \in W^{s,2}(\mathbb{R}^N)$ be such that $\tilde{z}|_{\mathbb{R}^N \setminus \Omega} = z$. A function $u \in W^{s,2}(\mathbb{R}^N)$ is said to be a weak solution to (3.1) if $u - \tilde{z} \in W_0^{s,2}(\overline{\Omega})$ and

$$\frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy = \langle f, v \rangle,$$

for every $v \in W_0^{s,2}(\overline{\Omega})$.

Firstly, we notice that since Ω is assumed to have a Lipschitz continuous boundary, we have that, for $z \in W^{s,2}(\mathbb{R}^N \setminus \Omega)$, there exists $\tilde{z} \in W^{s,2}(\mathbb{R}^N)$ such that $\tilde{z}|_{\mathbb{R}^N \setminus \Omega} = z$. Secondly, the existence and uniqueness of a weak solution u to (3.1) and the continuous dependence of u on the data f and z have been considered in [34] (see also [32, 50]). More precisely we have the following result.

Proposition 3.2. Let $f \in W^{-s,2}(\overline{\Omega})$ and $z \in W^{s,2}(\mathbb{R}^N \setminus \Omega)$. Then there exists a unique weak solution u to (3.1) in the sense of definition 3.1. In addition there is a constant $C > 0$ such that

$$\|u\|_{W^{s,2}(\mathbb{R}^N)} \leq C \left(\|f\|_{W^{-s,2}(\overline{\Omega})} + \|z\|_{W^{s,2}(\mathbb{R}^N \setminus \Omega)} \right). \quad (3.2)$$

Even though such a result is typically sufficient in most situations, nevertheless it is not directly useful in the current context of optimal control problem (1.3) since we are interested in taking the space $Z_D = L^2(\mathbb{R}^N \setminus \Omega)$. Thus we need existence of solutions (in some sense) to the fractional Dirichlet problem (3.1) when $z \in L^2(\mathbb{R}^N \setminus \Omega)$. In order to tackle this situation we introduce our notion of very-weak solution for (3.1).

Definition 3.3 (Very-weak solution to the Dirichlet problem). Let $z \in L^2(\mathbb{R}^N \setminus \Omega)$ and $f \in W^{-s,2}(\overline{\Omega})$. A function $u \in L^2(\mathbb{R}^N)$ is said to be a very-weak solution to (3.1) if the identity

$$\int_{\Omega} u(-\Delta)^s v dx = \langle f, v \rangle - \int_{\mathbb{R}^N \setminus \Omega} z \mathcal{N}_s v dx, \quad (3.3)$$

holds for every $v \in V := \{v \in W_0^{s,2}(\overline{\Omega}) : (-\Delta)^s v \in L^2(\Omega)\}$.

Remark 3.4. We mention the following facts.

- (a) We have shown in proposition 3.2 that if $z \in W^{s,2}(\mathbb{R}^N \setminus \Omega)$, then the Dirichlet problem (3.1) has a unique weak solution $u \in W^{s,2}(\mathbb{R}^N)$. In [30], letting

$$\mathcal{V}(\Omega) := \left\{ v : \mathbb{R}^N \rightarrow \mathbb{R}, v \in L^2(\Omega) \text{ and } \int_{\Omega} \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^2}{|x - y|^{N+2s}} dx dy < \infty \right\},$$

the authors have shown that if $z \in \mathcal{V}(\Omega)$ and $f \in \mathcal{V}(\Omega)^*$, then the Dirichlet problem (3.1) has a unique weak solution $u \in \mathcal{V}(\Omega)$. Notice that $W^{s,2}(\mathbb{R}^N) \hookrightarrow \mathcal{V}(\Omega) \hookrightarrow W^{s,2}(\Omega)$. The difference between the two notions is only the space where the exterior data and the solutions belong.

- (b) For the very-weak solution, we have just assumed that $z \in L^2(\mathbb{R}^N \setminus \Omega)$ (no additional regularity) and this has not been studied in [30] or elsewhere.

Next we prove the existence and uniqueness of a very-weak solution to (3.1).

Theorem 3.5. Let $f \in W^{-s,2}(\overline{\Omega})$ and $z \in L^2(\mathbb{R}^N \setminus \Omega)$. Then there exists a unique very-weak solution u to (3.1) according to definition 3.1 that fulfills

$$\|u\|_{L^2(\Omega)} \leq C \left(\|f\|_{W^{-s,2}(\overline{\Omega})} + \|z\|_{L^2(\mathbb{R}^N \setminus \Omega)} \right), \quad (3.4)$$

for a constant $C > 0$. In addition, if $z \in W^{s,2}(\mathbb{R}^N \setminus \Omega)$, then the following assertions hold.

- (a) Every weak solution of (3.1) is also a very-weak solution.
 (b) Every very-weak solution of (3.1) that belongs to $W^{s,2}(\mathbb{R}^N)$ is also a weak solution.

Proof. In order to show the existence of a very-weak solution we shall apply the Babuška–Lax–Milgram theorem.

Firstly, let $(-\Delta)_D^s$ be the realization of $(-\Delta)^s$ in $L^2(\Omega)$ with the zero Dirichlet exterior condition $u = 0$ in $\mathbb{R}^N \setminus \Omega$. More precisely,

$$D((-\Delta)_D^s) = V \text{ and } (-\Delta)_D^s u = (-\Delta)^s u \text{ in } \Omega.$$

Then a norm on V is given by $\|v\|_V = \|(-\Delta)_D^s v\|_{L^2(\Omega)}$ which follows from the fact that the operator $(-\Delta)_D^s$ is invertible (since by [45] $(-\Delta)_D^s$ has a compact resolvent and its first eigenvalue is strictly positive). Secondly, let $\mathcal{F}$ be the bilinear form defined on $L^2(\Omega) \times V$ by

$$\mathcal{F}(u, v) := \int_{\Omega} u (-\Delta)^s v \, dx.$$

Then $\mathcal{F}$ is clearly bounded on $L^2(\Omega) \times V$. More precisely there is a constant $C > 0$ such that

$$|\mathcal{F}(u, v)| \leq \|u\|_{L^2(\Omega)} \|(-\Delta)^s v\|_{L^2(\Omega)} \leq C \|u\|_{L^2(\Omega)} \|v\|_V, \quad \forall (u, v) \in L^2(\Omega) \times V.$$

Thirdly, we show the inf-sup conditions. From the definition of V , we have that

$$v \in W_0^{s,2}(\overline{\Omega}) \text{ and } (-\Delta)^s v \in L^2(\Omega) \iff v \in V.$$

Letting $u := \frac{(-\Delta)_D^s v}{\|(-\Delta)_D^s v\|_{L^2(\Omega)}} \in L^2(\Omega)$, we obtain that

$$\begin{aligned} \sup_{u \in L^2(\Omega), \|u\|_{L^2(\Omega)}=1} |(u, (-\Delta)_D^s v)_{L^2(\Omega)}| &\geq \frac{|((- \Delta)_D^s v, (- \Delta)_D^s v)_{L^2(\Omega)}|}{\|(- \Delta)_D^s v\|_{L^2(\Omega)}} \\ &\geq \|(- \Delta)_D^s v\|_{L^2(\Omega)} = \|v\|_V. \end{aligned}$$

Next we choose $v \in V$ as the unique weak solution of the Dirichlet problem

$$(-\Delta)_D^s v = \frac{u}{\|u\|_{L^2(\Omega)}} \quad \text{in } \Omega,$$

for some $0 \neq u \in L^2(\Omega)$. Then we readily obtain that

$$\sup_{v \in V, \|v\|_V=1} |(u, (-\Delta)_D^s v)_{L^2(\Omega)}| \geq \frac{|(u, u)_{L^2(\Omega)}|}{\|u\|_{L^2(\Omega)}} = \|u\|_{L^2(\Omega)} > 0,$$

for all $0 \neq u \in L^2(\Omega)$. Finally, we have to show that the right-hand-side in (3.3) defines a linear continuous functional on V . Indeed, applying the Hölder inequality in conjunction with lemma 2.1 we obtain that there is a constant $C > 0$ such that

$$\left| \int_{\mathbb{R}^N \setminus \Omega} z \mathcal{N}_s v \, dx \right| \leq \|z\|_{L^2(\mathbb{R}^N \setminus \Omega)} \|\mathcal{N}_s v\|_{L^2(\mathbb{R}^N \setminus \Omega)} \leq C \|z\|_{L^2(\mathbb{R}^N \setminus \Omega)} \|v\|_{W_0^{s,2}(\bar{\Omega})}, \quad (3.5)$$

where in the last step we have used the fact that $\|v\|_{W_0^{s,2}(\bar{\Omega})} = \|v\|_{W^{s,2}(\mathbb{R}^N)}$ for $v \in W_0^{s,2}(\bar{\Omega})$. Moreover

$$|\langle f, v \rangle| \leq \|f\|_{W^{-s,2}(\bar{\Omega})} \|v\|_{W_0^{s,2}(\bar{\Omega})}.$$

In view of the last two estimates, the right-hand-side in (3.3) defines a linear continuous functional on V . Therefore all the requirements of the Babuška–Lax–Milgram theorem hold. Thus, there exists a unique $u \in L^2(\Omega)$ satisfying (3.3). Let $u = z$ in $\mathbb{R}^N \setminus \Omega$. Then $u \in L^2(\mathbb{R}^N)$ and satisfies (3.3). We have shown the existence and uniqueness of a very-weak solution.

Next we show the estimate (3.4). Let $u \in L^2(\mathbb{R}^N)$ be a very-weak solution. Let $v \in V$ be a weak solution of $(-\Delta)_D^s v = u$ in Ω . Taking this v as a test function in (3.3) and using (3.5), we get that there is a constant $C > 0$ such that

$$\begin{aligned} \|u\|_{L^2(\Omega)}^2 &\leq \|f\|_{W^{-s,2}(\bar{\Omega})} \|v\|_{W_0^{s,2}(\bar{\Omega})} + \|z\|_{L^2(\mathbb{R}^N \setminus \Omega)} \|\mathcal{N}_s v\|_{L^2(\mathbb{R}^N \setminus \Omega)} \\ &\leq C \left(\|f\|_{W^{-s,2}(\bar{\Omega})} + \|z\|_{L^2(\mathbb{R}^N \setminus \Omega)} \right) \|v\|_{W_0^{s,2}(\bar{\Omega})} \\ &\leq C \left(\|f\|_{W^{-s,2}(\bar{\Omega})} + \|z\|_{L^2(\mathbb{R}^N \setminus \Omega)} \right) \|(-\Delta)_D^s v\|_{L^2(\Omega)} \\ &\leq C \left(\|f\|_{W^{-s,2}(\bar{\Omega})} + \|z\|_{L^2(\mathbb{R}^N \setminus \Omega)} \right) \|u\|_{L^2(\Omega)}. \end{aligned}$$

We have shown (3.4) and this completes the proof of the first part.

Next we prove the last two assertions of the theorem. Assume that $z \in W^{s,2}(\mathbb{R}^N \setminus \Omega)$.

- (a) Let $u \in W^{s,2}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$ be a weak solution of (3.1). It follows from the definition that $u = z$ in $\mathbb{R}^N \setminus \Omega$ and

$$\frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy = \langle f, v \rangle, \quad (3.6)$$

for every $v \in V$. Since $v = 0$ in $\mathbb{R}^N \setminus \Omega$, we have that

$$\begin{aligned} & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy \\ &= \int \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy. \end{aligned} \quad (3.7)$$

Using (3.6) and (3.7), the integration by parts formula (2.5) together with the fact that $u = z$ in $\mathbb{R}^N \setminus \Omega$, we get that

$$\begin{aligned} & \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy \\ &= \langle f, v \rangle \\ &= \int_{\Omega} u(-\Delta)^s v dx + \int_{\mathbb{R}^N \setminus \Omega} u \mathcal{N}_s v dx \\ &= \int_{\Omega} u(-\Delta)^s v dx + \int_{\mathbb{R}^N \setminus \Omega} z \mathcal{N}_s v dx. \end{aligned}$$

Thus u is a very-weak solution of (3.1).

- (b) Finally let u be a very-weak solution of (3.1) and assume that $u \in W^{s,2}(\mathbb{R}^N)$. Since $u = z$ in $\mathbb{R}^N \setminus \Omega$, we have that $z \in W^{s,2}(\mathbb{R}^N \setminus \Omega)$ and if $\tilde{z} \in W^{s,2}(\mathbb{R}^N)$ satisfies $\tilde{z}|_{\mathbb{R}^N \setminus \Omega} = z$, then clearly $(u - \tilde{z}) \in W_0^{s,2}(\Omega)$. Since u is a very-weak solution of (3.1), then by definition, for every $v \in V = D((-\Delta)_D^s)$, we have

$$\int_{\Omega} u(-\Delta)^s v dx = \langle f, v \rangle - \int_{\mathbb{R}^N \setminus \Omega} z \mathcal{N}_s v dx. \quad (3.8)$$

Since $u \in W^{s,2}(\mathbb{R}^N)$ and $v = 0$ in $\mathbb{R}^N \setminus \Omega$, then using (2.5) again we get that

$$\begin{aligned} & \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy \\ &= \frac{C_{N,s}}{2} \int \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy \\ &= \int_{\Omega} u(-\Delta)^s v dx + \int_{\mathbb{R}^N \setminus \Omega} u \mathcal{N}_s v dx \\ &= \int_{\Omega} u(-\Delta)^s v dx + \int_{\mathbb{R}^N \setminus \Omega} z \mathcal{N}_s v dx. \end{aligned} \quad (3.9)$$

It follows from (3.8) and (3.9) that for every $v \in V$, we have

$$\frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy = \langle f, v \rangle. \quad (3.10)$$

Since V is dense in $W_0^{s,2}(\overline{\Omega})$, we have that (3.10) remains true for every $v \in W_0^{s,2}(\overline{\Omega})$. We have shown that u is a weak solution of (3.1) and the proof is finished. $\square$

3.2. The Robin problem for the fractional Laplacian

In order to study the Robin problem (1.4b) we consider the Sobolev space introduced in [27]. For $g \in L^1(\mathbb{R}^N \setminus \Omega)$ fixed, we let

$$W_{\Omega,g}^{s,2} := \left\{ u : \mathbb{R}^N \rightarrow \mathbb{R} \text{ measurable, } \|u\|_{W_{\Omega,g}^{s,2}} < \infty \right\},$$

where

$$\|u\|_{W_{\Omega,g}^{s,2}} := \left(\|u\|_{L^2(\Omega)}^2 + \| |g|^{\frac{1}{2}} u \|_{L^2(\mathbb{R}^N \setminus \Omega)}^2 + \int \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}}. \quad (3.11)$$

Let μ be the measure on $\mathbb{R}^N \setminus \Omega$ given by $d\mu = |g|dx$. With this setting, the norm in (3.11) can be rewritten as

$$\|u\|_{W_{\Omega,g}^{s,2}} := \left(\|u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\mathbb{R}^N \setminus \Omega, \mu)}^2 + \int \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}}. \quad (3.12)$$

If $g = 0$, we shall let $W_{\Omega,0}^{s,2} = W_{\Omega}^{s,2}$. The following result has been proved in [27, proposition 3.1].

Proposition 3.6. *Let $g \in L^1(\mathbb{R}^N \setminus \Omega)$. Then $W_{\Omega,g}^{s,2}$ is a Hilbert space.*

Throughout the remainder of the article, for $g \in L^1(\mathbb{R}^N \setminus \Omega)$, we shall denote by $(W_{\Omega,g}^{s,2})^*$ the dual of $W_{\Omega,g}^{s,2}$.

Remark 3.7. We mention the following facts.

(a) Recall that

$$\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2 = (\Omega \times \Omega) \cup (\Omega \times (\mathbb{R}^N \setminus \Omega)) \cup ((\mathbb{R}^N \setminus \Omega) \times \Omega),$$

so that

$$\begin{aligned} \int \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy &= \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \\ &+ \int_{\Omega} \int_{\mathbb{R}^N \setminus \Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N \setminus \Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy. \end{aligned} \quad (3.13)$$

(b) If $g \in L^1(\mathbb{R}^N \setminus \Omega)$ and $u \in W_{\Omega,g}^{s,2}$, then using the Hölder inequality we get that

$$\begin{aligned} \left| \int_{\mathbb{R}^N \setminus \Omega} gu \, dx \right| &\leq \int_{\mathbb{R}^N \setminus \Omega} |g|^{\frac{1}{2}} |g|^{\frac{1}{2}} |u| \, dx \leq \left(\int_{\mathbb{R}^N \setminus \Omega} |g| \, dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N \setminus \Omega} |gu^2| \, dx \right)^{\frac{1}{2}} \\ &\leq \|g\|_{L^1(\mathbb{R}^N \setminus \Omega)}^{\frac{1}{2}} \|u\|_{L^2(\mathbb{R}^N \setminus \Omega, \mu)} \leq \|g\|_{L^1(\mathbb{R}^N \setminus \Omega)}^{\frac{1}{2}} \|u\|_{W_{\Omega,g}^{s,2}}. \end{aligned} \quad (3.14)$$

It follows from (3.14) that in particular, $L^1(\mathbb{R}^N \setminus \Omega, \mu) \hookrightarrow (W_{\Omega, g}^{s,2})^*$.

(c) By definition (using also (3.13)), $W_{\Omega, g}^{s,2} \hookrightarrow W_{\Omega}^{s,2} \hookrightarrow W^{s,2}(\Omega)$, so that we have the following continuous embeddings:

$$W_{\Omega, g}^{s,2} \hookrightarrow W_{\Omega}^{s,2} \hookrightarrow L^{\frac{2N}{N-2s}}(\Omega). \quad (3.15)$$

It follows from (3.15) that the embeddings $W_{\Omega, g}^{s,2} \hookrightarrow L^2(\Omega)$ and $W_{\Omega}^{s,2} \hookrightarrow L^2(\Omega)$ are compact.

We consider a generalized version of the system (1.4b) with nonzero right-hand-side f . That is, the problem:

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ \mathcal{N}_s u + \kappa u = \kappa z & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (3.16)$$

Throughout the following sections, the measure μ is defined with g replaced by κ . That is, $d\mu = \kappa dx$ (recall that κ is assumed to be non-negative). Here is our notion of weak solution.

Definition 3.8. Let $z \in L^2(\mathbb{R}^N \setminus \Omega, \mu)$ and $f \in (W_{\Omega, \kappa}^{s,2})^*$. A function $u \in W_{\Omega, \kappa}^{s,2}$ is said to be a weak solution of (3.16) if the identity

$$\begin{aligned} \frac{C_{N,s}}{2} \int \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N \setminus \Omega} \kappa u v dx \\ = \langle f, v \rangle_{(W_{\Omega, \kappa}^{s,2})^*, W_{\Omega, \kappa}^{s,2}} + \int_{\mathbb{R}^N \setminus \Omega} \kappa z v dx, \end{aligned} \quad (3.17)$$

holds for every $v \in W_{\Omega, \kappa}^{s,2}$.

We have the following existence result.

Proposition 3.9. Let $\kappa \in L^1(\mathbb{R}^N \setminus \Omega) \cap L^\infty(\mathbb{R}^N \setminus \Omega)$. Then for every $z \in L^2(\mathbb{R}^N \setminus \Omega, \mu)$ and $f \in (W_{\Omega, \kappa}^{s,2})^*$, there exists a weak solution $u \in W_{\Omega, \kappa}^{s,2}$ of (3.16).

Proof. Let $D(\mathcal{E}) = W_{\Omega, \kappa}^{s,2}$ and $\mathcal{E} : D(\mathcal{E}) \times D(\mathcal{E}) \rightarrow \mathbb{R}$ be given by

$$\mathcal{E}(u, v) := \frac{C_{N,s}}{2} \int \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N \setminus \Omega} \kappa u v dx. \quad (3.18)$$

Then $\mathcal{E}$ is a bilinear, symmetric, continuous and closed form on $L^2(\Omega)$. Hence, for every $z \in L^2(\mathbb{R}^N \setminus \Omega, \mu) \subset (W_{\Omega, \kappa}^{s,2})^*$ and $f \in (W_{\Omega, \kappa}^{s,2})^*$, there is a function $u \in W_{\Omega, \kappa}^{s,2}$ such that

$$\begin{aligned} \mathcal{E}(u, v) &= \langle f, v \rangle_{(W_{\Omega, \kappa}^{s,2})^*, W_{\Omega, \kappa}^{s,2}} + \langle z, v \rangle_{(W_{\Omega, \kappa}^{s,2})^*, W_{\Omega, \kappa}^{s,2}} \\ &= \langle f, v \rangle_{(W_{\Omega, \kappa}^{s,2})^*, W_{\Omega, \kappa}^{s,2}} + \int_{\mathbb{R}^N \setminus \Omega} \kappa z v dx, \end{aligned}$$

for every $v \in W_{\Omega, \kappa}^{s,2}$. That is, u satisfies (3.17). Thus u is a weak solution of (3.16). The proof is finished. $\square$

Remark 3.10. Notice that similarly to the classical Neumann problem when $\kappa \equiv 0$, proposition 3.9 only guarantees uniqueness of solutions to (1.4b) up to a constant. In case we assume κ to be strictly positive, uniqueness can be guaranteed under assumption 6.1 below. In that case we can also show that there is a constant $C > 0$ such that

$$\|u\|_{W_{\Omega,\kappa}^{s,2}} \leq C \left(\|f\|_{(W_{\Omega,\kappa}^{s,2})^*} + \|z\|_{L^2(\mathbb{R}^N \setminus \Omega, \mu)} \right). \quad (3.19)$$

4. Fractional Dirichlet exterior control problem

We begin by introducing the appropriate function spaces needed to study (1.3). We let

$$Z_D := L^2(\mathbb{R}^N \setminus \Omega), \quad U_D := L^2(\Omega).$$

In view of theorem 3.5 the following (solution-map) control-to-state map

$$S : Z_D \rightarrow U_D, \quad z \mapsto Sz = u,$$

is well-defined, linear and continuous. We also notice that for $z \in Z_D$, we have that $u := Sz \in L^2(\mathbb{R}^N)$. As a result we can write the *reduced fractional Dirichlet exterior control problem* as follows:

$$\min_{z \in Z_{ad,D}} \mathcal{J}(z) := J(Sz) + \frac{\xi}{2} \|z\|_{Z_D}^2, \quad (4.1)$$

where $\xi \geq 0$. The precise conditions on J depend on the result we would like to obtain. For this reason they will be given in the statements of our results.

We then have the following well-posedness result for (4.1) and equivalently (1.3).

Theorem 4.1. *Let $Z_{ad,D}$ be a closed and convex subset of Z_D . Let $\xi > 0$ with $J \geq 0$ or $Z_{ad,D}$ bounded and let $J : U_D \rightarrow \mathbb{R}$ be weakly lower-semicontinuous. Then there exists a solution $\bar{z}$ to (4.1) and equivalently to (1.3). If either J is convex and $\xi > 0$ or J is strictly convex and $\xi \geq 0$, then $\bar{z}$ is unique.*

Proof. The proof uses the so-called direct-method or the Weierstrass theorem [16, theorem 3.2.1]. We notice that for $\mathcal{J} : Z_{ad,D} \rightarrow \mathbb{R}$, we can construct a minimizing sequence $\{z_n\}_{n \in \mathbb{N}}$ (see [16, theorem 3.2.1] for a construction) such that

$$\inf_{z \in Z_{ad,D}} \mathcal{J}(z) = \lim_{n \rightarrow \infty} \mathcal{J}(z_n).$$

If $\xi > 0$ with $J \geq 0$ or $Z_{ad,D} \subset Z_D$ is bounded, then $\{z_n\}_{n \in \mathbb{N}}$ is a bounded sequence in Z_D which is a Hilbert space. Due to the reflexivity of Z_D , we have that (up to a subsequence if necessary) $z_n \rightharpoonup \bar{z}$ (weak convergence) in Z_D as $n \rightarrow \infty$. Since $Z_{ad,D}$ is closed and convex, hence is weakly closed, we have that $\bar{z} \in Z_{ad,D}$.

Since $S : Z_{ad,D} \rightarrow U_D$ is linear and continuous, we have that it is weakly continuous. This implies that $Sz_n \rightharpoonup S\bar{z}$ in U_D as $n \rightarrow \infty$. We have to show that $(S\bar{z}, \bar{z})$ fulfills the state equation according to definition 3.3. In particular we need to study the identity

$$\int_{\Omega} u_n (-\Delta)^s v \, dx = - \int_{\mathbb{R}^N \setminus \Omega} z_n \mathcal{N}_s v \, dx, \quad \forall v \in V, \quad (4.2)$$

as $n \rightarrow \infty$, where $u_n := Sz_n$. Since $u_n \rightharpoonup S\bar{z} =: \bar{u}$ in U_D as $n \rightarrow \infty$ and $z_n \rightharpoonup \bar{z}$ in Z_D as $n \rightarrow \infty$, we can immediately take the limit in (4.2) to obtain that

$$\int_{\Omega} \bar{u}(-\Delta)^s v \, dx = - \int_{\mathbb{R}^N \setminus \Omega} \bar{z} \mathcal{N}_s v \, dx, \quad \forall v \in V.$$

Thus $(\bar{u}, \bar{z}) \in U_D \times Z_{ad,D}$ fulfills the state equation in the sense of definition 3.3.

It then remains to show that $\bar{z}$ is the minimizer of (4.1). This is a consequence of the fact that $\mathcal{J}$ is weakly lower semicontinuous. Indeed, $\mathcal{J}$ is the sum of two weakly lower semicontinuous functions ($\|\cdot\|_{Z_D}^2$ is continuous and convex therefore weakly lower semicontinuous).

Finally, if $\xi > 0$ and J is convex, then $\mathcal{J}$ is strictly convex (sum of a strictly convex and convex functions). On the other hand, if J is strictly convex, then $\mathcal{J}$ is strictly convex. In either case we have that $\mathcal{J}$ is strictly convex and thus the uniqueness of $\bar{z}$ follows. $\square$

We next derive the first order necessary optimality conditions for (4.1). We begin by identifying the structure of the adjoint operator S^* .

Lemma 4.2. *For the state equation (1.3b) the adjoint operator $S^* : U_D \rightarrow Z_D$ is given by*

$$S^* w = -\mathcal{N}_s p \in Z_D,$$

where $w \in U_D$ and $p \in W_0^{s,2}(\bar{\Omega})$ is the weak solution to the problem

$$\begin{cases} (-\Delta)^s p = w & \text{in } \Omega, \\ p = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (4.3)$$

Proof. According to the definition of S^* , we have that for every $w \in U_D$ and $z \in Z_D$,

$$(w, Sz)_{L^2(\Omega)} = (S^* w, z)_{L^2(\mathbb{R}^N \setminus \Omega)}.$$

Next, testing the adjoint equation (4.3) with Sz and using the fact that Sz is a very-weak solution of (3.1) with $f = 0$, we arrive at

$$(w, Sz)_{L^2(\Omega)} = (Sz, (-\Delta)^s p)_{L^2(\Omega)} = -(z, \mathcal{N}_s p)_{L^2(\mathbb{R}^N \setminus \Omega)} = (z, S^* w)_{L^2(\mathbb{R}^N \setminus \Omega)}.$$

This yields the asserted result. $\square$

For the remainder of this section we will assume that $\xi > 0$.

Theorem 4.3. *Let the assumptions of theorem 4.1 hold. Let $\mathcal{Z}$ be an open set in Z_D such that $Z_{ad,D} \subset \mathcal{Z}$. Let $u \mapsto J(u) : U_D \rightarrow \mathbb{R}$ be continuously Fréchet differentiable with $J'(u) \in U_D$. If $\bar{z}$ is a minimizer of (4.1) over $Z_{ad,D}$, then the first order necessary optimality conditions are given by*

$$(-\mathcal{N}_s \bar{p} + \xi \bar{z}, z - \bar{z})_{L^2(\mathbb{R}^N \setminus \Omega)} \geq 0, \quad \forall z \in Z_{ad,D}, \quad (4.4)$$

where $\bar{p} \in W_0^{s,2}(\bar{\Omega})$ solves the adjoint equation

$$\begin{cases} (-\Delta)^s \bar{p} = J'(\bar{u}) & \text{in } \Omega, \\ \bar{p} = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (4.5)$$

Equivalently we can write (4.4) as

$$\bar{z} = \mathcal{P}_{Z_{ad,D}} \left(\frac{1}{\xi} \mathcal{N}_s \bar{p} \right), \quad (4.6)$$

where $\mathcal{P}_{Z_{ad,D}}$ is the projection onto the set $Z_{ad,D}$. If J is convex, then (4.4) is a sufficient condition.

Proof. The proof is a straightforward application of the differentiability properties of J and the chain rule in conjunction with lemma 4.2. Indeed, for a given direction $h \in Z_{ad,D}$, the directional derivative of $\mathcal{J}$ is given by

$$\begin{aligned} \mathcal{J}'(\bar{z})h &= (J'(S\bar{z}), Sh)_{L^2(\Omega)} + \xi(\bar{z}, h)_{L^2(\mathbb{R}^N \setminus \Omega)} \\ &= (S^* J'(S\bar{z}), h)_{L^2(\Omega)} + \xi(\bar{z}, h)_{L^2(\mathbb{R}^N \setminus \Omega)}, \end{aligned}$$

where in the first step we have used that $J'(S\bar{z}) \in \mathcal{L}(L^2(\Omega), \mathbb{R}) = L^2(\Omega)$ and in the second step we have used that S is linear and bounded, therefore S^* is well-defined. Then using lemma 4.2 we arrive at the asserted result. From lemma 2.1 we recall that $\mathcal{N}_s \bar{p} \in L^2(\mathbb{R}^N \setminus \Omega)$. Therefore the equivalence between (4.4) and (4.6) follows by using [16, theorem 3.3.5]. The proof is finished. $\square$

Remark 4.4 (Regularity for the optimization variables). We recall a rather surprising result for the adjoint equation (4.3). The standard maximal elliptic regularity that is known to hold for the classical Laplacian on smooth open sets does not hold in the case of the fractional Laplacian i.e. p does not always belong to $W^{2s,2}(\Omega)$. Notice that $w \in L^2(\Omega)$ and $p = [(-\Delta)_D^s]^{-1}w$. More precisely assume that Ω is a smooth bounded open set. If $0 < s < \frac{1}{2}$, then by [34, formula (7.4)] we have that $D((-\Delta)_D^s) = W_0^{2s,2}(\bar{\Omega})$ and hence, $p \in W^{2s,2}(\Omega)$ in that case. But if $\frac{1}{2} \leq s < 1$, an example has been given in [43, remark 7.2] where $D((-\Delta)_D^s) \not\subset W^{2s,2}(\Omega)$, thus in that case p does not always belong to $W^{2s,2}(\Omega)$. It has been shown in [18, 19] that only a local maximal elliptic regularity can be achieved. As a result, the best known result for $\mathcal{N}_s p$ is as given in lemma 2.1. Since $\mathcal{P}_{Z_{ad,D}}$ is a contraction (Lipschitz) we can conclude that $\bar{z}$ has the same regularity as $\mathcal{N}_s \bar{p}$, i.e. they are in $L^2(\mathbb{R}^N \setminus \Omega)$ globally and in $W_{loc}^{s,2}(\mathbb{R}^N \setminus \Omega)$ locally. As it is well-known, in case of the classical Laplacian, one can use a boot-strap argument to improve the regularity of $S\bar{z} = \bar{u}$ globally. However this is not the case for the fractional exterior value problems. We also notice that always for the case $\frac{1}{2} < s < 1$, Grubb [33, section 2] (see also [34]) has introduced some fractional order Sobolev spaces where a maximal elliptic regularity is obtained on these spaces. Of course these fractional order Sobolev spaces do not coincide with $W^{2s,2}(\Omega)$. She has also proved some maximal elliptic regularity on some certain spaces of Hölder continuous functions. We recall that our operator $(-\Delta)_D^s$ is different from the spectral Dirichlet fractional Laplacian (the fractional powers of the Laplace operator with the Dirichlet boundary condition). For the latter operator, Grubb [35] has shown that a maximal elliptic regularity can be achieved in some classical fractional order Sobolev spaces.

5. Fractional Robin exterior control problem

In this section we study the fractional Robin exterior control problem (1.4b). We begin by setting the functional analytic framework. We let

$$Z_R := L^2(\mathbb{R}^N \setminus \Omega, \mu), \quad U_R := W_{\Omega, \kappa}^{s,2}.$$

Notice that $d\mu = \kappa dx$. In addition we assume that $\kappa \in L^1(\mathbb{R}^N \setminus \Omega) \cap L^\infty(\mathbb{R}^N \setminus \Omega)$ and $\kappa > 0$ a.e. in $\mathbb{R}^N \setminus \Omega$. In view of proposition 3.9 the following (solution-map) control-to-state map

$$S : Z_R \rightarrow U_R, \quad z \mapsto u,$$

is well-defined. Moreover S is linear and continuous (by (3.19)). Since $U_R \hookrightarrow L^2(\Omega)$ with the embedding being continuous we can instead define

$$S : Z_R \rightarrow L^2(\Omega).$$

We can then write the so-called *reduced fractional Robin exterior control problem* as follows:

$$\min_{z \in Z_{ad,R}} \mathcal{J}(z) := J(Sz) + \frac{\xi}{2} \|z\|_{L^2(\mathbb{R}^N \setminus \Omega, \mu)}^2, \quad (5.1)$$

where $\xi \geq 0$. Here also, the precise conditions on J will be given in the statements of the results. We have the following well-posedness result.

Theorem 5.1. *Let $Z_{ad,R}$ be a closed and convex subset of Z_R . Let $\xi > 0$ with $J \geq 0$ or $Z_{ad,R} \subset Z_R$ bounded. Moreover, let $J : L^2(\Omega) \rightarrow \mathbb{R}$ be weakly lower-semicontinuous. Then there exists a solution $\bar{z}$ to (5.1) and equivalently to (1.4). If either J is convex and $\xi > 0$ or J is strictly convex and $\xi \geq 0$ then $\bar{z}$ is unique.*

Proof. We proceed as in the proof of theorem 4.1. Let $\{z_n\}_{n \in \mathbb{N}} \subset Z_{ad,R}$ be a minimizing sequence such that

$$\inf_{z \in Z_{ad,R}} \mathcal{J}(z) = \lim_{n \rightarrow \infty} \mathcal{J}(z_n).$$

If $\xi > 0$ with $J \geq 0$ or $Z_{ad,R} \subset Z_R$ is bounded, then after a subsequence, if necessary, we have that $z_n \rightharpoonup \bar{z}$ in $L^2(\mathbb{R}^N \setminus \Omega, \mu)$ as $n \rightarrow \infty$. Now since $Z_{ad,R}$ is a convex and closed subset of Z_R , it follows that $\bar{z} \in Z_{ad,R}$.

Next we show that the pair $(S\bar{z}, \bar{z})$ satisfies the state equation. Notice that $u_n := Sz_n$ is the weak solution of (1.4b) with exterior value z_n . Thus, by definition, $u_n \in W_{\Omega, \kappa}^{s,2}$ and the identity

$$\mathcal{E}(u_n, v) = \int_{\mathbb{R}^N \setminus \Omega} z_n v \, d\mu, \quad (5.2)$$

holds for every $v \in W_{\Omega, \kappa}^{s,2}$ where we recall that $\mathcal{E}$ is given in (3.18). We also notice that the mapping S is also bounded from Z_R into $W_{\Omega, \kappa}^{s,2}$ (by (3.19)). This shows that the sequence $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $W_{\Omega, \kappa}^{s,2}$. Thus, after a subsequence, if necessary, we have that $Sz_n = u_n \rightharpoonup S\bar{z} = \bar{u}$ in $W_{\Omega, \kappa}^{s,2}$ as $n \rightarrow \infty$. This implies that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left(\frac{C_{N,s}}{2} \int \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(u_n(x) - u_n(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx dy + \int_{\mathbb{R}^N \setminus \Omega} u_n v \, d\mu \right) \\ &= \frac{C_{N,s}}{2} \int \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(\bar{u}(x) - \bar{u}(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx dy + \int_{\mathbb{R}^N \setminus \Omega} \bar{u} v \, d\mu, \end{aligned}$$

for every $v \in W_{\Omega, \kappa}^{s,2}$. Since $z_n \rightharpoonup \bar{z}$ in $L^2(\mathbb{R}^N \setminus \Omega, \mu)$ as $n \rightarrow \infty$, it follows that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N \setminus \Omega} z_n v \, d\mu = \int_{\mathbb{R}^N \setminus \Omega} \bar{z} v \, d\mu,$$

for every $v \in W_{\Omega, \kappa}^{s,2}$. Therefore we can pass to the limit in (5.2) as $n \rightarrow \infty$ to obtain that

$$\mathcal{E}(\bar{u}, v) = \int_{\mathbb{R}^N \setminus \Omega} \bar{z} v \, dx, \quad \forall v \in W_{\Omega, \kappa}^{s,2}.$$

Thus, $(S\bar{z}, \bar{z}) = (\bar{u}, \bar{z})$ satisfies the state equation (1.4b). The rest of the steps are similar to the proof of theorem 4.1 and we omit them for brevity. $\square$

As in the case of the fractional Dirichlet exterior control problem (4.1) we identify next the adjoint of the control-to-state map S .

Lemma 5.2. *For the state equation (1.4b) the adjoint operator $S^* : L^2(\Omega) \rightarrow Z_R$ is given by*

$$(S^* w, z)_{Z_R} = \int_{\mathbb{R}^N \setminus \Omega} p z \, d\mu \quad \forall z \in Z_R,$$

where $w \in L^2(\Omega)$ and $p \in W_{\Omega, \kappa}^{s,2}$ is the weak solution to

$$\begin{cases} (-\Delta)^s p &= w & \text{in } \Omega, \\ \mathcal{N}_s p + \kappa p &= 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (5.3)$$

Proof. Let $w \in L^2(\Omega)$ and $z \in Z_R$. Then $Sz \in W_{\Omega, \kappa}^{s,2} \hookrightarrow L^2(\Omega)$ with the embedding being continuous. Then we can write

$$(w, Sz)_{L^2(\Omega)} = (S^* w, z)_{Z_R}.$$

Next we test (5.3) with Sz to arrive at

$$\begin{aligned} (w, Sz)_{L^2(\Omega)} &= \frac{C_{N,s}}{2} \int \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(u(x) - u(y))(p(x) - p(y))}{|x - y|^{N+2s}} \, dx dy + \int_{\mathbb{R}^N \setminus \Omega} u p \, d\mu \\ &= \int_{\mathbb{R}^N \setminus \Omega} z p \, d\mu = (S^* w, z)_{Z_R}, \end{aligned}$$

where we have used the fact that u solves the state equation according to definition 3.8. The proof is finished. $\square$

For the remainder of this section we will assume that $\xi > 0$. The proof of the next result is similar to the proof of theorem 4.3 and is omitted for brevity.

Theorem 5.3. *Let the assumptions of theorem 5.1 hold. Let $\mathcal{Z}$ be an open set in Z_R such that $Z_{ad,R} \subset \mathcal{Z}$. Let $u \mapsto J(u) : L^2(\Omega) \rightarrow \mathbb{R}$ be continuously Fréchet differentiable with $J'(u) \in L^2(\Omega)$. If $\bar{z}$ is a minimizer of (5.1) over $Z_{ad,R}$, then the first necessary optimality conditions are given by*

$$\int_{\mathbb{R}^N \setminus \Omega} (\bar{p} + \xi \bar{z})(z - \bar{z}) \, d\mu \geq 0, \quad z \in Z_{ad,R}, \quad (5.4)$$

where $\bar{p} \in W_{\Omega, \kappa}^{s,2}$ solves the adjoint equation

$$\begin{cases} (-\Delta)^s \bar{p} = J'(\bar{u}) & \text{in } \Omega, \\ \mathcal{N}_s \bar{p} + \kappa \bar{p} = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (5.5)$$

Equivalently we can write (5.4) as

$$\bar{z} = \mathcal{P}_{Z_{ad,R}} \left(-\frac{\bar{p}}{\xi} \right), \quad (5.6)$$

where $\mathcal{P}_{Z_{ad,R}}$ is the projection onto the set $Z_{ad,R}$. If J is convex, then (5.4) is a sufficient condition.

Remark 5.4 (Regularity of the optimization variables). As pointed out in remark 4.4 (Dirichlet case) the regularity for the integral fractional Laplacian is a delicate issue. In fact for the Robin problem, in $\mathbb{R}^N \setminus \Omega$ we can only guarantee that $\bar{p} \in L^2(\mathbb{R}^N \setminus \Omega, \mu)$. We further emphasize that the regularity for the fractional Robin problem is still open. Therefore due to the lack of such regularity results, we cannot use the classical boot-strap argument to further improve the regularity of the control $\bar{z}$.

6. Approximation of Dirichlet exterior value and control problems

We recall that the Dirichlet exterior value problem (1.2) in our case is only understood in the very-weak sense (see theorem 3.5). Moreover a numerical approximation of solutions to this problem will require a direct approximation of the interaction operator $\mathcal{N}_s$ which is challenging. Similar situations arise in the first order optimality conditions for the Dirichlet control problem (4.4).

The purpose of this section is to not only introduce a new approach to approximate weak and very-weak solutions to the nonhomogeneous Dirichlet exterior value problem (recall that if z is regular enough then a very-weak solution is a weak solution, and every weak solution is a very-weak solution, see theorem 3.5) but also to consider a regularized fractional Dirichlet exterior control problem. We begin by stating the regularized Dirichlet exterior value problem. Let $n \in \mathbb{N}$. Find $u_n \in W_{\Omega,\kappa}^{s,2}$ solving the elliptic problem

$$\begin{cases} (-\Delta)^s u_n = 0 & \text{in } \Omega, \\ \mathcal{N}_s u_n + n\kappa u_n = n\kappa z & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (6.1)$$

Notice that the fractional regularized Dirichlet exterior problem (6.1) is nothing but the fractional Robin exterior value problem (1.4b). We proceed by showing that the solution u_n to (6.1) converges to a function u , as $n \rightarrow \infty$, that solves the state equation (1.2) in the very weak sense (3.3). This is our new method to solve the non-homogeneous Dirichlet exterior value problem. Recall that the weak formulation of (6.1) does not require access to $\mathcal{N}_s$ (see definition (3.8)) and it is straightforward to implement.

In this section we are interested in solutions u_n to (6.1) that belong to $W_{\Omega,\kappa}^{s,2} \cap L^2(\mathbb{R}^N \setminus \Omega)$ which is endowed with the norm

$$\|u\|_{W_{\Omega,\kappa}^{s,2} \cap L^2(\mathbb{R}^N \setminus \Omega)} := \left(\|u\|_{W_{\Omega,\kappa}^{s,2}}^2 + \|u\|_{L^2(\mathbb{R}^N \setminus \Omega)}^2 \right)^{\frac{1}{2}}. \quad (6.2)$$

In addition, in our application we shall take κ such that its support $\text{supp}[\kappa] \subset \mathbb{R}^N \setminus \Omega$ has a positive Lebesgue measure. For this reason we shall assume the following.

Assumption 6.1. We assume that $\kappa \in L^1(\mathbb{R}^N \setminus \Omega) \cap L^\infty(\mathbb{R}^N \setminus \Omega)$ and satisfies $\kappa > 0$ almost everywhere in $K := \text{supp}[\kappa] \subset \mathbb{R}^N \setminus \Omega$, where the Lebesgue measure $|K| > 0$.

It follows from assumption 6.1 that $\int_{\mathbb{R}^N \setminus \Omega} \kappa \, dx > 0$.

To show the existence of weak solutions to the system (6.1) that belong to $W_{\Omega, \kappa}^{s,2} \cap L^2(\mathbb{R}^N \setminus \Omega)$, we need some preparation.

Lemma 6.2. Assume that assumption 6.1 holds. Then

$$\|u\|_W := \left(\int \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx dy + \int_{\mathbb{R}^N \setminus \Omega} |u|^2 \, dx \right)^{\frac{1}{2}} \quad (6.3)$$

defines an equivalent norm on $W_{\Omega, \kappa}^{s,2} \cap L^2(\mathbb{R}^N \setminus \Omega)$.

Proof. Firstly, it is readily seen that there is a constant $C > 0$ such that

$$\|u\|_W \leq C \|u\|_{W_{\Omega, \kappa}^{s,2} \cap L^2(\mathbb{R}^N \setminus \Omega)} \quad \text{for all } u \in W_{\Omega, \kappa}^{s,2} \cap L^2(\mathbb{R}^N \setminus \Omega). \quad (6.4)$$

Secondly, we claim that there is a constant $C > 0$ such that

$$\|u\|_{W_{\Omega, \kappa}^{s,2} \cap L^2(\mathbb{R}^N \setminus \Omega)} \leq C \|u\|_W \quad \text{for all } u \in W_{\Omega, \kappa}^{s,2} \cap L^2(\mathbb{R}^N \setminus \Omega). \quad (6.5)$$

It is clear that

$$\int_{\mathbb{R}^N \setminus \Omega} |u|^2 \, d\mu \leq \|\kappa\|_{L^\infty(\mathbb{R}^N \setminus \Omega)} \int_{\mathbb{R}^N \setminus \Omega} |u|^2 \, dx. \quad (6.6)$$

It suffices to show that there is a constant $C > 0$ such that for every $u \in W_{\Omega, \kappa}^{s,2} \cap L^2(\mathbb{R}^N \setminus \Omega)$,

$$\int_{\Omega} |u|^2 \, dx \leq C \left(\int \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx dy + \int_{\mathbb{R}^N \setminus \Omega} |u|^2 \, dx \right). \quad (6.7)$$

We prove (6.7) by contradiction. Assume to the contrary that for every $n \in \mathbb{N}$, there exists a sequence $\{u_n\}_{n \in \mathbb{N}} \subset W_{\Omega, \kappa}^{s,2} \cap L^2(\mathbb{R}^N \setminus \Omega)$ such that

$$\int_{\Omega} |u_n|^2 \, dx > n \left(\int \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N+2s}} \, dx dy + \int_{\mathbb{R}^N \setminus \Omega} |u_n|^2 \, dx \right). \quad (6.8)$$

By possibly dividing (6.8) by $\|u_n\|_{L^2(\Omega)}^2$ we may assume that $\|u_n\|_{L^2(\Omega)}^2 = 1$ for every $n \in \mathbb{N}$. Hence, by (6.8), there is a constant $C > 0$ (independent of n) such that for every $n \in \mathbb{N}$,

$$\int \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N+2s}} \, dx dy + \int_{\mathbb{R}^N \setminus \Omega} |u_n|^2 \, dx \leq C. \quad (6.9)$$

Since $\kappa \in L^\infty(\mathbb{R}^N \setminus \Omega)$, (6.9) and (6.6) imply that for every $n \in \mathbb{N}$,

$$\int_{\mathbb{R}^N \setminus \Omega} |u_n|^2 \, d\mu \leq C. \quad (6.10)$$

Now (6.9) and (6.10) together with $\|u_n\|_{L^2(\Omega)}^2 = 1$ imply that $\{u_n\}_{n \in \mathbb{N}}$ is a bounded sequence in the space $W_{\Omega, \kappa}^{s,2} \cap L^2(\mathbb{R}^N \setminus \Omega)$. Therefore, after passing to a subsequence, if necessary, we may assume that u_n converges weakly to some $u \in W_{\Omega, \kappa}^{s,2} \cap L^2(\mathbb{R}^N \setminus \Omega)$ and strongly to u in $L^2(\Omega)$, as $n \rightarrow \infty$ (as the embedding $W_{\Omega, \kappa}^{s,2} \hookrightarrow L^2(\Omega)$ is compact by remark 3.7(c)). It follows from (6.8) and the fact that $\|u_n\|_{L^2(\Omega)}^2 = 1$ that

$$\lim_{n \rightarrow \infty} \int \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N+2s}} dx dy = 0 \text{ and } \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N \setminus \Omega} |u_n|^2 dx = 0.$$

These identities imply that $u_n|_{\mathbb{R}^N \setminus \Omega}$ converges strongly to zero in $L^2(\mathbb{R}^N \setminus \Omega)$ as $n \rightarrow \infty$, and after passing to a subsequence, if necessary, we have that

$$\lim_{n \rightarrow \infty} |u_n(x) - u_n(y)| = 0 \text{ for a.e. } (x, y) \in \mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2, \quad (6.11)$$

and

$$u_n \rightarrow 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \text{ as } n \rightarrow \infty. \quad (6.12)$$

More precisely, (6.11) implies that

$$\begin{cases} \lim_{n \rightarrow \infty} |u_n(x) - u_n(y)| = 0 & \text{for a.e. } (x, y) \in \Omega \times \Omega, \\ \lim_{n \rightarrow \infty} |u_n(x) - u_n(y)| = 0 & \text{for a.e. } (x, y) \in \Omega \times (\mathbb{R}^N \setminus \Omega), \\ \lim_{n \rightarrow \infty} |u_n(x) - u_n(y)| = 0 & \text{for a.e. } (x, y) \in (\mathbb{R}^N \setminus \Omega) \times \Omega. \end{cases} \quad (6.13)$$

Using (6.13), we get that u_n converges a.e. to some constant function c in $\mathbb{R}^N$ as $n \rightarrow \infty$. From (6.12) and the uniqueness of the limit, we have that $c = 0$ a.e. in $\mathbb{R}^N$. Since (after passing to a subsequence, if necessary) u_n converges a.e. to u in Ω as $n \rightarrow \infty$, the uniqueness of the limit also implies that $c = u = 0$ a.e. on Ω . On the other hand, $\|u\|_{L^2(\Omega)}^2 = \lim_{n \rightarrow \infty} \|u_n\|_{L^2(\Omega)}^2 = 1$, and this is a contradiction. Hence, (6.8) is not possible and we have shown (6.7). Finally the lemma follows from (6.4) and (6.5). The proof is finished. $\square$

The following theorem is the main result of this section.

Theorem 6.3 (Approximation of solutions to the Dirichlet problem). *Assume that assumption 6.1 holds. Then the following assertions hold.*

- (a) Let $z \in W^{s,2}(\mathbb{R}^N \setminus \Omega)$ and $u_n \in W_{\Omega, \kappa}^{s,2} \cap L^2(\mathbb{R}^N \setminus \Omega)$ be the weak solution of (6.1). Let $u \in W^{s,2}(\mathbb{R}^N)$ be the weak solution to the state equation (1.3b). Then there is a constant $C > 0$ (independent of n) such that

$$\|u - u_n\|_{L^2(\mathbb{R}^N)} \leq \frac{C}{n} \|u\|_{W^{s,2}(\mathbb{R}^N)}. \quad (6.14)$$

In particular u_n converges strongly to u in $L^2(\mathbb{R}^N)$ as $n \rightarrow \infty$.

- (b) Let $z \in L^2(\mathbb{R}^N \setminus \Omega)$ and $u_n \in W_{\Omega, \kappa}^{s,2} \cap L^2(\mathbb{R}^N \setminus \Omega)$ be the weak solution of (6.1). Then there exist a subsequence that we still denote by $\{u_n\}_{n \in \mathbb{N}}$ and a function $\tilde{u} \in L^2(\mathbb{R}^N)$ such that $u_n \rightharpoonup \tilde{u}$ in $L^2(\mathbb{R}^N)$ as $n \rightarrow \infty$, and $\tilde{u}$ satisfies

$$\int_{\Omega} \tilde{u} (-\Delta)^s v dx = - \int_{\mathbb{R}^N \setminus \Omega} \tilde{u} \mathcal{N}_s v dx, \quad (6.15)$$

for all $v \in V$.

Remark 6.4 (Convergence to a very-weak solution). Notice that Part (a) of theorem 6.3 implies strong convergence to a weak solution (with rate). On the other hand, Part (b) ‘almost’ implies weak convergence to a very-weak solution (we still do not know if $\tilde{u}|_{\mathbb{R}^N \setminus \Omega} = z$). We emphasize that such an approximation of very-weak solutions using the Robin problem, to the best of our knowledge, is open even for the classical case $s = 1$ when the boundary function just belongs to $L^2(\partial\Omega)$.

Proof of theorem 6.3.

(a) Let $z \in W^{s,2}(\mathbb{R}^N \setminus \Omega)$. Firstly, recall that under our assumption $W^{s,2}(\mathbb{R}^N \setminus \Omega) \hookrightarrow L^2(\mathbb{R}^N \setminus \Omega) \hookrightarrow L^2(\mathbb{R}^N \setminus \Omega, \mu)$. Secondly, consider the system (6.1). A weak solution is a function $u_n \in W_{\Omega, \kappa}^{s,2} \cap L^2(\mathbb{R}^N \setminus \Omega)$ such that the identity

$$\begin{aligned} \frac{C_{N,s}}{2} \int \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(u_n(x) - u_n(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy \\ + n \int_{\mathbb{R}^N \setminus \Omega} u_n v d\mu = n \int_{\mathbb{R}^N \setminus \Omega} z v d\mu, \end{aligned} \quad (6.16)$$

holds for every $v \in W_{\Omega, \kappa}^{s,2} \cap L^2(\mathbb{R}^N \setminus \Omega)$. Proceeding as in the proof of proposition 3.9 we can easily deduce that for every $n \in \mathbb{N}$, there is a unique $u_n \in W_{\Omega, \kappa}^{s,2} \cap L^2(\mathbb{R}^N \setminus \Omega)$ satisfying (6.16).

For $v, w \in W_{\Omega, \kappa}^{s,2} \cap L^2(\mathbb{R}^N \setminus \Omega)$ we let

$$\mathcal{E}_n(v, w) := \frac{C_{N,s}}{2} \int \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(v(x) - v(y))(w(x) - w(y))}{|x - y|^{N+2s}} dx dy + n \int_{\mathbb{R}^N \setminus \Omega} v w d\mu.$$

We notice that proceeding as in the proof of lemma 6.2 we can deduce that there is a constant $C > 0$ such that

$$\frac{C_{N,s}}{2} \int \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N+2s}} dx dy + n \int_{\mathbb{R}^N \setminus \Omega} |u_n|^2 dx \leq C \mathcal{E}_n(u_n, u_n). \quad (6.17)$$

Next, let $u \in W^{s,2}(\mathbb{R}^N)$ be the weak solution of (3.1) and $v \in W_{\Omega, \kappa}^{s,2} \cap L^2(\mathbb{R}^N \setminus \Omega)$. Using the integration by parts formula (2.5) we get that

$$\begin{aligned} \mathcal{E}_n(u - u_n, v) &= \int_{\Omega} v(-\Delta)^s(u - u_n) dx + \int_{\mathbb{R}^N \setminus \Omega} v \mathcal{N}_s(u - u_n) dx \\ &\quad + n \int_{\mathbb{R}^N \setminus \Omega} v(u - u_n) d\mu \\ &= \int_{\Omega} v(-\Delta)^s(u - u_n) dx + \int_{\mathbb{R}^N \setminus \Omega} v \mathcal{N}_s u dx \\ &\quad - \int_{\mathbb{R}^N \setminus \Omega} v(\mathcal{N}_s u_n + n\kappa(u_n - z)) dx \\ &= \int_{\mathbb{R}^N \setminus \Omega} v \mathcal{N}_s u dx. \end{aligned} \quad (6.18)$$

Taking $v = u - u_n$ as a test function in (6.18) and using (6.17), we get that there is a constant $C > 0$ (independent of n) such that

$$\begin{aligned} n\|u - u_n\|_{L^2(\mathbb{R}^N \setminus \Omega)}^2 &\leq \mathcal{E}_n(u - u_n, u - u_n) = \int_{\mathbb{R}^N \setminus \Omega} (u - u_n) \mathcal{N}_s u \, dx \\ &\leq \|u - u_n\|_{L^2(\mathbb{R}^N \setminus \Omega)} \|\mathcal{N}_s u\|_{L^2(\mathbb{R}^N \setminus \Omega)} \\ &\leq C \|u - u_n\|_{L^2(\mathbb{R}^N \setminus \Omega)} \|u\|_{W^{s,2}(\mathbb{R}^N)}. \end{aligned}$$

We have shown that there is a constant $C > 0$ (independent of n) such that

$$\|u - u_n\|_{L^2(\mathbb{R}^N \setminus \Omega)} \leq \frac{C}{n} \|u\|_{W^{s,2}(\mathbb{R}^N)}. \quad (6.19)$$

Next, observe that

$$\|u - u_n\|_{L^2(\Omega)} = \sup_{\eta \in L^2(\Omega)} \frac{|\int_{\Omega} (u - u_n) \eta \, dx|}{\|\eta\|_{L^2(\Omega)}}. \quad (6.20)$$

For any $\eta \in L^2(\Omega)$, let $w \in W_0^{s,2}(\overline{\Omega})$ be the weak solution of the Dirichlet problem

$$(-\Delta)^s w = \eta \text{ in } \Omega, \quad w = 0 \text{ in } \mathbb{R}^N \setminus \Omega. \quad (6.21)$$

It follows from proposition 3.2 that there is a constant $C > 0$ such that

$$\|w\|_{W^{s,2}(\mathbb{R}^N)} \leq C \|\eta\|_{L^2(\Omega)}. \quad (6.22)$$

Since $w \in W_0^{s,2}(\overline{\Omega})$, then using (6.18) we get that

$$\begin{aligned} &\int_{\Omega} (u - u_n) (-\Delta)^s w \, dx \\ &= \frac{C_{N,s}}{2} \int \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{((u - u_n)(x) - (u - u_n)(y))(w(x) - w(y))}{|x - y|^{N+2s}} \, dx dy \\ &\quad - \int_{\mathbb{R}^N \setminus \Omega} (u - u_n) \mathcal{N}_s w \, dx \\ &= \mathcal{E}_n(u - u_n, w) - \int_{\mathbb{R}^N \setminus \Omega} (u - u_n) \mathcal{N}_s w \, dx \\ &= \int_{\mathbb{R}^N \setminus \Omega} w \mathcal{N}_s u \, dx - \int_{\mathbb{R}^N \setminus \Omega} (u - u_n) \mathcal{N}_s w \, dx \\ &= - \int_{\mathbb{R}^N \setminus \Omega} (u - u_n) \mathcal{N}_s w \, dx. \end{aligned}$$

It follows from the preceding identity, (6.19) and (6.22) that there is a constant $C > 0$ such that

$$\begin{aligned} \left| \int_{\Omega} (u - u_n) (-\Delta)^s w \, dx \right| &= \left| \int_{\mathbb{R}^N \setminus \Omega} (u - u_n) \mathcal{N}_s w \, dx \right| \\ &\leq \|u - u_n\|_{L^2(\mathbb{R}^N \setminus \Omega)} \|\mathcal{N}_s w\|_{L^2(\mathbb{R}^N \setminus \Omega)} \\ &\leq \frac{C}{n} \|u\|_{W^{s,2}(\mathbb{R}^N)} \|w\|_{W^{s,2}(\mathbb{R}^N)} \\ &\leq \frac{C}{n} \|u\|_{W^{s,2}(\mathbb{R}^N)} \|\eta\|_{L^2(\Omega)}. \end{aligned} \quad (6.23)$$

Using (6.20) and (6.23) we get that

$$\|u - u_n\|_{L^2(\Omega)} \leq \frac{C}{n} \|u\|_{W^{s,2}(\mathbb{R}^N)}. \quad (6.24)$$

Now the estimate (6.14) follows from (6.19) and (6.24). Observe that it follows from (6.14) that $u_n \rightarrow u$ in $L^2(\mathbb{R}^N)$ as $n \rightarrow \infty$ and this completes the proof of Part (a).

(b) Now let $z \in L^2(\mathbb{R}^N \setminus \Omega) \hookrightarrow L^2(\mathbb{R}^N \setminus \Omega, \mu)$. Notice that $\{u_n\}_{n \in \mathbb{N}}$ satisfies (6.16). Proceeding as in the proof of lemma 6.2 we can deduce that there is a constant $C > 0$ (independent of n) such that

$$n \|u_n\|_{L^2(\mathbb{R}^N \setminus \Omega)}^2 \leq C \mathcal{E}_n(u_n, u_n) \leq nC \|\kappa\|_{L^\infty(\mathbb{R}^N \setminus \Omega)} \|z\|_{L^2(\mathbb{R}^N \setminus \Omega)} \|u_n\|_{L^2(\mathbb{R}^N \setminus \Omega)},$$

and this implies that

$$\|u_n\|_{L^2(\mathbb{R}^N \setminus \Omega)} \leq C \|z\|_{L^2(\mathbb{R}^N \setminus \Omega)}. \quad (6.25)$$

Now we proceed as in the proof of (6.24). As in (6.20) we have that

$$\|u_n\|_{L^2(\Omega)} = \sup_{\eta \in L^2(\Omega)} \frac{|\int_{\Omega} u_n \eta \, dx|}{\|\eta\|_{L^2(\Omega)}}. \quad (6.26)$$

Let $\eta \in L^2(\Omega)$ and $w \in W_0^{s,2}(\overline{\Omega})$ the weak solution of (6.21). Since $w \in W_0^{s,2}(\overline{\Omega})$, then using (6.18) we have that

$$\begin{aligned} & \int_{\Omega} u_n (-\Delta)^s w \, dx \\ &= \frac{C_{N,s}}{2} \int \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(u_n(x) - u_n(y))(w(x) - w(y))}{|x - y|^{N+2s}} \, dx dy - \int_{\mathbb{R}^N \setminus \Omega} u_n \mathcal{N}_s w \, dx \\ &= - \int_{\mathbb{R}^N \setminus \Omega} u_n \mathcal{N}_s w \, dx. \end{aligned}$$

It follows from the preceding identity, (6.25) and (6.22) that there is a constant $C > 0$ such that

$$\begin{aligned} \left| \int_{\Omega} u_n (-\Delta)^s w \, dx \right| &= \left| \int_{\mathbb{R}^N \setminus \Omega} u_n \mathcal{N}_s w \, dx \right| \leq \|u_n\|_{L^2(\mathbb{R}^N \setminus \Omega)} \|\mathcal{N}_s w\|_{L^2(\mathbb{R}^N \setminus \Omega)} \\ &\leq C \|z\|_{L^2(\mathbb{R}^N \setminus \Omega)} \|w\|_{W^{s,2}(\mathbb{R}^N)}. \end{aligned} \quad (6.27)$$

Using (6.25), (6.27) and (6.22) we get that there is a constant $C > 0$ (independent of n) such that

$$\|u_n\|_{L^2(\Omega)} \leq C \|z\|_{L^2(\mathbb{R}^N \setminus \Omega)}. \quad (6.28)$$

Combining (6.25) and (6.28) we get that

$$\|u_n\|_{L^2(\mathbb{R}^N)} \leq C \|z\|_{L^2(\mathbb{R}^N \setminus \Omega)}. \quad (6.29)$$

Hence, the sequence $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $L^2(\mathbb{R}^N)$. Thus, after a subsequence, if necessary, we have that u_n converges weakly to some $\tilde{u}$ in $L^2(\mathbb{R}^N)$ as $n \rightarrow \infty$.

Using (6.16) we get that for every $v \in V := \{v \in W_0^{s,2}(\overline{\Omega}) : (-\Delta)^s v \in L^2(\Omega)\}$,

$$\frac{C_{N,s}}{2} \int \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(u_n(x) - u_n(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy = 0. \quad (6.30)$$

Using the integration by parts formula (2.5) we can deduce that

$$\begin{aligned} \frac{C_{N,s}}{2} \int \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(u_n(x) - u_n(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy \\ = \int_{\Omega} u_n(-\Delta)^s v dx + \int_{\mathbb{R}^N \setminus \Omega} u_n \mathcal{N}_s v dx, \end{aligned} \quad (6.31)$$

for every $v \in V$. Combining (6.30) and (6.31) we get that the identity

$$\int_{\Omega} u_n(-\Delta)^s v dx + \int_{\mathbb{R}^N \setminus \Omega} u_n \mathcal{N}_s v dx = 0, \quad (6.32)$$

holds for every $v \in V$. Passing to the limit in (6.32) as $n \rightarrow \infty$, we obtain that

$$\int_{\Omega} \tilde{u}(-\Delta)^s v dx + \int_{\mathbb{R}^N \setminus \Omega} \tilde{u} \mathcal{N}_s v dx = 0,$$

for every $v \in V$. We have shown (6.15) and the proof is finished. $\square$

Toward this end, for $\xi \geq 0$ we introduce the regularized fractional Dirichlet control problem:

$$\min_{u \in U_R, z \in Z_R} J(u) + \frac{\xi}{2} \|z\|_{L^2(\mathbb{R}^N \setminus \Omega)}^2, \quad (6.33a)$$

subject to the regularized exterior value problem (Robin problem): find $u_n \in U_R$ solving

$$\begin{cases} (-\Delta)^s u &= 0 & \text{in } \Omega \\ \mathcal{N}_s u + n\kappa u &= n\kappa z & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (6.33b)$$

and the control constraints

$$z \in Z_{ad,R}. \quad (6.33c)$$

Here $Z_R := L^2(\mathbb{R}^N \setminus \Omega)$, $Z_{ad,R}$ is a closed and convex subset of Z_R and $U_R := W_{\Omega,\kappa}^{s,2} \cap L^2(\mathbb{R}^N \setminus \Omega)$. We again remark that (6.33) is nothing but the fractional Robin exterior control problem.

Theorem 6.5 (Approximation of the Dirichlet control problem). *The regularized control problem (6.33) admits a minimizer $(z_n, u(z_n)) \in Z_{ad,R} \times (W_{\Omega,\kappa}^{s,2} \cap L^2(\mathbb{R}^N \setminus \Omega))$. Let $Z_R = W^{s,2}(\mathbb{R}^N \setminus \Omega)$ and $Z_{ad,R} \subset Z_R$ be bounded. Then for any sequence $\{n_\ell\}_{\ell=1}^\infty$ with $n_\ell \rightarrow \infty$, there exists a subsequence still denoted by $\{n_\ell\}_{\ell=1}^\infty$ such that $z_{n_\ell} \rightharpoonup \tilde{z}$ in $W^{s,2}(\mathbb{R}^N \setminus \Omega)$, $u(z_{n_\ell}) \rightarrow \tilde{u}$ in $L^2(\mathbb{R}^N)$ as $n_\ell \rightarrow \infty$ and $(\tilde{z}, \tilde{u})$ solves the Dirichlet control problem (1.3) with $Z_{ad,D}$ replaced by $Z_{ad,R}$.*

Proof. Since the regularized control problem (6.33) is nothing but the Robin control problem therefore the existence of minimizers follows by directly using theorem 5.1. Following the proof of theorem 5.1 and using the fact that $Z_{ad,R}$ is a bounded subset of the reflexive Banach space $W^{s,2}(\mathbb{R}^N \setminus \Omega)$, after a subsequence, if necessary, we have that $z_{n_\ell} \rightharpoonup \tilde{z}$ in $W^{s,2}(\mathbb{R}^N \setminus \Omega)$ as $n_\ell \rightarrow \infty$. Now since $Z_{ad,R}$ is closed and convex, then it is weakly closed. Thus $\tilde{z} \in Z_{ad,R}$.

Following the proof of theorem 6.3(a) we can deduce that there exists a subsequence $\{u_{n_\ell}\}$ such that $u_{n_\ell} \rightarrow \tilde{u}$ in $L^2(\mathbb{R}^N)$ as $n_\ell \rightarrow \infty$ and $\tilde{u} \in W^{s,2}(\mathbb{R}^N)$. Combining this convergence with the aforementioned convergence of z_{n_ℓ} we can conclude that $(\tilde{z}, \tilde{u}) \in Z_{ad,R} \times W^{s,2}(\mathbb{R}^N)$ solves the Dirichlet exterior value problem (1.3b).

It then remains to show that $(\tilde{z}, \tilde{u})$ is a minimizer of (1.3). Let (z', u') be any minimizer of (1.3). Let us consider the regularized state equation (6.33b) but with boundary datum z' . We denote the solution of the resulting state equation by u'_{n_ℓ} . By using the same limiting argument as above we can select a subsequence such that $u'_{n_\ell} \rightarrow u'$ in $L^2(\mathbb{R}^N)$ as $n_\ell \rightarrow \infty$. Letting $j(z, u) := J(u) + \frac{\xi}{2} \|z\|_{L^2(\mathbb{R}^N \setminus \Omega)}^2$, it then follows that

$$j(z', u') \leq j(\tilde{z}, \tilde{u}) \leq \liminf_{n_\ell \rightarrow \infty} j(z_{n_\ell}, u_{n_\ell}) \leq \liminf_{n_\ell \rightarrow \infty} j(z', u'_{n_\ell}) = j(z', u'),$$

where the second inequality is due to the weak-lower semicontinuity of J . The third inequality is due to the fact that $\{(z_{n_\ell}, u_{n_\ell})\}$ is a sequence of minimizers for (6.33). The proof is finished. $\square$

We conclude this section by writing the stationarity system corresponding to (6.33): find $(z, u, p) \in Z_{ad,R} \times (W_{\Omega,\kappa}^{s,2} \cap L^2(\mathbb{R}^N \setminus \Omega)) \times (W_{\Omega,\kappa}^{s,2} \cap L^2(\mathbb{R}^N \setminus \Omega))$ such that

$$\begin{cases} \mathcal{E}(u, v) = \int_{\mathbb{R}^N \setminus \Omega} n \kappa z v \, dx, \\ \mathcal{E}(w, p) = \int_{\Omega} J'(u) w \, dx, \\ \int_{\mathbb{R}^N \setminus \Omega} (n \kappa p + \xi z)(\tilde{z} - z) \, dx \geq 0, \end{cases} \quad (6.34)$$

for all $(\tilde{z}, v, w) \in Z_{ad,R} \times (W_{\Omega,\kappa}^{s,2} \cap L^2(\mathbb{R}^N \setminus \Omega)) \times (W_{\Omega,\kappa}^{s,2} \cap L^2(\mathbb{R}^N \setminus \Omega))$.

7. Numerical approximations

The purpose of this section is to introduce numerical approximations of the problems we have considered so far. We emphasize that the fractional PDEs are intrinsically expensive, since the underlying coefficient matrices require approximation of integrals with singular kernels and the resulting coefficient matrices are dense. In addition, the fractional control problems can be prohibitively expensive since they not only require solving the nonlocal state equation but also the nonlocal adjoint equation and in case of Dirichlet control problem one needs to approximate the nonlocal normal derivative for the adjoint variable to evaluate the control variable (4.6). The presented approach is a first of its kind to numerically solve the fractional Robin problem and to approximate the Dirichlet problem by the Robin problem. Further details on the numerical analysis of the underlying PDEs, the control problem and solvers for the fractional PDEs will be part of a forthcoming research paper.

The rest of the section is organized as follows: in section 7.1 we begin with a finite element approximation of the Robin problem (6.1) which is the same as the regularized Dirichlet problem. We approximate the Dirichlet problem using the Robin problem. In section 7.2 we introduce an external source identification problem where we clearly see the difference between the nonlocal case and the classical case ($s \sim 1$). Finally, section 7.3 is devoted to the optimal control problems.

7.1. Approximation of a nonhomogeneous Dirichlet problem via a Robin problem

In view of theorem 6.3 we can approximate the Dirichlet problem with the help of the Robin (regularized Dirichlet) problem (6.1). Therefore we begin by introducing a discrete scheme for the Robin problem. Let $\tilde{\Omega}$ be a bounded open set that contains Ω , the support of the control/

source z and the support of κ . We consider a conforming simplicial triangulation of Ω and $\tilde{\Omega} \setminus \Omega$ such that the resulting partition remains admissible. We shall assume that the support of z and κ is contained in $\tilde{\Omega} \setminus \Omega$. We let our finite element space $\mathbb{V}_h$ (on $\tilde{\Omega}$) to be a set of continuous piecewise linear functions. Then the discrete (weak) version of (6.33b) with nonzero right-hand-side is given as follows: find $u_h \in \mathbb{V}_h$ such that

$$\begin{aligned} & \frac{C_{N,s}}{2} \int \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(u_h(x) - u_h(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy + \int_{\tilde{\Omega} \setminus \Omega} n \kappa u_h v dx \\ & = \langle f, v \rangle_{(W_{\Omega, \kappa}^{s,2} \cap L^2(\mathbb{R}^N \setminus \Omega))^*, W_{\Omega, \kappa}^{s,2} \cap L^2(\mathbb{R}^N \setminus \Omega)} + \int_{\tilde{\Omega} \setminus \Omega} n \kappa z v dx \quad \forall v \in \mathbb{V}_h. \end{aligned} \quad (7.1)$$

We approximate the double integral over $\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2$ by using the approach from [1, 2]. The remaining integrals are computed using numerical quadrature which is accurate for polynomials of degree less than and equal to 4. All implementations are carried out in Matlab and the discrete system of equations corresponding to the state and adjoint equations are solved using direct solvers. Note that iterative solvers for the fractional Robin problem are part of our future work.

We next consider an example that has been taken from [3]. Let $\Omega = B_0(1/2) \subset \mathbb{R}^2$. Our goal is to find u solving

$$\begin{cases} (-\Delta)^s u = 2 & \text{in } \Omega, \\ u(\cdot) = \frac{2^{-2s}}{\Gamma(1+s)^2} (1 - |\cdot|^2)_+^s & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

The exact solution in this case is given by

$$u(x) = u_1(x) + u_2(x) = \frac{2^{-2s}}{\Gamma(1+s)^2} \left((1 - |x|^2)_+^s + \left(\frac{1}{4} - |x|^2 \right)_+^s \right),$$

where u_1 and u_2 solve the problems

$$\begin{cases} (-\Delta)^s u_1 = 1 & \text{in } \Omega, \\ u_1 = \frac{2^{-2s}}{\Gamma(1+s)^2} (1 - |\cdot|^2)_+^s & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad \begin{cases} (-\Delta)^s u_2 = 1 & \text{in } \Omega, \\ u_2 = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (7.2)$$

We let $\tilde{\Omega} = B_0(3/2)$. We next approximate (7.2) using (7.1) and we set $\kappa = 1$ on its support. At first we fix $s = 0.5$ and the Degrees of Freedom (DoFs) to be DoFs = 2920. For this configuration, we study the $L^2(\Omega)$ error $\|u - u_h\|_{L^2(\Omega)}$ with respect to n in figure 2 (left). As expected, from theorem 6.3(a) we observe an approximation rate of $1/n$.

Next for a fixed $s = 0.5$, we check the stability of our scheme with respect to n as we refine the mesh. We have plotted the L^2 -error as we refine the mesh (equivalently increase DoFs) for $n = 1 \times 10^2, 1 \times 10^3, 1 \times 10^4, 1 \times 10^5$. We notice that the error remains stable with respect to n and we observe the following convergence rate with respect to the DoFs:

$$\|u - u_h\|_{L^2(\Omega)} \approx (\text{DoFs})^{-\frac{1}{2}}.$$

In the right panel we have shown the L^2 -error for a fixed $n = 1 \times 10^5$ but for various $s = 0.2, 0.4, 0.6, 0.8$. When $0 < s < \frac{1}{2}$ we have observed a rate of $(\text{DoFs})^{-\frac{1}{2}(s+\frac{1}{2})}$ and for $\frac{1}{2} \leq s < 1$ we observe a convergence rate of $(\text{DoFs})^{-\frac{1}{2}}$.

7.2. External source identification problem

We next consider an inverse problem to identify a source that is located outside the observation domain Ω . The optimality system is as given in (6.34) where we have approximated the

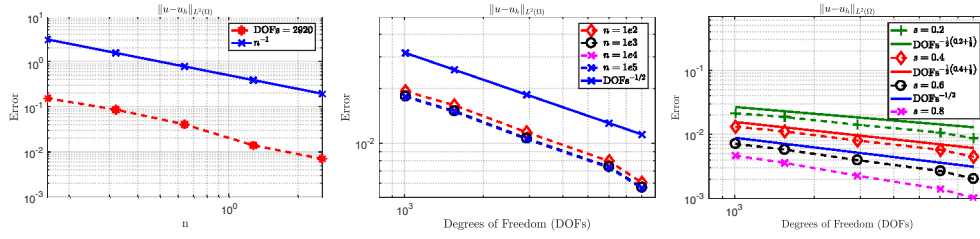


Figure 2. Left panel: Let $s = 0.5$ and DoFs = 2920 be fixed. We let $\kappa = 1$ and consider the L^2 -error between the actual solution u to the Dirichlet problem and its approximation u_h which solves the Robin problem. We have plotted the error with respect to n . The solid line denotes a reference line and the actual error. We observe a rate of $1/n$ which confirms our theoretical result (6.14). Middle panel: let $s = 0.5$ be fixed. For each $n = 1 \times 10^2, 1 \times 10^3, 1 \times 10^4, 1 \times 10^5$ we have plotted the L^2 -error with respect to the degrees of freedom (DOFs) as we refine the mesh. Notice that the error is stable with respect to n . Moreover, the observed rate of convergence is $(\text{DoFs})^{-\frac{1}{2}}$ and is independent of n . Right panel: Let $n = 1 \times 10^5$ be fixed. We plot the L^2 -error with respect to the DOFs for various values of s . The observed convergence rate is $(\text{DoFs})^{-\frac{1}{2}(s+\frac{1}{2})}$ for $0 < s < \frac{1}{2}$ and the observed rate is $(\text{DoFs})^{-\frac{1}{2}}$ for $\frac{1}{2} \leq s < 1$.

Dirichlet problem by the Robin problem. We use the continuous piecewise linear finite element discretization for all the optimization variables: state (u), control (z) and adjoint (p). We choose our objective function as

$$j(u, z) = J(u) + \frac{\xi}{2} \|z\|_{L^2(\mathbb{R}^N \setminus \Omega)}^2, \quad \text{with} \quad J(u) := \frac{1}{2} \|u - u_d\|_{L^2(\Omega)}^2,$$

and we let $Z_{ad,R} := \{z \in L^2(\mathbb{R}^N \setminus \Omega) : z \geq 0, \text{ a.e. in } \widehat{\Omega}\}$ where $\widehat{\Omega}$ is the support set of the control z that is contained in $\widetilde{\Omega} \setminus \Omega$. Moreover $u_d : L^2(\Omega) \rightarrow \mathbb{R}$ is the given data (observations). All the optimization problems below are solved using the projected-BFGS method with Armijo line search.

Our computational setup is shown in figure 3. The centered square region is $\Omega = [-0.4, 0.4]^2$ and the region inside the outermost ring is $\widetilde{\Omega} = B_0(3/2)$. The smaller square inside $\widetilde{\Omega} \setminus \Omega$ is $\widehat{\Omega}$ which is the support of the source/control. The right panel in figure 3 shows a finite element mesh with DoFs = 6103.

We define u_d as follows. For $z = 1$, we first solve the state equation for $\tilde{u}$ (first equation in (6.34)). We then add a normally distributed random noise with mean zero and standard deviation 0.02 to $\tilde{u}$. We call the resulting expression as u_d . Furthermore, we set $\kappa = 1$, and $n = 1e5$.

Our goal is then to identify the source $\bar{z}_h$. In figure 4, we first show the behavior of optimal $\bar{z}_h$ for different values of the regularization parameter $\xi = 1 \times 10^{-1}$ (4), 1×10^{-2} (4), 1×10^{-4} (4), 1×10^{-8} (2), 1×10^{-10} (2). The numbers in the bracket denote the total number of iterations that the BFGS has taken to achieve a stopping tolerance (for the projected gradient) of 1×10^{-7} . Notice that the Armijo line search has remained inactive in these cases. As expected the larger is the value of ξ , the smaller is the magnitude of $\bar{z}_h$, and this behavior saturates at $\xi = 1 \times 10^{-8}$.

Next, for a fixed $\xi = 1 \times 10^{-8}$, figure 5 shows the optimal $\bar{z}_h$ for $s = 0.1$ (4), 0.6 (2), 0.7 (2), 0.8 (2), 0.9 (2). The numbers in the bracket again denote the total number of iterations that the BFGS has taken to achieve a stopping tolerance (for the projected gradient) of 1×10^{-7} . Notice that the Armijo line search has remained inactive in these cases. We notice

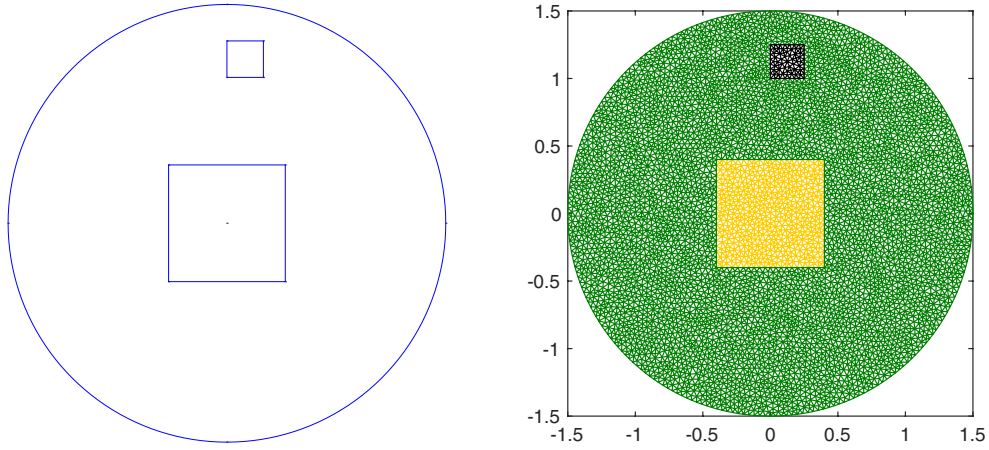


Figure 3. Left: computational domain where the inner square is Ω , the region inside the outer circle is $\tilde{\Omega}$ and the outer square inside $\tilde{\Omega} \setminus \Omega$ is $\hat{\Omega}$ which is the region where the source/control is supported. Right: a finite element mesh.

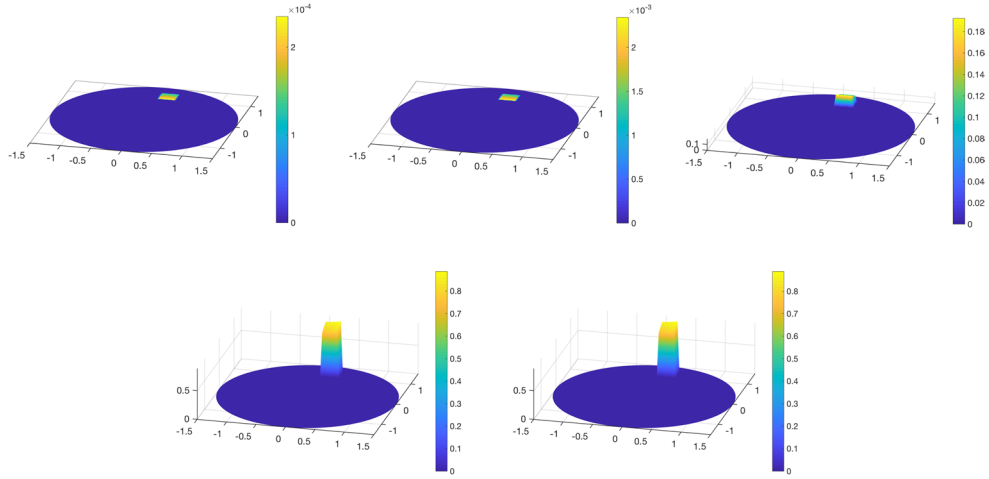


Figure 4. External source identification problem. The panels show the behavior of $\bar{z}_h$ with respect to the regularization parameter ξ : top row from left to right $\xi = 1 \times 10^{-1}, 1 \times 10^{-2}, 1 \times 10^{-4}$; bottom row from left to right: $\xi = 1 \times 10^{-8}, 1 \times 10^{-10}$. As it is expected, larger is ξ , smaller is the magnitude of $\bar{z}_h$, but this behavior saturates at $\xi = 1 \times 10^{-8}$.

that for large s , $\bar{z}_h \equiv 0$. This is expected as larger the s is, the more close we are to the classical Poisson problem case and we know that we cannot impose the external condition in that case.

7.3. Dirichlet control problem

We next consider two Dirichlet control problems. The setup is similar to section 7.2 except now we set $u_d \equiv 1$.

Example 7.1. The computational setup for the first example is shown in figure 6. Let $\Omega = B_0(1/2)$ (the region inside the innermost ring) and the region inside the outermost ring

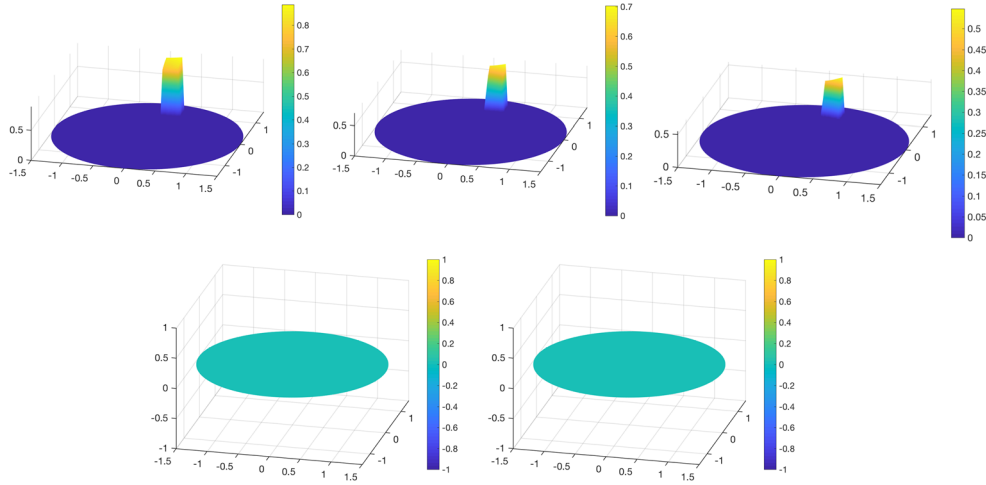


Figure 5. The panels show the behavior of $\bar{z}_h$ as we vary the exponent s . Top row from left to right: $s = 0.1, 0.6, 0.7$. Bottom row from left to right: $s = 0.8, 0.9$. For smaller values of s , the recovery of $\bar{z}_h$ is quite remarkable. However, for larger values of s , $\bar{z}_h \equiv 0$ as expected, the behavior of $\bar{u}_h$ for large s is close to the classical Poisson problem which does not allow external sources.

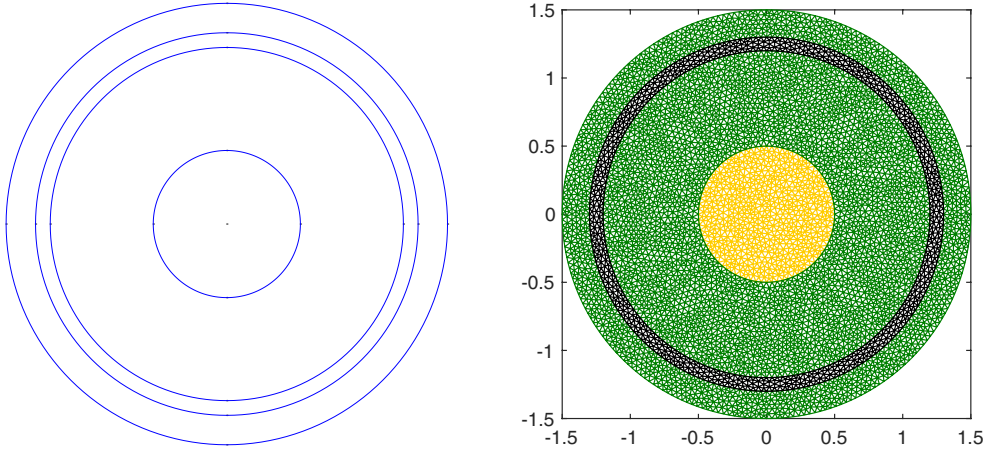


Figure 6. Left: computational domain where the inner circle is Ω , the region inside the outer circle is $\tilde{\Omega}$, and the annulus inside $\tilde{\Omega} \setminus \Omega$ is $\hat{\Omega}$ which is the region where the control is supported. Right: a finite element mesh.

is $\tilde{\Omega} = B_0(3/2)$. The annulus inside $\tilde{\Omega} \setminus \Omega$ is $\hat{\Omega}$ which is the support of the control. The right panel in figure 6 shows a finite element mesh with DoFs = 6069.

In figures 7 and 8 we have shown the optimization results for $s = 0.2$ (14) and $s = 0.8$ (4), respectively. Here again, the numbers in the bracket denote the total number of iterations that the BFGS has taken to achieve a stopping tolerance (for the projected gradient) of 1×10^{-7} . Notice that the Armijo line search has remained inactive in these cases. The top row shows the desired state u_d (left) and the optimal state $\bar{u}_h$ (right). The bottom row shows the optimal control $\bar{z}_h$ (left) and the optimal adjoint variable $\bar{p}_h$ (right). We notice

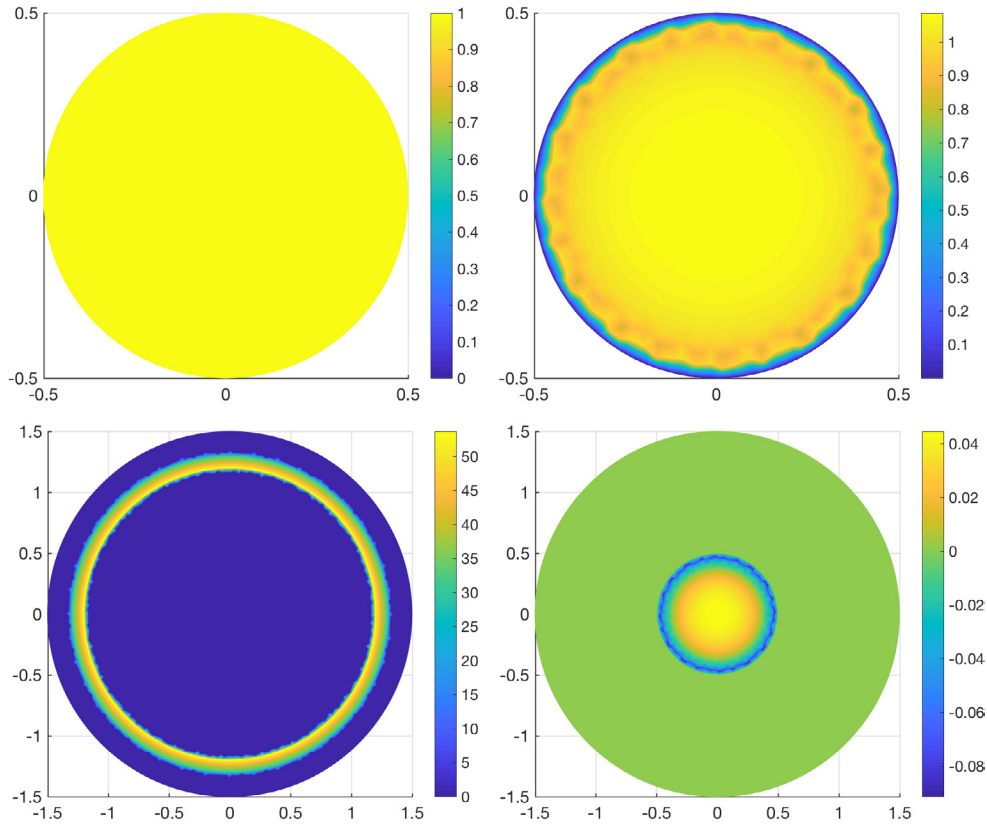


Figure 7. Example 1, $s = 0.2$: top row: left—desired state u_d ; right—optimal state $\bar{u}_h$. Bottom row: left—optimal control $\bar{z}_h$, right—optimal adjoint $\bar{p}_h$.

that in both cases we can approximate the desired state to a high accuracy but the approximation is slightly better for smaller s , especially close to the boundary. This is to be expected as for large values of s the regularity of the adjoint variable deteriorates significantly (see remark 4.4).

Example 7.2. The computational setup for our final example is shown in figure 9. The M-shape region is Ω and the region inside the outermost ring is $\tilde{\Omega} = B_0(0.6)$. The smaller region inside $\tilde{\Omega} \setminus \Omega$ is $\hat{\Omega}$ which is the support of the control. The right panel in figure 6 shows a finite element mesh with DoFs = 4462.

In figure 10 we have shown the optimization results for $s = 0.8$ (370). Again, the number in the bracket denotes the total number of iterations that the BFGS has taken to achieve a stopping tolerance (for the projected gradient) of 1×10^{-7} . Notice for this example, during most of the iterations, the Armijo line search has remained inactive but it got activated during a few number of iterations and has remained active for up to ten steps. The top row shows the desired state u_d (left) and the optimal state $\bar{u}_h$ (right). The bottom row shows the optimal control $\bar{z}_h$ (left) and the optimal adjoint variable $\bar{p}_h$ (right). Even though the control is applied in an extremely small region we can still match the desired state in certain parts of Ω .

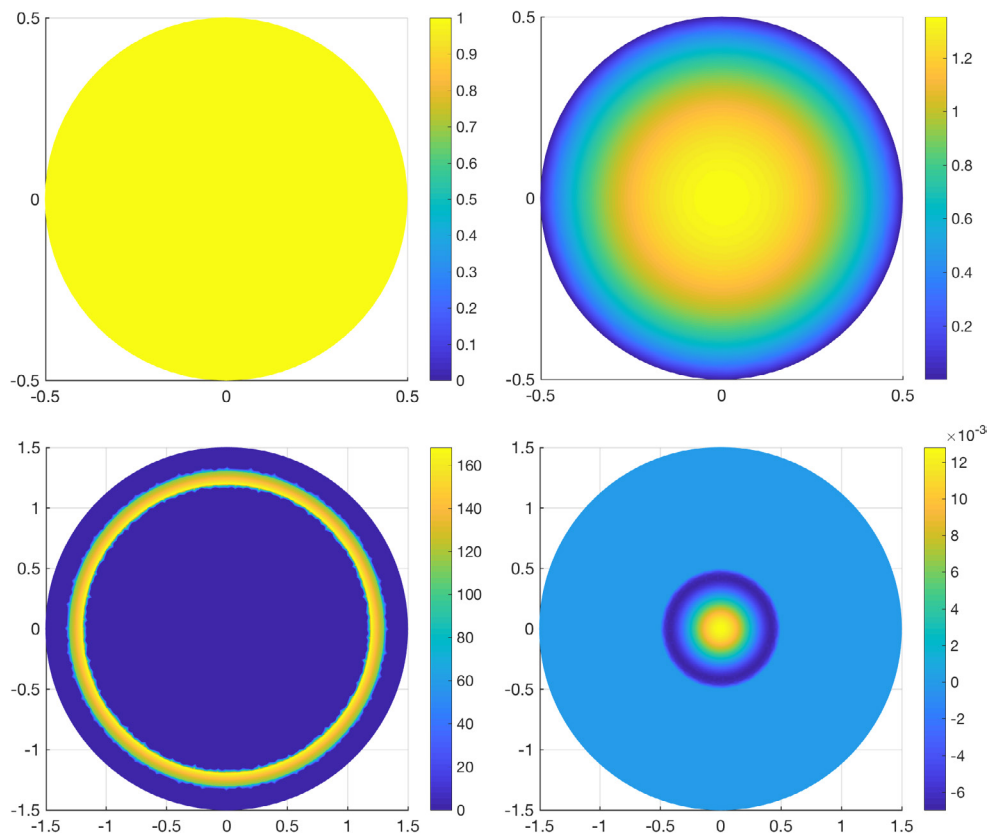


Figure 8. Example 1, $s = 0.8$: top row: left—desired state u_d ; right—optimal state $\bar{u}_h$. Bottom row: left—optimal control $\bar{z}_h$, right—optimal adjoint $\bar{p}_h$.

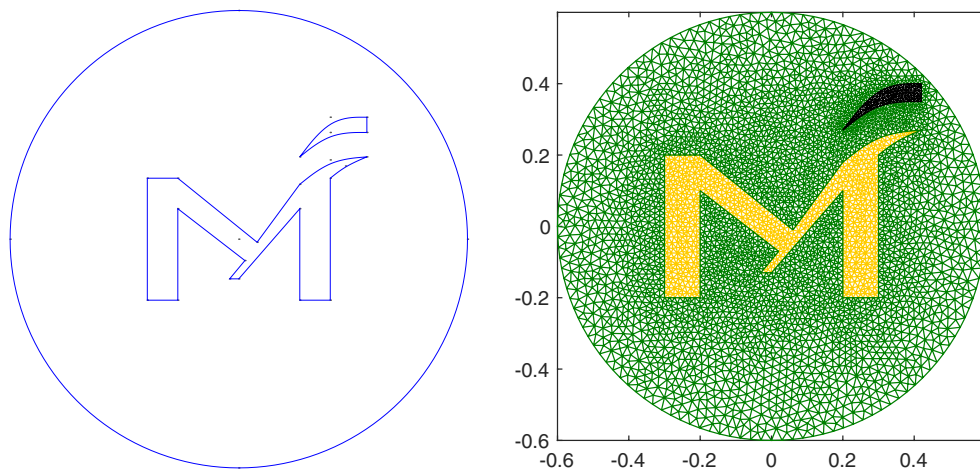


Figure 9. Left: computational domain where the M-shaped region is Ω , the region inside the outer circle is $\bar{\Omega}$ and the region inside $\bar{\Omega} \setminus \Omega$ is $\hat{\Omega}$ which is the region where control is supported. Right: a finite element mesh.

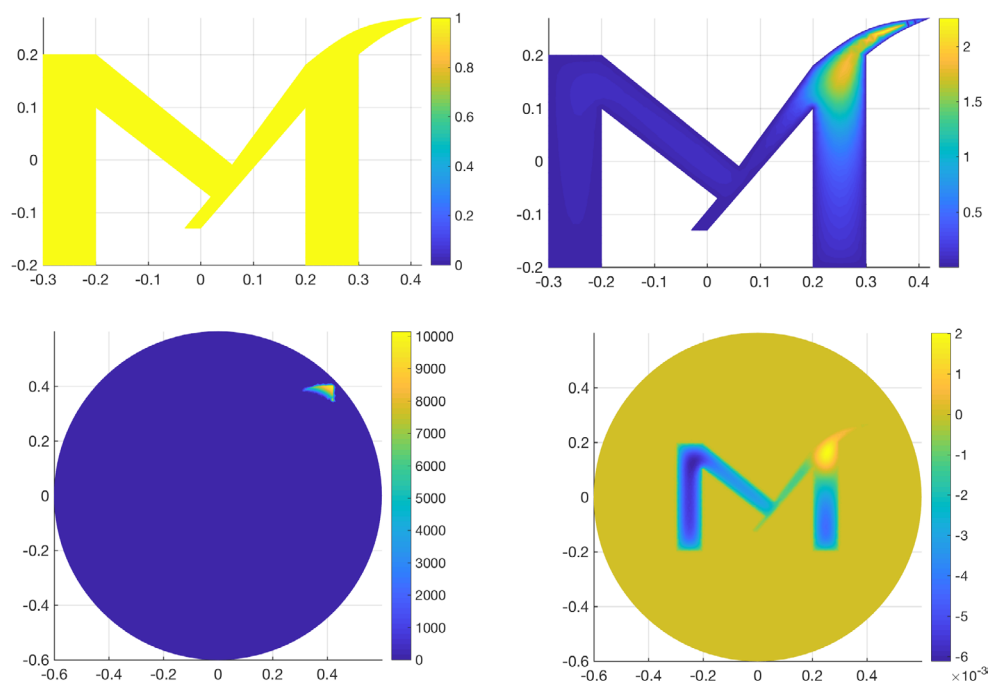


Figure 10. Example 3, $s = 0.8$: top row: left—desired state u_d ; right—optimal state $\bar{u}_h$. Bottom row: left—optimal control $\bar{z}_h$, right—optimal adjoint $\bar{p}_h$.

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References

- [1] Acosta G, Bersetche F M and Borthagaray J P 2017 A short FE implementation for a 2d homogeneous Dirichlet problem of a fractional Laplacian *Comput. Math. Appl.* **74** 784–816
- [2] Acosta G and Borthagaray J P 2017 A fractional Laplace equation: regularity of solutions and finite element approximations *SIAM J. Numer. Anal.* **55** 472–95
- [3] Acosta G, Borthagaray J P and Heuer N 2017 Finite element approximations of the nonhomogeneous fractional Dirichlet problem (arXiv:1709.06592)

- [4] Antil H and Bartels S 2017 Spectral approximation of fractional PDEs in image processing and phase field modeling *Comput. Methods Appl. Math.* **17** 661–78
- [5] Antil H, Berry T and Harlim J 2018 Fractional diffusion maps (arXiv:1810.03952)
- [6] Antil H, Kouri D P, Lacasse M D and Ridzal D (ed) 2018 *Frontiers in PDE-Constrained Optimization (The IMA Volumes in Mathematics and its Applications)* (New York: Springer)
- [7] Antil H, Nochetto R H and Venegas P 2018 Controlling the Kelvin force: basic strategies and applications to magnetic drug targeting *Optim. Eng.* **19** 559–89
- [8] Antil H, Nochetto R H and Venegas P 2018 Optimizing the Kelvin force in a moving target subdomain *Math. Models Methods Appl. Sci.* **28** 95–130
- [9] Antil H, Pfefferer J and Rogovs S 2018 Fractional operators with inhomogeneous boundary conditions: analysis, control, and discretization *Commun. Math. Sci.* **16** 1395–426
- [10] Antil H and Rautenberg C N 2019 Sobolev spaces with non-Muckenhoupt weights, fractional elliptic operators, and applications *SIAM J. Math. Anal.* (to appear)
- [11] Antil H and Warma M 2018 Optimal control of the coefficient for fractional p -Laplace equation: approximation and convergence *RIMS Kokyûroku* **2090** 102–16
- [12] Antil H and Warma M 2019 Optimal control of fractional semilinear PDEs *Control Optim. Calculus Variations (ESAIM: COCV)* to appear (<https://doi.org/10.1051/cocv/2019003>)
- [13] Antil H and Warma M 2019 Optimal control of the coefficient for regional fractional p -Laplace equations: approximation and convergence *Math. Control Relat. Fields* **9** 1–38
- [14] Apel T, Nicaise S and Pfefferer J 2016 Discretization of the Poisson equation with non-smooth data and emphasis on non-convex domains *Numer. Methods PDE* **32** 1433–54
- [15] Apel T, Nicaise S and Pfefferer J 2017 Adapted numerical methods for the Poisson equation with L^2 boundary data in nonconvex domains *SIAM J. Numer. Anal.* **55** 1937–57
- [16] Attouch H, Buttazzo G and Michaille G 2014 Applications to PDEs and optimization *Variational Analysis in Sobolev and BV Spaces (MOS-SIAM Series on Optimization vol 17)* 2nd edn (Philadelphia, PA: SIAM)
- [17] Berggren M 2004 Approximations of very weak solutions to boundary-value problems *SIAM J. Numer. Anal.* **42** 860–77 (electronic)
- [18] Biccari U, Warma M and Zuazua E 2017 Addendum: Local elliptic regularity for the Dirichlet fractional Laplacian *Adv. Nonlinear Stud.* **17** 837–9
- [19] Biccari U, Warma M and Zuazua E 2017 Local elliptic regularity for the Dirichlet fractional Laplacian *Adv. Nonlinear Stud.* **17** 387–409
- [20] Bjorland C, Caffarelli L and Figalli A 2012 Nonlocal tug-of-war and the infinity fractional Laplacian *Commun. Pure Appl. Math.* **65** 337–80
- [21] Brasco L, Parini E and Squassina M 2016 Stability of variational eigenvalues for the fractional p -Laplacian *Discrete Continuous Dyn. Syst.* **36** 1813–45
- [22] Caffarelli L and Silvestre L 2007 An extension problem related to the fractional Laplacian *Commun. PDE* **32** 1245–60
- [23] Caffarelli L A, Roquejoffre J-M and Sire Y 2010 Variational problems for free boundaries for the fractional Laplacian *J. Eur. Math. Soc.* **12** 1151–79
- [24] Caffarelli L A, Salsa S and Silvestre L 2008 Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian *Inventiones Math.* **171** 425–61
- [25] Carbotti A, Dipierro S and Valdinoci E 2018 Local density of solutions of time and space fractional equations (arXiv:1810.08448)
- [26] Di Nezza E, Palatucci G and Valdinoci E 2012 Hitchhiker’s guide to the fractional Sobolev spaces *Bull. Sci. Math.* **136** 521–73
- [27] Dipierro S, Ros-Oton X and Valdinoci E 2017 Nonlocal problems with Neumann boundary conditions *Rev. Mat. Iberoamericana* **33** 377–416
- [28] Dipierro S, Savin O and Valdinoci E 2019 Local approximation of arbitrary functions by solutions of nonlocal equations *J. Geom. Anal.* **29** 1428–55
- [29] Du Q, Gunzburger M, Lehoucq R B and Zhou K 2013 A nonlocal vector calculus, nonlocal volume-constrained problems, and nonlocal balance laws *Math. Models Methods Appl. Sci.* **23** 493–540
- [30] Felsinger M, Kassmann M and Voigt P 2015 The Dirichlet problem for nonlocal operators *Math. Z.* **279** 779–809
- [31] Ghosh T, Lin Y-H and Xiao J 2017 The Calderón problem for variable coefficients nonlocal elliptic operators *Commun. PDE* **42** 1923–61
- [32] Ghosh T, Salo M and Uhlmann G 2016 The Calderón problem for the fractional Schrödinger equation (arXiv:1609.09248)

- [33] Grubb G 2014 Local and nonlocal boundary conditions for μ -transmission and fractional elliptic pseudodifferential operators *Anal. PDE* **7** 1649–82
- [34] Grubb G 2015 Fractional Laplacians on domains, a development of Hörmander's theory of μ -transmission pseudodifferential operators *Adv. Math.* **268** 478–528
- [35] Grubb G 2016 Regularity of spectral fractional Dirichlet and Neumann problems *Math. Nachr.* **289** 831–44
- [36] Krylov N V 2018 On the paper: 'all functions are locally s-harmonic up to a small error' by Dipierro, Savin, and Valdinoci (arXiv:1810.07648)
- [37] Lai R-Y and Lin Y-H 2019 Global uniqueness for the fractional semilinear Schrödinger equation *Proc. Am. Math. Soc.* 147 1189–99
- [38] Larkin P A and Whalen M 1999 Direct, near field acoustic testing *Technical Report SAE technical paper* 01-5553, Harvard
- [39] Lions J-L 1971 *Optimal Control of Systems Governed by Partial Differential Equations (Die Grundlehren der Mathematischen Wissenschaften vol 170)* (New York: Springer) (transl. from the French by S K Mitter)
- [40] Louis-Rose C and Warma M 2018 Approximate controllability from the exterior of space-time fractional wave equations *Appl. Math. Optim.* **1**–44
- [41] Lübke A S et al 1996 Clinical experiences with magnetic drug targeting: a phase I study with 4'-epidoxorubicin in 14 patients with advanced solid tumors *Cancer Res.* **56** 4686–93
- [42] Niedermeyer E and da Silva F H L 2005 *Electroencephalography: Basic Principles, Clinical Applications, and Related Fields* (Baltimore, MD: Williams & Wilkins)
- [43] Ros-Oton X and Serra J 2014 The extremal solution for the fractional Laplacian *Calc. Var. PDE* **50** 723–50
- [44] Rüländ A and Salo M 2017 The fractional Calderón problem: low regularity and stability (arXiv:1708.06294)
- [45] Servadei R and Valdinoci E 2014 On the spectrum of two different fractional operators *Proc. R. Soc. Edinburgh A* **144** 831–55
- [46] Sprekels J and Valdinoci E 2017 A new type of identification problems: optimizing the fractional order in a nonlocal evolution equation *SIAM J. Control Optim.* **55** 70–93
- [47] Tröltzsch F 2010 Theory, methods and applications *Optimal Control of Partial Differential Equations (Graduate Studies in Mathematics vol 112)* (Providence, RI: American Mathematical Society) (transl. from the 2005 German original by J Sprekels)
- [48] Unsworth M 2005 New developments in conventional hydrocarbon exploration with electromagnetic methods *CSEG Recorder* **30** 34–8
- [49] Valdinoci E 2009 From the long jump random walk to the fractional Laplacian *Bol. Soc. Esp. Mat. Apl. SeMA* **49** 33–44
- [50] Visik M I and Èskin G I 1965 Convolution equations in a bounded region *Uspehi Mat. Nauk* **20** 89–152
- [51] Warma M 2015 A fractional Dirichlet-to-Neumann operator on bounded Lipschitz domains *Commun. Pure Appl. Anal.* **14** 2043–67
- [52] Warma M 2015 The fractional relative capacity and the fractional Laplacian with Neumann and Robin boundary conditions on open sets *Potential Anal.* **42** 499–547
- [53] Warma M 2019 Approximate controllability from the exterior of space-time fractional diffusive equations *SIAM J. Control Optim.* to appear
- [54] Warma M and Zamorano S 2018 Analysis of the controllability from the exterior of strong damping nonlocal wave equations *Control, Optimization and Calculus of Variations (ESAIM: COCV)*
- [55] Weiss C J, Waanders B G and Antil H 2019 Fractional operators applied to geophysical electromagnetics (arXiv:1902.05096)
- [56] Williams R L, Karacan I and Hirsch C J 1974 *Electroencephalography (EEG) of Human Sleep: Clinical Applications* (New York: Wiley)