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Inherently trap-free convex landscapes for fully quantum optimal control

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Abstract
A general quantum system may be steered by a control of either classical or quantum nature and the latter scenario is particularly important in many quantum engineering problems including coherent feedback and reservoir engineering. In this paper, we consider a quantum system steered by a quantum controller and explore the underlying Q–Q (quantum–quantum) control landscape features for the expectation value of an arbitrary observable of the system, with the control being the engineered initial state of the quantum controller. It is shown that the Q–Q control landscape is inherently convex, and hence devoid of local suboptima. Distinct from the landscapes for quantum systems controlled by time-dependent classical fields, the controllability is not a prerequisite for the Q–Q landscape to be trap-free, and there are no saddle points that generally exist with a classical controller. However, the forms of Hamiltonian, the flexibility in choosing initial state of the controller, as well as the control duration, can influence the reachable optimal value on the landscape. Moreover, we show that the optimal solution of the Q–Q control landscape can be readily extracted from a de facto landscape observable playing the role of an effective "observer". For illustration of the basic Q–Q landscape principles, we consider the Jaynes–Cummings model depicting a two-level atom in the presence of a cavity quantized radiation field.

Keywords Quantum control · Optimal control · Convex optimization

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1 Introduction

Quantum optimal control theories and experiments [1,2] typically involve tailored electromagnetic fields seeking the most favorable outcome of a specified physical or chemical processes via optimization of the control fields [3,4]. In the past decade, rigorous analyses of the *quantum control landscape*, defined as the expectation value of a desired system observable as a function(al) of the control [5], have been made to explain the widely observed experimental successes along with much larger numbers of almost perfect optimal control simulations. This advance is important not only for obtaining a good understanding of the landscape topology to establish the feasibility of finding globally optimal controls, but ultimately also for developing more efficient optimization algorithms exploiting these features. Of immediate relevance is the topological properties of the landscape *critical points*, where the landscape gradient with respect to the control is zero. At issue is the possible appearance of suboptimal local extrema as *traps*, which could halt a gradient search and impede reaching a global optimum.

A general control problem entails a physical system and a control that steers the system's dynamics. Both the system and the control may be either Classical or Quantum in nature, giving rise to a tetrad of control scenarios: C–C (a classical system steered by a classical control), Q–C (a quantum system steered by a classical control), C–Q (a classical system steered by a quantum control), and Q–Q (a quantum system steered by another quantum system—a quantum controller). The C–C, Q–C and C–Q control scenarios may be considered as approximations of the Q–Q control scenario, which then forms the more fundamental case for better understanding three other scenarios. Interestingly, the same tetrad of scenarios has been proposed in machine learning sciences, where the data and the algorithm can be either Quantum or Classical [6–8].

Most control studies, as well as the corresponding landscape analyses, have been done on the Q–C control scenario in the *semiclassical* limit. It has been proved that, with unconstrained classical control fields, the landscape for a controllable finite-level quantum system is almost always free of any local suboptima, thus all critical points are either global maxima or global minima, or possibly *saddle* points [5,9]. Imposing constraints on the properties of classical fields, for example, the pulse amplitude, bandwidth, and length, may result in additional undesirable features on the landscape, including possibly traps [10].

In this paper, we consider the Q–Q control landscape by taking a fully quantum perspective, assuming that a quantum system A (herein the system) is directly coupled to another quantum system B (the *control*), which can be arbitrarily manipulated [11–15]. In principle, both A and B have different attributes, for example, being quantized radiation fields in an optical cavity [16–20]), single atoms or spins, or bulk material systems, etc., representing a plethora of Q–Q control processes. Specifically, in the current Q–Q control scenario studied here, the initial density matrix of B is used to indirectly control the evolution of A, in contrast to the semiclassical Q–C perspective in which classical fields are used to control the quantum system. Many quantum engineering problems can be classified into this scenario, e.g., (1) in quantum-assisted optimization where the initial state of a quantum processor is considered as the decision variable of a minimization problem [21], (2) in quantum metrology where the initial quantum state of the probe is used as the control to improve the precision of measurement [6], (3) in a quantum interface where a flying single photon prepared at some certain state is used to transfer quantum information to a standing qubit [22].

The paper will perform an analysis of the Q–Q landscape with the density matrix of B considered as the control. The aim is to explore the fundamental Q–Q landscape features rather than design a new optimization algorithm. In the following, we demonstrate that the landscape expressed in the Q–Q context is *convex* without any additional assumptions, thus rigorously free of any local traps or saddles. Furthermore, we show that the Q–Q optimal solution can be directly extracted from a de facto "landscape observable" that underlies the Q–Q control landscape defined as a function of the initial density matrix of B.

The remainder of this paper is organized as follows. In Sect. 2, the Q–C and Q–Q control systems are introduced. In Sect. 3, the Q–Q control landscape is defined and proved to be convex. For illustration, Sect. 4 illustrates the basic Q–Q concepts with the Jaynes–Cummings model in which either an atom or an cavity field can be taken as a quantum controller. Finally, conclusions are made in Sect. 5.

2 From quantum-classical to quantum–quantum control systems

In the semiclassical Q–C scenario, the total Hamiltonian of a closed N-level quantum system, A, interacting with external classical fields can be written as:

$$H_{QC}(t) = H_A^0 + \sum_k u_k(t) H_A^k, \tag{1}$$

where H_A^0 is the drift Hamiltonian and H_A^k is the dipole-like control Hamiltonian associated with the k-th control field $u_k(t)$. The corresponding semiclassical unitary propagator $U_{QC}(t)$ satisfies the time-dependent Schrödinger equation

$$\dot{U}_{\mathrm{QC}}(t) = -i\left[H_A^0 + \sum_k u_k(t) H_A^k\right] U_{\mathrm{QC}}(t), \quad U_{\mathrm{QC}}(0) = \mathbb{I}_N, \tag{2}$$

where $\mathbb{I}_N$ is an N-dimensional identity matrix. The density matrix $\rho_A(t)$, i.e., the state of A at $t > 0$, can then be written as $\rho_A(t) = U_{QC}(t)\rho_A(0)U_{QC}^\dagger(t)$, where $\rho_A(0)$ is the initial state of A.

In the Q–Q scenario, the joint Hamiltonian of the composite system A/B can be expressed as

$$H_{AB} = H_A^0 \otimes \mathbb{I}_B + \mathbb{I}_A \otimes H_B^0 + \sum_k H_A^k \otimes H_B^k, \tag{3}$$

where $\mathbb{I}_A$ ($\mathbb{I}_B$) is the identity operator in the Hilbert space of A (B), which can be either finite or infinite dimensional, H_A^0 (H_B^0) is the Hamiltonian of A (B), and H_A^k (H_B^k) the interaction Hamiltonian associated with A (B). The unitary propagator associated with the joint Hamiltonian H_{AB} of the closed composite system A/B is simply $U_{AB}(t) = \exp(-\frac{i}{\hbar}H_{AB}t)$. Initially, the composite system A/B may be in a *separable* state, i.e., the initial joint density matrix $\rho = \rho_A \otimes \rho_B$ at $t = 0$, with $\rho_A = \rho_A(0)$ denoting the initial state of A (the system) and $\rho_B = \rho_B(0)$ the initial state of B (the control), or in an entangled state ρ subject to the constraint $\mathrm{Tr}_B(\rho) = \rho_A$, i.e., the reduced density matrix of A is still ρ_A. As soon as the initial state ρ is created, A and B will evolve in unison in accordance with the joint Hamiltonian H_{AB} and $\rho(t) = U_{AB}(t)\rho(0)U_{AB}^{\dagger}(t)$ will become entangled. The control scheme analyzed here assumes that the control B can be prepared at any admissible quantum state [23,24], prior to its interaction with A. In practice the manipulation or preparation of system B would be performed with another control of some nature which is not explicitly present in the analysis here. Even if this latter control is classical it would not change the resultant Q–Q interaction and its associated landscape for analysis.

The Q–C control system, Eq. (2), can be taken as an approximation of the Q–Q control system, Eq. (3), in the semiclassical limit when (1) the control B (here, quantized photon fields) is sufficiently large or has other appropriate characteristics so that it can be treated classically and (2) any back coupling of the system A on the control B is negligible so that A (the system) and B (the control) are virtually disentangled at all times [16,18]. This may be understood by rotating H_{AB} via the transformation $U_{\mathrm{rot}}(t) = \mathbb{I}_A \otimes \exp(-iH_B^0 t)$, giving rise to a time-dependent Hamiltonian, in the mixed Schröpdinger/Heisenberg picture [16]:

$$H_{AB}^{\mathrm{rot}}(t) = H_A^0 \otimes \mathbb{I}_B + \sum_k H_A^k \otimes \left[U_{\mathrm{rot}}^{\dagger}(t) H_B^k U_{\mathrm{rot}}(t)\right]. \tag{4}$$

The semiclassical approximation naturally arises in the so-called large N limit [25] (e.g., when the density matrix ρ_B is a coherent state with large mean photon number) such that the variance of $U_{\mathrm{tot}}^{\dagger}(t) H_B^k U_{\mathrm{tot}}(t)$ is much smaller than its expectation value, with

$$u_k(t) = \mathrm{Tr}\left[\rho_B U_{\mathrm{rot}}^{\dagger}(t) H_B^k U_{\mathrm{rot}}(t)\right] \tag{5}$$

then playing the role of classical fields in the Q–C control system (2). The approximation in Eq. 5 indicates that the semiclassical control $\{u_k(t)\}$ is actually inherited from the initial state ρ_B of the controller B, which can be engineered in advance (e.g., coherent or squeezed states of a laser). Therefore, it is natural to take ρ_B as the control variable, which can have arbitrary realizable states, in the Q–Q scenario.

3 The convex quantum–quantum control landscapes

In this section, after briefly reviewing the fundamental properties of the Q–C control landscape, we define the Q–Q control landscape and prove its trap-free property using convex optimization analysis.

3.1 The Q–C control landscape

In the Q–C scenario, the dynamic control landscape of the observable O_A is defined as the functional of a classical field $u(t)$, i.e.,

$$J_{\rm QC}[u(t)] = {\rm Tr}[U_{\rm QC}(T)\rho_A(0)U^{\dagger}_{\rm QC}(T)O_A], \tag{6}$$

where $\rho_A(0)$ is the initial density matrix of A. Alternatively, the kinematic landscape can be defined as the function of the unitary propagator $U_{\rm QC}(T)$, i.e.,

$$J_{\rm QC}[U_{\rm QC}(T)] = {\rm Tr}[U_{\rm QC}(T)\rho_A(0)U^{\dagger}_{\rm QC}(T)O_A]. \tag{7}$$

It has been shown that the dynamical landscape $J_{\rm QC}[u(t)]$ has no traps upon satisfaction of three basic assumptions:

(i) the system is *controllable* [26],
(ii) the local mapping $\delta u(t) \mapsto \delta U_{\rm QC}(T)$ bridging the kinematic and dynamical landscapes is *surjective* at any control [9,27],
(iii) the control field $u(t)$ is unconstrained, i.e., in practice there is sufficient freedom in the control field to exploit assumptions (i) and (ii) [10].

The landscape topologies based on these assumptions have been the focus of many previous studies [5]. We remark that assumptions (i) and (ii) can be shown as being "almost always" satisfied [9,28], consistent with the fact that the semiclassical Q–C landscape rarely exhibits traps. However, depending on the distributions of eigenvalues of $\rho_A(0)$ and O_A, the landscape may possess saddle points [29].

3.2 The Q–Q control landscape

The Q–Q control objective, analogous to the Q–C one, is to optimize the expectation value of the observable O_A at time T, given ρ_A the initial state of A. However, instead of tailoring the classical control field $u(t)$ in the Q–C control scenario, the Q–Q objective may be reached by tuning either the interaction with the two quantum systems [30] or the initial state of the quantum controller, as demonstrated in this paper. Since the interaction between quantum system is usually tuned by some classical parameters [30], which can be categorized into the Q–C control scenario, it is more natural in this paper to take the initial state as the control resource.

For a separable initial state $\rho_A \otimes \rho_B$, the Q–Q landscape can be defined as

$$J_{\rm QQ}[\rho_B] = {\rm Tr}[U_{AB}(T)(\rho_A \otimes \rho_B)U^{\dagger}_{AB}(T)(O_A \otimes \mathbb{I}_B)], \tag{8}$$

where the control ρ_B is a positive semi-definite matrix of trace one. Optimization of the Q–Q landscape $J_{\rm QQ}[\rho_B]$ can be posed as a semidefinite programming problem:

$$\begin{array}{rl} \min & J_{\rm QQ}[\rho_B] \\ \text{subject to :} & {\rm Tr}\rho_B = 1,\ \rho_B \succeq 0, \end{array} \tag{9}$$

Table 1 Summary of the properties of control landscapes in the Q–C and Q–C scenarios

Formulation	Q–C	Q–Q
Nature of control	$u(t)$	ρ_B
Landscape function	Nonlinear	linear
Landscape topology	Trap-free[a]	Convex, trap-free[b]

[a] Trap-free upon satisfaction of three key assumptions, and possibly with saddles
[b] No saddles or other suboptimal critical points present

where, without loss of generality, we consider only the minimization problem. From Eqs. (8) and (9), we note that (a) the landscape function J_{QQ} is *linear* with respect to the control variable ρ_B, thus both convex and concave, and (b) the admissible set of ρ_B is a closed convex set.

With conditions (a) and (b) above, and from the theory of convex optimization [31], it is readily seen that the landscape $J_{\mathrm{QQ}}[\rho_B]$ is trap-free, i.e., a local minimum of $J_{\mathrm{QQ}}[\rho_B]$ must also be a global one, because the admissible ρ_B's form a convex set. Note that this trap-free property holds under much milder conditions; in particular the controller B does not need to be controllable (i.e., the states defined in (9) do not need to be all reachable). As long as the admissible ρ_B's form a convex set, e.g., the convex ball $\mathcal{B}_c = \{\rho_B : 0 \leq \mathrm{Tr}(\rho_B^2) \leq c < 1\}$ or the set of separable states $\mathcal{S} = \{\rho_B : \rho_B = \sum_k \rho_{B_1} \otimes \rho_{B_2}\}$ when B is a bipartite system consisting of B_1 and B_2, any local optimum must be also globally optimal.

The evident convexity of the control landscape J_{QQ}, Eq. (8), in the Q–Q scenario is in sharp contrast to its Q–C counterpart J_{QC}, Eq. (7), which is a highly nonlinear functional of the control field $u(t)$. The control landscape properties in these two different scenarios are summarized in Table 1. Unlike the Q–C landscape, there is no controllability [14,15] requirement on the composite quantum system (see Eq. (3) for the Q–Q landscape to be trap-free, but the forms of interactions H_A^k and H_B^k, in Eq. (3), and initial states ρ_B, as well as final times T can influence the optimal value of J_{QQ} reachable in the Q–Q control scenario.

Since the function J_{QQ} is linear, see condition (a), it is easy to prove that all *level sets* (defined as the set of all controls associated with the same landscape function value) of the Q–Q landscape must also be convex sets, and thus must be *connected*. This can be seen from the fact that, given any two initial states $\rho_{B,1}$ and $\rho_{B,2}$ on the same level set of the Q–Q landscape, i.e., $J_{\mathrm{QQ}}[\rho_{B,1}] = J_{\mathrm{QQ}}[\rho_{B,2}] = J_0$, then any convex combination of $\rho_{B,1}$ and $\rho_{B,2}$ will also be on the same level set since J_{QQ} is a linear function of ρ_B, i.e.,

$$\begin{aligned} J_{\mathrm{QQ}}[\lambda\rho_{B,1} + (1-\lambda)\rho_{B,2}] &= \lambda J_{\mathrm{QQ}}[\rho_{B,1}] + (1-\lambda) J_{\mathrm{QQ}}[\rho_{B,2}] \\ &= \lambda J_0 + (1-\lambda) J_0 = J_0, \quad \lambda \in [0,1]. \end{aligned} \tag{10}$$

The topology (especially *connectivity*) of level sets has been extensively studied in the semiclassical Q–C context [32]. The level-set connectedness implies that all global optimal solutions for the Q–Q landscape are not isolated into disconnected "islands", and thus the optimization can be easier because desired optimal solution characteristics (e.g., seeking a high degree of robustness to variations in ρ_B) can be homotopically transformed from one local solution to another.

For an entangled initial state ρ of A/B, i.e., ρ cannot be separated as $\rho_A \otimes \rho_B$, the Q–Q landscape can instead be defined as

$$\max/\min J_{\mathrm{QQ}}[\rho] = \mathrm{Tr}[U_{AB}(T)\rho U_{AB}^{\dagger}(T)(O_A \otimes \mathbb{I}_B)], \tag{11}$$

where the control ρ is the density matrix of the composite A/B quantum system, and the optimization of the landscape $J_{\mathrm{QQ}}[\rho]$ is subject to the following constraints:

$$\mathrm{Tr}_B(\rho) = \rho_A, \quad \mathrm{Tr}\rho = 1, \quad \rho \succeq 0. \tag{12}$$

It can be easily verified that this circumstance also entails a convex optimization problem, since it again involves a linear landscape function in a convex admissible set of ρ, and thus the corresponding landscape $J_{\mathrm{QQ}}[\rho]$ is free of local traps.

The formulation leading to the convexity of Q–Q control landscape for two coupled, otherwise isolated, quantum systems A and B can be readily extended to quantum systems that are either (1) coupled to time-dependent external fields such that either the Hamiltonian H_A of A, or H_B of B, or both, becomes time-dependent, or (2) surrounded by a large baths such that the composite A/B system become an open system, because the nature of convexity remains unchanged. In the former time-dependent case, the unitary propagator $U_{AB}(T) = \exp(-\frac{i}{\hbar}H_{AB}T)$ in Eqs. (8) and (11) is replaced by its time-ordered counterpart [33]

$$U_{AB}(t) = \mathcal{T}\exp\left[-\frac{i}{\hbar}\int_0^T H_{AB}(t)\mathrm{d}t\right].$$

Whereas, in the latter open system case, the unitary propagator can be replaced by a set of Kraus operators [34], which may limit the range of achievable J_{QQ}, but do not alter the linearity of the controller state ρ_B in the landscape function J_{QQ}. Thus, in both extensions, the convexity is still preserved with full assurance of the trap-free property.

3.3 Optimal solutions for the Q–Q control landscape

A complete optimal solution for the landscape function minimization, Eq. (9), in the Q–Q formulation can be obtained by recasting Eq. (8) as

$$J_{\mathrm{QQ}}[\rho_B] = \mathrm{Tr}(\rho_B \mathcal{O}_B) \tag{13}$$

where

$$\mathcal{O}_B := \mathrm{Tr}_A[U_{AB}^{\dagger}(T)(O_A \otimes \mathbb{I}_B)U_{AB}(T)(\rho_A \otimes \mathbb{I}_B)] \tag{14}$$

with Tr_A denoting the partial trace over A. Here $\mathcal{O}_B$ defines a de facto landscape observable associated with control landscape $J_{\mathrm{QQ}}[\rho_B]$ in the Hilbert space spanned by the density matrix ρ_B. All the necessary information ρ_A, O_A and $U_{AB}(T)$ for determining the landscape optimum resides in the single observable $\mathcal{O}_B$, which plays the role of an effective "observer" enabling the landscape associated with the system A to be extracted [35]. The upper and lower bounds of $J_{\rm QQ}[\rho_B]$ in Eq. (13) can be given in terms of the eigenvalues of $\mathcal{O}_B$, i.e., $\mathcal{O}_B^{\min} \le {\rm Tr}(\rho_B \mathcal{O}_B) \le \mathcal{O}_B^{\max}$, where $\mathcal{O}_B^{\max}$ and $\mathcal{O}_B^{\min}$ are the maximal and minimal eigenvalues of $\mathcal{O}_B$, respectively. The bound values $\mathcal{O}_B^{\min}$ and $\mathcal{O}_B^{\max}$ depend on the initial state ρ_A and the observable $\mathcal{O}_A$ of the system and, the joint evolution $U_{AB}(T)$. These bounds represent an important feature of the Q–Q landscape since they characterize the ultimate limits that can be produced by a quantum controller [36].

To reach the global maximum (minimum) of $J_{\rm QQ}$, ρ_B must coincide with the eigenstate(s) of $\mathcal{O}_B$ corresponding to its maximal (minimal) eigenvalue. For a general degenerate extremal eigenvalue $\mathcal{O}_B^{\min}$, the landscape optimal solutions

$$\rho_B^* = \sum_i p_i |i\rangle\langle i|, \quad p_i \ge 0, \quad \sum_i p_i = 1, \tag{15}$$

form a convex set of mixed states, with $\{|i\rangle\}$ being the subspace of degenerate eigenstates of $\mathcal{O}_B$ associated with $\mathcal{O}_B^*$, thus leading to the optimal landscape function value $J_{\rm QQ}[\rho_B^*] = \mathcal{O}_B^{\min}$. If the extremal eigenvalue $\mathcal{O}_B^{\min}$ is nondegenerate, the optimal control $\rho_B^* = |i\rangle\langle i|$ can only be a pure state, which is an extremal point on the boundary of the admissible set. We remark that in general, at the semiclassical Q–C dynamical landscape optimum, there are infinitely many distinct control fields that require identification, of at least one field, by deterministic or stochastic searching algorithms [37]. In contrast, the optimal solution ρ_B^* of the Q–Q control landscape $J_{\rm QQ}$, Eq. (8), can be determined directly from Eq. (15).

4 Illustration: Q–Q control landscape in the Jaynes–Cummings model

As an illustration of the key principle in Sect. 3, we consider the Jaynes–Cummings (JC) model [38] that depicts a two-level atom with a ground state $|g\rangle$ and an excited state $|e\rangle$, in a quantized radiation field containing a single bosonic mode with countably infinite number states $|n\rangle$, $n = 0, 1, \ldots$. In the rotating wave approximation, the total Hamiltonian of the JC model is written as

$$H_{AB} = \frac{\omega}{2}\sigma_z + \nu a^\dagger a + \frac{\Omega}{2}(\sigma_+ a + \sigma_- a^\dagger) \tag{16}$$

where ω and ν are the frequencies of the atom and the field, respectively, and Ω is the coupling strength. $a^\dagger$ and a are the creation and annihilation operators of the field, while $\sigma_+ = |e\rangle\langle g|$, $\sigma_- = |g\rangle\langle e|$, and $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$ are operators of the atom. We remark that in the Q–Q scenario, either the atom or the cavity mode can be taken as the quantum controller while the other is the system to be controlled. In the following, we will analyze these two cases separately.

4.1 Field controlled by atom

Consider the control landscape given in Eq. (9), where the initial state ρ_B of the atom (B) is utilized as the control to manipulate the photon number (corresponding to the observable $O_A = a^\dagger a$) of the cavity field (A) that is initially prepared in an arbitrary Fock state $\rho_A = |n\rangle\langle n|$. Using Eq. (14), the resultant $\mathcal{O}_B$ is a diagonal matrix with

$$\mathcal{O}_A = n\mathbb{I}_2 - \frac{1}{\Delta^2+1}\sin^2\left(\frac{\Omega T}{2}\sqrt{\Delta^2+1}\right)\sigma_z \tag{17}$$

where $\Delta = \frac{\nu-\omega}{\Omega}$ is a dimensionless parameter and T is the total evolution time. This result shows that by properly choosing the initial state ρ_B of the atom, one can change the photon number by at most $\pm\frac{1}{\Delta^2+1}\sin^2\left(\frac{\Omega T}{2}\sqrt{\Delta^2+1}\right)$. Only when the atom and the cavity is on resonance, i.e., $\Delta = 0$, and the time is chosen such that $\Omega T = \pi$, can the photon be added (or subtracted) by 1 when the control density matrix $\rho_B = |g\rangle\langle g|$ (or $\rho_B = |e\rangle\langle e|$).

4.2 Atom controlled by field

Consider the opposite case that the atom (A) is controlled by the quantized field (B) (e.g., in a superconducting resonator [39]), where the initial state ρ_B of the quantized field is utilized as the control to optimize the transition probability from the ground to an excited state in the atom, i.e., we specify that $\rho_A = |g\rangle\langle g|$ and $O_A = |e\rangle\langle e|$. Using Eq. (14), the resultant $\mathcal{O}_B$ is a diagonal matrix with

$$\langle n|\mathcal{O}_B|n\rangle = \frac{n}{\Delta^2+n}\sin^2\left(\frac{\Omega T}{2}\sqrt{\Delta^2+n}\right) \tag{18}$$

for $n = 0, 1, 2, \ldots$. The eigenvalues of $\mathcal{O}_B$ are distributed within the interval [0, 1], the maximum and minimum among which will determine the range of the landscape $J_{\rm QQ}$ defined by Eq. 13 and we have $\Omega T = \frac{(2k+1)\pi}{\sqrt{\Delta^2+n}}$, $k = 0, 1, 2, \ldots$. In the on-resonance case of $\Delta = 0$, when only one eigenvalue of the matrix $\mathcal{O}_B$ is 1, then the full transition from $|g\rangle$ to $|e\rangle$ can be accomplished by the control $\rho_B^* = |n\rangle\langle n|$ at time $t = T$. If there are multiple eigenvalues equal to 1, the optimal control state can be any density matrix over the corresponding subspace of Fock states. In the off-resonance case that $\Delta \neq 0$, however, the upper bound for the eigenvalues of $\mathcal{O}_B$ is always less than 1 for any finite n, and the full transition may only be asymptotically approached in the limit that $n \to \infty$, i.e., at infinite field strength; however, the landscape is still convex, thus trap-free even without access to a maximum yield.

Suppose that quantized field states with the photon number up to value $N_{\rm ph}$ form a convex subset of all admissible physical states. As stated above, the resulting convex optimization is always trap-free, no matter how limited the control resource is (e.g., when $N_{\rm ph}$ is a finite integer). Owing to the diagonal structure of $\mathcal{O}_B$, $J_{\rm QQ} = \langle n|\mathcal{O}_B|n\rangle$ is maximized in the eigenspace of $\mathcal{O}_B$ associated with its largest eigenvalue. The optimal controller state is associated with the highest eigenvalue of $\mathcal{O}_B$.

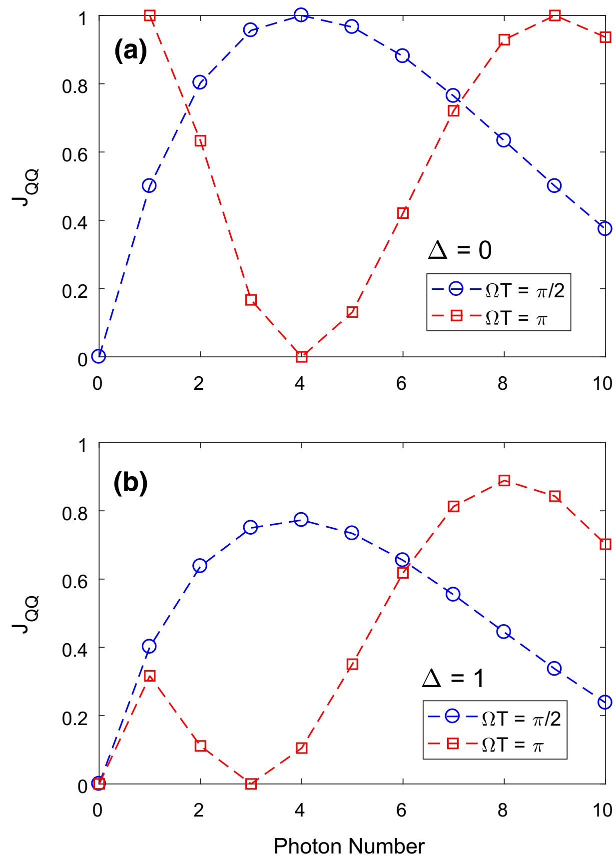

Fig. 1 (color online). State transition $|g\rangle \rightarrow |e\rangle$ in a truncated JC model: the optimal controller state at different ΩT, with first $N_{\rm ph}$ levels of the quantized field selected as the control resource. The frequency detuning $\Delta = (\nu - \omega)/\Omega$ is set to **a** $\Delta = 0$ (on-resonance) and **b** $\Delta = 1$ (off-resonance). In the on-resonance case, perfect state transfer can be achieved ($J_{QQ}^{\max} = 1$) and there exists one ($\Omega T = \pi/2$) or multiple ($\Omega T = \pi$) optimal controller states. In the off-resonance case, there is only one optimal controller state and no perfect state transfer from $|g\rangle$ to $|e\rangle$. Note that the "local peak" to the left in **b** is not a trap, as the plot is with respect to the photon number and not the control ρ_B

As an example we take $N_{\rm ph} = 10$ and plot the yield $J_{\rm QQ}$ at each Fock state from $n = 0$ to $n = 10$ in Fig. 1, among which the highest peak indicates the global maximum. In the on-resonance case $\Delta = 0$, there exists only one maximum $|4\rangle\langle 4|$ when $\Omega T = \pi/2$, that achieves perfect state transfer $J_{QQ} = 1$. However, when $\Omega T = \pi$, $J_{\rm QQ} = 1$ at both $|1\rangle\langle 1|$ and $|9\rangle\langle 9|$, which implies that the optimal controller state can be any superposition or classical mixture of the two states, as discussed above. In the off-resonance case with $\Delta = 1$, there is only one optimal (i.e., maximal) value for $J_{\rm QQ}$ for both $\Omega T = \pi/2$ at state $|4\rangle\langle 4|$ and $\Omega T = \pi$ at state $|8\rangle\langle 8|$. It is also easy to see that the uniqueness of optimal state holds for any finite $N_{\rm ph}$. Figure 2 shows the numerical results using the CVX, convex optimization package [40] for the

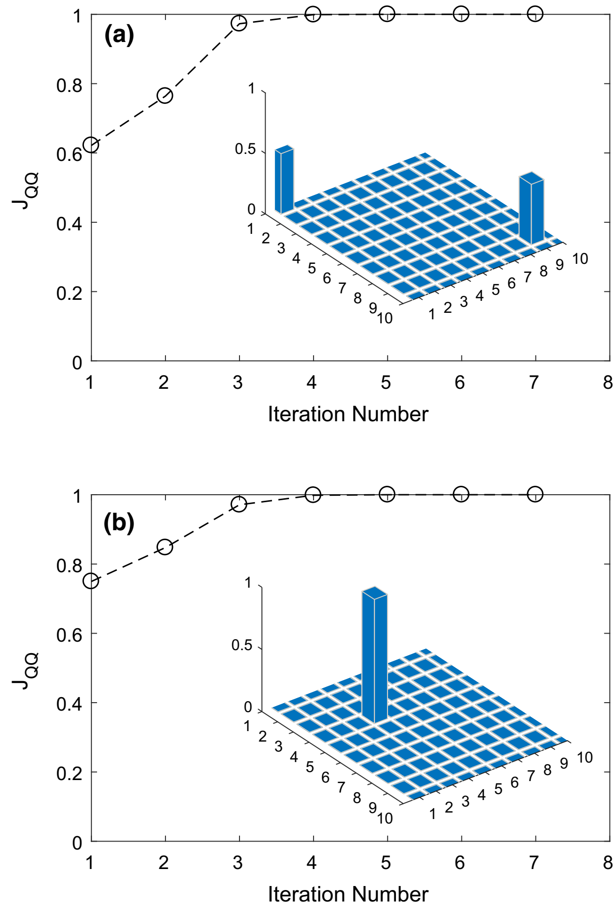

Fig. 2 (color online). Convex optimization of quantized field states with $N_{\mathrm{ph}} = 10$, where the field is on resonance with the atom. In **a** $\Omega T = \pi/2$, and the searches finds an optimal mixed state at $\rho_B = 0.5|1\rangle\langle 1| + 0.5|9\rangle\langle 9|$. In **b** $\Omega T = \pi$, the search finds the optimal state at $\rho_B = |4\rangle\langle 4|$. The insets show the finally obtained density matrix elements of ρ_B

above resonance cases. The optimization is with respect the controller state ρ_B, which finds optimal solutions without being trapped as expected from the formal analysis.

5 Conclusions

In conclusion, this paper provides a fully quantum (Q–Q) formulation for the control landscape aiming to optimize the expectation value of an observable associated with the system. The system A and the control B are both treated quantum mechanically, which together undergo free evolution governed by the joint time-independent Hamiltonian of the coupled quantum systems. The control consists of the initial density matrix ρ_B of B, which may be prepared by any available means. Within this framework, optimization over the landscape $J_{QQ}[\rho_B]$ with respect to the density matrix ρ_B presents a convex problem with a convex admissible set of controls. Therefore, the Q–Q control landscape is rigorously free of any suboptimal local extrema as either traps or saddles if no additional constraints are imposed on ρ_B to violate its convexity. The mathematical simplicity of the Q–Q control problem permits ready extraction of the landscape optimal solutions ρ_B^* from a de facto landscape observable, and it has been shown that the landscape optimum can always be achieved by some pure (or mixed, for general cases) initial state ρ_B^* of the control. The conclusions here imply that the search for optimal solutions over a Q–Q control landscape in the laboratory will be efficient provided that an appropriate initial state of the control can be prepared. Full exploitation of this option may call for advanced technological capabilities, and the analysis here will be compelling for such developments.

As a step towards better understanding of controlling generic quantum systems, this work explores the Q–Q scenario to broaden the scope and foundation of existing control landscape studies. The disparate control landscape properties in the Q–C, C–Q and C–C scenarios can be understood as rooted in taking the appropriate classical limit [18,25] of either or both of quantum systems in the Q–Q scenario. However, the detailed nature of making these transitions remains a challenge for the future as they need to also provide insight into the landscape topological feature involved (e.g., what is the origin of the saddle features in many Q–C landscapes [41–43]). Further theoretical studies of the properties of these additional landscapes are also of practical importance, with the common C–C scenario only partially explored to date [44], while the C–Q scenario has yet to be physically defined. A full understanding of these issues appears complex to unravel but fundamentally important, and we expect that the present Q–Q landscape analysis may provide a foundation for future research to draw together the full tetrad of classical and quantum mechanical control scenarios in a seamless fashion. Additionally, there are other Q–Q control scenarios for exploration as mentioned in the introduction.

We emphasize that the objective of this work is a fundamental exploration of the Q–Q landscape features, including the evident distinction from that of the Q–C landscape. Practical laboratory implementation of Q–Q control calls for a detailed assessment in each particular scenario, which is beyond the scope of the present work. However, we note that the Q–Q control regime is most relevant when the quantum system A can be controlled, quickly and efficiently, via its interaction with the quantum controller B. At present time the Q–C control offers the most accessible laboratory scenario, but the compelling Q–Q regime offers an operationally compelling option, as evolving technologies more readily permit. Besides considering the initial density matrix of B as the control, the Q–Q control scenario can be also constructed in other ways that include taking the interaction Hamiltonian H_B^k [in Eq. (3)] as control variables. In the unrestricted case that the latter operators are allowed to be non-constant , arbitrary functions of time, we obtain a mathematically equivalent Q–C control landscape by taking the system and the controller as a joint closed quantum system. The time-dependent matrix elements of H_B^k can thus be taken as independent control functions. The corresponding landscape is expected to be trap-free [9], although not convex. The case that all H_B^k are restricted to be time-independent is also non-convex, and we leave this case to future studies.

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