ON THE p-ADIC VARIATION OF HEEGNER POINTS

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ABSTRACT. In this paper, we prove an "explicit reciprocity law" relating Howard's system of big Heegner points to a two-variable p-adic L-function (constructed here) interpolating the p-adic Rankin L-series of Bertolini–Darmon–Prasanna in Hida families. As applications, we obtain a direct relation between classical Heegner cycles and the higher weight specializations of big Heegner points, refining earlier work of the author, and prove the vanishing of Selmer groups of CM elliptic curves twisted by 2-dimensional Artin representations in cases predicted by the equivariant Birch and Swinnerton-Dyer conjecture.

CONTENTS

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1. Introduction

Let $f = \sum_{n=1}^{\infty} a_n q^n \in S_{2r}(\Gamma_0(N))$ be a newform of weight $2r > 2$, fix a prime $p \nmid 6N$, and let L be a finite extension of $\mathbf{Q}_p$ with ring of integers $\mathfrak{O}$ containing the image of the Fourier coefficients of f under a fixed embedding $\imath_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. Denote by

$$\rho_f : G_{\mathbf{Q}} := \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \mathrm{Aut}_L(V_f(r)) \simeq \mathrm{GL}_2(L)$$

the Kummer self-dual twist of the p-adic Galois representation associated with f. Let K be an imaginary quadratic field of odd discriminant $-D_K < -3$. Let $\mathcal{O}_K$ be the ring of integers of K, and assume that K satisfies the classical *Heegner hypothesis* relative to N:

(heeg) there is an integral ideal $\mathfrak{N}$ of K with $\mathcal{O}_K/\mathfrak{N} \simeq \mathbf{Z}/N\mathbf{Z}$;

equivalently, every prime $q \mid N$ either splits or ramifies in K, with $q^2 \nmid N$ in the latter case.

The first purpose of this paper is to complete earlier work of the author [Cas13] comparing two natural constructions of a cohomology class of "Heegner-type" attached to the pair (f, K). For the first one of these classes, let $\mathrm{Sel}(K, V_f(r)) \subset H^1(G_K, V_f(r))$ be the Bloch–Kato Selmer group for $V_f(r)|_{\mathrm{Gal}(\overline{\mathbf{Q}}/K)}$. By [Nek00], the image under the p-adic étale Abel–Jacobi map of classical Heegner cycles [Nek95] on the $(2r-1)$-dimensional Kuga–Sato variety of level N give rise to a class

$$\Phi^{\text{ét}}_{f,K}(\Delta_r^{\text{heeg}}) \in \mathrm{Sel}(K, V_f(r)).$$

For the second class, assume that f is *ordinary at* $\imath_p$, i.e.:

(ord) $$a_p \in \mathfrak{O}^{\times}.$$

Fix a $G_{\mathbf{Q}}$-stable $\mathfrak{O}$-lattice $T_f \subset V_f$, let $\bar{\rho}_f : G_{\mathbf{Q}} \to \mathrm{GL}_2(\kappa_L)$ be the associated semi-simple residual representation, where κ_L is the residue field of L, and assume that

(irred) $$\bar{\rho}_f \text{ is irreducible.}$$

Let $D_p \subset G_{\mathbf{Q}}$ be a decomposition group at p. By hypothesis (ord), the restriction $\rho_f|_{D_p}$ can be made upper-triangular, and we shall assume in addition that

(dist) $$\bar{\rho}_f \text{ is } D_p\text{-distinguished;}$$

i.e., the semi-simplification of $\bar{\rho}_f|_{D_p}$ is the direct sum of two *distinct* characters. Suppose that $r \equiv 1 \pmod{p-1}$, and let

$$\mathbf{f} = \sum_{n=1}^{\infty} \mathbf{a}_n q^n \in \mathbb{I}[[q]]$$

be the Hida family passing through f. Thus $\mathbb{I}$ is a finite flat extension of $\mathfrak{O}[[X]]$, and for every continuous $\mathfrak{O}$-algebra homomorphism $\nu : \mathbb{I} \to \overline{\mathbf{Q}}_p$ satisfying $\nu(1+X) = (1+p)^{k_\nu - 2}$ for some integer $k_\nu > 2$ with $k_\nu \equiv 2 \pmod{p-1}$, the q-series $\mathbf{f}_\nu := \sum_{n=1}^{\infty} \nu(\mathbf{a}_n) q^n$ is such that

$$\mathbf{f}_\nu = f_\nu(q) - \frac{p^{k_\nu - 1}}{\nu(\mathbf{a}_p)} f_\nu(q^p)$$

for some p-ordinary newform $f_\nu \in S_{k_\nu}(\Gamma_0(N))$, with $f = f_\nu$ for a unique $\nu = \nu_f$ with $k_\nu = 2r$. Under the above hypotheses, Howard's construction of big Heegner points [How07b] produces a class

$$\mathfrak{Z}_0 \in H^1(G_K, \mathbf{T}^\dagger),$$

where $\mathbf{T}^\dagger$ is a self-dual twist of the big Galois representation associated to $\mathbf{f}$. Under some additional hypotheses on $\bar\rho_f$ when $(D_K, N) > 1$, one can show that $\mathfrak{Z}_0$ lies in the so-called *strict Greenberg Selmer group* $\mathrm{Sel}_{\mathrm{Gr}}(K, \mathbf{T}^\dagger) \subset H^1(G_K, \mathbf{T}^\dagger)$, and so its image under the specialization map ν_f yields a second class $\nu_f(\mathfrak{Z}_0) \in \mathrm{Sel}(K, V_f(r))$.

Theorem A (Theorem 6.5)**.** *Assume in addition that $p = \mathfrak{p}\overline{\mathfrak{p}}$ splits in K, $\bar\rho_f|_{G_K}$ is irreducible, and $\bar\rho_f$ is ramified at every prime $q \mid N$ which is non-split in K. Then*

$$\nu_f(\mathfrak{Z}_0) = \left(1 - \frac{p^{r-1}}{\nu_f(\mathbf{a}_p)}\right)^2 \cdot \frac{\Phi^{\text{ét}}_{f,K}(\Delta_r^{\text{heeg}})}{u_K(2\sqrt{-D_K})^{r-1}},$$

where $u_K = |\mathcal{O}_K^\times|/2$.

This subsumes the main result of [Cas13], which only implies the equality in Theorem A under the assumption of Howard's "horizontal nonvanishing conjecture" [How07b, Conj. 2.2.2] and the nondegeneracy of the cyclotomic p-adic height pairing. The class $\mathfrak{Z}_0$ is obtained from Howard's big Heegner point $\mathfrak{X}_1$ of conductor 1, and more generally Theorem 6.5 establishes the relation between the Selmer classes constructed from clasical Heegner cycles of conductor $c > 0$ prime to Np and the corresponding higher weight specializations of the big Heegner point $\mathfrak{X}_c$. Thus Theorem 6.5 answers a question raised by Howard (see [How07b, p. 93]).

As in [Cas13], the proof of Theorem A follows from relating the cohomology classes under consideration to special values of L-functions. More precisely, extending work of Bertolini–Darmon–Prasanna [BDP13] and Brakočević [Bra11], in [CH18] we constructed an anticyclotomic p-adic L-function $\mathscr{L}_{\mathfrak{p},\psi}(f)$ interpolating central critical values of the L-function of f twisted by certain Hecke characters of K. Moreover, we constructed a compatible system of cohomology classes $\mathbf{z}_f$ interpolating the p-adic étale Abel–Jacobi images of (generalized) Heegner cycles of p-power conductor, and extending the p-adic Gross–Zagier formula of [BDP13] we obtained an "explicit reciprocity law"

$$\langle \mathcal{L}_{\mathfrak{p},\psi}(\mathbf{z}_f), \omega_f \otimes t^{1-2r} \rangle = -\mathscr{L}_{\mathfrak{p},\psi}(f) \tag{1.1}$$

relating $\mathscr{L}_{\mathfrak{p},\psi}(f)$ to the image of $\mathbf{z}_f$ under a Perrin-Riou logarithm map. Let $H_{p^\infty} = \bigcup_n H_{p^n}$ be the union of the ring class fields of K of p-power conductor. Denote by $\mathcal{W}$ the completion of the ring of integers of the maximal unramified extension of $\mathbf{Q}_p$, and set $\mathbb{I}_{\mathcal{W}} := \mathbb{I}\widehat{\otimes}_{\mathbf{Z}_p}\mathcal{W}$. In Section 2 of this paper, we construct a two-variable p-adic L-function

$$\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f}) \in \mathbb{I}_{\mathcal{W}}[[\mathrm{Gal}(H_{p^\infty}/K)]],$$

where $\boldsymbol{\xi}$ is a certain $\mathbb{I}$-adic anticyclotomic character of G_K, interpolating the p-adic L-functions of [CH18] attached to the different specializations f_ν of $\mathbf{f}$; in particular,

$$\nu_f(\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f})) = \mathscr{L}_{\mathfrak{p},\psi}(f). \tag{1.2}$$

The key new ingredient in our proof of Theorem A is then the connection that we find between $\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f})$ and the system

$$\mathfrak{Z}_\infty \in H^1_{\mathrm{Iw}}(H_{p^\infty}/H_1, \mathbf{T}^\dagger) = \varprojlim_n H^1(H_{p^n}, \mathbf{T}^\dagger)$$

of Howard's big Heegner points of p-power conductor. To simply state that result, we suppose that $H_1 = K$ in the next paragraph. By ordinarity, for each place v of K above p there is a G_{K_v}-stable $\mathbb{I}$-submodule $\mathscr{F}^+\mathbf{T}^\dagger \subset \mathbf{T}^\dagger$ of rank 1, and as shown by Howard, the image of $\mathfrak{Z}_\infty$ under the restriction map $\mathrm{res}_v : H^1_{\mathrm{Iw}}(H_{p^\infty}/K, \mathbf{T}^\dagger) \to H^1_{\mathrm{Iw}}(H_{p^\infty,v}/K_v, \mathbf{T}^\dagger)$ lands in the image of the natural map $H^1_{\mathrm{Iw}}(H_{p^\infty,v}/K_v, \mathscr{F}^+\mathbf{T}^\dagger) \to H^1_{\mathrm{Iw}}(H_{p^\infty,v}/K_v, \mathbf{T}^\dagger)$. In particular, the twist $\mathfrak{Z}_\infty^{\boldsymbol{\xi}^{-1}}$ of $\mathfrak{Z}_\infty$ by the character $\boldsymbol{\xi}^{-1}$ yields a class

$$\mathrm{res}_{\mathfrak{p}}(\mathfrak{Z}_\infty^{\boldsymbol{\xi}^{-1}}) \in H^1_{\mathrm{Iw}}(H_{p^\infty,\mathfrak{p}}/K_{\mathfrak{p}}, \mathscr{F}^+\mathbb{T}),$$

where $\mathscr{F}^+\mathbb{T} := \mathscr{F}^+\mathbf{T}^\dagger \otimes \boldsymbol{\xi}^{-1}$. Let

$$\lambda = \mathbf{a}_p \cdot \varepsilon_{\mathrm{cyc}}\Theta\boldsymbol{\xi}^{-1}(\mathrm{Frob}_p) - 1,$$

where $\mathrm{Frob}_p \in G_{K_{\mathfrak{p}}}$ is a geometric Frobenius element, and set $\widetilde{\mathbb{I}}_{\mathcal{W}} := \mathbb{I}[\lambda^{-1}]\widehat{\otimes}_{\mathbf{Z}_p}\mathcal{W}$.

Theorem B (Theorem 5.3)**.** *There is a Perrin-Riou big logarithm map*

$$\mathcal{L}^\Gamma_{\omega_{\mathbf{f}}} : H^1(H_{p^\infty,\mathfrak{p}}/K_{\mathfrak{p}}, \mathscr{F}^+\mathbb{T}) \longrightarrow \widetilde{\mathbb{I}}_{\mathcal{W}}[[\mathrm{Gal}(H_{p^\infty}/K)]]$$

for the local extension $H_{p^\infty,\mathfrak{p}}/K_{\mathfrak{p}}$ such that

$$\mathcal{L}^\Gamma_{\omega_{\mathbf{f}}}(\mathrm{res}_{\mathfrak{p}}(\mathfrak{Z}_\infty^{\boldsymbol{\xi}^{-1}})) = \mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f}) \cdot \sigma_{-1,\mathfrak{p}},$$

where $\sigma_{-1,\mathfrak{p}} := \mathrm{rec}_{\mathfrak{p}}(-1)|_{H_{p^\infty}} \in \mathrm{Gal}(H_{p^\infty}/K)$.

The construction of the two-variable Perrin-Riou map $\mathcal{L}^\Gamma_{\omega_{\mathbf{f}}}$ is given in Section 3, building upon work of Ochiai [Och03] and Loeffler–Zerbes [LZ14], and the proof of the "explicit reciprocity law" of Theorem B is obtained in Section 5 after a suitable extension of the calculations in [Cas13]. With this result in hand, the proof of Theorem A follows easily by specializing the equality in Theorem 5.3 at ν_f, using (1.2) and the interpolation property of the map $\mathcal{L}^\Gamma_{\omega_{\mathbf{f}}}$, and comparing it with the equality in (1.1).

The second purpose of this paper is to exploit the p-adic variation of Heegner points in Hida families to establish certain new cases of the equivariant Birch–Swinnerton-Dyer conjecture for rational elliptic curves with complex multiplication. More precisely, let $A/\mathbf{Q}$ be an elliptic curve with CM, and let

$$\varrho : G_{\mathbf{Q}} \longrightarrow \mathrm{Aut}_E(V_\varrho) \simeq \mathrm{GL}_2(E)$$

be a 2-dimensional odd and irreducible Artin representation factoring through a finite quotient $\mathrm{Gal}(F/\mathbf{Q})$ and with values in a finite extension $E \subset \mathbf{C}$ of $\mathbf{Q}$. Let $T_p(A)$ be the p-adic Tate module of A, and set $V_p(A) := \mathbf{Q}_p \otimes_{\mathbf{Z}_p} T_p(A)$. Associated to the compatible system $V_p(A) \otimes \imath_p V_\varrho$ of p-adic representations of $G_{\mathbf{Q}}$ is a Artin–Hasse–Weil L-function $L(A/\mathbf{Q}, \varrho, s)$. This is defined for $\mathrm{Re}(s) > 3/2$ by an absolutely convergent Euler product of degree 4, and by [Hec27] and [KW09] it is known to admit analytic continuation to the entire complex plane, with a functional equation relating its values at s and $2 - s$. The equivariant Birch–Swinnerton-Dyer conjecture predicts that

$$\mathrm{ord}_{s=1}L(A/\mathbf{Q}, \varrho, s) \stackrel{?}{=} \dim_E \mathrm{Hom}_{G_{\mathbf{Q}}}(V_\varrho, A(F)_E), \tag{1.3}$$

and that

$$\mathrm{Hom}_{G_{\mathbf{Q}}}(V_\varrho, Ш_{p^\infty}(A/F)_E) \stackrel{?}{=} \{0\} \tag{1.4}$$

for all primes p, where $Ш_{p^\infty}(A/F)$ is the p-primary component of the Tate–Shafarevich group of A/F, and for any abelian group M we have set $M_E := M \otimes_{\mathbf{Z}} E$. Let N_A and N_ϱ be the

conductor of A and ϱ, respectively, and denote by $\mathrm{Sel}(F, V_pA) \subset H^1(G_F, V_p(A))$ the Bloch–Kato Selmer group of $V_p(A)|_{\mathrm{Gal}(\overline{\mathbf{Q}}/F)}$.

Theorem C. *Let $A/\mathbf{Q}$ be an elliptic curve of conductor N_A and with complex multiplication by an imaginary quadratic field K, let $p \nmid 6N_\varrho N_A$ be a prime, and let $\mathfrak{P}$ be a prime of E above p. Assume that:*

- *N_ϱ and N_A are coprime;*
- *$p = \mathfrak{p}\overline{\mathfrak{p}}$ splits in K;*
- *K satisfies hypothesis* (heeg) *relative to N_ϱ;*
- *$\varrho(\mathrm{Frob}_p)$ has distinct eigenvalues modulo $\mathfrak{P}$.*

If $L(A/\mathbf{Q}, \varrho, 1) \neq 0$, then

$$\mathrm{Hom}_{G_{\mathbf{Q}}}(V_\varrho, \mathrm{Sel}(F, V_p(A))_E) = \{0\}.$$

In particular, (1.3) *and* (1.4) *hold.*

The conclusion that (1.3) holds under the nonvanishing of $L(A/\mathbf{Q}, \varrho, s)$ at $s = 1$ was already contained in earlier work of Bertolini–Darmon–Rotger [BDR15, Thm. A], while recent work of Kings–Loeffler–Zerbes [KLZ17, Thm. 11.7.4] establishes an analog of Theorem C for rational elliptic curves *without* complex multiplication (the CM case is excluded in [KLZ17] by the "big image hypothesis" of [*loc.cit.*, §11.1]). Thus the new content of Theorem C is the vanishing of the ϱ-isotypical component of $Ш_{p^\infty}(A/F)_E$ for "half" of the primes p under the nonvanishing of $L(A/\mathbf{Q}, \varrho, 1)$.

Let us conclude this Introduction with a few words about the proof of Theorem C. Denote by $L(f/K, \chi, s)$ the Rankin–Selberg L-function for the convolution of a cusp form $f \in S_k(\Gamma_1(N))$ with a Hecke character χ of K. From the explicit reciprocity law of Theorem B, we deduce a proof of the implication

$$L(f_\nu/K, \chi\mathbf{N}^{k_\nu/2}, 0) \neq 0 \implies \nu(\mathfrak{Z}_\infty)^{\chi^{-1}} \neq 0,$$

for $\nu : \mathbb{I} \to \overline{\mathbf{Q}}_p$ of weight $k_\nu > 0$ and certain anticyclotomic Hecke characters χ. Since Howard's systems of big Heegner points satisfies the compatibilities of an anticyclotomic Euler system, one can deduce from Kolyvagin's methods (as extended in [CH18, §7.2] to the anticyclotomic setting) a proof of the implication

$$L(f_\nu/K, \chi\mathbf{N}^{k_\nu/2}, 0) \neq 0 \implies \mathrm{Sel}(K, V_{\nu,\chi}) = \{0\},$$

where $\mathrm{Sel}(K, V_{\nu,\chi})$ is the Bloch–Kato Selmer group for $V_{f_\nu}(k_\nu/2)|_{G_K} \otimes \chi$. Since by [KW09] any Artin representation ϱ as in Theorem C is attached to some $g \in S_1(\Gamma_1(N_\varrho))$, taking χ so that $\chi\mathbf{N}^{1/2}$ corresponds to the grossencharacter of A, $\mathbf{f}$ to be a Hida family passing through g, and specializing the resulting $\mathfrak{Z}_\infty$ to weight one, the proof of Theorem C follows.

Some notations and definitions. For any place v of a number field E, let $\mathrm{rec}_v : E_v^\times \to G_{E_v}^{\mathrm{ab}}$ and $\mathrm{rec}_E : E^\times\backslash\mathbb{A}_E^\times \to G_E^{\mathrm{ab}}$ be the local and global reciprocity maps, respectively, with geometric normalizations. If $\phi : \mathbf{Z}_p^\times \to \mathbf{C}^\times$ is a continuous character of conductor p^n, the Gauss sum of ϕ is defined by

$$\mathfrak{g}(\phi) = \sum_{u \in (\mathbf{Z}/p^n\mathbf{Z})^\times} \phi(u)\mathbf{e}(u/p^n),$$

where $\mathbf{e}(z) = \exp(2\pi iz)$, and if $\chi : \mathbf{Q}_p \to \mathbf{C}^\times$ is a continuous character of conductor p^n, we define the ε-factor of χ by $\varepsilon(\chi) = p^n\chi^{-1}(p^n)\mathfrak{g}(\chi^{-1})^{-1}$.

2. p-adic Rankin L-series

In this section, we give the construction of a two-variable anticyclotomic p-adic L-function $\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f})$ attached to a Hida family $\mathbf{f}$ and an imaginary quadratic field K in which $p=\mathfrak{p}\overline{\mathfrak{p}}$ splits. Such construction closely parallels the one-variable construction by Brakočević [Bra11], and was essentially contained in [Bra12].

2.1. Geometric modular forms.

Fix a prime p, and let $N\geqslant 3$ be an integer prime to p.

Definition 2.1. Let k be an integer and let B be a $\mathbf{Z}_{(p)}$-algebra. A *geometric modular form* f of weight k on $\Gamma_1(Np^\infty)$ defined over B is a rule which assigns, for every B-algebra C and every triple $(A,\eta,\omega)_{/C}$ consisting of:

- an elliptic curve A/C;
- a $\Gamma_1(Np^\infty)$-level structure η on A, i.e., an immersion
$$\eta=(\eta^{(p)},\eta_p):\boldsymbol{\mu}_N\oplus\boldsymbol{\mu}_{p^\infty}\hookrightarrow A[N]\oplus A[p^\infty]$$
of group schemes over C;
- a C-basis ω of $H^0(A,\Omega^1_{A/C})$,

a value $f(A,\eta,\omega)\in C$ depending only on the isomorphism class of (A,η,ω) over C and such that:

(1) For any B-algebra homomorphism $\varphi:C\to C'$, we have
$$f((A,\eta,\omega)\otimes_C C')=\varphi(f(A,\eta,\omega));$$
(2) For all $\lambda\in C^\times$, we have
$$f(A,\eta,\lambda\omega)=\lambda^{-k}f(A,\eta,\omega);$$
(3) Letting $(\mathrm{Tate}(q),\eta_{\mathrm{can}},\omega_{\mathrm{can}})_{/B((q))}$ be the Tate elliptic curve $\mathbf{G}_m/q^{\mathbf{Z}}$ equipped with its canonical level structure η_{can} and differential ω_{can}, we have
$$f(\mathrm{Tate}(q),\eta_{\mathrm{can}},\omega_{\mathrm{can}})\in B[[q]].$$

Let $\mathrm{Ig}(N)_{/\mathbf{Z}_{(p)}}$ be the Igusa scheme parameterizing isomorphism classes of pairs $(A,\eta)_{/S}$ consisting of an elliptic curve A equipped with $\Gamma_1(Np^\infty)$-level structure η over arbitrary locally Noetherian $\mathbf{Z}_{(p)}$-schemes S. The generic fiber $\mathrm{Ig}(N)_{/\mathbf{Q}}$ of $\mathrm{Ig}(N)$ is given by

$$\mathrm{Ig}(N)_{/\mathbf{Q}}=\varprojlim_s Y_1(Np^s)_{/\mathbf{Q}},\tag{2.1}$$

where $Y_1(Np^s)_{/\mathbf{Q}}$ is the usual open modular curve of level $\Gamma_1(Np^s)$, and a geometric modular form f as in Definition 2.1 can be viewed as a section of a certain sheaf on $\mathrm{Ig}(N)_{/\mathbf{Z}_{(p)}}$.

2.2. p-adic modular forms.

For any p-adic ring R (i.e., $R\simeq\varprojlim_m R/p^mR$), let $\widehat{\mathrm{Ig}}(N)_{/R}$ be the completion of $\mathrm{Ig}(N)_{/R}$ along the closed subscheme $\mathrm{Ig}(N)_{/R}\otimes_R R/pR$.

Definition 2.2. Let R be a p-adic ring. A *p-adic modular form* of tame level N defined over R is a function on $\widehat{\mathrm{Ig}}(N)_{/R}$. Let $V_p(N;R)$ be the space of such functions, so that

$$V_p(N;R):=H^0(\widehat{\mathrm{Ig}}(N)_{/R},\mathcal{O}_{\widehat{\mathrm{Ig}}(N)_{/R}}).$$

Denote by Γ^{wt} the group $1+p\mathbf{Z}_p\subset\mathbf{Z}_p^\times$. For $k\in\mathbf{Z}_p$ and $\varepsilon:\Gamma^{\mathrm{wt}}\to\boldsymbol{\mu}_{p^\infty}(R)$, we say that a p-adic modular form $f\in V_p(N;R)$ has weight (k,ε) if it satisfies

$$f|\langle u\rangle_p(A,\eta):=f(A,\eta^{(p)},\eta_p u)=\varepsilon(u)u^k f(A,\eta),$$

for all $u \in \Gamma^{\mathrm{wt}}$ and any point $(A, \eta) = (A, \eta^{(p)}, \eta_p)$ of $\widehat{\mathrm{Ig}}(N)_{/R}$ valued in a p-adic R-algebra.

Associated with a geometric modular form f on $\Gamma_1(Np^\infty)$ defined over R there is a p-adic modular form $\widehat{f} \in V_p(N; R)$ defined by the rule

$$\widehat{f}(A, \eta) = f(A, \eta, \widehat{\omega}(\eta_p)),$$

where $\widehat{\omega}(\eta_p)$ is the differential on A arising from the isomorphism of formal groups $\widehat{\eta}_p : \widehat{\mathbf{G}}_m \simeq \widehat{A}$ induced by $\eta_p : \boldsymbol{\mu}_{p^\infty} \hookrightarrow A[p^\infty]$.

2.3. **$\mathbb{I}$-adic modular forms.** Let $\mathfrak{O}$ be the ring of integers of a finite extension of L of $\mathbf{Q}_p$, and set $\Lambda_{\mathfrak{O}}^{\mathrm{wt}} = \mathfrak{O}[[\Gamma^{\mathrm{wt}}]]$.

Definition 2.3. Let $\mathbb{I}$ be a finite flat $\Lambda_{\mathfrak{O}}^{\mathrm{wt}}$-algebra, and denote by $\mathcal{X}_{\mathfrak{O}}(\mathbb{I})$ the set of $\mathfrak{O}$-algebra homomorphisms $\nu : \mathbb{I} \to \overline{\mathbf{Q}}_p$. For any $k \in \mathbf{Z}$ and $\varepsilon : \Gamma^{\mathrm{wt}} \to \boldsymbol{\mu}_{p^\infty}$ let

$$\nu_{k,\varepsilon} : \Lambda_{\mathfrak{O}}^{\mathrm{wt}} \longrightarrow \overline{\mathbf{Q}}_p$$

be the $\mathfrak{O}$-algebra homomorphism defined by $u \mapsto \varepsilon(u)u^{k-2}$ for $u \in \Gamma^{\mathrm{wt}}$. We say that $\nu \in \mathcal{X}_{\mathfrak{O}}(\mathbb{I})$ has *weight* (k, ε) if the composition

$$\Lambda_{\mathfrak{O}}^{\mathrm{wt}} \longrightarrow \mathbb{I} \xrightarrow{\nu} \overline{\mathbf{Q}}_p$$

is of the form $\nu_{k,\varepsilon}$, and we say that $\nu \in \mathcal{X}_{\mathfrak{O}}(\mathbb{I})$ is an *arithmetic prime* if it has weight (k, ε) for some $k \in \mathbf{Z}_{\geqslant 2}$ and $\varepsilon : \Gamma^{\mathrm{wt}} \to \boldsymbol{\mu}_{p^\infty}$.

Denote by $\mathcal{X}_{\mathfrak{O}}^a(\mathbb{I})$ the set of arithmetic primes of $\mathbb{I}$, which we may view (just as $\mathcal{X}_{\mathfrak{O}}(\mathbb{I})$ itself) as a subset of $\mathrm{Spec}(\mathbb{I})(\overline{\mathbf{Q}}_p)$. For each $\nu \in \mathcal{X}_{\mathfrak{O}}(\mathbb{I})$, let F_ν be the residue field of $\ker(\nu) \subset \mathbb{I}$, and $\mathcal{O}_\nu \subset F_\nu$ be the valuation ring.

Definition 2.4. Let $\psi_0 : (\mathbf{Z}/Np\mathbf{Z})^\times \to \mathfrak{O}^\times$ be a Dirichlet character modulo Np, and let $\mathbb{I}$ be a finite flat $\Lambda_{\mathfrak{O}}^{\mathrm{wt}}$-algebra.

(1) An $\mathbb{I}$-*adic modular form* of tame level N is a formal q-expansion

$$\mathbf{f} = \sum_{n=0}^{\infty} \mathbf{a}_n q^n \in \mathbb{I}[[q]]$$

such that for all but finitely many $\nu \in \mathcal{X}_{\mathfrak{O}}^a(\mathbb{I})$ of weight (k, ε), the q-series $\sum_{n=0}^{\infty} \nu(\mathbf{a}_n)q^n$ is the q-expansion of a p-adic modular form $\mathbf{f}_\nu \in V_p(N; \mathcal{O}_\nu)$ of weight (k, ε). We denote by $G(N; \mathbb{I})$ the module of $\mathbb{I}$-adic modular forms of tame level N.

(2) We say that $\mathbf{f} \in G(N; \mathbb{I})$ is *arithmetic* with tame character ψ_0 if for all but finitely many $\nu \in \mathcal{X}_{\mathfrak{O}}^a(\mathbb{I})$ of weight (k, ε), the p-adic modular form $\mathbf{f}_\nu$ is the p-adic avatar $\widehat{f}_\nu$ of a classical modular form

$$f_\nu \in M_k(\Gamma_0(Np^s), \psi_0\varepsilon\omega^{2-k}),$$

where $s = \max\{1, \mathrm{ord}_p(\mathrm{cond}(\varepsilon)\}$, and $\omega : (\mathbf{Z}/p\mathbf{Z})^\times \to \mathbf{Z}_p^\times$ is the Teichmüller character; and we say that $\mathbf{f}$ is *cuspidal* if f_ν is a cusp form for all such ν. Denote by $S^a(N, \psi_0; \mathbb{I}) \subset G(N; \mathbb{I})$ the submodule of cuspidal arithmetic $\mathbb{I}$-adic modular forms of tame character ψ_0.

(3) We say that $\mathbf{f} \in S^a(N, \psi_0; \mathbb{I})$ is *ordinary* if $\mathbf{f}_\nu$ is a U_p-eigenvector for all but finitely many $\nu \in \mathcal{X}_{\mathfrak{O}}^a(\mathbb{I})$, with the U_p-eigenvalue being a p-adic unit, and we let $S^{\mathrm{ord}}(N, \psi_0; \mathbb{I}) \subset S^a(N, \psi_0; \mathbb{I})$ be the corresponding submodule. Finally, we say that $\mathbf{f}$ is an *ordinary* $\mathbb{I}$-*adic newform*[1] if for all but finitely many ν as above, f_ν is a p-stabilized newform of tame

[1] or alternatively, a *primitive cuspidal Hida family*, or just a *Hida family* in this paper.

level N, i.e., either f_ν is a newform of level Np^s, or is the ordinary p-stabilization of a p-ordinary newform of level N.

Define

$$V_p(N;\mathbb{I}) := V_p(N;\mathfrak{O})\widehat{\otimes}_{\mathfrak{O}}\mathbb{I}, \tag{2.2}$$

and let $[z] : \mathbf{Z}_p^\times \to \mathfrak{O}[[\mathbf{Z}_p^\times]]^\times$ be the natural inclusion as group-like elements. The space $V_p(N;\mathbb{I})$ is equipped with two different actions of $z \in \Gamma^{\mathrm{wt}}$: one via the diamond operators $\langle z\rangle_p$ acting on the first factor of (2.2), and the other via multiplication by $[z]$ on the second factor, composed with the structure map $\mathfrak{O}[[\mathbf{Z}_p^\times]]^\times \to \mathbb{I}^\times$.

Proposition 2.5. *There is a canonical $\mathbb{I}$-module isomorphism*

$$G(N;\mathbb{I}) = \{\mathbf{f} \in V_p(N;\mathbb{I}) :\ \mathbf{f}|\langle z\rangle_p = [z]\mathbf{f}, \quad \forall z \in \mathbf{Z}_p^\times\}.$$

Proof. See [Hid00, Thm. 3.2.16]. □

Thus, in light of Proposition 2.5, we may evaluate any $\mathbb{I}$-adic modular form $\mathbf{f} \in G(N;\mathbb{I})$ at a point $x \in \widehat{\mathrm{Ig}}(N)(\mathbb{I})$, producing an element $\mathbf{f}(x) \in \mathbb{I}$ such that

$$\nu(\mathbf{f}(x)) = \mathbf{f}_\nu(x) \tag{2.3}$$

for all $\nu \in \mathcal{X}_{\mathfrak{O}}(\mathbb{I})$. (Indeed, this follows from the q-expansion principle, since by definition the specialization property (2.3) holds when x is coming from a Tate curve.) This will be used in §2.4 to define measures associated with $\mathbf{f}$ which, for appropriate choices of x (defined in §2.5), interpolate special values of L-functions.

2.4. **Modular measures.** For a compact totally disconnected topological space X (which in our application will be $X \simeq \Delta \times \mathbf{Z}_p$ with Δ a finite group) and a p-adic ring R, we denote by $\mathrm{Cont}(X,R)$ the space of continuous R-valued functions on X. Let

$$\mathrm{Meas}(X,R) := \mathrm{Hom}_{\mathrm{cts}}(\mathrm{Cont}(X,R),R)$$

be the space of R-valued measures on X. As usual, if $\mu \in \mathrm{Meas}(X,R)$ and $\phi \in \mathrm{Cont}(X,R)$, we denote by $\int_X \phi(z)\mathrm{d}\mu(z) \in R$ the value of μ at ϕ. For $X = \mathbf{Z}_p$, the *Amice transform* of a measure $\mu \in \mathrm{Meas}(\mathbf{Z}_p,R)$ is the power series $\mathscr{A}_\mu(T) \in R[[T]]$ given by

$$\mathscr{A}_\mu(T) = \sum_{m=0}^{\infty} c_m(\mu)T^m,$$

where $c_m(\mu) = \int_{\mathbf{Z}_p}\binom{z}{m}\mathrm{d}\mu(z)$. One easily checks that

$$\int_{\mathbf{Z}_p} z^n \mathrm{d}\mu(z) = \left(T\frac{d}{dT}\right)^n \mathscr{A}_\mu(T)\big|_{T=0}$$

for all $n \geqslant 0$, and by Mahler's theorem the rule $\mu \mapsto \mathscr{A}_\mu(T)$ defines an isomorphism $\mathrm{Meas}(\mathbf{Z}_p,R) \cong R[[T]]$ of p-adic Banach algebras.

Let d be the operator on $V_p(N;R)$ given by

$$d : \sum_{n=0}^{\infty} a_n q^n \mapsto \sum_{n=0}^{\infty} n a_n q^n,$$

and for each $m \in \mathbf{Z}_{\geqslant 0}$ let $\binom{d}{m}$ denote the operator given by $\sum_n a_n q^n \mapsto \sum_n \binom{n}{m} a_n q^n$.

Definition 2.6. For $g \in V_p(N;R)$ and $x \in \mathrm{Ig}(N)(R)$, let $\mu_{g,x} \in \mathrm{Meas}(\mathbf{Z}_p, R)$ be the measure determined by

$$\int_{\mathbf{Z}_p} \binom{z}{m} \mathrm{d}\mu_{g,x}(z) = \binom{d}{m} g(x),$$

for all $m \geqslant 0$.

Let $U = U_p$ and V be the operators on $V_p(N;R)$ given by $\sum_n a_n q^n \mapsto \sum_n a_{np} q^n$ and $\sum_n a_n q^n \mapsto \sum_n a_n q^{np}$, respectively, in terms of q-expansions. If $g \in V_p(N;R)$ has q-expansion $\sum_n a_n q^n$, setting

$$g^\flat := g|(1-UV) = \sum_{(n,p)=1} a_n q^n \in V_p(N;R),$$

it is easily seen that the associated measure $\mu_{g^\flat,x}$ is supported on $\mathbf{Z}_p^\times$.

2.5. **CM points.** Let K be an imaginary quadratic field of odd discriminant $-D_K < -3$, let $p > 2$ be a prime split in K, and write

$$p\mathcal{O}_K = \mathfrak{p}\overline{\mathfrak{p}},$$

where $\mathfrak{p}$ is the prime of K above p induced by our fixed embedding $\imath_p : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$. We shall assume throughout that K satisfies the following *Heegner hypothesis* relative to a fixed integer $N > 0$ prime to p:

(heeg) there is an ideal $\mathfrak{N} \subset \mathcal{O}_K$ with $\mathcal{O}_K/\mathfrak{N} \simeq \mathbf{Z}/N\mathbf{Z}$.

The existence of such $\mathfrak{N}$, which will be fixed from now on, amounts to the requirement that every prime $q \mid N$ is either split or ramified in K, with $q^2 \nmid N$ in the latter case.

For each positive integer c let $\mathcal{O}_c = \mathbf{Z} + c\mathcal{O}_K$ be the order of K of that conductor, and let H_c be the corresponding ring class field, so that $\mathrm{Gal}(H_c/K) \simeq \mathrm{Pic}(\mathcal{O}_c)$ by the Artin reciprocity map. For each invertible $\mathcal{O}_c$-ideal $\mathfrak{a}$ prime to $\mathfrak{N}\mathfrak{p}$, let $A_\mathfrak{a}/H_c$ be the CM elliptic curve with the complex uniformization $A_\mathfrak{a}(\mathbf{C}) = \mathbf{C}/\mathfrak{a}^{-1}$. Let $a \in \widehat{K}^\times$ be such that $a\widehat{\mathcal{O}}_c \cap K = \mathfrak{a}$, and equip $A_\mathfrak{a}$ with the $\Gamma_1(Np^\infty)$-level structure

$$\eta_a : \boldsymbol{\mu}_N \oplus \boldsymbol{\mu}_{p^\infty} \hookrightarrow A_\mathfrak{a}[N] \oplus A_\mathfrak{a}[p^\infty]$$

defined in [CH18, p. 576]. The pair $(A_\mathfrak{a}, \eta_a)$ defines a point $x_\mathfrak{a} \in \mathrm{Ig}(N)(\mathcal{V})$ over the valuation ring

$$\mathcal{V} := \imath_p^{-1}(\mathcal{O}_{\mathbf{C}_p}) \cap K^{\mathrm{ab}},$$

where K^{ab} is the maximal abelian extension of K. For the ease of notation, set $x_c := x_{\mathcal{O}_c}$.

Write $c = c_o p^n$ with $p \nmid c_o$, and decompose $c_o = c_o^+ c_o^-$ with c_o^+ (resp. c_o^-) only divisible by primes which are split (resp. non-split) in K. We similarly decompose $N = N^+N^-$, and set $\mathfrak{C}^+ := c_o^+\mathcal{O}_K$ and $\mathfrak{N}^+ := N^+\mathcal{O}_K$. Fix a square-root $\sqrt{-D_K} \in K$, and set

$$\vartheta := (D_K + \sqrt{-D_K})/2.$$

Following [CH18, §2.4], we define the matrix $\varsigma^{(\infty)} = (\varsigma_q) \in \mathrm{GL}_2(\widehat{\mathbf{Q}})$ by

- $\varsigma_q = 1$, if $q \nmid c_o^+ N^+ p$,
- $\varsigma_q = (\overline{\vartheta} - \vartheta)^{-1} \begin{pmatrix} \overline{\vartheta} & \vartheta \\ 1 & 1 \end{pmatrix}$, if $q\mathcal{O}_K = \mathfrak{q}\overline{\mathfrak{q}}$ with $\mathfrak{q} \mid \mathfrak{C}^+\mathfrak{N}^+\mathfrak{p}$,

and the matrix $\gamma_c = (\gamma_{c,q}) \in \mathrm{GL}_2(\widehat{\mathbf{Q}})$ by

- $\gamma_{c,q} = 1$, if $q \nmid cNp$,

- $\gamma_{c,q} = \begin{pmatrix} q^{\mathrm{ord}_q(c)} & 1 \\ 0 & 1 \end{pmatrix}$, if $q\mathcal{O}_K = \mathfrak{q}\overline{\mathfrak{q}}$ with $\mathfrak{q} \mid \mathfrak{C}^+\mathfrak{N}^+\mathfrak{p}$,
- $\gamma_{c,q} = \begin{pmatrix} 1 & 0 \\ 0 & q^{\mathrm{ord}_q(c)-\mathrm{ord}_q(N)} \end{pmatrix}$, if $q \mid c_o^- N^-$,

and set $\xi_c := \varsigma^{(\infty)}\gamma_c$. Under the complex uniformization

$$[\cdot] : \mathfrak{H} \times \mathrm{GL}_2(\widehat{\mathbf{Q}}) \longrightarrow \mathrm{Ig}(N)(\mathbf{C})$$

coming from (2.1) and the complex uniformization of $Y_1(Np^s)$, we have $[(\vartheta, \xi_c)] = x_c$. Moreover, by Shimura's reciprocity law, if $\mathfrak{a}$ is an invertible $\mathcal{O}_c$-ideal prime to $\mathfrak{N}\mathfrak{p}$ and $a \in \widehat{K}^{(cp)\times}$ is such that $\mathfrak{a} = a\widehat{\mathcal{O}}_c \cap K$, then

$$x_{\mathfrak{a}} = [(\vartheta, \overline{a}^{-1}\xi_c)] = x_c^{\sigma_{\mathfrak{a}}} \in \mathrm{Ig}(N)(H_c(\mathfrak{p}^\infty)),$$

where $\sigma_{\mathfrak{a}} = \mathrm{rec}_K(a^{-1})|_{H_c(\mathfrak{p}^\infty)} \in \mathrm{Gal}(H_c(\mathfrak{p}^\infty)/K)$ is the Artin symbol of $\mathfrak{a}$ over the compositum of H_c with the ray class field of K of conductor $\mathfrak{p}^\infty$, and $a \mapsto \overline{a}$ denotes the action of the non-trivial automorphism $\tau \in \mathrm{Gal}(K/\mathbf{Q})$ on $\mathbb{A}_K$.

2.6. **Anticyclotomic Hecke characters.** We say that a Hecke character $\psi : K^\times\backslash\mathbb{A}_K^\times \to \mathbf{C}^\times$ has infinity type (ℓ_1, ℓ_2), with $\ell_1, \ell_2 \in \frac{1}{2}\mathbf{Z}$ such that $\ell_1 - \ell_2 \in \mathbf{Z}$, if

$$\psi_\infty(z) = z^{\ell_1-\ell_2}(z\overline{z})^{\ell_2},$$

where for each place v of K, we let $\psi_v : K_v^\times \to \mathbf{C}^\times$ be the component of ψ at v. The conductor of ψ is the largest ideal $\mathfrak{c} \subset \mathcal{O}_K$ such that $\psi_{\mathfrak{q}}(u) = 1$ for all $u \in (1 + \mathfrak{c}\mathcal{O}_{K,\mathfrak{q}})^\times \subset K_{\mathfrak{q}}^\times$. If ψ has conductor $\mathfrak{c}_\psi$ and $\mathfrak{a}$ is any fractional ideal of K prime to $\mathfrak{c}_\psi$, we write $\psi(\mathfrak{a})$ for $\psi(a)$, where $a \in \widehat{K}^{(\mathfrak{c}_\psi)\times}$ is such that $a\widehat{\mathcal{O}}_K \cap K = \mathfrak{a}$. As a function on fractional ideals, ψ satisfies

$$\psi((\alpha)) = \alpha^{\ell_2-\ell_1}(\alpha\overline{\alpha})^{-\ell_2}$$

for all $\alpha \in K^\times$ with $\alpha \equiv 1 \pmod{\mathfrak{c}_\psi}$.

Definition 2.7. Let $\psi = \psi_{\mathrm{fin}}\psi_\infty$ be a Hecke character of K with infinity type (ℓ_1, ℓ_2). The *p-adic avatar* $\widehat{\psi} : K^\times\backslash\widehat{K}^\times \to \mathbf{C}_p^\times$ of ψ is defined by

$$\widehat{\psi}(z) = \imath_p\imath_\infty^{-1}(\psi_{\mathrm{fin}}(z))z_{\mathfrak{p}}^{\ell_1}z_{\overline{\mathfrak{p}}}^{\ell_2}.$$

Via the reciprocity map rec_K, we shall often regard $\widehat{\psi}$ as a Galois character $\widehat{\psi} : G_K \to \mathbf{C}_p^\times$.

We say that a Hecke character $\psi : K^\times\backslash\mathbb{A}_K^\times \to \mathbf{C}^\times$ is *anticyclotomic* if $\psi|_{\mathbb{A}_{\mathbf{Q}}^\times} = \mathbb{1}$. The infinity type of an anticyclotomic ψ is of the form $(\ell, -\ell)$, and the correspondence $\psi \mapsto \widehat{\psi}$ establishes a bijection between the set of anticyclotomic Hecke characters of K of conductor dividing p^∞ and the set of locally algebraic $\mathbf{C}_p$-valued characters of $\mathrm{Gal}(H_{p^\infty}/K)$, for H_{p^∞} the union of the ring class fields of K of conductor p-power conductor.

2.7. **A two-variable anticyclotomic p-adic L-function.** Let $\mathbf{f} \in S^a(N, \psi_0; \mathbb{I})$ be an ordinary $\mathbb{I}$-adic newform of tame level N and character $\psi_0 : (\mathbf{Z}/Np\mathbf{Z})^\times \to \mathfrak{O}^\times$ as in Definition 2.4. Recall the Teichmüller character $\omega : (\mathbf{Z}/p\mathbf{Z})^\times \to \mathbf{Z}_p^\times$, and let $\varepsilon_{\mathrm{cyc}} : G_{\mathbf{Q}} \to \mathbf{Z}_p^\times$ be the cyclotomic character. By composing it with $\varepsilon_{\mathrm{cyc}} \bmod p$, we shall also view ω as a Galois character $\omega : G_{\mathbf{Q}} \to \mathbf{Z}_p^\times$.

Let $\lambda : K^\times\backslash\mathbb{A}_K^\times \to \mathfrak{O}^\times$ be the p-adic avatar of a fixed Hecke character of infinity type $(1, 0)$ and conductor $\mathfrak{c}\mathfrak{p}$ for some ideal $\mathfrak{c} \subset \mathcal{O}_K$ with $(\mathfrak{c}, Np) = 1$. Let $\mathfrak{O}^\times_{/\mathrm{tor}}$ be the maximal $\mathbf{Z}_p$-free quotient

of $\mathfrak{O}$, and let $W \subset \mathfrak{O}^\times_{/\mathrm{tors}}$ be the subset topologically generated by the values of λ. Then W is isomorphic to $\mathbf{Z}_p$, and it naturally contains (the image of) $\Gamma^{\mathrm{wt}} = 1 + p\mathbf{Z}_p$. Write $p^b = [W : \Gamma^{\mathrm{wt}}]$ and let $\mathbb{J} = \mathfrak{O}[[S]]$ be the extension of $\Lambda^{\mathrm{wt}}_{\mathfrak{O}}$ defined by $(1+S)^{p^b} = 1 + p$. Upon enlarging $\mathbb{I}$ if necessary, we shall assume that $\mathbb{I} \supset \mathbb{J}$.

Definition 2.8.

(1) Let $i \in \mathbf{Z}/(p-1)\mathbf{Z}$ be such that $\psi_0|_{(\mathbf{Z}/p\mathbf{Z})^\times} = \omega^i$, and define the *critical character* $\Theta : G_{\mathbf{Q}} \to \Lambda^{\mathrm{wt},\times}$ by
$$\Theta(\sigma) := \omega^{i/2}(\sigma) \cdot [\langle \varepsilon_{\mathrm{cyc}}(\sigma)\rangle^{1/2}],$$
where $\langle\cdot\rangle^{1/2} : \mathbf{Z}_p^\times \to \Gamma^{\mathrm{wt}}$ is the composition of the projection $\langle\cdot\rangle : \mathbf{Z}_p^\times \twoheadrightarrow \Gamma^{\mathrm{wt}}$ with the map $x \mapsto x^{1/2}$, and $[\cdot] : \Gamma^{\mathrm{wt}} \hookrightarrow \Lambda^{\mathrm{wt},\times}$ is the inclusion of group-like elements.

(2) Take a finite order Hecke character χ_0 of K of conductor dividing $\mathfrak{N}$ such that
$$\chi_0|_{\mathbb{A}_{\mathbf{Q}}^\times} = \psi_0^{-1}|_{(\mathbf{Z}/N\mathbf{Z})^\times},$$
an define the $\mathbb{I}$-adic character $\boldsymbol{\chi} : K^\times\backslash\mathbb{A}_K^\times \to \mathbb{I}^\times$ by
$$\boldsymbol{\chi}(x) := \psi_0(x)\Theta(\mathrm{rec}_{\mathbf{Q}}(\mathrm{N}_{K/\mathbf{Q}}(x))),$$
where Θ is viewed as taking values in $\mathbb{I}^\times$ by composition with the structure morphism $\Lambda^{\mathrm{wt}} \to \mathbb{I}$.

(3) Denote by $\langle\lambda\rangle$ the composition of λ with the projection onto $\mathfrak{O}^\times_{/\mathrm{tors}}$. Let $w \in W$ be a topological generator, and define the $\mathbb{I}$-adic character $\Xi : K^\times\backslash\mathbb{A}_K^\times \to \mathbb{J}^\times \to \mathbb{I}^\times$ by
$$\Xi(x) = \lambda(x)(1+S)^{l(x)}, \quad \langle\lambda(x)\rangle = w^{l(x)}.$$
Finally, define $\boldsymbol{\xi} : K^\times\backslash\mathbb{A}_K^\times \to \mathbb{I}^\times$ by
$$\boldsymbol{\xi}(x) = \Xi(x)\Xi^{-1}(\overline{x}).$$

Remark 2.9. Recall that we assume $p > 2$ and note that implicit in Definition 2.8 is the choice of a lift of i to $\mathbf{Z}/2(p-1)\mathbf{Z}$; we fix either one of the two possible choices, *cf.* [How07b, Rem. 2.1.3].

Let $c_o\mathcal{O}_K$ be the prime-to-p part of the conductor of the anticyclotomic character $\lambda(x)\lambda^{-1}(\overline{x})$, and for any $\mathcal{O}_{c_o}$-ideal $\mathfrak{a}$ prime to $\mathfrak{N}\mathfrak{p}$, let $x_{\mathfrak{a}} \in \mathrm{Ig}(N)(\mathcal{V})$ be the CM point constructed in §2.5. Since $p \nmid c_o$, the point $x_{\mathfrak{a}}$ admits a model over the discrete valuation ring
$$\mathcal{V}^{\mathrm{ur}} := \mathcal{W} \cap K^{\mathrm{ab}},$$
where $\mathcal{W} = W(\overline{\mathbb{F}}_p)$ is the ring of integers of the completion of the maximal unramified extension of $\mathbf{Q}_p$. Let
$$\mathbb{I}_{\mathcal{W}} := \mathbb{I}\widehat{\otimes}_{\mathbf{Z}_p}\mathcal{W}.$$
In light of Proposition 2.5, extending scalars we view $\mathbf{f}$ as an element in $V_p(N;\mathbb{I}_{\mathcal{W}})$. Letting $x_{\mathfrak{a}}$ still denote the pullback of the above point $x_{\mathfrak{a}}$ under the structure map $\mathrm{Spec}(\mathbb{I}_{\mathcal{W}}) \to \mathrm{Spec}(\mathcal{W})$, we let $\mu_{\mathbf{f}^\flat,x_{\mathfrak{a}}} \in \mathrm{Meas}(\mathbf{Z}_p,\mathbb{I}_{\mathcal{W}})$ be the measure of Definition 2.6, and let $\mu_{\mathbf{f}^\flat_{\mathfrak{a}}}$ be the measure on $\mathbf{Z}_p$ characterized by
$$\mathscr{A}_{\mu_{\mathbf{f}^\flat_{\mathfrak{a}}}}(T) = \mathscr{A}_{\mu_{\mathbf{f}^\flat,\mathfrak{a}}}((1+T)^{\mathbf{N}(\mathfrak{a})^{-1}\sqrt{-D_K}^{-1}} - 1).$$
Since $\mu_{\mathbf{f}^\flat,x_{\mathfrak{a}}}$ is supported on $\mathbf{Z}_p^\times$, so is the measure $\mu_{\mathbf{f}^\flat_{\mathfrak{a}}}$.

For each integer $c > 0$, let $\widetilde{H}_c$ denote the composition of the ring class field H_c with the ray class field of K of conductor $\mathfrak{N}$, and set $\widetilde{\Gamma} := \mathrm{Gal}(\widetilde{H}_{p^\infty}/K)$. Let $\mathrm{rec}_K : K^\times\backslash\mathbb{A}_K^\times \to G_K^{\mathrm{ab}} \to \widetilde{\Gamma}$ and $\mathrm{rec}_{\mathfrak{p}} : \mathbf{Q}_p^\times = K_{\mathfrak{p}}^\times \to G_K^{\mathrm{ab}} \to \widetilde{\Gamma}$ be the global and local-at-$\mathfrak{p}$ reciprocity maps, respectively.

Definition 2.10. The *two-variable anticyclotomic p-adic L-function* attached to $\mathbf{f}$ and $\boldsymbol{\xi}$ is the $\mathbb{I}_{\mathcal{W}}$-valued measure $\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f})$ on $\widetilde{\Gamma}$ given, for all $\phi : \widetilde{\Gamma} \to \mathcal{O}_{\mathbf{C}_p}^\times$, by

$$\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f})(\phi) = \sum_{\sigma \in \mathrm{Gal}(\widetilde{H}_{c_o}/K)} \boldsymbol{\xi}\boldsymbol{\chi}^{-1}(\mathfrak{a})\mathbf{N}(\mathfrak{a})^{-1} \int_{\mathbf{Z}_p^\times} (\phi|[\mathfrak{a}])(z)\mathrm{d}\mu_{\mathbf{f}_\mathfrak{a}^\flat}(z),$$

where $\mathfrak{a}$ corresponds to σ under the Artin map, and $\phi|[\mathfrak{a}]$ is the character on $\mathbf{Z}_p^\times$ defined by

$$\phi|[\mathfrak{a}](z) := \phi(\sigma \mathrm{rec}_\mathfrak{p}(z)).$$

Now we describe the interpolation property satisfied by $\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f})$. For the statement, recall that if $f = \sum_{n=1}^\infty a_n q^n$ is a normalized newform of weight k and nebentypus ε_f, χ is a Hecke character of K with central character $\chi|_{\mathbb{A}_\mathbf{Q}^\times} = \varepsilon_f^{-1}$, and ψ is an anticyclotomic Hecke character of conductor $c\mathcal{O}_K$, the Rankin L-series $L(f/K, \chi\psi, s)$ is given in terms of automorphic L-functions by the equality

$$L(f/K, \chi\psi, s) = L\left(s - \frac{k-1}{2}, \pi_K \otimes \chi\psi\right), \tag{2.4}$$

where π_K is the base change to K of the automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbf{Q})$ generated by f. Thus since $\pi_K \otimes \chi\psi$ is self-dual, $L(f/K, \chi\psi, s)$ satisfies a functional equation relating is values at s and $k-s$.

By the calculation in [How07a, p. 808] and our definition of the $\mathbb{I}$-adic character $\boldsymbol{\chi}$ (which differs from that in [How07a, §3] by the factor χ_0 in order to allow non-trivial N-part of the nebentypus), for every $\nu \in \mathcal{X}_\mathfrak{D}(\mathbb{I})$ such that f_ν classical, the specialization χ_ν of the $\mathbb{I}$-adic character $\boldsymbol{\chi}$ in Definition 2.8 is such that $\chi_\nu|_{\mathbb{A}_\mathbf{Q}^\times} = \varepsilon_{f_\nu}^{-1}$. For such ν, and ψ an anticyclotomic Hecke character of K of conductor $c_o p^n \mathcal{O}_K$ with $p \nmid c_o$, define the p-adic multiplier $\mathscr{E}_\mathfrak{p}(f_\nu, \chi_\nu\psi)$ by

$$\mathscr{E}_\mathfrak{p}(f_\nu, \chi_\nu\psi) = \begin{cases} \big(1 - \nu(\mathbf{a}_p)(\chi_\nu\psi)_{\overline{\mathfrak{p}}}(p)p^{-k_\nu/2}\big)\big(1 - (\chi_\nu\psi)_{\overline{\mathfrak{p}}}(p)p^{k_\nu/2-1}\nu(\mathbf{a}_p)^{-1}\big) & \text{if } n = 0, \\ \varepsilon((\chi_\nu\psi)_\mathfrak{p}^{-1})p^{-n} & \text{if } n \geqslant 1, \end{cases}$$

and set

$$L^{\mathrm{alg}}(f_\nu/K, \chi_\nu\psi, k_\nu - 1) := \frac{\Gamma(k_\nu + \ell)\Gamma(\ell+1)}{(2\pi)^{k_\nu+2\ell+1}(\mathrm{Im}\ \vartheta)^{k_\nu+2\ell}} \cdot \frac{L(f_\nu/K, \chi_\nu\psi, k_\nu - 1)}{\Omega_K^{2k_\nu+4\ell}},$$

where $\Omega_K \in \mathbf{C}^\times$ is a complex period attached to K as in [CH18, §2.5].

Theorem 2.11. *Let $\nu \in \mathcal{X}_\mathfrak{D}(\mathbb{I})$ of weight $(k, \mathbb{1})$ with $k \geqslant 1$ be such that f_ν is classical, and let $\widehat{\phi}$ be the p-adic avatar of an anticyclotomic Hecke character ϕ of K of infinity type $(\ell, -\ell)$ with $\ell \geqslant 0$ and conductor $c_o p^n \mathcal{O}_K$ with $p \nmid c_o$. Then:*

$$\frac{\nu(\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f}))(\widehat{\phi})^2}{\Omega_p^{2k+4\ell}} = L^{\mathrm{alg}}(f_\nu/K, \chi_\nu\xi_\nu\phi, k_\nu - 1) \cdot \mathscr{E}_\mathfrak{p}(f_\nu, \chi_\nu\xi_\nu\phi)^2 \cdot \phi(\mathfrak{N}^{-1}) \cdot 2^3 \cdot c_o\varepsilon(f_\nu) \cdot w_K^2\sqrt{D_K},$$

where $\varepsilon(f_\nu)$ is the global root number of f_ν, $w_K := |\mathcal{O}_K^\times|$, and $\Omega_p \in \mathcal{W}^\times$ is a p-adic period as in [CH18, §2.5].

Proof. Let ν be as in the statement and set $f = f_\nu$. Then $\Theta_\nu(z) = z^{k/2-1}$ for all $z \in \mathbf{Z}_p^\times$, and hence

$$\chi_\nu(\mathfrak{a}) = \mathbf{N}(\mathfrak{a})^{k/2-1}. \tag{2.5}$$

From (2.3), it follows that $\nu\big(\binom{d}{m}\mathbf{f}(x)\big) = \binom{d}{m}\mathbf{f}_\nu(x)$ for all $m \geqslant 0$, and hence for the measure $\mu_{\mathbf{f}_\nu,x}$ of Definition 2.6 we have

$$\nu\Big(\int_{\mathbf{Z}_p} \rho(z)\mathrm{d}\mu_{\mathbf{f},x}(z)\Big) = \int_{\mathbf{Z}_p} \rho(z)\mathrm{d}\mu_{\mathbf{f}_\nu,x}(z)$$

for all $\rho : \mathbf{Z}_p \to \mathcal{O}_{\mathbf{C}_p}$. Thus specializing $\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f})$ at ν we see that for any ramified character ρ on $\mathbf{Z}_p^\times$:

$$\nu(\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f}))(\rho) = \sum_{\sigma\in\mathrm{Gal}(\widetilde{H}_{c_0}/K)} \xi_\nu(\mathfrak{a})\mathbf{N}(\mathfrak{a})^{-k/2}\int_{\mathbf{Z}_p^\times}(\rho|[\mathfrak{a}])(z)\mathrm{d}\mu_{\widehat{f^\flat_\mathfrak{a}}}(z).$$

where $\widehat{f^\flat}$ is the p-adic avatar of $f^\flat$. Let $\widehat{f^\flat}\otimes(\rho|[\mathfrak{a}])$ be the p-adic modular form $\widehat{f^\flat}$ twisted by $\rho|[\mathfrak{a}]$. Setting $T = t-1$ and tracing through the definitions, we see that

$$\int_{\mathbf{Z}_p^\times}(\rho|[\mathfrak{a}])(z)\mathrm{d}\mu_{\widehat{f^\flat_\mathfrak{a}}}(z) = \mathscr{A}_{\mu_{\widehat{f^\flat}\otimes(\rho|[\mathfrak{a}])}}(t^{\mathbf{N}(\mathfrak{a})^{-1}\sqrt{-D_K}^{-1}})|_{t=1}. \tag{2.6}$$

Since the right-hand side of (2.6) agrees with the expression $\big(\widehat{f^\flat_\mathfrak{a}}\otimes\rho|[\mathfrak{a}]\big)(A_\mathfrak{a},\eta_\mathfrak{a})$ appearing in [CH18, Def. 3.7] and ξ_ν is the p-adic avatar of an anticyclotomic Hecke character of infinity type $(k/2,-k/2)$, the above shows that $\nu(\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f}))$ agrees with the $\mathcal{W}$-valued measure $\mathscr{L}_{\mathfrak{p},\xi_\nu}(f)$ on $\widetilde{\Gamma}$ constructed in [CH18, §3.3] (or rather its immediate extension in the slightly more general setting considered here). The result this follows from [*loc.cit.*, Prop. 3.8]. (Note that in [CH18] only cusp form of even weights $k \geqslant 2$ are considered, but the construction of $\mathscr{L}_{\mathfrak{p},\xi_\nu}(f)$ readily extends to any $k \in \mathbf{Z}_{\geqslant 1}$, and the results quoted from [Hsi14] are available in this level of generality.) □

Remark 2.12. Note that by (2.5) we have $L(f_\nu/K,\chi_\nu\xi_\nu\phi,k_\nu-1) = L(f_\nu/K,\xi_\nu\phi,k_\nu/2)$, and so the L-values appearing in Theorem 2.11 are central critical values.

Corollary 2.13. *For every $\nu \in \mathcal{X}_\mathfrak{D}(\mathbb{I})$ of weight $(k,\mathbb{1})$ with $k \geqslant 1$ such that f_ν is classical, the p-adic L-function $\nu(\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f}))$ is not identically zero.*

Proof. As shown in the proof of Theorem 2.11, the specialization $\nu(\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f}))$ agrees with (the natural extension of) the p-adic L-function $\mathscr{L}_{\mathfrak{p},\xi_\nu}(f)$ constructed in [CH18, §3.3] with $f = f_\nu$, and so the result similarly follows from [*loc.cit.*, Thm. 3.9]. □

3. Big logarithm maps

In this section we construct a Perrin-Riou big logarithm map adapted to our global anticyclotomic setting. Starting with [PR94], the cyclotomic theory of these maps has been widely studied in the literature; see e.g. [Ber03] and the references therein. The construction we give here combines work of Ochiai [Och03] and Loeffler–Zerbes [LZ14].

3.1. **Review of p-adic Hodge theory.** Let F and L be finite extensions of $\mathbf{Q}_p$. For a finite-dimensional L-vector space V equipped with a continuous linear action of G_F, we denote by $\mathbf{D}_{\mathrm{dR},F}(V)$ the filtered $(L\otimes_{\mathbf{Q}_p}F)$-module

$$\mathbf{D}_{\mathrm{dR},F}(V) := (V\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{dR}})^{G_F},$$

where $\mathbf{B}_{\mathrm{dR}}$ is Fontaine's ring of p-adic de Rham periods. If V is a de Rham G_F-representation (i.e., $\dim_F\mathbf{D}_{\mathrm{dR},F}(V) = \dim_L V$), then for any finite extension E/F there is a canonical isomorphism $\mathbf{D}_{\mathrm{dR},E}(V) \cong E\otimes_F\mathbf{D}_{\mathrm{dR},F}(V)$. Denote by $\langle\ ,\ \rangle$ the de Rham pairing

$$\langle\ ,\ \rangle : \mathbf{D}_{\mathrm{dR},F}(V)\times\mathbf{D}_{\mathrm{dR},F}(V^*(1)) \longrightarrow L\otimes_{\mathbf{Q}_p}F \longrightarrow \mathbf{C}_p,$$

where $V^* = \mathrm{Hom}_L(V, L)$. Denote by F_0 the maximal unramified subfield of F. Let $\mathbf{B}_{\mathrm{cris}} \subset \mathbf{B}_{\mathrm{dR}}$ be the crystalline period ring and define

$$\mathbf{D}_{\mathrm{cris},F}(V) := (V \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{cris}})^{G_F};$$

this is an $(L \otimes_{\mathbf{Q}_p} F_0)$-module equipped with the action of a semi-linear crystalline Frobenius Φ. If V is a crystalline G_F-representation (i.e., $\dim_{F_0} \mathbf{D}_{\mathrm{cris},F}(V) = \dim_L V$), we have a canonical isomorphism $F \otimes_{F_0} \mathbf{D}_{\mathrm{cris},F}(V) \cong \mathbf{D}_{\mathrm{dR},F}(V)$. Suppose further that

$$\mathbf{D}_{\mathrm{cris},F}(V)^{\Phi=1} = \{0\}.$$

We denote by $\log_p$ the Bloch–Kato logarithm map

$$\log_{F,V} : H^1_{\mathrm{f}}(F, V) \longrightarrow \frac{\mathbf{D}_{\mathrm{dR},F}(V)}{\mathrm{Fil}^0 \mathbf{D}_{\mathrm{dR},F}(V)} \cong \mathrm{Fil}^0 \mathbf{D}_{\mathrm{dR},F}(V^*(1))^\vee,$$

where $H^1_{\mathrm{f}}(F, V) \subset H^1(F, V)$ is the Bloch–Kato finite subspace [BK90, (3.7.2)], which under the above hypothesis agrees with the image of the Bloch–Kato exponential map

$$\exp_{F,V} : \frac{\mathbf{D}_{\mathrm{dR},F}(V)}{\mathrm{Fil}^0 \mathbf{D}_{\mathrm{dR},F}(V)} \longrightarrow H^1(F, V),$$

that we shall denote by $\exp_p$. Also, let $\exp_p^*$ denote the dual exponential map

$$\exp^*_{F,V^*(1)} : H^1(F, V) \longrightarrow \mathrm{Fil}^0 \mathbf{D}_{\mathrm{dR},F}(V)$$

obtained by dualizing $\exp_{F,V^*(1)}$ with respect to the de Rham and local Tate pairings (see e.g. [LZ14, §2.4]).

For the ease of notation, we shall write $\mathbf{D}_{\mathrm{dR}}(V)$ and $\mathbf{D}_{\mathrm{cris}}(V)$ for $\mathbf{D}_{\mathrm{dR},\mathbf{Q}_p}(V)$ and $\mathbf{D}_{\mathrm{cris},\mathbf{Q}_p}(V)$, respectively.

3.2. **Ochiai's map for nearly p-ordinary deformations.** We keep the notations introduced in §2.3 and §2.7; in particular, $\mathfrak{O}$ denotes the ring of integers of finite extension of L of $\mathbf{Q}_p$ and $\mathbb{I}$ is a finite flat extension of $\Lambda_{\mathfrak{O}}^{\mathrm{wt}} = \mathfrak{O}[[\Gamma^{\mathrm{wt}}]]$. We also identify $G_{\mathbf{Q}_p} := \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ with the decomposition group $D_p \subset G_{\mathbf{Q}}$ determined by our fixed embedding $\imath_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$.

Definition 3.1. Let $\mathbb{T}$ be a free $\mathbb{I}$-module of rank 2 equipped with a continuous linear action of $G_{\mathbf{Q}}$. We say that $\mathbb{T}$ is a *p-ordinary deformation* if:

(i) the action of $G_{\mathbf{Q}}$ on $\det(\mathbb{T})$ is given by

$$\Theta^{-2}\varepsilon_{\mathrm{cyc}}^{-1} : G_{\mathbf{Q}} \longrightarrow \mathbb{I}^\times,$$

where $\varepsilon_{\mathrm{cyc}} : G_{\mathbf{Q}} \to \mathbf{Z}_p^\times$ is the p-adic cyclotomic character, viewed as taking values in $\mathbb{I}^\times$ by the inclusion of scalars $\mathbf{Z}_p^\times \subset \mathfrak{O}^\times \subset \Lambda_{\mathfrak{O}}^{\mathrm{wt},\times} \subset \mathbb{I}^\times$;

(ii) there exists a filtration as $G_{\mathbf{Q}_p}$-modules

$$0 \longrightarrow \mathscr{F}^+\mathbb{T} \longrightarrow \mathbb{T} \longrightarrow \mathscr{F}^-\mathbb{T} \longrightarrow 0 \tag{3.1}$$

with $\mathscr{F}^\pm\mathbb{T}$ free of rank 1 over $\mathbb{I}$, and with the action on $\mathscr{F}^+\mathbb{T}$ being unramified.

Fix a p-ordinary deformation $\mathbb{T}$ as in Definition 3.1. Let Γ_{cyc} be the Galois group of the cyclotomic $\mathbf{Z}_p$-extension of $\mathbf{Q}_p$, and let Λ_{cyc} be the free $\mathbf{Z}_p[[\Gamma_{\mathrm{cyc}}]]$-module of rank 1 where $G_{\mathbf{Q}_p}$ acts through the tautological character $G_{\mathbf{Q}_p} \twoheadrightarrow \Gamma_{\mathrm{cyc}} \hookrightarrow \mathbf{Z}_p[[\Gamma_{\mathrm{cyc}}]]^\times$.

Definition 3.2. Set $\mathcal{I} := \mathbb{I}\widehat{\otimes}\mathbf{Z}_p[[\Gamma_{\mathrm{cyc}}]]$. The *nearly p-ordinary deformation* associated to a p-ordinary deformation $\mathbb{T}$ is the $\mathcal{I}$-module

$$\mathcal{T} := \mathbb{T}\widehat{\otimes}_{\mathbf{Z}_p}\Lambda_{\mathrm{cyc}}$$

equipped with the diagonal $G_{\mathbf{Q}_p}$-action. From (3.1), $\mathcal{T}$ fits in an exact sequence of $\mathcal{I}[G_{\mathbf{Q}_p}]$-modules

$$0 \longrightarrow \mathscr{F}^+\mathcal{T} \longrightarrow \mathcal{T} \longrightarrow \mathscr{F}^-\mathcal{T} \longrightarrow 0$$

with $\mathscr{F}^\pm\mathcal{T} := \mathscr{F}^\pm\mathbb{T}\widehat{\otimes}_{\mathbf{Z}_p}\Lambda_{\mathrm{cyc}}$.

Let $\epsilon : \Gamma_{\mathrm{cyc}} \simeq 1 + p\mathbf{Z}_p$ be the isomorphism induced by the p-adic cyclotomic character. We denote by $\mathcal{X}^a_{\mathfrak{O}}(\Gamma_{\mathrm{cyc}})$ the set of continuous characters $\sigma : \Gamma_{\mathrm{cyc}} \to \overline{\mathbf{Q}}_p^\times$ of the form $\sigma = \epsilon^{w_\sigma}\sigma_o$ for some $w_\sigma \in \mathbf{Z}$, called the *weight* of σ, and some finite order character σ_o. We say that σ has *conductor* p^n if σ_o has conductor p^n when seen as a character of $\mathbf{Z}_p^\times$.

Recall the set $\mathcal{X}^a_{\mathfrak{O}}(\mathbb{I})$ from Definition 2.3. For every pair $(\nu,\sigma) \in \mathcal{X}^a_{\mathfrak{O}}(\mathbb{I}) \times \mathcal{X}^a_{\mathfrak{O}}(\Gamma_{\mathrm{cyc}})$ let $\mathcal{O}_{\nu,\sigma}$ be the extension of $\mathcal{O}_\nu$ generated by the values of σ, and let $\mathcal{O}_{\nu,\sigma}(\sigma)$ be the free $\mathcal{O}_{\nu,\sigma}$-module of rank 1 where $G_{\mathbf{Q}_p}$ acts via the character σ. For a p-ordinary deformation $\mathbb{T}$ define

$$\begin{aligned} T_\nu &:= \mathbb{T}\otimes_{\mathbb{I},\nu}\mathcal{O}_\nu, & V_\nu &:= T_\nu\otimes_{\mathbf{Z}_p}\mathbf{Q}_p,\\ T_{\nu,\sigma} &:= \mathcal{T}\otimes_{\mathcal{I},(\nu,\sigma)}\mathcal{O}_{\nu,\sigma}(\sigma), & V_{\nu,\sigma} &:= T_{\nu,\sigma}\otimes_{\mathbf{Z}_p}\mathbf{Q}_p,\\ \mathscr{F}^\pm T_{\nu,\sigma} &:= \mathscr{F}^\pm\mathcal{T}\otimes_{\mathcal{I},(\nu,\sigma)}\mathcal{O}_{\nu,\sigma}(\sigma), & \mathscr{F}^\pm V_{\nu,\sigma} &:= \mathscr{F}^\pm T_{\nu,\sigma}\otimes_{\mathbf{Z}_p}\mathbf{Q}_p, \end{aligned}$$

and for every finite extension F of $\mathbf{Q}_p$ let

$$\mathrm{Sp}_{\nu,\sigma} : H^1(F,\mathscr{F}^+\mathcal{T}) \longrightarrow H^1(F,\mathscr{F}^+T_{\nu,\sigma}) \longrightarrow H^1(F,\mathscr{F}^+V_{\nu,\sigma}) \tag{3.2}$$

be the induced maps on cohomology.

Definition 3.3. Let $\mathbb{T}$ be a p-ordinary deformation, and set

$$\mathbb{D} := (\mathscr{F}^+\mathbb{T}\widehat{\otimes}_{\mathbf{Z}_p}\widehat{\mathbf{Z}}_p^{\mathrm{ur}})^{G_{\mathbf{Q}_p}}, \tag{3.3}$$

where the $G_{\mathbf{Q}_p}$-action on $\mathscr{F}^+\mathbb{T}\widehat{\otimes}_{\mathbf{Z}_p}\widehat{\mathbf{Z}}_p^{\mathrm{ur}}$ is the diagonal one. Also set

$$\mathcal{D} := \mathbb{D}\widehat{\otimes}_{\mathbf{Z}_p}\mathbf{Z}_p[[\Gamma_{\mathrm{cyc}}]].$$

Let F be a finite unramified extension of $\mathbf{Q}_p$ with ring of integers $\mathcal{O}_F$. Since $\mathscr{F}^+V_\nu$ is an unramified $G_{\mathbf{Q}_p}$-representation, we have $\mathbf{D}_{\mathrm{dR}}(\mathscr{F}^+V_\nu) \cong (\mathscr{F}^+V_\nu\otimes\widehat{\mathbf{Q}}_p^{\mathrm{ur}})^{G_{\mathbf{Q}_p}}$. Let

$$\mathrm{Sp}_\nu : \mathbb{D}\otimes_{\mathbf{Z}_p}\mathcal{O}_F \longrightarrow \mathbf{D}_{\mathrm{dR},F}(\mathscr{F}^+V_\nu) \tag{3.4}$$

be the specialization map induced by the G_F-invariants of the natural map $\mathscr{F}^+\mathbb{T}\widehat{\otimes}_{\mathbf{Z}_p}\widehat{\mathbf{Z}}_p^{\mathrm{ur}} \to \mathscr{F}^+T_\nu\widehat{\otimes}_{\mathbf{Z}_p}\widehat{\mathbf{Z}}_p^{\mathrm{ur}}$. Fix a compatible system $\zeta = (\zeta_{p^n})_n$ of p-power roots of unity; this defines a basis vector e of $\mathbf{Q}_p(1)$ and an element $t \in \mathbf{B}_{\mathrm{dR}}$ (Fontaine's p-adic analogue of $2\pi i$), so that $\delta_{\mathbf{Q}_p(1)} := t^{-1}\otimes e$ gives a $\mathbf{Q}_p$-basis of $\mathbf{D}_{\mathrm{dR}}(\mathbf{Q}_p(1))$. For $\sigma \in \mathcal{X}^a_{\mathfrak{O}}(\Gamma_{\mathrm{cyc}})$ of weight w and conductor p^n, let $\mathrm{Sp}_\sigma : \mathbf{Z}_p[[\Gamma_{\mathrm{cyc}}]] \to \mathbf{D}_{\mathrm{dR}}(K_\sigma(\sigma))$ be defined by

$$\begin{aligned} \mathrm{Sp}_\sigma : \mathbf{Z}_p[[\Gamma_{\mathrm{cyc}}]] &\longrightarrow \mathbf{D}_{\mathrm{dR}}(\mathbf{Q}_p(w))\otimes_{\mathbf{Q}_p}\mathbf{Q}_p(\mu_{p^n}) \cong \mathbf{D}_{\mathrm{dR}}(\mathbf{Q}_p(w)\otimes_{\mathbf{Q}_p}\mathbf{Q}_p[(\mathbf{Z}/p^n\mathbf{Z})^\times])\\ &\xrightarrow{\sigma_0} \mathbf{D}_{\mathrm{dR}}(\mathbf{Q}_p(w)\otimes_{\mathbf{Q}_p}K_\sigma(\sigma_0)) = \mathbf{D}_{\mathrm{dR}}(K_\sigma(\sigma)), \end{aligned}$$

where the first arrow is given by $g \mapsto \delta_{\mathbf{Q}_p(1)}^{\otimes w} \otimes \zeta_{p^n}^g$ and the isomorphism is given by Shapiro's lemma. For every pair $(\nu,\sigma) \in \mathcal{X}_{\mathfrak{D}}^a(\mathbb{I}) \times \mathcal{X}_{\mathfrak{D}}^a(\Gamma_{\mathrm{cyc}})$ we thus obtain the specialization map

$$\begin{aligned} \mathrm{Sp}_{\nu,\sigma} : \mathcal{D} \otimes_{\mathbf{Z}_p} \mathcal{O}_F &\xrightarrow{\mathrm{Sp}_\nu \otimes 1} \mathbf{D}_{\mathrm{dR},F}(\mathscr{F}^+V_\nu) \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{Z}_p[[\Gamma_{\mathrm{cyc}}]] \\ &\xrightarrow{1 \otimes \mathrm{Sp}_\sigma} \mathbf{D}_{\mathrm{dR},F}(\mathscr{F}^+V_\nu) \otimes_{\mathbf{Q}_p} \mathbf{D}_{\mathrm{dR}}(K_\sigma(\sigma)) \cong \mathbf{D}_{\mathrm{dR},F}(\mathscr{F}^+V_{\nu,\sigma}). \end{aligned} \tag{3.5}$$

Theorem 3.4. *Let $\mathbb{T}$ be a p-ordinary deformation, and define*

$$\mathcal{J} := (\Psi(\mathrm{Fr}_p) - 1, \gamma_o - 1) \subseteq \mathcal{I},$$

where $\Psi : G_{\mathbf{Q}_p} \to \mathbb{I}^\times$ is the unramified character by which $G_{\mathbf{Q}_p}$ acts on $\mathscr{F}^+\mathbb{T}$ and $\gamma_o \in \Gamma_{\mathrm{cyc}}$ be a topological generator. Then for every finite unramified extension F of $\mathbf{Q}_p$ there exists an injective $\mathcal{I}$-linear map

$$\mathcal{E}_F^{\Gamma_{\mathrm{cyc}}} : \mathcal{J}(\mathcal{D} \otimes_{\mathbf{Z}_p} \mathcal{O}_F) \longrightarrow H^1(F, \mathscr{F}^+\mathcal{T})$$

with pseudo-null cokernel and such that for every $\nu \in \mathcal{X}_{\mathfrak{D}}^a(\mathbb{I})$ and $\sigma \in \mathcal{X}_{\mathfrak{D}}^a(\Gamma_{\mathrm{cyc}})$ of weight $w > 0$ and conductor p^n, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{J}(\mathcal{D} \otimes_{\mathbf{Z}_p} \mathcal{O}_F) & \xrightarrow{\mathcal{E}_F^{\Gamma_{\mathrm{cyc}}}} & H^1(F, \mathscr{F}^+\mathcal{T}) \\ \downarrow{\scriptstyle \mathrm{Sp}_{\nu,\sigma}} & & \downarrow{\scriptstyle \mathrm{Sp}_{\nu,\sigma}} \\ \mathbf{D}_{\mathrm{dR},F}(\mathscr{F}^+V_{\nu,\sigma}) & \longrightarrow & H^1(F, \mathscr{F}^+V_{\nu,\sigma}), \end{array}$$

where the bottom horizontal map is given by

$$(-1)^{w-1}(w-1)! \cdot \exp_p \times \begin{cases} \big(1 - \frac{p^{w-1}}{\Psi_\nu(\mathrm{Fr}_p)}\big)\big(1 - \frac{\Psi_\nu(\mathrm{Fr}_p)}{p^w}\big)^{-1} & \text{if } n = 0; \\ \mathfrak{g}(\sigma_o^{-1})\big(\frac{p^{w-1}}{\Psi_\nu(\mathrm{Fr}_p)}\big)^n & \text{if } n \geqslant 1, \end{cases}$$

with $\Psi_\nu(\mathrm{Fr}_p) \in F_\nu$ the image of $\Psi(\mathrm{Fr}_p) \in \mathbb{I}$ under ν.

Proof. See [Och03, Prop. 5.3]. □

3.3. **Going up the unramified $\mathbf{Z}_p$-extension.** Let F be a finite unramified extension of $\mathbf{Q}_p$, and let F_∞/F be an infinite unramified p-adic Lie extension with Galois group U (so U is isomorphic to $\mathbf{Z}_p^\times \times \Delta$ with Δ finite). Write $F_\infty = \bigcup_{m \geqslant 0} F_m$ with F_0/F a finite extension and F_m/F_0 having degree p^m. Set $U_m := \mathrm{Gal}(F_\infty/F_m)$. Let $y_m : \mathcal{O}_{F_m} \to \mathcal{O}_{F_m}[U/U_m]$ be the $\mathbf{Z}_p$-linear map defined by

$$y_m(x) = \sum_{\sigma \in U/U_m} x^\sigma [\sigma^{-1}],$$

and let $\mathcal{S}_m \subset \mathcal{O}_{F_m}[U/U_m]$ be the image of y_m.

For any $x \in \mathcal{O}_{F_{m+1}}$, it is readily seen that the image of $y_{m+1}(x)$ in $\mathcal{O}_{F_{m+1}}[U/U_m]$ agrees with the image of $y_m(\mathrm{Tr}_{F_{m+1}/F_m}(x))$, and hence passing to the inverse limits with respect to the trace maps, we obtain an isomorphism

$$\varprojlim_m y_m : \varprojlim_m \mathcal{O}_{F_m} \xrightarrow{\simeq} \mathcal{S}_\infty := \varprojlim_m \mathcal{S}_m. \tag{3.6}$$

Let $\widehat{\mathcal{O}}_{F_\infty}$ be the completion of the ring of integers of F_∞.

Proposition 3.5. *The module $\mathcal{S}_\infty$ is free of rank 1 over $\mathbf{Z}_p[[U]]$, and it is identified with*

$$\{g \in \widehat{\mathcal{O}}_{F_\infty}[[U]] : \ g^u = [u]g \ \text{ for all } u \in U\},$$

where g^u denotes the action of u on the coefficients $\widehat{\mathcal{O}}_{F_\infty}$ and $[u]g$ denotes the action of u via multiplication as group-like element.

Proof. See [LZ14, Prop. 3.2, Prop. 3.6]. □

3.4. **A two-variable regulator map for p-ordinary deformations.** Let $F/\mathbf{Q}_p$ and F_∞/F be unramified extensions as in §3.3, set $L_\infty := F_\infty(\mu_{p^\infty})$, and let $G := \mathrm{Gal}(L_\infty/F) \cong U \times \mathbf{Z}_p^\times$.

As in §3.2, we let $\mathbb{T}$ be a p-ordinary deformation in the sense of Definition 3.1, and let $\Psi : G_{\mathbf{Q}_p} \to \mathbb{I}^\times$ be the unramified character giving the $G_{\mathbf{Q}_p}$-action on the subspace $\mathscr{F}^+\mathbb{T} \subset \mathbb{T}$.

Definition 3.6. An arithmetic prime $\nu \in \mathcal{X}^a_{\mathfrak{D}}(\mathbb{I})$ is *exceptional* for $\mathbb{T}$ if ν has weight $(k, \varepsilon) = (2, \mathbb{1})$, and $\Psi_\nu(\mathrm{Fr}_p) = 1$.

For any finite extension F' of $\mathbf{Q}_p$ contained in L_∞ and any subquotient $\mathbb{M}$ of $\mathbb{T}$ define

$$H^1_{\mathrm{Iw}}(L_\infty/F', \mathbb{M}) := \varprojlim_L H^1(L, \mathbb{M}),$$

where L runs over the finite extensions of F' contained in L_∞, and the transition maps are given by corestriction. By Shapiro's lemma, we have $H^1_{\mathrm{Iw}}(L_\infty/F, \mathscr{F}^+\mathbb{T}) \cong H^1(F, \mathbb{T}\widehat{\otimes}_{\mathbf{Z}_p}\mathbf{Z}_p[[G]])$, and in the same manner as in (3.2) and (3.5), for every $\nu \in \mathcal{X}^a_{\mathbb{I}}(\mathbb{I})$ and Hodge–Tate character ϕ of G we have specialization maps

$$\mathrm{Sp}_{\nu,\phi} : H^1_{\mathrm{Iw}}(L_\infty/F, \mathscr{F}^+\mathbb{T}) \longrightarrow H^1(F, \mathscr{F}^+V_{\nu,\phi})$$

and

$$\mathrm{Sp}_{\nu,\phi} : \mathbb{D}\widehat{\otimes}_{\mathbf{Z}_p}\mathbf{Z}_p[[G]] \longrightarrow \mathbf{D}_{\mathrm{dR},F}(\mathscr{F}^+V_{\nu,\phi}).$$

Note that if ϕ has weight Hodge–Tate weight[2] $w > 0$, then $\mathrm{Fil}^0\mathbf{D}_{\mathrm{dR},F}(\mathscr{F}^+V_{\nu,\phi}) = \{0\}$, and the Bloch–Kato logarithm becomes an isomorphism

$$\log_p : H^1_{\mathrm{f}}(F, \mathscr{F}^+V_{\nu,\phi}) = H^1(F, \mathscr{F}^+V_{\nu,\phi}) \xrightarrow{\simeq} \mathbf{D}_{\mathrm{dR},F}(\mathscr{F}^+V_{\nu,\phi}).$$

Theorem 3.7. *Let $\mathbb{T}$ be a p-ordinary deformation, and set $\lambda := \Psi(\mathrm{Fr}_p) - 1 \in \mathbb{I}$. Then there is an injective $\mathbb{I}[[G]]$-linear map*

$$\mathcal{L}^G : H^1_{\mathrm{Iw}}(L_\infty/F, \mathscr{F}^+\mathbb{T}) \longrightarrow \lambda^{-1} \cdot \mathcal{J}(\mathbb{D}\widehat{\otimes}_{\mathbf{Z}_p}\widehat{\mathcal{O}}_{F_\infty}[[G]])$$

such that for every non-exceptional $\nu \in \mathcal{X}^a_{\mathfrak{D}}(\mathbb{I})$ and every Hodge–Tate character $\phi : G \to L^\times$ of conductor p^n and Hodge–Tate weight $w > 0$, the following diagram commutes

$$\begin{array}{ccc} H^1_{\mathrm{Iw}}(L_\infty/F, \mathscr{F}^+\mathbb{T}) & \xrightarrow{\mathcal{L}^G} & \lambda^{-1} \cdot \mathcal{J}(\mathbb{D}\widehat{\otimes}_{\mathbf{Z}_p}\widehat{\mathcal{O}}_{F_\infty}[[G]]) \\ \downarrow{\scriptstyle \mathrm{Sp}_{\nu,\phi}} & & \downarrow{\scriptstyle \mathrm{Sp}_{\nu,\phi}} \\ H^1(F, \mathscr{F}^+V_{\nu,\phi}) & \longrightarrow & \mathbf{D}_{\mathrm{dR},F}(\mathscr{F}^+V_{\nu,\phi}), \end{array}$$

where the bottom horizontal map is given by

$$\frac{(-1)^{w-1}}{(w-1)!} \cdot \log_p \cdot \begin{cases} \big(1 - \frac{\Psi_\nu(\mathrm{Fr}_p)}{p^w}\big)\big(1 - \frac{p^{w-1}}{\Psi_\nu(\mathrm{Fr}_p)}\big)^{-1} & \text{if } n = 0, \\ \varepsilon(\phi)\Psi_\nu(\mathrm{Fr}_p)^n & \text{if } n \geqslant 1. \end{cases}$$

[2]In this paper, we adopt the convention that the Hodge–Tate weight of $\varepsilon_{\mathrm{cyc}}$ is $+1$. Thus the Hodge–Tate weights of a p-adic de Rham representation V are the integers w such that $\mathrm{Fil}^{-w}\mathbf{D}_{\mathrm{dR}}(V) \supsetneq \mathrm{Fil}^{-w+1}\mathbf{D}_{\mathrm{dR}}(V)$.

Proof. For each $m \geqslant 0$, let

$$\mathcal{E}_{F_m}^{\Gamma_{\mathrm{cyc}}} : \mathcal{J}(\mathcal{D} \otimes_{\mathbf{Z}_p} \mathcal{O}_{F_m}) \longrightarrow H^1(F_m, \mathscr{F}^+\mathcal{T})$$

be the big exponential map of Theorem 3.4 for the unramified extension $F_m/\mathbf{Q}_p$, and using (3.6) define

$$\mathcal{E}^G := \varprojlim_m \mathcal{E}_{F_m}^{\Gamma_{\mathrm{cyc}}} : \mathcal{J}(\mathcal{D}\widehat{\otimes}_{\mathbf{Z}_p}\mathcal{S}_\infty) \longrightarrow H^1_{\mathrm{Iw}}(F_\infty/F, \mathscr{F}^+\mathcal{T}).$$

By Shapiro's lemma, we view $\mathcal{E}^G$ as taking values in $H^1_{\mathrm{Iw}}(L_\infty/F, \mathscr{F}^+\mathbb{T})$. Since each $\mathcal{E}_{F_m}^{\Gamma_{\mathrm{cyc}}}$ has cokernel killed by λ, it is readily seen that $\mathcal{E}^G$ is an injective $\mathbb{I}[[G]]$-linear map with cokernel killed by λ, and hence given any $\mathfrak{Y}_\infty \in H^1_{\mathrm{Iw}}(L_\infty/F, \mathscr{F}^+\mathbb{T})$, the product

$$\mathcal{L}^G(\mathfrak{Y}_\infty) := \lambda^{-1} \cdot (\mathcal{E}^G)^{-1}(\lambda \cdot \mathfrak{Y}_\infty)$$

is a well-defined element in

$$\lambda^{-1} \cdot \mathcal{J}(\mathcal{D}\widehat{\otimes}_{\mathbf{Z}_p}\mathcal{S}_\infty) \hookrightarrow \lambda^{-1} \cdot \mathcal{J}(\mathcal{D}\widehat{\otimes}_{\mathbf{Z}_p}\widehat{\mathcal{O}}_{F_\infty}[[U]]) \simeq \lambda^{-1} \cdot \mathcal{J}(\mathbb{D}\widehat{\otimes}_{\mathbf{Z}_p}\widehat{\mathcal{O}}_{F_\infty}[[G]]).$$

Thus constructed, the claimed interpolation properties of $\mathcal{L}^G$ for each non-exceptional $\nu \in \mathcal{X}^a_{\mathfrak{I}}(\mathbb{I})$ follow as in [LZ14, Thm. 4.15]. □

Next we consider the specialization of the map $\mathcal{L}^G$ of Theorem 3.7 at Hodge–Tate characters of ϕ of G of weight $w \leqslant 0$.

Definition 3.8. Let $\mathbf{f} \in \mathbb{I}[[q]]$ be an ordinary $\mathbb{I}$-adic newform of tame level N (prime to p). We say that an arithmetic prime $\nu \in \mathcal{X}^a_{\mathfrak{I}}(\mathbb{I})$ is *p-old* if f_ν is the p-stabilization of a p-ordinary newform of level N.

If $\nu \in \mathcal{X}^a_{\mathfrak{I}}(\mathbb{I})$ has weight $(k, \mathbb{1})$ with $k > 2$, then ν is p-old (see [How07b, Lem. 2.1.5]). Note also that any p-old arithmetic prime is necessarily non-exceptional. For any p-old $\nu \in \mathcal{X}^a_{\mathfrak{I}}(\mathbb{I})$ and any Hodge–Tate character ϕ of G of weight $w \leqslant 0$, the Bloch–Kato dual exponential map becomes an isomorphism

$$\exp_p^* : H^1(F, \mathscr{F}^+V_{\nu,\phi}) \xrightarrow{\simeq} \mathrm{Fil}^0\mathbf{D}_{\mathrm{dR},F}(\mathscr{F}^+V_{\nu,\phi}) = \mathbf{D}_{\mathrm{dR},F}(\mathscr{F}^+V_{\nu,\phi}).$$

Corollary 3.9. *Let $\nu \in \mathcal{X}^a_{\mathfrak{I}}(\mathbb{I})$ be a p-old arithmetic prime. If $\phi : G \to L^\times$ is a Hodge–Tate character of weight $w \leqslant 0$ and conductor p^n, then the following diagram commutes*

$$\begin{array}{ccc} H^1_{\mathrm{Iw}}(L_\infty/F, \mathscr{F}^+\mathbb{T}) & \xrightarrow{\mathcal{L}^G} & \lambda^{-1} \cdot \mathcal{J}(\mathbb{D}\widehat{\otimes}_{\mathbf{Z}_p}\widehat{\mathcal{O}}_{F_\infty}[[G]]) \\ \downarrow{\scriptstyle \mathrm{Sp}_{\nu,\phi}} & & \downarrow{\scriptstyle \mathrm{Sp}_{\nu,\phi}} \\ H^1(F, \mathscr{F}^+V_{\nu,\phi}) & \longrightarrow & \mathbf{D}_{\mathrm{dR},F}(\mathscr{F}^+V_{\nu,\phi}), \end{array}$$

where the bottom horizontal map is given by

$$(-w)! \cdot \exp_p^* \cdot \begin{cases} \big(1 - \frac{\Psi_\nu(\mathrm{Fr}_p)}{p^w}\big)\big(1 - \frac{p^{w-1}}{\Psi_\nu(\mathrm{Fr}_p)}\big)^{-1} & \text{if } n = 0, \\ \varepsilon(\phi)\Psi_\nu(\mathrm{Fr}_p)^n & \text{if } n \geqslant 1. \end{cases}$$

Proof. Since ν is non-exceptional, the composition of the map $\mathcal{L}^G$ of Theorem 3.7 with the specialization map (3.4) at ν factors through $H^1_{\mathrm{Iw}}(L_\infty/F, \mathscr{F}^+\mathbb{T}) \to H^1_{\mathrm{Iw}}(L_\infty/F, \mathscr{F}^+\mathbb{T}) \otimes_{\mathbb{I}} F_\nu$ giving rise to an $F_\nu[[G]]$-linear map

$$\mathcal{L}^G_\nu : H^1_{\mathrm{Iw}}(L_\infty/F, \mathscr{F}^+\mathbb{T}) \otimes_{\mathbb{I}} F_\nu \longrightarrow \mathbf{D}_{\mathrm{dR},F}(\mathscr{F}^+V_\nu) \otimes_{\mathbf{Z}_p} \widehat{\mathcal{O}}_{F_\infty}[[G]].$$

By Theorem 3.7, this map enjoys the same interpolation properties at a dense set of characters of G as the restriction via

$$H^1_{\mathrm{Iw}}(L_\infty/F, \mathscr{F}^+\mathbb{T}) \otimes_{\mathbb{I}} F_\nu \hookrightarrow H^1_{\mathrm{Iw}}(L_\infty/F, \mathscr{F}^+V_\nu)$$

of the map $\mathcal{L}^G_V$ constructed in [LZ14, Thm. 4.7] for $V = \mathscr{F}^+V_\nu$. (Note that since ν is assumed to be p-old, $\mathscr{F}^+V_\nu$ is a "good crystalline" G_F-representation in the sense of [LZ14, Def. 4.1].) Since $\mathcal{L}^G_{\mathscr{F}^+V_\nu}$ is uniquely determined by its values at such characters (for every given class in $H^1_{\mathrm{Iw}}(L_\infty/F, \mathscr{F}^+V_\nu)$), the result follows from [LZ14, Thm. 4.15]. □

4. Big Heegner points

Fix a prime $p > 3$, and let $f = \sum_{n=1}^\infty a_n q^n \in S_k(\Gamma_1(N))$ be a p-ordinary newform of weight $k \geqslant 1$, level N prime to p, and nebentypus ε_f. Let K be an imaginary quadratic field as in §2.5. (However, note that the assumption that p splits in K will not needed in this section.) Let L be a finite extension of $\mathbf{Q}_p$ with ring of integers $\mathfrak{O}$ containing the Fourier coefficients of f. In this section, we briefly recall Howard's construction of big Heegner points associated to the ordinary $\mathbb{I}$-adic newform passing through f.

4.1. **Galois representations associated to Hida families.** Denote by $\widetilde{X}_{s/\mathbf{Q}}$ the compactified modular curve whose non-cuspidal points classify isomorphism classes of triples (E, t_N, t_p) with:

- E an elliptic curve over an arbitrary $\mathbf{Q}$-scheme S;
- t_N a point of E of exact order N;
- t_p a point of E of exact order p^s.

For any field extension $M/\mathbf{Q}$, set

$$\widetilde{J}_s(M) := \mathrm{Jac}(\widetilde{X}_s)(M) \otimes_{\mathbf{Z}} \mathfrak{O},$$

where $\mathrm{Jac}(\widetilde{X}_s)$ is the Jacobian variety of $\widetilde{X}_s$. Denote by $\mathfrak{h}_s$ the $\mathfrak{O}$-algebra generated by the Hecke operators T_ℓ for $\ell \nmid Np$), the operators U_ℓ for $\ell \mid Np$, and the diamond operators $\langle a \rangle_{Np}$ for $a \in (\mathbf{Z}/Np^s\mathbf{Z})^\times$, acting on $\widetilde{J}_s(\mathbf{C})$ by Albanese functoriality, and let

$$e^{\mathrm{ord}} := \lim_{m \to \infty} U_p^{m!}$$

be Hida's ordinary projector. Following the convention in [How07b, §2.1], we make $\mathfrak{h}_s$ into a $\mathfrak{O}[[\mathbf{Z}_p^\times]]$-algebra via $[z] \mapsto \langle z \rangle_p$, where $[z] \in \mathfrak{O}[[\mathbf{Z}_p^\times]]^\times$ is the group-like element corresponding to $z \in \mathbf{Z}_p^\times$ and $\langle \cdot \rangle_p$ denotes the p-part of the diamond operator $\langle \cdot \rangle_{Np}$. By [Hid86b, Thm. 3.1], the algebra $\mathfrak{h}^{\mathrm{ord}} := \varprojlim_s e^{\mathrm{ord}} \mathfrak{h}_s$ is finite flat over $\Lambda^{\mathrm{wt}}_{\mathfrak{O}}$; in particular, $\mathfrak{h}^{\mathrm{ord}}$ is a semi-local ring equal to the product of its localizations at its maximal ideals. Our fixed newform f defines an algebra homomorphism $\lambda_f : \mathfrak{h}^{\mathrm{ord}} \to \mathfrak{O}$, and we let $\mathfrak{h}^{\mathrm{ord}}_{\mathfrak{m}}$ be the direct summand of $\mathfrak{h}^{\mathrm{ord}}$ through which λ_f factors.

Definition 4.1. Let α_p and β_p be the roots of the Hecke polynomial $X^2 - a_p X + \varepsilon_f(p)p^{k-1}$. We say that f is *regular* at p if $\alpha_p \neq \beta_p$.

Of course, since f is assumed to be ordinary at p, it can be non-regular at p only if $k = 1$.

Lemma 4.2. *Assume that either:*

(a) $k \geqslant 2$;
(b) $k = 1$ *and* f *is regular at* p.

Then the localization of $\mathfrak{h}^{\mathrm{ord}}_{\mathfrak{m}}$ *at* $\ker(\lambda_f)$ *is a discrete valuation ring.*

Proof. In case (a), this is a classical result of Hida (see [Hid86a, Cor. 1.4]). The result in case (b) is recent work of Bellaïche–Dimitrov [BD16, Thm. 1.1]. □

Assume from now on that one of the conditions in Lemma 4.2 holds. Thus there is a unique minimal prime $\mathfrak{a} \subset \mathfrak{h}_{\mathfrak{m}}^{\mathrm{ord}}$ containing $\ker(\lambda_f)$, and we set

$$\mathbb{I} := \mathfrak{h}_{\mathfrak{m}}^{\mathrm{ord}}/\mathfrak{a}.$$

For each $i \in \mathbf{Z}/(p-1)\mathbf{Z}$, let e_i be the idempotent of $\mathfrak{O}[[\mathbf{Z}_p^\times]]$ projecting onto the ω^i-isotypical component for the action of $(\mathbf{Z}/p\mathbf{Z})^\times \subset \mathbf{Z}_p^\times$, and note that $\mathfrak{h}_{\mathfrak{m}}^{\mathrm{ord}} = e_{k-2}\mathfrak{h}_{\mathfrak{m}}^{\mathrm{ord}}$. Letting $\mathbf{a}_n \in \mathbb{I}$ be the image of T_n, the formal q-expansion

$$\mathbf{f} = \sum_{n=1}^{\infty} \mathbf{a}_n q^n \in \mathbb{I}[[q]]$$

is an ordinary $\mathbb{I}$-adic newform of tame level N and character $\varepsilon_f \omega^{k-2}$ in the sense of Definition 2.4.

Let κ_L be the residue field of L and denote by $\bar{\rho}_f : G_{\mathbf{Q}} \to \mathrm{GL}_2(\kappa_L)$ the semi-simple residual representation associated with f.

Theorem 4.3. *Assume that $\bar{\rho}_f$ is irreducible and p-distinguished. Then the following hold:*

(1) *The module*

$$\mathbf{T} := \left(\varprojlim_s e^{\mathrm{ord}}(\mathrm{Ta}_p(J_s) \otimes_{\mathbf{Z}_p} \mathfrak{O}) \right) \otimes_{\mathfrak{h}^{\mathrm{ord}}} \mathbb{I}$$

is free of rank 2 over $\mathbb{I}$, and the resulting Galois representation

$$\rho_{\mathbf{f}} : G_{\mathbf{Q}} \longrightarrow \mathrm{Aut}_{\mathbb{I}}(\mathbf{T}) \simeq \mathrm{GL}_2(\mathbb{I})$$

is unramified outside Np with

$$\mathrm{trace}\ \rho_{\mathbf{f}}(\mathrm{Fr}_\ell^{-1}) = \mathbf{a}_\ell, \qquad \det \rho_{\mathbf{f}}(\mathrm{Fr}_\ell^{-1}) = \varepsilon_f(\ell)[\ell]\ell,$$

for all $\ell \nmid Np$, where Fr_ℓ^{-1} is an arithmetic Frobenius.

(2) *There is an exact sequence of $\mathbb{I}[G_{\mathbf{Q}_p}]$-modules*

$$0 \longrightarrow \mathscr{F}^+\mathbf{T} \longrightarrow \mathbf{T} \longrightarrow \mathscr{F}^-\mathbf{T} \longrightarrow 0 \tag{4.1}$$

with $\mathscr{F}^\pm\mathbf{T} \simeq \mathbb{I}$, and with the action of $G_{\mathbf{Q}_p}$ on $\mathscr{F}^-\mathbf{T}$ given by the unramified character $\alpha : G_{\mathbf{Q}_p} \to \mathbb{I}^\times$ sending Fr_p^{-1} to $\mathbf{a}_p$.

Proof. Part (1) follows from [MT90, Thm. 7], and part (2) from [Wil88, Thm. 2.2.2]. □

4.2. **Howard's big Heegner points.** Fix a positive integer c_o prime to Np. For $n \geqslant s$, the CM points $x_{c_o p^n} \in \mathrm{Ig}(N)(\mathbf{C})$ constructed in §2.5 descend to points $P_{c_o p^n, s} \in \widetilde{X}_s(\widetilde{H}_{c_o p^n}(\mu_{p^s}))$, where $\widetilde{H}_{c_o p^n}$ is the composition of $H_{c_o p^n}$ with the ray class field of K of conductor $\mathfrak{N}$.

Proposition 4.4. *The following hold:*

(1) *Let $n \geqslant s > 0$. For all $\sigma \in \mathrm{Gal}(\widetilde{H}_{c_o p^n}(\mu_{p^s})/\widetilde{H}_{c_o p^n})$, we have*

$$P_{c_o p^n, s}^{\sigma} = \langle \vartheta(\sigma) \rangle_p \cdot P_{c_o p^n, s},$$

where $\vartheta : \mathrm{Gal}(\widetilde{H}_{c_o p^n}(\mu_{p^s})/\widetilde{H}_{c_o p^n}) \to \mathbf{Z}_p^\times/\{\pm 1\}$ is such that $\vartheta^2 = \varepsilon_{\mathrm{cyc}}$.

(2) *If $n \geqslant s > 1$, then*

$$\sum_{\sigma \in \mathrm{Gal}(\widetilde{H}_{c_0p^n}(\mu_{p^s})/\widetilde{H}_{c_0p^{n-1}}(\mu_{p^s}))} \widetilde{\alpha}_s(P^{\sigma}_{c_0p^n,s}) = U_p \cdot P_{c_0p^n,s-1},$$

where $\widetilde{\alpha}_s : \widetilde{X}_s \to \widetilde{X}_{s-1}$ is the degeneracy map given by $(E, t_N, t_p) \mapsto (E, t_N, p \cdot t_p)$ on non-cuspidal moduli.

(3) *If $n \geqslant s \geqslant 1$, then*

$$\sum_{\sigma \in \mathrm{Gal}(\widetilde{H}_{c_0p^n}(\mu_{p^s})/\widetilde{H}_{c_0p^{n-1}}(\mu_{p^s}))} P^{\sigma}_{c_0p^n,s} = U_p \cdot P_{c_0p^{n-1},s}.$$

Proof. Let X_s be the compactified modular curve for the congruence subgroup $\Gamma_0(N) \cap \Gamma_1(p^s)$, and consider the degeneracy map $\beta_N : \widetilde{X}_s \to X_s$ given by $(E, t_N, t_p) \mapsto (E, C_N, t_p)$ on non-cuspidal moduli, where C_N denotes the cyclic subgroup of $E[N]$ generated by t_N. From the construction of $x_{c_0p^n}$ given in §2.5, it is immediate to see that for $n \geqslant s$ the image $\beta_N(P_{c_0p^n,s})$ agrees with the point $h_{c_0p^{n-s},s} \in X_s(\mathbf{C})$ constructed in [How07b, §2.2], i.e., corresponding to the triple $(A_{c_0p^{n-s},s}, \mathfrak{n}_{c_0p^{n-s},s}, \pi_{c_0p^{n-s},s})$ with:

- $A_{c_0p^{n-s},s}(\mathbf{C}) = \mathbf{C}/\mathcal{O}_{c_0p^n}$;
- $\mathfrak{n}_{c_0p^{n-s},s} = A_{c_0p^{n-s},s}[\mathfrak{N} \cap \mathcal{O}_{c_0p^n}]$;
- $\pi_{c_0p^{n-s},s}$ a generator of the kernel of the cyclic p^s-isogeny $\mathbf{C}/\mathcal{O}_{c_0p^n} \to \mathbf{C}/\mathcal{O}_{c_0p^{n-s}}$.

Thus properties (1), (2), and (3) follow immediately from Corollary 2.2.2, Lemma 2.2.4, and Proposition 2.3.1 of [How07b], respectively. □

Set $\widetilde{L}_{c,s} := \widetilde{H}_{cp^s}(\mu_{p^s})$, and keep the notations from Proposition 4.4. As in [How07b, p. 100], one easily checks that for $t \geqslant 0$ and $\sigma \in \mathrm{Gal}(\widetilde{L}_{c_0p^t,s}/\widetilde{H}_{c_0p^{t+s}})$ we have the equality $\Theta(\sigma) = \langle \vartheta(\sigma) \rangle_p$ as endomorphisms of $e_{k-2}e^{\mathrm{ord}}\widetilde{J}_s(\widetilde{L}_{c_0p^t,s})$, and so (using that U_p has degree p and we are taking ordinary parts) the points $e_{k-2}e^{\mathrm{ord}}P_{c_0p^{t+s},s}$ define classes

$$y_{c_0p^t,s} \in e^{\mathrm{ord}}\widetilde{J}_s(\widetilde{L}_{c_0p^t,s})$$

which satisfy

$$y^{\sigma}_{c_0p^t,s} = \Theta(\sigma) \cdot y_{c_0p^t,s} \tag{4.2}$$

for all $\sigma \in \mathrm{Gal}(\widetilde{L}_{c_0p^t,s}/\widetilde{H}_{c_0p^{t+s}})$.

Definition 4.5. For any $\Lambda^{\mathrm{wt}}_{\mathfrak{O}}$-module M equipped with a linear $G_{\mathbf{Q}}$-action, we let $M^{\dagger}$ denote its twist by the character Θ^{-1}.

Thus (4.2) amounts to the statement that

$$y_{c_0p^t,s} \in H^0(\widetilde{H}_{c_0p^{t+s}}, e^{\mathrm{ord}}\widetilde{J}_s(\widetilde{L}_{c_0p^t,s})^{\dagger}).$$

For any number field F let $\mathfrak{G}_F$ be the Galois group of the maximal extension of F unramified outside the primes above Np. By Proposition 4.4, the image of $y_{c_0p^t,s}$ under the composite map

$$\begin{aligned} H^0(\widetilde{H}_{c_0p^{t+s}}, e^{\mathrm{ord}}\widetilde{J}_s(\widetilde{L}_{c_0p^t,s})^{\dagger}) &\xrightarrow{\mathrm{Cor}} H^0(\widetilde{H}_{c_0p^t}, e^{\mathrm{ord}}\widetilde{J}_s(\widetilde{L}_{c_0p^t,s})^{\dagger}) \\ &\xrightarrow{\mathrm{Kum}} H^1(\mathfrak{G}_{\widetilde{H}_{c_0p^t}}, e^{\mathrm{ord}}\mathrm{Ta}_p(\widetilde{J}_s)^{\dagger}) \end{aligned}$$

defines a class $\mathfrak{X}_{c_0p^t,s}$ satisfying

$$\alpha_{s*}\mathfrak{X}_{c_0p^t,s} = U_p \cdot \mathfrak{X}_{c_0p^t,s-1}$$

under the map

$$\widetilde{\alpha}_{s*} : H^1(\mathfrak{G}_{\widetilde{H}_{c_o p^t}}, e^{\mathrm{ord}}\mathrm{Ta}_p(\widetilde{J}_s)^\dagger) \longrightarrow H^1(\mathfrak{G}_{\widetilde{H}_{c_o p^t}}, e^{\mathrm{ord}}\mathrm{Ta}_p(\widetilde{J}_{s-1})^\dagger)$$

induced by $\widetilde{\alpha}_s : \widetilde{X}_s \to \widetilde{X}_{s-1}$ by Albanese functoriality.

Definition 4.6. Let $c > 0$ be an integer prime to N. The *big Heegner point of conductor c* is the class

$$\mathfrak{X}_c \in H^1(\mathfrak{G}_{\widetilde{H}_c}, \mathbf{T}^\dagger)$$

defined as the image of $\varprojlim_s U_p^{-s} \cdot \mathfrak{X}_{c,s}$ under the natural map

$$\varprojlim_s H^1(\mathfrak{G}_{\widetilde{H}_c}, e^{\mathrm{ord}}\mathrm{Ta}_p(\widetilde{J}_s)^\dagger) \longrightarrow H^1(\mathfrak{G}_{\widetilde{H}_c}, \mathbf{T}^\dagger).$$

By inflation, we shall view $\mathfrak{X}_c$ as a class in $H^1(\widetilde{H}_c, \mathbf{T}^\dagger)$.

As in [How07b, Prop. 2.3.1], it follows easily from Proposition 4.4 that the classes

$$\mathfrak{Z}_{c_o,t} := U_p^{1-t} \cdot \mathfrak{X}_{c_o p^t} \in H^1(\widetilde{H}_{c_o p^t}, \mathbf{T}^\dagger) \tag{4.3}$$

are compatible under the corestriction maps, thus defining a class

$$\mathfrak{Z}_{c_o,\infty} := \varprojlim_t \mathfrak{Z}_{c_o,t} \in H^1_{\mathrm{Iw}}(\widetilde{H}_{c_o p^\infty}/\widetilde{H}_{c_o}, \mathbf{T}^\dagger) = \varprojlim_t H^1(\mathfrak{G}_{\widetilde{H}_{c_o p^t}}, \mathbf{T}^\dagger).$$

We conclude this section by recalling some of the local conditions satisfied by these classes.

Lemma 4.7. *Let F be a finite extension of K, and let v be a prime of F above a prime ℓ dividing (D_K, N). If $\bar{\rho}_f$ is ramified at ℓ, then $H^1(F_v^{\mathrm{ur}}, \mathbf{T}^\dagger)$ is $\mathbb{I}$-torsion free.*

Proof. This is well-known; see e.g. [Büy14, Lem. 3.12]. □

For F a finite extension K, let $\mathrm{Sel}_{\mathrm{Gr}}(F, \mathbf{T}^\dagger) \subset H^1(\mathfrak{G}_F, \mathbf{T}^\dagger)$ be the strict Greenberg Selmer group of [How07b, Def. 2.4.2].

Proposition 4.8. *If $\bar{\rho}_f$ is ramified at every prime ℓ dividing (D_K, N), then $\mathfrak{X}_c \in \mathrm{Sel}_{\mathrm{Gr}}(\widetilde{H}_c, \mathbf{T}^\dagger)$ for all positive integers c prime to N.*

Proof. The proof of [How07b, Prop. 2.4.5] shows that the localization $\mathrm{loc}_v(\mathfrak{X}_c)$ of $\mathfrak{X}_c$ at any place v of H_c lies in the local subspace $H^1_{\mathrm{Gr}}(\widetilde{H}_{c,v}, \mathbf{T}^\dagger) \subset H^1(\widetilde{H}_{c,v}, \mathbf{T}^\dagger)$ defining $\mathrm{Sel}_{\mathrm{Gr}}(\widetilde{H}_c, \mathbf{T}^\dagger)$, except possibly at primes $v \mid \ell \mid N$ which are non-split in K, in which case it is shown that

$$\mathrm{loc}_v(\mathfrak{X}_c) \in \ker\left(H^1(\widetilde{H}_{c,v}, \mathbf{T}^\dagger) \longrightarrow \frac{H^1(\widetilde{H}^{\mathrm{ur}}_{c,v}, \mathbf{T}^\dagger)}{H^1(\widetilde{H}^{\mathrm{ur}}_{c,v}, \mathbf{T}^\dagger)_{\mathrm{tors}}} \right),$$

where $H^1(\widetilde{H}^{\mathrm{ur}}_{c,v}, \mathbf{T}^\dagger)_{\mathrm{tors}} \subset H^1(\widetilde{H}^{\mathrm{ur}}_{c,v}, \mathbf{T}^\dagger)$ is the $\mathbb{I}$-torsion submodule. In light of Lemma 4.7, the result follows. □

5. Explicit reciprocity law

In this section is we prove Theorem 5.3, the main technical result of this paper. We keep the setting introduced at the beginning of Section 4.

5.1. **Regulator map for the anticyclotomic $\mathbf{Z}_p$-extension of K.** Recall the $\mathbb{I}$-adic Hecke character $\boldsymbol{\xi} : K^\times\backslash\mathbb{A}_K^\times \to \mathbb{I}^\times$ introduced in Definition 2.8. With a slight abuse of notation, we also let $\boldsymbol{\xi} : G_K \to \mathbb{I}^\times$ be corresponding Galois character, and set

$$\mathbb{T} := \mathbf{T}|_{G_K} \otimes \Theta^{-1}\boldsymbol{\xi}^{-1}. \tag{5.1}$$

Since $p = \mathfrak{p}\overline{\mathfrak{p}}$ splits in K, by Theorem 4.3 the restriction of $\mathbb{T}$ to a decomposition group at $\mathfrak{p}$ takes the form

$$\mathbb{T}|_{G_{K_\mathfrak{p}}} \ : \ \begin{pmatrix} \alpha^{-1}\varepsilon_f\varepsilon_{\mathrm{cyc}}\Theta\boldsymbol{\xi}^{-1} & * \\ 0 & \alpha\Theta^{-1}\boldsymbol{\xi}^{-1} \end{pmatrix} \tag{5.2}$$

on a suitable $\mathbb{I}$-basis. Since

$$\Psi := \alpha^{-1}\varepsilon_f\varepsilon_{\mathrm{cyc}}\Theta\boldsymbol{\xi}^{-1} \tag{5.3}$$

is an unramified character of $G_{K_\mathfrak{p}}$,the local representation (5.2) is a p-ordinary deformation in the sense of Definition 3.1, and so associated with it we may consider the regulator map $\mathcal{L}^G$ of Theorem 3.7. Here, we take F to be the completion at a prime above $\mathfrak{p}$ of the ring class field H_{c_o} of K of conductor c_o (prime-to-p), F_∞/F an infinite unramified extension as in §3.3, and

$$G = \mathrm{Gal}(L_\infty/F), \quad \text{where } L_\infty = F_\infty(\mu_{p^\infty}).$$

Recall the $\mathbb{I}$-module $\mathbb{D}$ of Definition 3.5, which by [Och03, Lem. 3.3] is free of rank 1.

Lemma 5.1. *There exists a canonical isomorphism of $\mathbb{I}$-modules $\omega_\mathbf{f} : \mathbb{D} \to \mathbb{I}$ such that for every $\nu \in \mathcal{X}_\mathfrak{D}^a(\mathbb{I})$ and every Hodge–Tate character ϕ of G of weight $0 < w < k_\nu - 1$ the following diagram commutes*

$$\begin{array}{ccc} \mathbb{D}\widehat{\otimes}_{\mathbf{Z}_p}\mathcal{O}_F[[G]] & \xrightarrow{\omega_\mathbf{f}\otimes 1} & \mathbb{I}\widehat{\otimes}_{\mathbf{Z}_p}\mathcal{O}_F[[G]] \\ \downarrow{\scriptstyle \mathrm{Sp}_{\nu,\phi}} & & \downarrow{\scriptstyle \mathrm{Sp}_{\nu,\phi}} \\ \mathbf{D}_{\mathrm{dR},F}(\mathscr{F}^+V_{\nu,\phi}) & \longrightarrow & F_{\nu,\phi}\otimes_{\mathbf{Q}_p} F, \end{array}$$

where the bottom horizontal map is given by pairing with the differential $\omega_{f_\nu}\otimes\phi^{-1}$ under the canonical identification

$$\mathbf{D}_{\mathrm{dR},F}(\mathscr{F}^+V_{\nu,\phi}) \cong \frac{\mathbf{D}_{\mathrm{dR},F}(V_{\nu,\phi})}{\mathrm{Fil}^0\mathbf{D}_{\mathrm{dR},F}(\mathscr{F}^+V_{\nu,\phi})} \cong \mathrm{Fil}^1\mathbf{D}_{\mathrm{dR},F}(V_{\nu,\phi}^*)^\vee.$$

Proof. The first isomorphism in the last part of the statement is explained in [Och03, Lem. 3.2] and the second isomorphism is given by the de Rham pairing $\langle,\rangle_{\mathrm{dR}}$. The result thus follows from [KLZ17, Prop. 10.1.1(1)]. □

Set $\lambda := \Psi(\mathrm{Fr}_p) - 1 \in \mathbb{I}$, and define $\widetilde{\mathbb{I}}_\mathcal{W} := \mathbb{I}[\lambda^{-1}]\widehat{\otimes}_{\mathbf{Z}_p}\mathcal{W}$.

Proposition 5.2. *Let $\mathcal{K}_\infty/F$ be a $\mathbf{Z}_p^\times$-extension contained in L_∞ obtained by adjoining the torsion points of a relative Lubin–Tate formal group over $F/\mathbf{Q}_p$, and let $\Gamma = \mathrm{Gal}(\mathcal{K}_\infty/F)$ be the corresponding quotient of G. There exists an injective $\widetilde{\mathbb{I}}_\mathcal{W}[[\Gamma]]$-linear map*

$$\mathcal{L}^\Gamma_{\omega_\mathbf{f}} : H^1_{\mathrm{Iw}}(\mathcal{K}_\infty/F, \mathscr{F}^+\mathbb{T}) \longrightarrow \widetilde{\mathbb{I}}_\mathcal{W}[[\Gamma]]$$

with pseudo-null cokernel such that for every Hodge–Tate character ϕ of Γ of weight $w > 0$ and conductor p^n, if $\mathfrak{Y}_\infty \in H^1_{\mathrm{Iw}}(\mathcal{K}_\infty/F, \mathscr{F}^+\mathbb{T})$ then

$$\mathrm{Sp}_{\nu,\phi}(\mathcal{L}^\Gamma_{\omega_\mathbf{f}}(\mathfrak{Y}_\infty)) = \frac{\varepsilon(\phi)p^{-n}}{\nu(\mathbf{a}_p^n)\varepsilon_f(p^n)\chi_\nu\xi_\nu^{-1}(p_\mathfrak{p}^n)} \cdot \begin{cases} \frac{(1-\nu(\mathbf{a}_p)\chi_\nu\xi_\nu^{-1}\phi^{-1}(\mathfrak{p}))}{(1-p\nu(\mathbf{a}_p)^{-1}\chi_\nu\xi_\nu^{-1}\phi(\mathfrak{p}))} & \text{if } n = 0, \\ 1 & \text{if } n \geqslant 1. \end{cases}$$
$$\times \frac{(-1)^{w-1}}{(w-1)!} \cdot \langle \log_p(\mathrm{Sp}_{\nu,\phi}(\mathfrak{Y}_\infty)), \omega_{f_\nu} \otimes \phi^{-1} \rangle_{\mathrm{dR}}.$$

Proof. By (5.2), the action of $G_{\mathbf{Q}_p}$ on $\mathscr{F}^+\mathbb{T}$ is given by the unramified character sending Fr_p to $\mathbf{a}_p \cdot \varepsilon_f\varepsilon_{\mathrm{cyc}}\Theta\boldsymbol{\xi}^{-1}(\mathrm{Fr}_p) = \mathbf{a}_p \cdot \boldsymbol{\chi}\boldsymbol{\xi}^{-1}(p_\mathfrak{p})\varepsilon_f(p)p$, where $p_\mathfrak{p}$ is the idèle of K with $\mathfrak{p}$-component equal to p and 1 everywhere else. Thus the map $\mathcal{L}^G$ of Theorem 3.7 can be applied to $\mathscr{F}^+\mathbb{T}$, and we define $\mathcal{L}^G_{\omega_\mathbf{f}} : H^1_{\mathrm{Iw}}(L_\infty/F, \mathscr{F}^+\mathbb{T}) \to \widetilde{\mathbb{I}}_\mathcal{W}[[G]]$ by the composition

$$\mathcal{L}^G_{\omega_\mathbf{f}} : H^1_{\mathrm{Iw}}(L_\infty/F, \mathscr{F}^+\mathbb{T}) \xrightarrow{\mathcal{L}^G} \lambda^{-1} \cdot \mathcal{J}(\mathbb{D}\widehat{\otimes}_{\mathbf{Z}_p}\widehat{\mathcal{O}}_{F_\infty}[[G]]) \xrightarrow{\omega_\mathbf{f}\otimes 1} \widetilde{\mathbb{I}}_\mathcal{W}[[G]],$$

where $\omega_\mathbf{f} \otimes 1$ is given by Lemma 5.1. Let $\mathbb{J}$ be the kernel of the natural projection $\mathbb{I}[[G]] \twoheadrightarrow \mathbb{I}[[\Gamma]]$. The corestriction map

$$H^1_{\mathrm{Iw}}(L_\infty/F, \mathscr{F}^+\mathbb{T})/\mathbb{J} \longrightarrow H^1_{\mathrm{Iw}}(\mathcal{K}_\infty/F, \mathscr{F}^+\mathbb{T})$$

is injective, and its cokernel is contained in the $\mathbb{J}$-torsion submodule of $H^2(L_\infty, \mathscr{F}^+\mathbb{T})$, which vanishes since $H^0(\mathcal{K}_\infty, \mathscr{F}^+\mathbb{T}) = \{0\}$ (as one can see e.g. by the argument right before [CH18, Lem. 5.5]). Quotienting $\mathcal{L}^G_{\omega_\mathbf{f}}$ by $\mathbb{J}$ we thus obtain a map

$$\mathcal{L}^\Gamma_{\omega_\mathbf{f}} : H^1_{\mathrm{Iw}}(\mathcal{K}_\infty/F, \mathscr{F}^+\mathbb{T}) \cong H^1_{\mathrm{Iw}}(L_\infty/F, \mathscr{F}^+\mathbb{T})/\mathbb{J} \longrightarrow \widetilde{\mathbb{I}}_\mathcal{W}[[\Gamma]]$$

having the desired properties by virtue of Theorem 3.7 and Corollary 3.9. □

5.2. **Explicit reciprocity law for big Heegner points.** Recall the character λ used in the construction of $\boldsymbol{\xi}$, and let $c_o\mathcal{O}_K$ be the prime-to-p part of the conductor of $\lambda(x)\lambda^{-1}(\overline{x})$. Let $\mathfrak{Z}_{c_o,\infty} \in H^1_{\mathrm{Iw}}(\widetilde{H}_{c_op^\infty}/\widetilde{H}_{c_o}, \mathbf{T}^\dagger)$ be Howard's system of big Heegner points, as recalled in §4.2. Since $\mathbf{T}^\dagger \otimes \boldsymbol{\xi}^{-1} = \mathbb{T}$ by (5.1), the twist $\mathfrak{Z}^{\boldsymbol{\xi}^{-1}}_{c_o,\infty}$ lies in $H^1_{\mathrm{Iw}}(\widetilde{H}_{c_op^\infty}/\widetilde{H}_{c_o}, \mathbb{T})$.

Let F be the completion of $\widetilde{H}_{c_o}$ at a prime v above $\mathfrak{p}$ (so F is a finite unramified extension of $\mathbf{Q}_p$), and let $\widetilde{H}_{c_op^\infty,v}$ be the completion of $\widetilde{H}_{c_op^\infty}$ at the unique prime above v. By [How07b, Prop. 2.4.5], the class $\mathrm{res}_v(\mathfrak{Z}^{\boldsymbol{\xi}^{-1}}_{c_o,\infty})$ goes to zero under the second arrow in the exact sequence

$$H^1_{\mathrm{Iw}}(\widetilde{H}_{c_op^\infty,v}/F, \mathscr{F}^+\mathbb{T}) \longrightarrow H^1_{\mathrm{Iw}}(\widetilde{H}_{c_op^\infty,v}/F, \mathbb{T}) \longrightarrow H^1_{\mathrm{Iw}}(\widetilde{H}_{c_op^\infty,v}/F, \mathscr{F}^-\mathbb{T})$$

induced by (4.1); since the first arrow is injective by [How07b, Lem. 2.4.4], we naturally have

$$\mathrm{res}_v(\mathfrak{Z}^{\boldsymbol{\xi}^{-1}}_{c_o,\infty}) \in H^1_{\mathrm{Iw}}(\widetilde{H}_{c_op^\infty,v}/F, \mathscr{F}^+\mathbb{T}).$$

The extension $\widetilde{H}_{c_op^\infty,v}/F$ is totally ramified and is well-known to agree with the $\mathbf{Z}_p^\times$-extension obtained by adjoining the torsion points of a Lubin–Tate formal group relative to the extension $F/\mathbf{Q}_p$ (see e.g. [Shn16, Prop. 39]). Thus for $\Gamma = \mathrm{Gal}(H_{c_op^\infty,v}/F)$ we have the regulator map $\mathcal{L}^\Gamma_{\omega_\mathbf{f}}$ of Proposition 5.2, and we may let $\mathcal{L}^\Gamma_{\omega_\mathbf{f}}(\mathrm{res}_\mathfrak{p}(\mathfrak{Z}^{\boldsymbol{\xi}^{-1}}_{c_o,\infty})) \in \widetilde{\mathbb{I}}_\mathcal{W}[[\widetilde{\Gamma}]]$ be the image of $\mathrm{res}_v(\mathfrak{Z}^{\boldsymbol{\xi}^{-1}}_{c_o,\infty})$ under

the composition

$$\begin{aligned} H^1_{\mathrm{Iw}}(\widetilde{H}_{c_o p^\infty,v}/F, \mathscr{F}^+\mathbb{T}) &\xrightarrow{\mathcal{L}^\Gamma_{\omega_{\mathbf{f}}}} \widetilde{\mathbb{I}}_{\mathcal{W}}[[\Gamma]] \cong \widetilde{\mathbb{I}}_{\mathcal{W}}[[\mathrm{Gal}(\widetilde{H}_{c_o p^\infty}/\widetilde{H}_{c_o})]] \\ &\xrightarrow{\mathrm{cor}_{\widetilde{H}_{c_o}/K}} \widetilde{\mathbb{I}}_{\mathcal{W}}[[\mathrm{Gal}(\widetilde{H}_{c_o p^\infty}/K)]] \\ &\xrightarrow{\mathrm{res}_{\widetilde{H}_{p^\infty}}} \widetilde{\mathbb{I}}_{\mathcal{W}}[[\widetilde{\Gamma}]], \end{aligned}$$

where $\widetilde{\Gamma} = \mathrm{Gal}(\widetilde{H}_{p^\infty}/K)$. We can now state and prove the explicit reciprocity law for Howard's big Heegner points, where we let $\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f})$ be the two-variable p-adic L-function constructed in §2.7.

Theorem 5.3. *The following equality holds in* $\widetilde{\mathbb{I}}_{\mathcal{W}}[[\widetilde{\Gamma}]]$*:*

$$\mathcal{L}^\Gamma_{\omega_{\mathbf{f}}}(\mathrm{res}_{\mathfrak{p}}(\mathfrak{Z}^{\boldsymbol{\xi}^{-1}}_{c_o,\infty})) = \mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f}) \cdot \sigma_{-1,\mathfrak{p}},$$

where $\sigma_{-1,\mathfrak{p}} := \mathrm{rec}_{\mathfrak{p}}(-1)|_{H_{p^\infty}} \in \widetilde{\Gamma}$.

We shall deduce Theorem 5.3 easily after the proof of the following result.

Proposition 5.4. *Let* $\nu \in \mathcal{X}^a_{\mathfrak{O}}(\mathbb{I})$ *be an arithmetic prime of weight* $(2,\varepsilon)$ *with* $\varepsilon : \Gamma^{\mathrm{wt}} \to \boldsymbol{\mu}_{p^\infty}$ *of conductor* p^s*, and let* $\widehat{\phi} : \widetilde{\Gamma} \to L^\times$ *be the* p*-adic avatar of an anticyclotomic Hecke character* ϕ *of* K *of infinity type* $(1,-1)$ *and conductor* p^n *with* $n \geqslant s$ *Then*

$$\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f})(\nu,\widehat{\phi}^{-1}) = \frac{\phi_{\mathfrak{p}}(-1)\varepsilon(\phi_{\mathfrak{p}})}{\nu(\mathbf{a}^n_p)\varepsilon_f \chi_\nu \xi_\nu^{-1}(\mathrm{Fr}^n_p)} \cdot \langle \log_p(\mathrm{res}_{\mathfrak{p}}(\mathrm{Sp}_{\nu,\phi^{-1}}(\mathfrak{Z}^{\boldsymbol{\xi}^{-1}}_{c_o,\infty}))), \omega_{f_\nu} \otimes \phi \rangle_{\mathrm{dR}}.$$

Proof. Our hypotheses imply that the character $\xi_\nu\phi^{-1}$ has finite order and it factors through the $\mathrm{Gal}(H_{c_o p^{n+1}}/K)$. By the same calculation as in the proof of [CH18, Thm. 4.9] (see esp. [*loc.cit.*, (4.8)]) we obtain

$$\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f})(\nu,\widehat{\phi}^{-1}) = \mathfrak{g}(\phi_{\mathfrak{p}}^{-1}) p^{-n} \phi_{\mathfrak{p}}(p^n) \sum_{\sigma \in \mathrm{Gal}(\widetilde{H}_{c_o p^{n+1}}/K)} \xi_\nu^{-1}\phi(\sigma)\chi_\nu^{-1}(\sigma) \cdot d^{-1}\widehat{f}^{\flat}_\nu(x^\sigma_{c_o p^{n+1},s}), \tag{5.4}$$

where $d^{-1}\widehat{f}_\nu$ is the p-adic modular form of weight 0 given by

$$d^{-1}\widehat{f}^{\flat}_\nu := \lim_{t \to -1} d^t \widehat{f}^{\flat}_\nu = \sum_{(n,p)=1} \nu(\mathbf{a}_n) n^{-1} q^n.$$

To proceed with the proof, we need to recall the definition of the Frobenius operator Frob on the space $V_p(N;R)$ of p-adic modular forms, where we take R to be a complete discrete valuation ring containing $\mathcal{O}_\nu$. If $x = [(A, \eta^{(p)}, \eta_p)]$ is a point in $\widehat{\mathrm{Ig}}(N)_{/R}$ with

$$(\eta^{(p)}, \eta_p) : \boldsymbol{\mu}_N \oplus \boldsymbol{\mu}_{p^\infty} \hookrightarrow A[N] \oplus A[p^\infty],$$

then η_p amounts to giving an isomorphism $\widehat{\eta}_p : \widehat{\mathbf{G}}_m \simeq \widehat{A}$ of formal groups, and we set

$$\mathrm{Frob}(x) := (A_0, \eta_0^{(p)}, \eta_{0,p}),$$

where:

- $A_0 := A/\eta_p(\boldsymbol{\mu}_p)$ is the quotient of A by its canonical subgroup, and we let $\lambda_0 : A \to A_0$ is the natural projection;
- $\eta_0^{(p)} := \lambda_0 \circ \eta^{(p)} : \boldsymbol{\mu}_N \hookrightarrow A_0[N]$;

- $\eta_{0,p} : \boldsymbol{\mu}_{p^\infty} \hookrightarrow A_0[p^\infty]$ induces $\widehat{\eta}_{0,p} := \widehat{\eta}_p \circ \widehat{\mu}_0$, where $\widehat{\mu}_0 : \widehat{A}_0 \simeq \widehat{A}$ is the isomorphism of formal groups induced by the dual isogeny $\mu_0 = \lambda_0^\vee$.

The action of Frob on $V_p(N;R)$ is then defined in the obvious manner, setting

$$\mathrm{Frob}(g)(x) := g(\mathrm{Frob}(x)),$$

for every $g \in V_p(N;R)$ and $x \in \widehat{\mathrm{Ig}}(N)_{/R}$.

Now let $F_{\omega_{f_\nu}}$ be the Coleman primitive of the differential ω_{f_ν}, normalized so that it vanishes at the cusp ∞; this is a locally analytic p-adic modular form (as defined in [BDP13, p. 1083]) of weight 0 satisfying

$$dF_{\omega_{f_\nu}} = \omega_{f_\nu}$$

and characterized by the further requirement that

$$F_{\omega_{f_\nu}} - \frac{\nu(\mathbf{a}_p)}{p}\mathrm{Frob}(F_{\omega_{f_\nu}}) = d^{-1}\widehat{f}_\nu^\flat \tag{5.5}$$

(*cf.* [Cas13, Cor. 2.8]). In particular, note that $U_pF_{\omega_{f_\nu}} = \frac{\nu(\mathbf{a}_p)}{p}F_{\omega_{f_\nu}}$.

Let $F_{n,s}$ be a finite extension of $\imath_p(\widetilde{L}_{c_0p^{n+1},s})$ in $\overline{\mathbf{Q}}_p$ such that the base-change of $\widetilde{X}_{s/\mathbf{Q}_p}$ to $F_{n,s}$ admits a stable model. The calculation in [Cas13, Prop. 2.9] applies to f and the classes

$$\Delta_{c_0p^{n+1},s} := (P_{c_0p^{n+1},s}) - (\infty), \quad \Delta_{c_0p^{n+1+s},s} := (P_{c_0p^{n+1+s},s}) - (\infty)$$

in $\widetilde{J}_s(F_{n,s})$, yielding the formulae

$$\log_{\omega_{f_\nu}}(\Delta_{c_0p^{n+1},s}) = F_{\omega_{f_\nu}}(P_{c_0p^{n+1},s}), \quad \log_{\omega_{f_\nu}}(\Delta_{c_0p^{n+1+s},s}) = F_{\omega_{f_\nu}}(P_{c_0p^{n+1+s},s}), \tag{5.6}$$

where $\log_{\omega_{f_\nu}} : \widetilde{J}_s(F_{n,s}) \to \mathbf{C}_p$ is the formal group logarithm associated with ω_{f_ν}.

Now define $Q_{c_0p^{n+1},s} \in \widetilde{J}_s(\widetilde{L}_{c_0p^{n+1},s}) \otimes_{\mathbf{Z}} F_\nu$ by

$$Q_{c_0p^{n+1},s} = \sum_{\sigma \in \mathrm{Gal}(\widetilde{H}_{c_0p^{n+1+s}}/\widetilde{H}_{c_0p^{n+1}})} \Delta_{c_0p^{n+1+s},s}^{\tilde{\sigma}} \otimes \chi_\nu^{-1}(\tilde{\sigma}), \tag{5.7}$$

where for each $\sigma \in \mathrm{Gal}(\widetilde{H}_{c_0p^{n+1+s}}/\widetilde{H}_{c_0p^{n+1}})$, $\tilde{\sigma}$ is an arbitrary lift of σ to $\mathrm{Gal}(\widetilde{L}_{c_0p^{n+1},s}/\widetilde{H}_{c_0p^{n+1}})$; by (4.2), the point $Q_{c_0p^{n+1},s}$ does not depend on the particular choice of lift. Taking lifts $\tilde{\sigma}$ in (5.7) which act trivially on μ_{p^s} (as we may, since $\widetilde{H}_{c_0p^{n+1+s}} \cap \widetilde{H}_{c_0p^{n+1}}(\mu_{p^s}) = \widetilde{H}_{c_0p^{n+1}}$) and extending the map $\log_{\omega_f}$ by F_ν-linearity, we deduce from (5.6) that

$$\begin{aligned}
\log_{\omega_{f_\nu}}(Q_{c_0p^{n+1},s}) &= \sum_{\tau \in \mathrm{Gal}(\widetilde{L}_{c_0p^{n+1},s}/\widetilde{H}_{c_0p^{n+1}}(\mu_{p^s}))} F_{\omega_{f_\nu}}(P_{c_0p^{n+1+s},s}^\tau) \\
&= F_{\omega_{f_\nu}}(U_p^s \cdot P_{c_0p^{n+1+s},s}) \\
&= \left(\frac{\nu(\mathbf{a}_p)}{p}\right)^s \cdot F_{\omega_{f_\nu}}(P_{c_0p^{n+1},s}),
\end{aligned} \tag{5.8}$$

using Proposition 4.4 for the second equality. Since as noted at the beginning of §4.2 the points $x_{c_0p^n,s} \in \mathrm{Ig}(N)(\mathbf{C})$ descend to the points $P_{c_0p^n,s} \in \widetilde{X}_s(\widetilde{H}_{c_0p^n}(\mu_{p^s}))$ for $n \geqslant s$, substituting (5.8) into (5.4) and using (5.5) we thus arrive at

$$\begin{aligned}
\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f})(\nu, \widehat{\phi}^{-1}) &= \mathfrak{g}(\phi_{\mathfrak{p}}^{-1})p^{-n}\phi_{\mathfrak{p}}(p^n) \\
&\quad \times \left(\frac{p}{\nu(\mathbf{a}_p)}\right)^s \sum_{\sigma \in \mathrm{Gal}(\widetilde{H}_{c_0p^{n+1}}/K)} \xi_\nu^{-1}\phi\chi_\nu^{-1}(\sigma) \cdot \log_{\omega_f}(Q_{c_0p^{n+1},s}^\sigma).
\end{aligned} \tag{5.9}$$

Recall that $\mathbf{T}^\dagger$ denotes the twist $\mathbf{T}\otimes\Theta^{-1}$, and note that $\mathbf{T}^\dagger\otimes_{\mathbb{I}}F_\nu\simeq\mathbf{T}\otimes_{\mathbb{I}}F_\nu$ as $G_{\mathbf{Q}(\mu_{p^s})}$-representations. By Hida's control theorem (see e.g. [Hid86a, Thm. 3.1(i)]), the natural map $\mathbf{T}\to\mathbf{T}\otimes_{\mathbb{I}}F_\nu$ factors as

$$\mathbf{T}\longrightarrow e^{\mathrm{ord}}\mathrm{Ta}_p(\widetilde{J}_s)\longrightarrow\mathbf{T}\otimes_{\mathbb{I}}F_\nu,$$

and tracing through the definition of $\mathfrak{X}_{c_op^{n+1}}$ in §4.2 we see that the image of $Q_{c_op^{n+1},s}$ under the induced map

$$\widetilde{J}_s(\widetilde{L}_{c_op^{n+1},s})\otimes F_\nu\xrightarrow{\mathrm{Kum}\circ e^{\mathrm{ord}}}H^1(\widetilde{L}_{c_op^{n+1},s},e^{\mathrm{ord}}\mathrm{Ta}_p(\widetilde{J}_s)\otimes F_\nu)\longrightarrow H^1(\widetilde{L}_{c_op^{n+1},s},\mathbf{T}\otimes_{\mathbb{I}}F_\nu)$$

agrees with the image of $U_p^s\cdot\nu(\mathfrak{X}_{c_op^{n+1}})$ under the restriction

$$H^1(\widetilde{H}_{c_op^{n+1+s}},\mathbf{T}^\dagger\otimes_{\mathbb{I}}F_\nu)\longrightarrow H^1(\widetilde{L}_{c_op^{n+1},s},\mathbf{T}^\dagger\otimes_{\mathbb{I}}F_\nu)\simeq H^1(\widetilde{L}_{c_op^{n+1},s},\mathbf{T}\otimes_{\mathbb{I}}F_\nu),$$

and hence

$$\log_{\omega_f}(Q_{c_op^{n+1},s})=\left(\frac{\nu(\mathbf{a}_p)}{p}\right)^s\cdot\log_{\omega_f}(\mathrm{res}_{\mathfrak{p}}(\nu(\mathfrak{X}_{c_op^{n+1}}))).\tag{5.10}$$

Note that $\varepsilon(\phi_{\mathfrak{p}})=\mathfrak{g}(\phi_{\mathfrak{p}}^{-1})\phi_{\mathfrak{p}}(-p^n)$. Thus substituting (5.10) into (5.9) and using (4.3) for the second equality, we conclude that

$$\begin{aligned}\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f})(\nu,\widehat{\phi}^{-1})&=\phi_{\mathfrak{p}}(-1)\varepsilon(\phi_{\mathfrak{p}})p^{-n}\sum_{\sigma\in\mathrm{Gal}(\widetilde{H}_{c_op^{n+1}}/K)}\xi_\nu^{-1}\phi(\sigma)\cdot\log_{\omega_f}(\mathrm{res}_{\mathfrak{p}}(\nu(\mathfrak{X}_{c_op^{n+1}})^\sigma))\\&=\frac{\phi_{\mathfrak{p}}(-1)\varepsilon(\phi_{\mathfrak{p}})p^{-n}}{\nu(\mathbf{a}_p^n)\varepsilon_f(p^n)\chi_\nu\xi_\nu^{-1}(p_{\mathfrak{p}}^n)}\cdot\langle\log_p(\mathrm{res}_{\mathfrak{p}}(\mathrm{Sp}_{\nu,\phi^{-1}}(\mathfrak{Z}_{c_o,\infty}^{\boldsymbol{\xi}^{-1}}))),\omega_f\otimes\phi\rangle_{\mathrm{dR}},\end{aligned}$$

as was to be shown. □

Proof of Theorem 5.3. In light of Proposition 5.2, the content of Proposition 5.4 amounts to the equality

$$\mathcal{L}^\Gamma_{\omega_{\mathbf{f}}}(\mathrm{res}_{\mathfrak{p}}(\mathfrak{Z}_{c_o,\infty}^{\boldsymbol{\xi}^{-1}}))(\nu,\widehat{\phi}^{-1})=(\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f})\cdot\sigma_{-1,\mathfrak{p}})(\nu,\widehat{\phi}^{-1}),$$

for all pairs (ν,ϕ) as in the statement of that result. Since an element in $\widetilde{\mathbb{I}}_{\mathcal{W}}[[\widetilde{\Gamma}]]$ is uniquely determined by values at such a collection of pairs, the result follows. □

An immediate consequence of Theorem 5.3 is the following nontriviality statement for the classes $\mathfrak{Z}_{c_o,\infty}$. For $c_o=1$ and under the additional hypotheses that $(D_K,N)=1$ and $p\nmid\varphi(N)$ (Euler's totient function), this result was first shown by Howard (see [How07b, §3.1]) building on the methods of Cornut–Vatsal.

Corollary 5.5. *Let c_o be a positive integer prime to p, and let $\widetilde{\Gamma}_{c_o}=\mathrm{Gal}(\widetilde{H}_{c_op^\infty}/\widetilde{H}_{c_o})$. Then the class $\mathfrak{Z}_{c_o,\infty}$ is not $\mathbb{I}[[\widetilde{\Gamma}_{c_o}]]$-torsion.*

Proof. Note that it suffices to show the nontriviality statement for a character twist of $\mathfrak{Z}_{c_o,\infty}$. Let $\nu\in\mathcal{X}_{\mathfrak{D}}(\mathbb{I})$ have weight $(k,\mathbb{1})$ with $k\geqslant 1$ and be such that f_ν is classical, and let $\mathfrak{P}$ be the kernel of the map $\mathbb{I}[[\widetilde{\Gamma}_{c_o}]]\to\mathbb{I}[[\widetilde{\Gamma}_{c_o}]]\otimes_{\mathbb{I}}F_\nu$. Then $\mathfrak{P}$ is a height 1 prime of $\mathbb{I}[[\widetilde{\Gamma}_{c_o}]]$ at which the specialization of $\mathfrak{Z}_{c_o,\infty}^{\boldsymbol{\xi}^{-1}}$ is nontrivial by Theorem 5.3 and Corollary 2.13. Since there are infinitely many such $\mathfrak{P}$, it follows that $\mathfrak{Z}_{c_o,\infty}^{\boldsymbol{\xi}^{-1}}$ is not $\mathbb{I}[[\widetilde{\Gamma}_{c_o}]]$-torsion, whence the result. □

6. Arithmetic applications

In the following, for a prime $p > 3$, we let $f \in S_k(\Gamma_1(N))$ be a p-ordinary newform of weight $k \geqslant 1$ and level N prime to p with associated Galois representations

$$\rho_f : G_{\mathbf{Q}} \longrightarrow \mathrm{Aut}_L(V_f) \simeq \mathrm{GL}_2(L),$$

where L is a finite extension of $\mathbf{Q}_p$ with ring of integers $\mathfrak{O}$. Also, we let K be an imaginary quadratic field of odd discriminant $-D_K < 0$ satisfying hypothesis (heeg) relative to N and in which $p = \mathfrak{p}\overline{\mathfrak{p}}$ splits.

6.1. **Preparations.** Let χ be the p-adic avatar of an anticyclotomic Hecke character of K of infinity type $(j, -j)$ with $j - k/2 \in \mathbf{Z}$, and set

$$V_{f,\chi} := V_f(k/2)|_{G_K} \otimes \chi.$$

For S a finite set of places of K containing the primes above Np, and for every finite extension F of K, let $\mathfrak{G}_{F,S}$ be the Galois group of the maximal extension of F unramified outside the places above S. Recall that the *Bloch–Kato Selmer group* $\mathrm{Sel}(F, V_{f,\chi})$ is defined by

$$\mathrm{Sel}(F, V_{f,\chi}) = \ker\left(H^1(\mathfrak{G}_{F,S}, V_{f,\chi}) \to \prod_v \frac{H^1(F_v, V_{f,\chi})}{H^1_{\mathrm{f}}(F_v, V_{f,\chi})} \right), \tag{6.1}$$

where v runs over all places of F, and

$$H^1_{\mathrm{f}}(F_v, V_{f,\chi}) := \begin{cases} \ker\left(H^1(F_v, V_{f,\chi}) \to H^1(F_v^{\mathrm{ur}}, V_{f,\chi}) \right) & \text{if } v \nmid p; \\ \ker\left(H^1(F_v, V_{f,\chi}) \to H^1(F_v, V_{f,\chi} \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{cris}}) \right) & \text{if } v \mid p. \end{cases}$$

Fix a $G_{\mathbf{Q}}$-stable $\mathfrak{O}$-lattice $T_f \subset V_f$ and set $T_{f,\chi} := T_f(k/2)|_{G_K} \otimes \chi$. We define $\mathrm{Sel}(F, T_{f,\chi})$ by the same recipe (6.1), replacing $H^1_{\mathrm{f}}(F_v, V_{f,\chi})$ by their natural preimages in $H^1(F_v, T_{f,\chi})$. Let $F_{\mathfrak{p}}$ denote the completion of F at any place above $\mathfrak{p}$, and similarly for $F_{\overline{\mathfrak{p}}}$.

Lemma 6.1. *If the infinity type of χ is $(j, -j)$ with $j - k/2 \in \mathbf{Z}$ and $j \geqslant k/2$, then:*

$$H^1_{\mathrm{f}}(F_{\overline{\mathfrak{p}}}, V_{f,\chi}) = \{0\}, \qquad H^1_{\mathrm{f}}(F_{\mathfrak{p}}, V_{f,\chi}) = H^1(F_{\mathfrak{p}}, V_{f,\chi}).$$

In particular, the classes in the Bloch–Kato Selmer group $\mathrm{Sel}(F, V_{f,\chi})$ are trivial at all primes above $\overline{\mathfrak{p}}$ and satisfy no local condition at the primes above $\mathfrak{p}$.

Proof. From our conventions (see the footnote in Theorem 3.7), we find that the Hodge–Tate weights of $V_{\overline{\mathfrak{p}}} := V_{f,\chi}|_{G_{F_{\overline{\mathfrak{p}}}}}$ are $k/2 - j$ and $1 - k/2 - j$; since these are non-positive integers under the above hypotheses, it follows that $\mathrm{Fil}^0 \mathbf{D}_{\mathrm{dR},F_{\overline{\mathfrak{p}}}}(V_{\overline{\mathfrak{p}}}) = \mathbf{D}_{\mathrm{dR},F_{\overline{\mathfrak{p}}}}(V_{\overline{\mathfrak{p}}})$. Similarly, the Hodge–Tate weights of $V_{\mathfrak{p}} := V_{f,\chi}|_{G_{F_{\mathfrak{p}}}}$ are the strictly positive integers $k/2 + j$ and $1 - k/2 + j$, and therefore $\mathrm{Fil}^0 \mathbf{D}_{\mathrm{dR},F_{\mathfrak{p}}}(V_{\mathfrak{p}}) = \{0\}$. The result thus follows from [BK90, Thm. 4.1(ii)]. □

We will also have use for the following generalized Selmer groups obtained by changing in definition (6.1) the local condition at the places above p. For $v \mid p$ and $\mathcal{L}_v \in \{\emptyset, \mathrm{Gr}, 0\}$, set

$$H^1_{\mathcal{L}_v}(F_v, V_{f,\chi}) := \begin{cases} H^1(F_v, V_{f,\chi}) & \text{if } \mathcal{L}_v = \emptyset; \\ H^1(F_v, \mathscr{F}^+ V_{f,\chi}) & \text{if } \mathcal{L}_v = \mathrm{Gr}; \\ \{0\} & \text{if } \mathcal{L}_v = 0, \end{cases} \tag{6.2}$$

and for $\mathcal{L} = \{\mathcal{L}_v\}_{v|p}$, define

$$H^1_{\mathcal{L}}(F, V_{f,\chi}) := \ker\left(H^1(\mathfrak{G}_{F,S}, V_{f,\chi}) \to \prod_{v \nmid p} \frac{H^1(F_v, V_{f,\chi})}{H^1_{\mathrm{f}}(F_v, V_{f,\chi})} \times \prod_{v|p} \frac{H^1(F_v, V_{f,\chi})}{H^1_{\mathcal{L}_v}(F_v, V_{f,\chi})} \right).$$

In addition, we define $H^1_{\mathcal{L}}(F_v, T_{f,\chi})$ taking preimages just as before.

Remark 6.2. By Lemma 6.1 we have

$$\mathrm{Sel}(F, V_{f,\chi}) = H^1_{\emptyset,0}(F, V_{f,\chi}). \tag{6.3}$$

Remark 6.3. Taking $\mathcal{L}_v = \mathrm{Gr}$ for all $v \mid p$ we get

$$H^1_{\mathcal{L}}(F, V_{f,\chi}) = \mathrm{Sel}_{\mathrm{Gr}}(F, V_{f,\chi}), \tag{6.4}$$

where $\mathrm{Sel}_{\mathrm{Gr}}(L, V_{f,\chi})$ is the Greenberg Selmer group considered in [How07b, Def. 2.4.2].

6.2. **Higher weight specializations of big Heegner points.** In this section we relate the higher weight specializations of Howard's big Heegner points to the étale Abel–Jacobi images of classical Heegner cycles [Nek95]. A first result in this spirit was obtained in [Cas13] under a certain nonvanishing hypothesis (see [*loc.cit.*, Thm. 5.11]). In Theorem 6.5 below we remove that hypothesis, and find a relation between the global cohomology classes themselves, rather than just their cyclotomic p-adic heights.

Assume that f has even weight $k = 2r \geqslant 2$ and trivial nebentypus. Fix an integer c_o prime to p, and set

$$\mathrm{Sel}_{\mathrm{Gr}}(H_{c_o p^\infty}/H_{c_o}, T_f(r)) := \varprojlim_n \mathrm{Sel}_{\mathrm{Gr}}(H_{c_o p^n}, T_f(r)),$$

where $\mathrm{Sel}_{\mathrm{Gr}}(H_{c_o p^n}, T_f(r)) = H^1_{\mathcal{L}}(H_{c_o p^n}, T_f(r))$ as in Remark 6.3. In particular, for every place v of $H_{c_o p^n}$ above p, the restriction map $\mathrm{res}_v : \mathrm{Sel}_{\mathrm{Gr}}(H_{c_o p^n}, T_f(r)) \to H^1(H_{c_o p^n, v}, T_f(r))$ has image contained in $H^1(H_{c_o p^n, v}, \mathscr{F}^+ T_f(r)) \subset H^1(H_{c_o p^n, v}, T_f(r))$.

Let $\mathbf{f}$ be the ordinary $\mathbb{I}$-adic newform of tame level N passing through f, and let

$$\mathfrak{Z}_{c_o,\infty} \in H^1_{\mathrm{Iw}}(H_{c_o p^\infty}/H_{c_o}, \mathbf{T}^\dagger)$$

be Howard's system of big Heegner points attached to $\mathbf{f}$ and K. (Note that since we assume here that f has trivial nebentypus, the classes $\mathfrak{Z}_{c_o,t}$ are defined over the ring class fields $H_{c_o p^t}$ rather than their extensions $\widetilde{H}_{c_o p^t}$ considered in §4.2.)

Lemma 6.4. *Assume that $\bar\rho_f|_{G_K}$ is irreducible. Then for every place v of H_{c_o} above $\mathfrak{p}$ the restriction map*

$$\mathrm{res}_v : \mathrm{Sel}_{\mathrm{Gr}}(H_{c_o p^\infty}/H_{c_o}, T_f(r)) \longrightarrow H^1_{\mathrm{Iw}}(H_{c_o p^\infty, v}/H_{c_o, v}, \mathscr{F}^+ T_f(r))$$

is injective.

Proof. Let $\Lambda_{\mathfrak{O}} = \mathfrak{O}[[\mathrm{Gal}(H_{c_o p^\infty}/H_{c_o})]]$. Since $\mathrm{Sel}_{\mathrm{Gr}}(H_{c_o p^\infty}/H_{c_o}, T_f(r))$ is $\Lambda_{\mathfrak{O}}$-torsion-free by our irreducibility hypothesis (see [How04, Lem. 2.2.9] and [PR00, §1.3.3]), it suffices to show that the kernel of res_v is $\Lambda_{\mathfrak{O}}$-torsion; for this, it will suffice to show that for infinitely many $\phi : \mathrm{Gal}(H_{c_o p^\infty}/H_{c_o}) \to \mathcal{O}^\times_{\mathbf{C}_p}$, the ϕ-specialized map

$$\mathrm{res}_v : \mathrm{Sel}_{\mathrm{Gr}}(H_{c_o}, V_f(r) \otimes \phi) \longrightarrow H^1(H_{c_o, v}, \mathscr{F}^+ V_f(r) \otimes \phi)$$

is injective. By considering twists for each of the characters of $\mathrm{Gal}(H_{c_o}/K)$, it will suffice to show that for infinitely many $\phi : \widetilde{\Gamma} = \mathrm{Gal}(H_{p^\infty}/K) \to \mathcal{O}^\times_{\mathbf{C}_p}$, the restriction map

$$\mathrm{res}_{\mathfrak{p}} : \mathrm{Sel}_{\mathrm{Gr}}(K, V_f(r) \otimes \xi_\nu^{-1}\phi) \longrightarrow H^1(K_{\mathfrak{p}}, \mathscr{F}^+ V_f(r) \otimes \xi_\nu^{-1}\phi) \tag{6.5}$$

is injective. Let $\nu \in \mathcal{X}^a_{\mathfrak{O}}(\mathbb{I})$ be such that $\mathbf{f}_\nu$ is the ordinary p-stabilization of f. By Corollary 2.13, we have $\nu(\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f}))(\phi) \neq 0$ for all but finitely many characters ϕ of $\widetilde{\Gamma}$, and by Theorem 5.3 this

shows that $\mathrm{res}_{\mathfrak{p}}(\mathrm{Sp}_{\nu,\phi}(\mathfrak{Z}^{\boldsymbol{\xi}^{-1}}_{c_o,\infty})) \neq 0$ for every such ϕ, where $\mathrm{Sp}_{\nu,\phi}(\mathfrak{Z}^{\boldsymbol{\xi}^{-1}}_{c_o,\infty})$ is the image of $\mathfrak{Z}^{\boldsymbol{\xi}^{-1}}_{c_o,\infty}$ under the composition

$$H^1_{\mathrm{Iw}}(H_{c_op^\infty}/H_{c_o}, \mathbf{T}^\dagger \otimes \boldsymbol{\xi}^{-1}) \xrightarrow{\mathrm{Sp}_\nu} H^1(H_{c_op^\infty}/H_{c_o}, T_f(r)\otimes \xi_\nu^{-1}) \xrightarrow{\phi} H^1(K, T_f(r)\otimes \xi_\nu^{-1}\phi),$$

and we view ϕ as a character on $\mathrm{Gal}(H_{c_op^\infty}/H_{c_o})$ via $\mathrm{Gal}(H_{c_op^\infty}/H_{c_o}) \subset \mathrm{Gal}(H_{c_op^\infty}/K) \twoheadrightarrow \widetilde{\Gamma}$. By the results of [How07b, §2.3], the class $\mathrm{Sp}_{\nu,\phi}(\mathfrak{Z}^{\boldsymbol{\xi}^{-1}}_{c_o,\infty})$ is the base class of an anticyclotomic Euler system for $T_f(r)\otimes\xi_\nu^{-1}\phi$ with the Bloch–Kato local condition (see [CH18, Def. 7.2]), and so by [CH18, Thm. 7.7] we have the implication

$$\mathrm{Sp}_{\nu,\phi}(\mathfrak{Z}^{\boldsymbol{\xi}^{-1}}_{c_o,\infty}) \neq 0 \quad \Longrightarrow \quad \mathrm{Sel}_{\mathrm{Gr}}(K, V_f(r)\otimes\xi_\nu^{-1}\phi) = L.\mathrm{Sp}_{\nu,\phi}(\mathfrak{Z}^{\boldsymbol{\xi}^{-1}}_{c_o,\infty}),$$

noting the equality between the Greenberg and the Bloch–Kato Selmer groups in our setting (see e.g. [How07b, Eq. (23)]). We thus conclude that (6.5) is injective, whence the result. □

We are now ready to prove Theorem A in the Introduction, the strenghtening of the main result of [Cas13] advanced in [CH18, §1].

For a p-ordinary newform $g = f_\nu$ of even weight $k_\nu = 2r_\nu \geqslant 2$, let

$$\mathbf{z}_{f_\nu,c_o,\alpha} \in H^1_{\mathrm{Iw}}(H_{c_op^\infty}/H_{c_o}, T_g(r_\nu))$$

be the $\Lambda_{\mathfrak{O}}$-adic class constructed in [CH18, §5.2], which by its geometric construction and the equality (6.4) lands in $\mathrm{Sel}_{\mathrm{Gr}}(H_{c_op^\infty}/H_{c_o}, T_{f_\nu}(r_\nu))$. We refer the reader to [Cas13, p. 1250] and [CH18, Eq. (4.6)] for the definition of the p-adic étale Abel–Jacobi images

$$\Phi^{\text{ét}}_{f_\nu,H_{c_o}}(\Delta^{\mathrm{heeg}}_{c_o,r_\nu}),\ \Phi^{\text{ét}}_{f_\nu,H_{c_o}}(\Delta^{\mathrm{BDP}}_{c_o,r_\nu}) \ \in \mathrm{Sel}(H_{c_o}, T_{f_\nu}(r_\nu)) \tag{6.6}$$

of classical [Nek95] and generalized [BDP13] Heegner cycles, respectively, attached to f_ν and K. On the other hand, as before let $\mathfrak{Z}_{c_o,\infty} \in H^1_{\mathrm{Iw}}(H_{c_op^\infty}/H_{c_o}, \mathbf{T}^\dagger)$ be Howard's system of big Heegner points attached to K and the Hida family $\mathbf{f} = \sum_{n=1}^\infty \mathbf{a}_n q^n \in \mathbb{I}[[q]]$ passing through f.

Theorem 6.5. *Assume that:*

- $k \equiv 2 \pmod{p-1}$*;*
- $\bar\rho_f$ *is ramified at every prime* $q \mid (D_K, N)$*;*
- $\bar\rho_f$ *p-distinguished;*
- $\bar\rho_f|_{G_K}$ *is irreducible.*

Then for all $\nu \in \mathcal{X}^a_{\mathfrak{O}}(\mathbb{I})$ *of weight* $2r_\nu > 2$ *with* $2r_\nu \equiv k \pmod{2(p-1)}$ *and trivial character, we have*

$$\nu(\mathfrak{Z}_{c_o,\infty})\cdot c_o^{r_\nu - 1} = \mathbf{z}_{f_\nu,c_o,\alpha}$$

as elements in $\mathrm{Sel}_{\mathrm{Gr}}(H_{c_op^\infty}/H_{c_o}, T_{f_\nu}(r_\nu))$*, where* $\alpha = \nu(\mathbf{a}_p)$*. In particular, for all such* ν *we have*

$$\nu(\mathfrak{Z}_{c_o,0}) = \left(1 - \frac{p^{r_\nu-1}}{\nu(\mathbf{a}_p)}\right)^2 \cdot \frac{\Phi^{\text{ét}}_{f_\nu,H_{c_o}}(\Delta^{\mathrm{heeg}}_{c_o,r_\nu})}{u_{c_o}(2\sqrt{-D_K})^{r_\nu-1}}, \tag{6.7}$$

where $u_{c_o} = |\mathcal{O}_{c_o}^\times|/2$.

Proof. Let $\psi := \xi_\nu$ be the specialization of $\boldsymbol{\xi}$ as ν, and let

$$\mathcal{L}^{\psi^{-1}}_{\nu,\mathfrak{p}} := \left\langle \mathcal{L}_{\mathfrak{p},\psi}(-), \omega_{f_\nu}\otimes t^{1-2r_\nu}\right\rangle : H^1_{\mathrm{Iw}}(H_{c_op^\infty}/H_{c_o}, \mathscr{F}^+V_{f_\nu}(r_\nu)\otimes\psi^{-1}) \longrightarrow \Lambda_{\mathcal{W}}[[\widetilde{\Gamma}]]$$

be the map introduced in [CH18, §5.3]. By construction, the map $\mathcal{L}^\Gamma_{\omega_{\mathbf{f}}}$ in Proposition 5.2 specializes at ν to the map $\mathcal{L}^{\psi^{-1}}_{\nu,\mathfrak{p}}$, and using Theorem 5.3 for the second equality we have

$$\mathcal{L}^{\psi^{-1}}_{\nu,\mathfrak{p}}(\nu(\mathfrak{Z}^{\boldsymbol{\xi}^{-1}}_{c_o,\infty})) = \nu(\mathcal{L}^\Gamma_{\omega_{\mathbf{f}}}(\mathrm{res}_{\mathfrak{p}}(\mathfrak{Z}^{\boldsymbol{\xi}^{-1}}_{c_o,\infty}))) = \nu(\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f})\cdot\sigma_{-1,\mathfrak{p}}). \tag{6.8}$$

On the other hand, as shown in the proof of Theorem 2.11, $\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f})$ specializes at ν to the p-adic L-function $\mathscr{L}_{\mathfrak{p},\psi}(f_\nu)$ of [CH18, §3.3], and so by the explicit reciprocity law of [*loc.cit.*, Thm. 5.7] we have

$$\nu(\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f})\cdot\sigma_{-1,\mathfrak{p}}) = \mathscr{L}_{\mathfrak{p},\psi}(f_\nu)\cdot\sigma_{-1,\mathfrak{p}} = \mathcal{L}_{\nu,\mathfrak{p}}^{\psi^{-1}}(\mathbf{z}_{f_\nu,c_o,\alpha}\otimes\psi^{-1})\cdot c_o^{1-r_\nu}, \tag{6.9}$$

where $\alpha=\nu(\mathbf{a}_p)$, since this is the U_p-eigenvalue of the p-stabilized newform f_ν.

Comparing (6.8) and (6.9), the proof of the first statement in Theorem 6.5 follows from Lemma 6.4 and the injectivity of $\mathcal{L}_{\nu,\mathfrak{p}}^{\psi^{-1}}$. (The injectivity of this map is not explicitly stated in [CH18, §5.3], but it follows from the construction in [*loc.cit.*, Thm. 5.1] and [LZ14, Prop. 4.11].) In particular, by the construction of $\mathbf{z}_{f_\nu,c_o,\alpha}$ in [CH18, §5.2] (see [*loc.cit.*, Def. 5.2]), we obtain the relation

$$\nu(\mathfrak{Z}_{c_o,0}) = \frac{1}{u_{c_o}}\left(1-\frac{p^{r_\nu-1}}{\nu(\mathbf{a}_p)}\right)^2\cdot\Phi^{\text{ét}}_{f_\nu,H_{c_o}}(\Delta^{\text{BDP}}_{c_o,r_\nu})\cdot c_o^{1-r_\nu},$$

where $u_{c_o}=|\mathcal{O}_{c_o}^\times|/2$, and by [BDP17, Prop. 4.1.2] (with $r_1=2r_\nu-2$, $r_2=0$, and so $u=r_\nu-1$) the equality of classes (6.7) follows. □

Remark 6.6. For f_ν of weight 2 and trivial character, the classes (6.6) both reduce to Kummer images of classical Heegner points, and if $\mathbf{f}_\nu$ is the ordinary p-stabilization of f_ν, the argument in the proof of Theorem 6.5 applies verbatim, yielding the same relation between classes. This excludes the case of arithmetic primes ν of weight 2 and trivial character for which $\mathbf{f}_\nu$ has conductor divisible p, which is the subject of [Cas18].

6.3. **Proof of Theorem C.** Keeping the notations as in the statement of Theorem C in the Introduction, let $V_\varrho^\vee$ br the contragredient of the representation V_ϱ, and let $g\in S_1(\Gamma_1(N_\varrho))$ be an eigenform whose associated Deligne–Serre representation V_g is isomorphic to $V_\varrho^\vee$. (Note that the existence of g is a consequence of the proof [KW09] of Serre's modularity conjecture.) For $\mathfrak{P}$ a prime of E above p, we shall view g and V_g as defined over the finite extension of $\mathbf{Q}_p$ given by the completion $L:=E_{\mathfrak{P}}$, and let $T_g\subset V_g$ be any $G_{\mathbf{Q}}$-stable $\mathfrak{O}$-lattice.

Let $g_p\in S_1(\Gamma_0(p)\cap\Gamma_1(N_\varrho))$ be a p-stabilization of g. By [Wil88, Thm. 3], there exists a ordinary $\mathbb{I}$-adic newform $\mathbf{f}$ of tame conductor N_ϱ such that $\nu(\mathbf{f})=g_p$ for some $\nu\in\mathcal{X}_{\mathfrak{O}}(\mathbb{I})$ of weight 1. Note that our hypotheses on ϱ guarantee that the associated residual representation $\bar\rho_{\mathbf{f}}$ is irreducible and p-distinguished; in particular, $\mathbf{f}$ is unique by Lemma 4.2.

Let λ be the grossencharacter of K associated to A by the theory of complex multiplication, and let $\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f})$ be the two-variable p-adic L-function of §2.7 constructed with the corresponding $\mathbb{I}$-adic character $\boldsymbol{\xi}$. As usual, let $c_o\mathcal{O}_K$ be the prime-to-p conductor of $\lambda(x)\lambda^{-1}(\overline{x})$, and let $\mathfrak{Z}_{c_o,\infty}\in H^1_{\text{Iw}}(H_{c_op^\infty}/H_{c_o},\mathbf{T}^\dagger)$ be Howard's system of big Heegner points attached to $\mathbf{f}$ and K. As already noted (see the comments right before the statement of Theorem 2.11), the specialization of the $\mathbb{I}$-adic character $\boldsymbol{\chi}$ at ν has central character $\chi_\nu|_{\mathbb{A}_{\mathbf{Q}}^\times}=\varepsilon_g^{-1}$. Noting that $\lambda(\mathfrak{a})\lambda(\bar{\mathfrak{a}})=\mathbf{N}(\mathfrak{a})$, we thus see from Theorem 2.11 and Theorem 5.3 that

$$\begin{aligned} L(A/\mathbf{Q},\varrho,1)\neq 0 &\implies L(g/K,\chi_\nu\xi_\nu\mathbf{N}^{-1/2},0)\neq 0\\ &\implies \nu(\mathscr{L}_{\mathfrak{p},\boldsymbol{\xi}}(\mathbf{f}))(\mathbb{1})\neq 0\\ &\implies \text{res}_{\mathfrak{p}}(\text{Sp}_{\nu,\mathbb{1}}(\mathfrak{Z}_{c_o,\infty}^{\boldsymbol{\xi}^{-1}}))\neq 0, \end{aligned} \tag{6.10}$$

and so $\text{res}_{\bar{\mathfrak{p}}}(\text{Sp}_{\nu,\mathbb{1}}(\mathfrak{Z}_{c_o,\infty}^{\boldsymbol{\xi}}))\neq 0$ by the action of complex conjugation.

Let $\phi:=\xi_\nu$. The Euler system relations established in [How07b, §2.3] imply that $\text{Sp}_{\nu,\mathbb{1}}(\mathfrak{Z}_{c_o,\infty}^{\boldsymbol{\xi}^{-1}})$ is the base class of an anticyclotomic Euler system for $T_{g,\phi}:=T_g(1/2)\otimes\chi_\nu\phi$ in the sense of

[CH18, Def. 7.2] for the local conditions defining the generalized Selmer group $H^1_{\mathrm{Gr,Gr}}(K, V_{g,\phi})$ of (6.2), where $V_{g,\phi} = T_{g,\phi}[1/p]$. Thus as in the proof of [CH18, Thm. 7.9] the last nonvanishing in (6.10) implies that

$$H^1_{\mathrm{Gr,Gr}}(K, V_{g,\phi}) = L \cdot \mathrm{Sp}_{\nu,\mathbb{1}}(\mathfrak{Z}^{\boldsymbol{\xi}^{-1}}_{c_o,\infty})^\tau = L \cdot \mathrm{Sp}_{\nu,\mathbb{1}}(\mathfrak{Z}^{\boldsymbol{\xi}}_{c_o,\infty}),$$

and since $\mathrm{res}_{\bar{\mathfrak{p}}}(\mathrm{Sp}_{\nu,\mathbb{1}}(\mathfrak{Z}^{\boldsymbol{\xi}}_{c_o,\infty})) \neq 0$, this implies that

$$H^1_{\mathrm{Gr},0}(K, V_{g,\phi}) = \{0\}. \tag{6.11}$$

From Poitou–Tate duality we obtain the exact sequence

$$\begin{aligned} 0 \longrightarrow H^1_{0,\emptyset}(K, V_{g,\phi^{-1}}) \longrightarrow H^1_{\mathrm{Gr},\emptyset}(K, V_{g,\phi^{-1}}) &\xrightarrow{\mathrm{res}_{\mathfrak{p}}} H^1(K_{\mathfrak{p}}, \mathscr{F}^+ V_{g,\phi^{-1}}) \\ &\longrightarrow H^1_{\emptyset,0}(K, V_{g,\phi})^\vee \longrightarrow H^1_{\mathrm{Gr},0}(K, V_{g,\phi})^\vee, \end{aligned}$$

and since $H^1(K_{\mathfrak{p}}, \mathscr{F}^+ V_{g,\phi^{-1}})$ is one-dimensional, combining (6.10) and (6.11) we conclude that

$$H^1_{\emptyset,0}(K, V_{g,\phi}) = \{0\}, \tag{6.12}$$

and so $\mathrm{Sel}(K, V_p(A) \otimes V_\varrho^\vee)$ vanishes by Lemma 6.1.

Now let F be the splitting field of ϱ, and set $H = \mathrm{Gal}(F/\mathbf{Q})$. Since $\mathrm{Hom}_{G_{\mathbf{Q}}}(V_\varrho, \mathrm{Sel}(F, V_p(A))_L)$ is naturally identified with the space of H-invariant classes in $\mathrm{Sel}(F, V_p(A)) \otimes V_\varrho^\vee = \mathrm{Sel}(F, V_p(A) \otimes V_\varrho^\vee)$ and the restriction map

$$\mathrm{Sel}(\mathbf{Q}, V_p(A) \otimes V_\varrho^\vee) \longrightarrow \mathrm{Sel}(F, V_p(A) \otimes V_\varrho^\vee)^H$$

is as isomorphism, the proof of Theorem C follows immediately from (6.12).

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