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ON A TWISTED VERSION OF LINNIK AND SELBERG'S CONJECTURE ON SUMS OF KLOOSTERMAN SUMS

RAPHAEL S. STEINER

Abstract. We generalize the work of Sarnak and Tsimerman to twisted sums of Kloosterman sums and thus give evidence towards the twisted Linnik–Selberg conjecture.

§1. *Introduction.* The study of Kloosterman sums

$$S(m,n;c)=\sum_{\substack{a \bmod (c)\\(a,c)=1}} e\left(\frac{ma+n\overline{a}}{c}\right),\quad \text{where } e(z)=\mathrm{e}^{2\pi \mathrm{i} z} \text{ and } a\overline{a}\equiv 1 \bmod (c)$$

is interesting for a variety of reasons. One of these reasons is their connection to the spectral theory of automorphic forms. In particular the sign changes of $S(m,n;c)$, for c varying in the arithmetic progression $c\equiv 0 \bmod (s)$, are related to the Selberg conjecture about the smallest positive eigenvalue of the Laplacian on the space $\Gamma_0(s)\backslash\mathbb{H}$. Concretely, we have that the smallest positive eigenvalue $\lambda_1^s \geqslant \frac{1}{4}$ if and only if the following conjecture holds (see [**14**, Theorem 16.9]).

CONJECTURE 1 (Smooth Linnik in arithmetic progression). *Let* $m, s\in\mathbb{N}$, $g\in C^3(\mathbb{R}^+,\mathbb{R}_0^+)$ *a compactly supported bump function with* $|g^{(a)}|\leqslant 1$ *for* $a=0$, $1,2,3$, *and* $C\geqslant 1$. *Then we have for every* $\epsilon>0$,

$$\sum_{c\equiv 0 \bmod (s)} \frac{1}{c}S(m,m;c)g\left(\frac{C}{c}\right)\ll_{\epsilon,m,s} C^{\epsilon}.$$

In this paper, however, we are interested in the sharp cut-off variant of the above conjecture. The first non-trivial progress towards this conjecture was made by Kuznetsov [**18**], who managed to prove that

$$\sum_{c\leqslant C}\frac{1}{c}S(m,n;c)\ll_{m,n} C^{1/6}\log(2C)^{1/3} \tag{1.1}$$

by exploiting the Kuznetsov trace formula (see Proposition 6), which was established in the same paper. The bound (1.1) is still the best known bound to date and the Kuznetsov trace formula has become a very powerful tool in a variety of contexts.

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In their paper [23], Sarnak and Tsimerman have made the dependence on m, n in (1.1) explicit and moreover achieved a non-trivial bound in the harder "Selberg" range ($C \leqslant \sqrt{|mn|}$). Their result has further been generalized to the arithmetic progressions $c \equiv 0 \bmod (s)$ by Ganguly and Sengupta [10]. Blomer and Milićević [1] have considered the different congruence restriction $c \equiv a \bmod (r)$ with $(a, r) = 1$ in (1.1), though in their work they only treated the "Linnik" range ($C \geqslant \sqrt{|mn|}$). In a different context, Kiral and Young [17] recently remarked how Kuznetsov trace formulas for different congruence subgroups and characters can be combined to incorporate both congruence conditions $c \equiv 0 \bmod (s)$ and $c \equiv a \bmod (r)$ in (1.1) simultaneously (assuming that $(r, as) = 1$) in a simplified manner.

Motivated by an application to the efficiency of a certain universal set of quantum gates, Browning, Kumaraswamy and Steiner [3] have proposed the following twisted version of the Linnik–Selberg conjecture.

CONJECTURE 2 (Twisted Linnik–Selberg). *Let $B, C \geqslant 1$ and let $m, n \in \mathbb{Z}$ be non-zero. Let $s \in \mathbb{N}$ and let $a \in \mathbb{Z}/s\mathbb{Z}$. Then, for any $\alpha \in [-B, B]$, we have*

$$\sum_{\substack{c \equiv a \bmod (s) \\ c \leqslant C}} \frac{1}{c} S(m, n; c) e\left(\frac{2\sqrt{mn}}{c}\alpha\right) \ll_{\epsilon, s, B} (|mn|C)^{\epsilon}$$

for any $\epsilon > 0$.

The same exponential twist arises also in a different context when establishing a spectral large sieve; compare [7, Proposition 3 and Theorem 2]. Albeit, there an additional average over m and n is present, which they made extensive use of and then summed over c trivially.

In this paper we are concerned only with the sum over c. In other words we shall establish some progress towards Conjecture 2. Before we state our results we shall introduce some simplifying notation: $F \lesssim G$ means $|F| \leqslant K_{\epsilon}(Cmns(1+|\alpha|))^{\epsilon} G$ for some positive constant K_{ϵ}, depending on ϵ, and every $\epsilon > 0$.

THEOREM 1. *Let $C \geqslant 1$, $\alpha \in \mathbb{R}$, $s \in \mathbb{N}$, and $m, n \in \mathbb{Z}$ with $mn > 0$, $s \ll \min\{(mn)^{1/4}, C^{1/2}\}$, and $(m, n, s) = 1$. Then we have*

$$\begin{aligned}
&\sum_{\substack{c \leqslant C \\ c \equiv 0 \bmod (s)}} \frac{1}{c} S(m, n; c) e\left(\frac{2\sqrt{mn}}{c}\alpha\right) \\
&\quad + 2\pi \sum_{t_h \in \mathrm{i}[0,\theta]} \frac{\sqrt{mn} \cdot \overline{\rho_h(m)} \rho_h(n)}{\cos(\pi |t_h|)} \int_{4\pi\sqrt{mn}/C}^{\infty} Y_{2|t_h|}(x) \mathrm{e}^{\mathrm{i}\alpha x} \frac{dx}{x} \\
&\lesssim \frac{C^{1/6}}{s^{1/3}} + |\alpha| + (1 + |\alpha|^{1/3}) \frac{(mn)^{1/6}}{s^{2/3}} + \frac{m^{1/4}(m, s)^{1/4} + n^{1/4}(n, s)^{1/4}}{s^{1/2}} \\
&\quad + \min\left\{\frac{(mn)^{1/8+\theta/2}(mn, s)^{1/8}}{s^{1/2}}, \frac{(mn)^{1/4}(mn, s)^{1/4}}{s}\right\},
\end{aligned}$$

where Y_t is the Bessel function of the second kind of order t, θ is the best known progress towards the Ramanujan–Selberg conjecture, and the summation t_h is over all exceptional eigenfunctions h with eigenvalue $\frac{1}{4}+t_h^2$ of the Laplacian for the manifold $\Gamma_0(s)\backslash\mathbb{H}$, where $\rho_h(n)$ denotes its n-th L^2-normalized Fourier coefficient.

A few remarks are in order about this theorem. First, we should remark that one has $\theta \leqslant \frac{7}{64}$ by the work of Kim and Sarnak [**16**]. Next, we observe the appearance of a main term, which is contrary to [**10**]. Indeed, the latter has an erroneous treatment of the exceptional spectrum[1]. One may further analyse the main term by making use of asymptotics of the Bessel function of the second kind $Y_t(y)$ for $y \to 0$. However, the reader familiar with Bessel functions may know that these asymptotics behave quite differently for $t = 0$ and $t > 0$ and therefore it would generate uniformity issues in the parameter s. If one disregards these uniformity issues by fixing all parameters except for C, one finds that each exceptional eigenvalue t_h gives rise to a main term of the size $C^{2|t_h|}$ assuming that $\rho_h(m)\rho_h(n) \neq 0$. One may also bound the main term altogether. In this case one gets the following corollary.

COROLLARY 2. *Assume the same assumptions as in Theorem 1. Then we have*

$$\sum_{\substack{c\leqslant C\\ c\equiv 0 \bmod (s)}} \frac{1}{c} S(m,n;c) e\left(\frac{2\sqrt{mn}}{c}\alpha\right)$$
$$\lesssim \frac{C^{1/6}}{s^{1/3}} + C^{2\theta} + (1+|\alpha|^{1/3})\frac{(mn)^{1/6}}{s^{2/3}} + \frac{m^{1/4}(m,s)^{1/4}+n^{1/4}(n,s)^{1/4}}{s^{1/2}}$$
$$+\min\left\{\frac{(mn)^{1/8+\theta/2}(mn,s)^{1/8}}{s^{1/2}}, \frac{(mn)^{1/4}(mn,s)^{1/4}}{s}\right\}.$$

As far as the restrictions go in Theorem 1, they are not very limiting. Indeed, if $s \geqslant C^{1/2}$, then the Weil bound, which gives the bound $s^{-1+\epsilon}C^{1/2+\epsilon}$, is more than sufficient and, if $(mn)^{1/4} \leqslant s \leqslant C^{1/2}$, then one is automatically in the easier Linnik range and for instance the holomorphic contribution is negligible. One may also consider $mn < 0$, which would lead one to analyse different Bessel transforms, or incorporate the further restriction $c \equiv a \bmod (r)$ with $(a,r) = 1$. However, for the latter, an analogue to Proposition 9 for the group $\Gamma_0(s)\cap\Gamma_1(r)$ is required. In fact, the associated Kloosterman sums for this group admit further cancellation. This can be observed in [**12**], for example, thereby leading to stronger results in terms of the parameter r. Investigations of this sort shall be considered by the author in future work.

[1] At the end of page 161, the compact domain to which they apply the mean value theorem of calculus varies with x. In order to make their argument rigorous, they would need to consider the domain $(\nu, y) \in [-\frac{7}{32}, \frac{7}{32}] \times [0,1]$, say. However, the function $J_\nu(y)$ has a singularity at $y = 0$ for $\nu < 0$, which in turn is responsible for the occurrence of a main term.

The parameter α will play an important role in the discussion as it introduces a phase in the Bessel transforms. Since the Bessel functions naturally possess a phase of their own, some interesting phenomena will occur. In particular, the case $|\alpha| = 1$ stands out as this is where the stationary phase is at infinity and transitions from being a point of stationary phase for the Bessel transform corresponding to the holomorphic spectrum to the one corresponding to the non-holomorphic or Maass spectrum. Therefore, it should come as no surprise that the cases $|\alpha| < 1$ and $|\alpha| > 1$ behave differently in nature. In the case $|\alpha| < 1$, we are able to slightly improve upon Theorem 1, thereby recovering the results of [**23**] and [**10**].

THEOREM 3. *Let $C \geqslant 1$, $\alpha \in \mathbb{R}$ with $|\alpha| < 1$, $s \in \mathbb{N}$, $m, n \in \mathbb{Z}$ with $mn > 0$, $s \ll \min\{(mn)^{1/4}, C^{1/2}\}$, and $(m, n, s) = 1$. Then we have*

$$\begin{aligned}
&\sum_{\substack{c\leqslant C\\ c\equiv 0 \bmod (s)}} \frac{1}{c} S(m,n;c) e\left(\frac{2\sqrt{mn}}{c}\alpha\right) \\
&\quad + 2\pi \sum_{t_h \in \mathrm{i}[0,\theta]} \frac{\sqrt{mn}\cdot\overline{\rho_h(m)}\rho_h(n)}{\cos(\pi|t_h|)} \int_{4\pi\sqrt{mn}/C}^{\infty} Y_{2|t_h|}(x) \mathrm{e}^{\mathrm{i}\alpha x} \frac{dx}{x} \\
&\lesssim (1-|\alpha|)^{-1/2-\epsilon} \left(\frac{C^{1/6}}{s^{1/3}} + \frac{m^{1/8}(m,s)^{1/8} + n^{1/8}(n,s)^{1/8}}{s^{1/4}} \right. \\
&\quad \times \min\left\{ (mn)^{\theta/2}, \frac{m^{1/8}(m,s)^{1/8} + n^{1/8}(n,s)^{1/8}}{s^{1/4}} \right\} \\
&\quad \left. + \frac{(mn)^{1/6}}{s^{2/3}} + \min\left\{ \frac{(mn)^{1/16+3\theta/4}(mn,s)^{1/16}}{s^{1/4}}, \frac{(mn)^{1/4}(mn,s)^{1/4}}{s} \right\} \right)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{\substack{c\leqslant C\\ c\equiv 0 \bmod (s)}} \frac{1}{c} S(m,n;c) e\left(\frac{2\sqrt{mn}}{c}\alpha\right) \\
&\quad \lesssim (1-|\alpha|)^{-1/2-\epsilon} \left(\frac{C^{1/6}}{s^{1/3}} + \frac{m^{1/8}(m,s)^{1/8} + n^{1/8}(n,s)^{1/8}}{s^{1/4}} \right. \\
&\quad \times \min\left\{ (mn)^{\theta/2}, \frac{m^{1/8}(m,s)^{1/8} + n^{1/8}(n,s)^{1/8}}{s^{1/4}} \right\} + \frac{(mn)^{1/6}}{s^{2/3}} \\
&\quad \left. + \min\left\{ \frac{(mn)^{1/16+3\theta/4}(mn,s)^{1/16}}{s^{1/4}}, \frac{(mn)^{1/4}(mn,s)^{1/4}}{s} \right\} \right) + C^{2\theta}.
\end{aligned}$$

The main goal in [**3**] was to show that it is possible to improve Sardari's work on covering exponents for S^3 [**22**] under the assumption that Conjecture 2 holds. It is unfortunate that the derived upper bounds in Theorems 1 and 3 are not strong enough to offer any unconditional improvement. The reason behind this is that in the application one is very deep in the Selberg range, for which the trivial bound is still the best known bound. Discussions on exactly why the Selberg range poses great difficulties can be found in [23].

Finally, we would like to point out a little gem that is hidden inside Theorem 1.

COROLLARY 4. *Let* $C \in \mathbb{R}^+$ *and* $Q(T) = mT^2 + lT + n \in \mathbb{Z}[T]$ *with* $mn > 0$. *Then we have*

$$\sum_{c \leqslant C} \frac{1}{c} \sum_{\substack{a \bmod (c)\\ (a,c)=1}} e\left(\frac{Q(a)\overline{a}}{c}\right) \ll_\epsilon C^{1/6+\epsilon} + \max\{|l|, |m|, |n|\}^{23/64+\epsilon}.$$

Upon noting that the inner sum is equal to $S(m, n; c)e(l/c)$, this is essentially a consequence of Theorem 1. Since there is no congruence restriction, the relevant group is $\mathrm{SL}_2(\mathbb{Z})$, which has no exceptional spectrum. One should further remark that as there is no exceptional spectrum the term $|\alpha|$ in Theorem 1 can be omitted as in the proof of Corollary 2. As a consequence, we find that either there is cancellation in the sign or very often the inner exponential sum is much smaller than $\sqrt{c}$.

§2. *Holomorphic and Maass forms.* In this section, we set up some notation and recall necessary facts about holomorphic and Maass forms.

Let $\mathbb{H}$ be the upper half-plane and let $\mathrm{SL}_2(\mathbb{R})$ act on it by Möbius transformations:

$$\gamma \cdot z = \gamma z = \frac{az+b}{cz+d}, \qquad j(\gamma, z) = cz + d, \quad \text{where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

We consider the following congruence subgroup:

$$\Gamma_0(s) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \bmod (s) \right\}.$$

For a given cusp $\mathfrak{a}$ of $\Gamma_0(s)$, we fix a matrix $\sigma_\mathfrak{a} \in \mathrm{SL}_2(\mathbb{R})$ such that $\sigma_\mathfrak{a}\infty = \mathfrak{a}$ and, if $\Gamma_\mathfrak{a}$ denotes the stabilizer of $\mathfrak{a}$, then $\sigma_\mathfrak{a}^{-1}\Gamma_\mathfrak{a}\sigma_\mathfrak{a} = \Gamma_\infty$, where $\Gamma_\infty = \{\pm T^n | n \in \mathbb{Z}\}$ is the stabilizer at ∞ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Such a matrix is called a scaling matrix for the cusp $\mathfrak{a}$.

The space of cuspidal Maass forms is spanned by the real-analytic square-integrable eigenfunctions of the Laplacian on the space $L^2(\Gamma_0(s)\backslash\mathbb{H})$ with respect to the inner product

$$\langle h_1, h_2 \rangle = \int_{\Gamma_0(s)\backslash\mathbb{H}} h_1(z)\overline{h_2(z)} \frac{dx\,dy}{y^2}. \tag{2.1}$$

Such a Maass cusp form h possesses a Fourier expansion of the shape

$$h(z) = \sum_{\substack{n \in \mathbb{Z}\\ n \neq 0}} \rho_h(n) W_{0,\mathrm{i}t_h}(4\pi|n|y)e(nx), \tag{2.2}$$

where $W_{a,b}$ is the Whittaker function, $z = x + iy$, and $\frac{1}{4} + t_h^2$ ($t_h \in [0, \infty) \cup i[0, 1/2]$) is the eigenvalue with respect to the Laplacian. A theory of Hecke operators as well as Atkin–Lehner theory can be developed for this space. In particular for a newform h we have

$$\sqrt{n}\rho_h(n) = \lambda_h(n)\rho_h(1) \quad \text{for all } n \in \mathbb{N},$$

where $\lambda_h(n)$ is the eigenvalue with respect to the n-th Hecke operator, which furthermore satisfies $\lambda_h(n) \ll_\epsilon n^{\theta+\epsilon}$, where $\theta = \frac{7}{64}$ is admissible by the work of Kim and Sarnak [16].

We shall require a special basis of this space, which has been worked out in [2][2]. For a Maass newform of level $r|s$, define the arithmetic functions

$$r_h(c) = \sum_{b|c} \frac{\mu(b)\lambda_h(b)^2}{b}\left(\sum_{d|b}\frac{\chi_0(d)}{d}\right)^{-2}, \quad A(c) = \sum_{b|c}\frac{\mu(b)\chi_0(b)^2}{b^2},$$
$$B(c) = \sum_{b|c}\frac{\mu(b)^2\chi_0(b)}{b},$$

where χ_0 is the trivial character modulo r. For $l|d$, define

$$\xi'_{d,h}(l) = \frac{\mu(d/l)\lambda_h(d/l)}{r_h(d)^{1/2}(d/l)^{1/2}B(d/l)}, \qquad \xi''_{d,h}(l) = \frac{\mu_h(d/l)}{r_h(d)^{1/2}(d/l)^{1/2}A(d)^{1/2}},$$

where $\mu_h(c)$ is a multiplicative function defined by the following equation of Dirichlet series:

$$\left(\sum_{c\geqslant 1}\frac{\lambda_h(c)}{c^z}\right)^{-1} = \sum_{c\geqslant 1}\frac{\mu_h(c)}{c^z}.$$

Write $d = d_1d_2$ with d_1 square-free and d_2 square-full and $(d_1, d_2) = 1$. Then, for $l|d$, define

$$\xi_{d,h}(l) = \xi'_{d_1,h}((d_1, l))\xi''_{d_2,h}((d_2, l)) \ll_\epsilon d^\epsilon. \tag{2.3}$$

Then an orthonormal basis of Maass forms of level s is given by

$$\bigcup_{r|s}\ \bigcup_{\substack{h \text{ new} \\ \text{of level } r}} \left\{h^d(z) = \sum_{l|d}\xi_{d,h}(l)h(lz)\ \middle|\ d\,\middle|\,\frac{s}{r}\right\}. \tag{2.4}$$

We furthermore need a bound on the size of the Fourier coefficient of an element of the above basis. We have

$$\sqrt{n}\rho_{h^d}(n) = \sum_{l|(d,n)}\sqrt{l}\xi_{d,h}(l)\lambda_h\left(\frac{n}{l}\right)\rho_h(1)$$

[2] Corrections can be found at http://www.uni-math.gwdg.de/blomer/corrections.pdf.

$$\begin{aligned} &\ll_\epsilon (ns)^\epsilon n^\theta |\rho_h(1)| \sum_{l|(d,n)} l^{1/2-\theta} \\ &\ll_\epsilon (ns)^\epsilon n^\theta \left(\frac{s}{r}\right)^{1/2} |\rho_h(1)|, \end{aligned} \tag{2.5}$$

where we have made use of (2.3) and $\lambda_h(n) \ll_\epsilon n^{\theta+\epsilon}$. Since h is new of level r, but normalized with respect to the inner product of level s (2.1), we further have

$$|\rho_h(1)| \ll_\epsilon (s(1+|t_h|))^\epsilon \left(\frac{\cosh(\pi t_h)}{s}\right)^{1/2}, \tag{2.6}$$

due to Hoffstein and Lockhart [11].

Other Maass forms which are important in our discussion are the Eisenstein series associated to a cusp $\mathfrak{c}$. They are defined for $\mathrm{Re}(\tau) > 1$ as

$$E_\mathfrak{c}(z, \tau) = \sum_{\gamma \in \Gamma_\infty \backslash \sigma_\mathfrak{c}^{-1} \Gamma_0(s)} \mathrm{Im}(\gamma z)^\tau$$

and admit a meromorphic extension to the whole complex plane. They also admit a Fourier expansion of the same shape (2.2), which at the point $\tau = \frac{1}{2} + \mathrm{i}t$ we write as

$$E_\mathfrak{c}(z, \tfrac{1}{2} + \mathrm{i}t) = \varphi_\mathfrak{c}(0, t; z) + \sum_{n \neq 0} \varphi_\mathfrak{c}(n, t) W_{0,\mathrm{i}t}(4\pi |n| y) e(nx).$$

For holomorphic forms the situation is quite analogous. A holomorphic cusp form of weight $k \in \mathbb{N}$ of level s is a holomorphic function $h : \mathbb{H} \to \mathbb{C}$ that satisfies $j(\gamma, z)^{-k} h(\gamma z) = h(z)$ for all $\gamma \in \Gamma_0(s)$ and is square-integrable with respect to the inner product

$$\langle h_1, h_2 \rangle = \int_{\Gamma_0(s) \backslash \mathbb{H}} h_1(z) \overline{h_2(z)} y^k \frac{dx\, dy}{y^2}. \tag{2.7}$$

Holomorphic cusp forms also admit a Fourier expansion of a different shape

$$h(z) = \sum_{n \geqslant 1} \psi_h(n) e(nz)$$

and there is a theory of Hecke and Atkin–Lehner operators. For h a newform, we have

$$\psi_h(n) = \lambda_h(n) \psi_h(1),$$

where $\lambda_h(n)$ is the eigenvalue of the n-th Hecke operator, which furthermore satisfies the bound $\lambda_h(n) \ll_\epsilon n^{(k-1)/2+\epsilon}$ due to Deligne [4, 5] and Deligne and Serre [6]. Analogous to the Maass case we have a nice orthonormal basis of the space $S_k(s)$ of holomorphic cusp forms of level s and weight k:

$$\bigcup_{r|s} \bigcup_{\substack{h \text{ new} \\ \text{of level } r}} \left\{ h^d(z) = \sum_{l|d} \xi_{d,h}(l) l^{k/2} h(lz) \;\middle|\; d \,\middle|\, \frac{s}{r} \right\}. \tag{2.8}$$

We furthermore need a bound on the size of the Fourier coefficients of an element of the above basis. We have

$$\begin{aligned}\psi_{h^d}(n) &= \sum_{l\mid(d,n)} \xi_{d,h}(l) l^{k/2} \lambda_h\left(\frac{n}{l}\right)\psi_h(1)\\ &\ll_\epsilon (ns)^\epsilon n^{(k-1)/2} |\psi_h(1)| \sum_{l\mid(d,n)} l^{1/2}\\ &\ll_\epsilon (ns)^\epsilon n^{(k-1)/2}\left(\frac{s}{r}\right)^{1/2} |\psi_h(1)|, \end{aligned} \tag{2.9}$$

where we have made use of the Deligne bound as well as (2.3). We further have the bound

$$|\psi_h(1)| \ll_\epsilon \frac{(4\pi)^{(k-1)/2}}{s^{1/2}\Gamma(k)^{1/2}}(ks)^\epsilon, \tag{2.10}$$

when h is new of level r, but normalized with respect to (2.7); see for example [**19**, pp. 41 and 42].

§3. *Proof of the theorem.* We shall prove a dyadic version of Theorem 1 from which we shall deduce Theorem 1.

THEOREM 5. *Let* $\alpha \in \mathbb{R}$, $s \in \mathbb{N}$, $m, n \in \mathbb{Z}$ *with* $mn > 0$, *and* $(m, n, s) = 1$. *Assume that* $s \ll \min\{(mn)^{1/4}, C^{1/2}\}$. *Then we have*

$$\begin{aligned} &\sum_{\substack{C\leqslant c<2C\\ c\equiv 0 \bmod (s)}} \frac{1}{c} S(m,n;c) e\left(\frac{2\sqrt{mn}}{c}\alpha\right)\\ &\quad + 2\pi \sum_{t_h\in \mathrm{i}[0,\theta]} \frac{\sqrt{mn}\cdot\overline{\rho_h(m)}\rho_h(n)}{\cos(\pi|t_h|)} \int_{2\pi\sqrt{mn}/C}^{4\pi\sqrt{mn}/C} Y_{2|t_h|}(x) e^{\mathrm{i}\alpha x}\frac{dx}{x}\\ &\lesssim \frac{C^{1/6}}{s^{1/3}} + (1+|\alpha|)\frac{(mn)^{1/2}}{C} + \frac{m^{1/4}(m,s)^{1/4}+n^{1/4}(n,s)^{1/4}}{s^{1/2}}\\ &\quad + \min\left\{\frac{(mn)^{\theta/2+1/8}(mn,s)^{1/8}}{s^{1/2}}, \frac{(mn)^{1/4}(mn,s)^{1/4}}{s}\right\}. \end{aligned}$$

For $|\alpha| < 1$, *we can do slightly better:*

$$\begin{aligned} &\sum_{\substack{C\leqslant c<2C\\ c\equiv 0 \bmod (s)}} \frac{1}{c} S(m,n;c) e\left(\frac{2\sqrt{mn}}{c}\alpha\right)\\ &\quad + 2\pi \sum_{t_h\in \mathrm{i}[0,\theta]} \frac{\sqrt{mn}\cdot\overline{\rho_h(m)}\rho_h(n)}{\cos(\pi|t_h|)} \int_{2\pi\sqrt{mn}/C}^{4\pi\sqrt{mn}/C} Y_{2|t_h|}(x) e^{\mathrm{i}\alpha x}\frac{dx}{x}\\ &\lesssim (1-|\alpha|)^{-1/2-\epsilon}\left(\frac{C^{1/6}}{s^{1/3}} + \frac{m^{1/8}(m,s)^{1/8}+n^{1/8}(n,s)^{1/8}}{s^{1/4}}\right. \end{aligned}$$

$$\times \min\left\{(mn)^{\theta/2}, \frac{m^{1/8}(m,s)^{1/8}+n^{1/8}(n,s)^{1/8}}{s^{1/4}}\right\}$$
$$+\frac{(mn)^{1/2}}{C}+\min\left\{\frac{(mn)^{3\theta/4+1/16}(mn,s)^{1/16}}{s^{1/4}}, \frac{(mn)^{1/4}(mn,s)^{1/4}}{s}\right\}\bigg).$$

We follow the argument in [23] and [10], and replace the sharp cut-off with a smooth cut-off and then use the Kuznetsov trace formula. We shall require the following version of the Kuznetsov trace formula.

PROPOSITION 6 (Kuznetsov trace formula). *Let $s\in\mathbb{N}$ and $m,n\in\mathbb{Z}$ be two integers with $mn>0$. Then, for any C^3-class function f with compact support in $]0,\infty)$, one has*

$$\sum_{c\equiv 0 \bmod (s)} \frac{1}{c}S(m,n;c)f\left(\frac{4\pi\sqrt{mn}}{c}\right)$$
$$= \mathcal{H}^s(m,n;f)+\mathcal{M}^s(m,n;f)+\mathcal{E}^s(m,n;f),$$

where

$$\mathcal{H}^s(m,n;f)=\frac{1}{\pi}\sum_{\substack{k\equiv 0 \bmod (2)\\ k>0}}\ \sum_{\substack{\{h_{j,k}\}_j\ ONB\\ of\ S_k(s)}} \frac{\mathrm{i}^k\Gamma(k)}{(4\pi\sqrt{mn})^{k-1}}\overline{\psi_{h_{j,k}}(m)}\psi_{h_{j,k}}(n)\widetilde{f}(k-1),$$
$$\mathcal{M}^s(m,n;f)=4\pi\sum_h \frac{\sqrt{mn}}{\cosh\pi t_h}\overline{\rho_h(m)}\rho_h(n)\widehat{f}(t_h),$$
$$\mathcal{E}^s(m,n;f)=\sum_{\mathfrak{c}\ \mathrm{cusp}}\int_{-\infty}^{\infty}\frac{\sqrt{mn}}{\cosh(\pi t)}\overline{\varphi_{\mathfrak{c}}(m,t)}\varphi_{\mathfrak{c}}(n,t)\widehat{f}(t)\,dt.$$

Here $\sum_h$ is a sum over an orthonormal basis of Maass forms with respect to the group $\Gamma_0(s)$ and the Bessel transforms are given by

$$\widetilde{f}(t)=\int_0^\infty J_t(y)f(y)\frac{dy}{y},$$
$$\widehat{f}(t)=\frac{\mathrm{i}}{\sinh\pi t}\int_0^\infty \frac{J_{2\mathrm{i}t}(x)-J_{-2\mathrm{i}t}(x)}{2}f(x)\frac{dx}{x},$$

where $J_t(y)$ is the Bessel function of the first kind of order t.

Proof. See [21] or [7]. □

From now on let $f(x)=\mathrm{e}^{\mathrm{i}\alpha x}g(x)$ with $g\in C^\infty([0,\infty),\mathbb{R}_0^+)$ a smooth real-valued bump function satisfying the following properties:

(i) $g(x)=1$ for $2\pi\sqrt{mn}/C\leqslant x\leqslant 4\pi\sqrt{mn}/C$;
(ii) $g(x)=0$ for $x\leqslant 2\pi\sqrt{mn}/(C+T)$ and $x\geqslant 4\pi\sqrt{mn}/(C-T)$;
(iii) $\|g'\|_1\ll 1$ and $\|g''\|_1\ll C/(X\cdot T)$,

where

$$X = \frac{4\pi\sqrt{mn}}{C} \quad \text{and} \quad 1 \leqslant T \leqslant \frac{C}{2}. \tag{3.1}$$

The parameter T will be chosen at a later point. Note that we have $\operatorname{Supp} g \subseteq [X/3, 2X]$.

We now wish to compare the smooth sum

$$\sum_{c \equiv 0 \bmod (s)} \frac{1}{c} S(m, n; c) f\left(\frac{4\pi\sqrt{mn}}{c}\right) \tag{3.2}$$

with the sum in Theorem 5 having the sharp cut-off. By making use of the Weil bound for the Kloosterman sum, we find that their difference is bounded by

$$\begin{aligned}\sum_{\substack{C-T\leqslant c\leqslant C \text{ or}\\ 2C\leqslant c\leqslant 2C+2T,\\ c\equiv 0 \bmod (s)}} \frac{1}{c}|S(m,n;c)| &\leqslant \sum_{\substack{C-T\leqslant c\leqslant C \text{ or}\\ 2C\leqslant c\leqslant 2C+2T\\ c\equiv 0 \bmod (s)}} \frac{\tau(c)}{\sqrt{c}}(m,n,c)^{1/2} \\ &\leqslant \frac{\tau(s)}{\sqrt{s}} \sum_{e|(m,n)} \sum_{\substack{(C-T)/se\leqslant c'\leqslant C/se \text{ or}\\ 2C/se\leqslant c'\leqslant (2C+2T)/se}} \frac{\tau(ec')}{\sqrt{ec'}} \mathrm{e}^{1/2} \\ &\lesssim \frac{1}{\sqrt{s}} \sum_{e|(m,n)} \frac{\sqrt{se}}{\sqrt{C}}\left(1 + \frac{T}{se}\right) \\ &\lesssim \frac{1}{\sqrt{C}}\left((m,n)^{1/2} + \frac{T}{s}\right).\end{aligned} \tag{3.3}$$

Now we apply Kuznetsov (see Proposition 6) to the smooth sum (3.2). This leads to the expression

$$\begin{aligned}&\sum_{c\equiv 0 \bmod (s)} \frac{1}{c} S(m,n;c) f\left(\frac{4\pi\sqrt{mn}}{c}\right) \\ &\quad = \mathcal{H}^s(m,n;f) + \mathcal{M}^s(m,n;f) + \mathcal{E}^s(m,n;f).\end{aligned}$$

We shall deal with each of these terms separately in §§3.2, 3.3 and 3.1, respectively.

In what follows we will use many estimates on the Bessel transforms of f, which we summarize here, but postpone their proof until §4.

LEMMA 7. *Let f be defined as it is immediately preceding (3.1). Then we have*

$$\widehat{f}(t), \widetilde{f}(t) \ll \frac{1 + |\log(X)| + \log^+(|\alpha|)}{1 + X^{1/2} + ||\alpha|^2 - 1|^{1/2} X} \quad \textit{for all } t \in \mathbb{R}, \tag{3.4}$$

$$\widehat{f}(\mathrm{i}t) = -\frac{1}{2}\int_{X/2}^{X} Y_{2t}(x)\mathrm{e}^{\mathrm{i}\alpha x}\frac{dx}{x} + O_{\epsilon,\delta}\left(1 + \frac{T}{C}X^{-2t-\epsilon}\right)$$

$$\textit{for all } 0 \leqslant t \leqslant \frac{1}{4} - \delta, \tag{3.5}$$

where $\log^+(x) = \max\{0, \log(x)\}$. *For* $t \geqslant 8$, *we have*

$$\int_0^{t/2} J_t(y) f(y) \frac{dy}{y} \ll \mathbb{1}_{[2X/3,\infty)}(t) \cdot t^{-1/2} \mathrm{e}^{-(2/5)t}, \tag{3.6}$$

$$\int_{t/2}^{t-t^{1/3}} J_t(y) f(y) \frac{dy}{y} \ll \mathbb{1}_{[1/4,\infty)}(X) \mathbb{1}_{[X/3,4X]}(t) \cdot t^{-1} (\log(t))^{2/3}, \tag{3.7}$$

$$\int_{t-t^{1/3}}^{t+t^{1/3}} J_t(y) f(y) \frac{dy}{y} \ll \mathbb{1}_{[1/4,\infty)}(X) \mathbb{1}_{[3X/16,3X]}(t) \cdot t^{-1}, \tag{3.8}$$

$$\int_{t+t^{1/3}}^{\infty} J_t(y) f(y) \frac{dy}{y} \ll \mathbb{1}_{[1/4,\infty)}(X) \mathbb{1}_{[0,2X]}(t) \cdot t^{-1} \times \min\left\{1 + |1 - |\alpha||^{-1/4}, \left(\frac{X}{t}\right)^{1/2}\right\}, \tag{3.9}$$

where $\mathbb{1}_{\mathcal{I}}$ *is the characteristic function of the interval* $\mathcal{I}$. *Finally, when* $|t| \geqslant 1$ *and either* $|t| \notin [(\frac{1}{12})||\alpha|^2 - 1|^{1/2} X, 2||\alpha|^2 - 1|^{1/2} X]$ *or* $|\alpha| \leqslant 1$, *we have*

$$\widehat{f}(t) \ll |t|^{-3/2} \left(1 + \min\left\{\left(\frac{X}{|t|}\right)^{1/2}, ||\alpha|^2 - 1|^{-1} \left(\frac{X}{|t|}\right)^{-3/2}\right\}\right), \tag{3.10}$$

$$\widehat{f}(t) \ll \frac{C}{T} |t|^{-5/2} \left(1 + \min\left\{\left(\frac{X}{|t|}\right)^{3/2}, ||\alpha|^2 - 1|^{-2} \left(\frac{X}{|t|}\right)^{-5/2}\right\}\right). \tag{3.11}$$

One should mention that similar estimates have been derived previously by Jutila [**15**, Lemma 3 and Remarks 1 and 2] for a slightly different class of functions and ranges.

3.1. *The continuous spectrum.* The goal of this section is to prove the following bound on the continuous contribution:

$$\mathcal{E}^s(m, n; f) \lesssim 1. \tag{3.12}$$

For this endeavour, we need the following lemma.

LEMMA 8. *Let* $s = s_\star s_\square^2$ *with* $s_\star$ *square-free and let* m, n *be positive integers. We have*

$$\sum_{\mathfrak{c} \text{ cusp}} \frac{\sqrt{mn}}{\cosh(\pi t)} \overline{\varphi_{\mathfrak{c}}(m, t)} \varphi_{\mathfrak{c}}(n, t) \ll_\epsilon \frac{(m, s_\star s_\square)^{1/2} (n, s_\star s_\square)^{1/2}}{s_\star s_\square} (mns(1 + |t|))^\epsilon.$$

Proof. This is part of [**1**, Lemma 1]. □

Substituting this inequality into the definition of $\mathcal{E}^s(m,n;f)$ (see Proposition 6) yields the bound

$$\mathcal{E}^s(m,n;f) \lesssim \frac{(m, s_\star s_\square)^{1/2}(n, s_\star s_\square)^{1/2}}{s_\star s_\square}\int_{-\infty}^{\infty}(1+|t|)^\epsilon|\widehat{f}(t)|\,dt$$
$$\lesssim \int_{-\infty}^{\infty}(1+|t|)^\epsilon|\widehat{f}(t)|\,dt.$$

We split the integral up into three parts:

$$\mathcal{I}_1 = \pm[\tfrac{1}{12}||\alpha|^2-1|^{1/2}X, 2||\alpha|^2-1|^{1/2}X],$$
$$\mathcal{I}_2 = [-\max\{1,X^{1/2}\},\max\{1,X^{1/2}\}]\backslash\mathcal{I}_1,$$
$$\mathcal{I}_3 = \pm[\max\{1,X^{1/2}\},\infty)\backslash\mathcal{I}_1.$$

For $\mathcal{I}_1$, we use (3.4) and arrive at

$$\int_{\mathcal{I}_1}(1+|t|)^\epsilon|\widehat{f}(t)|\,dt \ll_\epsilon \int_{\mathcal{I}_1}(1+|t|)^\epsilon\frac{1+|\log(X)|+\log^+(|\alpha|)}{||\alpha|^2-1|^{1/2}X}\,dt$$
$$\ll_\epsilon (1+X)^\epsilon(1+|\alpha|)^\epsilon(1+|\log(X)|+\log^+(|\alpha|))$$
$$\lesssim 1.$$

For $\mathcal{I}_2$, we use (3.4) again and arrive at

$$\int_{\mathcal{I}_2}(1+|t|)^\epsilon|\widehat{f}(t)|\,dt \ll_\epsilon \int_{\mathcal{I}_2}(1+|t|)^\epsilon\frac{1+|\log(X)|+\log^+(|\alpha|)}{1+X^{1/2}}dt$$
$$\ll_\epsilon (1+X)^\epsilon(1+|\log(X)|+\log^+(|\alpha|))$$
$$\lesssim 1.$$

For $\mathcal{I}_3$, we use (3.10) and arrive at

$$\int_{\mathcal{I}_3}(1+|t|)^\epsilon|\widehat{f}(t)|\,dt \ll_\epsilon \int_{\mathcal{I}_3}|t|^{-3/2+\epsilon}\left(1+\left(\frac{X}{|t|}\right)^{1/2}\right)dt$$
$$\ll_\epsilon \min\{1,X^{-1/4+\epsilon}\}+X^{1/2}\min\{1,X^{-1/2+\epsilon}\}$$
$$\lesssim 1.$$

This concludes the proof of (3.12).

3.2. *The holomorphic spectrum.* The goal of this section is to prove the following inequality:

$$\mathcal{H}^s(m,n;f) \lesssim 1+X. \tag{3.13}$$

In order to prove this inequality, we choose our orthonormal basis as in (2.8). Then

$$\mathcal{H}^s(m,n;f) = \frac{1}{\pi}\sum_{\substack{k\equiv 0 \bmod (2)\\ k>0}}\sum_{r|s}\sum_{\substack{h\in S_k(r)\\ \text{new}}}\sum_{d|\frac{s}{r}}\frac{i^k\Gamma(k)}{(4\pi\sqrt{mn})^{k-1}}\overline{\psi_{h^d}(m)}\psi_{h^d}(n)\widetilde{f}(k-1)$$

$$\begin{aligned}
&\lesssim \sum_{\substack{k\equiv 0 \bmod (2)\\ k>0}} \sum_{r\mid s} \sum_{\substack{h\in S_k(r)\\ \text{new}}} \sum_{d\mid \frac{s}{r}} \frac{\Gamma(k)}{(4\pi)^{k-1}} \frac{s}{r}|\psi_h(1)|^2|\widetilde{f}(k-1)| \\
&\lesssim \sum_{\substack{k\equiv 0 \bmod (2)\\ k>0}} \sum_{r\mid s} \sum_{\substack{h\in S_k(r)\\ \text{new}}} \frac{1}{r}|\widetilde{f}(k-1)| \\
&\lesssim \sum_{\substack{k\equiv 0 \bmod (2)\\ k>0}} k^{1+\epsilon}|\widetilde{f}(k-1)|,
\end{aligned}$$

where we have made use of (2.9), (2.10), and $\dim S_k(r) \ll rk$. The latter sum we split up into $k \leqslant 9$ and $k > 9$. Using (3.4), we find that

$$\sum_{\substack{k\equiv 0 \bmod (2)\\ 9\geqslant k>0}} k^{1+\epsilon}|\widetilde{f}(k-1)| \ll 1 + |\log(X)| + \log^+(|\alpha|) \lesssim 1.$$

We also find that

$$\sum_{\substack{k\equiv 0 \bmod (2)\\ k>9}} k^{1+\epsilon}|\widetilde{f}(k-1)| \leqslant \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4,$$

where

$$\begin{aligned}
\mathcal{S}_1 &= \sum_{\substack{k\equiv 0 \bmod (2)\\ k>9}} k^{1+\epsilon}\left|\int_0^{(k-1)/2} J_{k-1}(y) f(y)\frac{dy}{y}\right|, \\
\mathcal{S}_2 &= \sum_{\substack{k\equiv 0 \bmod (2)\\ k>9}} k^{1+\epsilon}\left|\int_{(k-1)/2}^{(k-1)-(k-1)^{1/3}} J_{k-1}(y) f(y)\frac{dy}{y}\right|, \\
\mathcal{S}_3 &= \sum_{\substack{k\equiv 0 \bmod (2)\\ k>9}} k^{1+\epsilon}\left|\int_{(k-1)-(k-1)^{1/3}}^{(k-1)+(k-1)^{1/3}} J_{k-1}(y) f(y)\frac{dy}{y}\right|, \\
\mathcal{S}_4 &= \sum_{\substack{k\equiv 0 \bmod (2)\\ k>9}} k^{1+\epsilon}\left|\int_{(k-1)+(k-1)^{1/3}}^{\infty} J_{k-1}(y) f(y)\frac{dy}{y}\right|.
\end{aligned}$$

Using (3.6), we find that

$$\mathcal{S}_1 \ll_\epsilon \sum_{k>9} k^{1/2+\epsilon} \mathrm{e}^{-(2/5)k} \ll_\epsilon 1.$$

Using (3.7), we find that

$$\mathcal{S}_2 \ll_\epsilon \sum_{X/3\leqslant k-1\leqslant 4X} k^\epsilon \lesssim 1 + X.$$

Using (3.8), we find that

$$\mathcal{S}_3 \ll_\epsilon \sum_{3X/16 \leqslant k-1 \leqslant 3X} k^\epsilon \lesssim 1 + X.$$

Using (3.9), we find that

$$\mathcal{S}_4 \ll_\epsilon \sum_{3X/2 \geqslant k-1 > 8} k^\epsilon \left(\frac{X}{k}\right)^{1/2} \lesssim 1 + X.$$

The claim (3.13) now follows.

3.3. *The non-holomorphic spectrum.* In this section, we shall prove the following two estimates:

$$\begin{aligned}
&\mathcal{M}^s(m,n;\, f) + 2\pi \sum_{t_h \in \mathrm{i}[0,\theta]} \frac{\sqrt{mn}\cdot\overline{\rho_h(m)}\rho_h(n)}{\cos(\pi|t_h|)} \int_{X/2}^{X} Y_{2|t_h|}(x)\mathrm{e}^{\mathrm{i}\alpha x}\frac{dx}{x} \\
&\quad \lesssim \left(\frac{C}{T}\right)^{1/2} + (1+|\alpha|)X + \left(1 + \frac{T}{C}X^{-2\theta}\right)\Bigg(1 + \frac{m^{1/4}(m,s)^{1/4} + n^{1/4}(n,s)^{1/4}}{s^{1/2}} \\
&\qquad + \min\left\{\frac{(mn)^{\theta/2+1/8}(mn,s)^{1/8}}{s^{1/2}}, \frac{(mn)^{1/4}(mn,s)^{1/4}}{s}\right\}\Bigg)
\end{aligned} \tag{3.14}$$

and for $|\alpha| < 1$ also

$$\begin{aligned}
&\mathcal{M}^s(m,n;\, f) + 2\pi \sum_{t_h \in \mathrm{i}[0,\theta]} \frac{\sqrt{mn}\cdot\overline{\rho_h(m)}\rho_h(n)}{\cos(\pi|t_h|)} \int_{X/2}^{X} Y_{2|t_h|}(x)\mathrm{e}^{\mathrm{i}\alpha x}\frac{dx}{x} \\
&\quad \lesssim (1-|\alpha|)^{-1/2-\epsilon}\Bigg[\left(\frac{C}{T}\right)^{1/2} + \left(1 + \frac{T}{C}X^{-2\theta}\right) \\
&\qquad \times \Bigg(1 + \frac{m^{1/8}(m,s)^{1/8} + n^{1/8}(n,s)^{1/8}}{s^{1/4}} \\
&\qquad \times \min\left\{(mn)^{\theta/2}, \frac{m^{1/8}(m,s)^{1/8} + n^{1/8}(n,s)^{1/8}}{s^{1/4}}\right\} \\
&\qquad + \min\left\{\frac{(mn)^{3\theta/4+1/16}(mn,s)^{1/16}}{s^{1/4}}, \frac{(mn)^{1/4}(mn,s)^{1/4}}{s}\right\}\Bigg)\Bigg].
\end{aligned} \tag{3.15}$$

We shall require the following proposition.

PROPOSITION 9. *Let $A \geqslant 1$ and $n \in \mathbb{N}$. Then we have for the group $\Gamma_0(s)$*

$$\sum_{|t_h| \leqslant A} \frac{n}{\cosh(\pi t_h)}|\rho_h(n)|^2 \ll_\epsilon A^2 + \frac{\sqrt{n}}{s}(n,s)^{1/2}(ns)^\epsilon.$$

Proof. For the full modular group this is due to Kuznetsov [**18**, equation (5.19)] and only minor modifications yield the above; see for example [**24**, Lemma 2.9] or [**10**, Theorem 9][3]. □

Let us first prove (3.14). We split the summation over t_h in $\mathcal{M}^s(m, n; f)$ into various ranges $\mathcal{I}_1, \ldots, \mathcal{I}_4$, which are treated individually. They are

$$\begin{aligned}
\mathcal{I}_1 &= [0, \max\{1, X^{1/2}\}],\\
\mathcal{I}_2 &= [\tfrac{1}{12}||\alpha|^2 - 1|^{1/2}X, 2||\alpha|^2 - 1|^{1/2}X]\backslash\mathcal{I}_1,\\
\mathcal{I}_3 &= [\max\{1, X^{1/2}\}, \infty)\backslash\mathcal{I}_2,\\
\mathcal{I}_4 &= i[0, \tfrac{1}{2}].
\end{aligned}$$

The first way to treat the range $\mathcal{I}_1$ is to choose the basis (2.4) and use (2.5) as well as (2.6):

$$\begin{aligned}
&\sum_{t_h \in \mathcal{I}_1} \frac{\sqrt{mn}}{\cosh(\pi t_h)} \overline{\rho_h(m)} \rho_h(n) \widehat{f}(t_h)\\
&\quad \lesssim (mn)^{\theta} \sum_{r|s} \frac{1}{r} \sum_{\substack{t_h \in \mathcal{I}_1\\ \text{new of level } r}} \sum_{d|\frac{s}{r}} (1 + |t_h|)^{\epsilon} \sup_{t \in \mathcal{I}_1} |\widehat{f}(t)|.
\end{aligned}$$

Next, we use (3.4) to bound the transform and a uniform Weyl law to bound the number of Maass forms h of level r with $t_h \leqslant T$ by $r^{1+\epsilon}T^2$ (see for example [**20**, Corollary 3.2.3]). We arrive at the bound

$$\lesssim (mn)^{\theta}(1 + X^{1/2}). \tag{3.16}$$

A second way to treat the range $\mathcal{I}_1$ is to apply the Cauchy–Schwarz inequality in conjunction with Proposition 9 and (3.4):

$$\begin{aligned}
&\sum_{t_h \in \mathcal{I}_1} \frac{\sqrt{mn}}{\cosh(\pi t_h)} \overline{\rho_h(m)} \rho_h(n) \widehat{f}(t_h)\\
&\leqslant \left(\sum_{t_h \in \mathcal{I}_1} \frac{m}{\cosh(\pi t_h)} |\rho_h(m)|^2 \right)^{1/2} \left(\sum_{t_h \in \mathcal{I}_1} \frac{n}{\cosh(\pi t_h)} |\rho_j(n)|^2 \right)^{1/2} \sup_{t \in \mathcal{I}_1} |\widehat{f}(t)|\\
&\lesssim \left(1 + X + \frac{\sqrt{m}}{s}(m, s)^{1/2} \right)^{1/2} \left(1 + X + \frac{\sqrt{n}}{s}(n, s)^{1/2} \right)^{1/2} \frac{1}{1 + X^{1/2}}\\
&\lesssim 1 + X^{1/2} + \frac{m^{1/4}(m, s)^{1/4} + n^{1/4}(n, s)^{1/4}}{s^{1/2}}\\
&\quad + \frac{(mn)^{1/4}(mn, s)^{1/4}}{s(1 + X^{1/2})}.
\end{aligned} \tag{3.17}$$

[3] The factor $(n, s)^{1/2}$ is missing in this reference. The author assumes this is due to the ongoing assumption in the paper that n is coprime to the level of the congruence subgroup, i.e. $(n, s) = 1$, since the author of the current paper is unaware of a proof that allows omitting this factor.

The range $\mathcal{I}_2$ we treat in exactly the same manner and we arrive at the inequalities

$$\sum_{t_h \in \mathcal{I}_2} \frac{\sqrt{mn}}{\cosh(\pi t_h)} \overline{\rho_h(m)} \rho_h(n) \widehat{f}(t_h) \lesssim (mn)^\theta \frac{(1+||\alpha|^2-1|^{1/2}X)^2}{1+||\alpha|^2-1|^{1/2}X}$$
$$\lesssim (mn)^\theta (1+||\alpha|^2-1|^{1/2}X) \quad (3.18)$$

and

$$\sum_{t_h \in \mathcal{I}_2} \frac{\sqrt{mn}}{\cosh(\pi t_h)} \overline{\rho_h(m)} \rho_h(n) \widehat{f}(t_h)$$
$$\lesssim \frac{(1+||\alpha|^2-1|^{1/2}X + m^{1/4}(m,s)^{1/4}/s^{1/2})(1+||\alpha|^2-1|^{1/2}X + n^{1/4}(n,s)^{1/4}/s^{1/2})}{1+||\alpha|^2-1|^{1/2}X}$$
$$\lesssim 1+||\alpha|^2-1|^{1/2}X + \frac{m^{1/4}(m,s)^{1/4}+n^{1/4}(n,s)^{1/4}}{s^{1/2}}$$
$$+ \frac{(mn)^{1/4}(mn,s)^{1/4}}{s(1+||\alpha|^2-1|^{1/2}X)}. \quad (3.19)$$

The range $\mathcal{I}_3$ we further split into dyadic ranges

$$\mathcal{I}_3(l) = [2^l \max\{1, X^{1/2}\}, 2^{l+1} \max\{1, X^{1/2}\}] \backslash \mathcal{I}_2, \quad l \geqslant 0.$$

Again, we can estimate

$$\sum_{t_h \in \mathcal{I}_3(l)} \frac{\sqrt{mn}}{\cosh(\pi t_h)} |\rho_h(m)\rho_h(n)| \lesssim (mn)^\theta 2^{2l}(1+X) \quad (3.20)$$

and

$$\sum_{t_h \in \mathcal{I}_3(l)} \frac{\sqrt{mn}}{\cosh(\pi t_h)} |\rho_h(m)\rho_h(n)|$$
$$\lesssim 2^{2l}(1+X) + 2^l(1+X^{1/2}) \frac{m^{1/4}(m,s)^{1/4}+n^{1/4}(n,s)^{1/4}}{s^{1/2}}$$
$$+ \frac{(mn)^{1/4}(mn,s)^{1/4}}{s}. \quad (3.21)$$

However, this time we use (3.10) and (3.11) to deal with the transform. We have

$$\sup_{t \in \mathcal{I}_3(l)} |\widehat{f}(t)|$$
$$\lesssim \begin{cases} \min\left\{ \dfrac{1+X^{1/2}}{2^{2l}(1+X)}, \dfrac{C}{T} \dfrac{1+X^{3/2}}{2^{4l}(1+X)^2} \right\} & \text{for } l \leqslant \log_2(\max\{1, X^{1/2}\}), \\ \min\left\{ \dfrac{1}{2^{(3/2)l}(1+X)^{3/4}}, \dfrac{C}{T} \dfrac{1}{2^{(5/2)l}(1+X)^{5/4}} \right\} & \text{for } l > \log_2(\max\{1, X^{1/2}\}). \end{cases}$$
$$(3.22)$$

Combining (3.20), (3.21), and (3.22), we find that the contribution stemming from $l \leqslant \log_2(\max\{1, X^{1/2}\})$ is

$$
\begin{aligned}
&\lesssim \sum_{l \leqslant \log_2(\max\{1, X^{1/2}\})} \Bigg(1 + X^{1/2} + 2^{-l} \frac{m^{1/4}(m,s)^{1/4} + n^{1/4}(n,s)^{1/4}}{s^{1/2}} \\
&\qquad + \min\left\{(mn)^{\theta}(1+X)^{1/2}, 2^{-2l} \frac{(mn)^{1/4}(mn,s)^{1/4}}{s(1+X)^{1/2}}\right\}\Bigg) \\
&\lesssim 1 + X^{1/2} + \frac{m^{1/4}(m,s)^{1/4} + n^{1/4}(n,s)^{1/4}}{s^{1/2}} \\
&\qquad + \min\left\{\frac{(mn)^{\theta/2+1/8}(mn,s)^{1/8}}{s^{1/2}}, \frac{(mn)^{1/4}(mn,s)^{1/4}}{s}\right\}
\end{aligned} \tag{3.23}
$$

and the contribution from $l > \log_2(\max\{1, X^{1/2}\})$ is

$$
\begin{aligned}
&\lesssim \sum_{l > \log_2(\max\{1, X^{1/2}\})} \Bigg(\left(\frac{C}{T}\right)^{1/2+\delta} 2^{-\delta l} + 2^{-l/2} \frac{m^{1/4}(m,s)^{1/4} + n^{1/4}(n,s)^{1/4}}{s^{1/2}} \\
&\qquad + 2^{-l/2} \min\left\{\frac{(mn)^{\theta/2+1/8}(mn,s)^{1/8}}{s^{1/2}}, \frac{(mn)^{1/4}(mn,s)^{1/4}}{s}\right\}\Bigg) \\
&\lesssim \left(\frac{C}{T}\right)^{1/2} + \frac{m^{1/4}(m,s)^{1/4} + n^{1/4}(n,s)^{1/4}}{s^{1/2}} \\
&\qquad + \min\left\{\frac{(mn)^{\theta/2+1/8}(mn,s)^{1/8}}{s^{1/2}}, \frac{(mn)^{1/4}(mn,s)^{1/4}}{s}\right\}
\end{aligned} \tag{3.24}
$$

for a sufficiently small $\delta > 0$.

For the contribution from $\mathcal{I}_4$, we first note that we have $|t_h| \leqslant \theta$ for $t_h \in \mathcal{I}_4$ by [16]. We first insert (3.5) and further find that

$$
\begin{aligned}
&4\pi \sum_{t_h \in \mathrm{i}[0,\theta]} \frac{\sqrt{mn}}{\cosh(\pi t_h)} \overline{\rho_h(m)} \rho_h(n) \\
&\qquad \times \left(-\frac{1}{2} \int_{X/2}^{X} Y_{2|t_h|}(x) \mathrm{e}^{\mathrm{i}\alpha x} \frac{dx}{x} + O_\epsilon\left(1 + \frac{T}{C} X^{-2|t_h|-\epsilon}\right)\right) \\
&= -2\pi \sum_{t_h \in \mathrm{i}[0,\theta]} \frac{\sqrt{mn} \cdot \overline{\rho_h(m)} \rho_h(n)}{\cos(\pi |t_h|)} \int_{X/2}^{X} Y_{2|t_h|}(x) \mathrm{e}^{\mathrm{i}\alpha x} \frac{dx}{x} \\
&\qquad + O_\epsilon\Bigg(\left(1 + \frac{T}{C} X^{-2\theta-\epsilon}\right) \\
&\qquad \times \min\left\{(mn)^{\theta}, 1 + \frac{m^{1/4}(m,s)^{1/4} + n^{1/4}(n,s)^{1/4}}{s^{1/2}} \right. \\
&\qquad \left. + \frac{(mn)^{1/4}(mn,s)^{1/4}}{s}\right\}\Bigg).
\end{aligned} \tag{3.25}
$$

Combining the minimum of (3.16) and (3.17), the minimum of (3.18) and (3.19), (3.23), and (3.24) with (3.25) gives (3.14).

Let us now turn our attention to (3.15). This time we split up into the intervals

$$\mathcal{I}_1 = [0, 1],$$
$$\mathcal{I}_2 = [1, \infty),$$
$$\mathcal{I}_3 = \mathrm{i}\big[0, \tfrac{1}{2}\big].$$

By making use of (3.4), we find that the contribution from $\mathcal{I}_1$ is bounded by

$$\lesssim \min\left\{(mn)^{\theta}, 1 + \frac{m^{1/4}(m,s)^{1/4} + n^{1/4}(n,s)^{1/4}}{s^{1/2}} + \frac{(mn)^{1/4}(mn,s)^{1/4}}{s}\right\}. \tag{3.26}$$

As before, we split up $\mathcal{I}_2$ into dyadic ranges $\mathcal{I}_2(l) = [2^l, 2^{l+1}]$, $l \geqslant 0$, and use

$$\sup_{t\in\mathcal{I}_2(l)} |\widehat{f}(t)| \lesssim \min\left\{(1-|\alpha|)^{-1/4}2^{-(3/2)l}, \frac{C}{T}(1-|\alpha|)^{-3/4}2^{-(5/2)l}\right\},$$

which follows from (3.10) and (3.11). Thus, we find that the contribution from $\mathcal{I}_2$ is bounded by

$$\begin{aligned}
&\lesssim (1-|\alpha|)^{-1/2-\delta/2}\sum_{l\geqslant 0}\Bigg(\left(\frac{C}{T}\right)^{1/2+\delta}2^{-\delta l}\\
&\quad + \min\left\{(mn)^{\theta/2-\theta\delta}\frac{m^{1/8+\delta/4}(m,s)^{1/8+\delta/4} + n^{1/8+\delta/4}(n,s)^{1/8+\delta/4}}{s^{1/4+\delta/2}}2^{-\delta l},\right.\\
&\quad\quad \left.\frac{m^{1/4}(m,s)^{1/4} + n^{1/4}(n,s)^{1/4}}{s^{1/2}}2^{-(1/2)l}\right\}\\
&\quad + \min\left\{(mn)^{3\theta/4-\theta\delta}\frac{(mn)^{1/16+\delta/4}(mn,s)^{1/16+\delta/4}}{s^{1/4+\delta}}2^{-2\delta l},\right.\\
&\quad\quad \left.\frac{(mn)^{1/4}(mn,s)^{1/4}}{s}2^{-(3/2)l}\right\}\Bigg)\\
&\lesssim (1-|\alpha|)^{-1/2-\epsilon}\Bigg(\left(\frac{C}{T}\right)^{1/2} + \frac{m^{1/8}(m,s)^{1/8} + n^{1/8}(n,s)^{1/8}}{s^{1/4}}\\
&\quad \times \min\left\{(mn)^{\theta/2}, \frac{m^{1/8}(m,s)^{1/8} + n^{1/8}(n,s)^{1/8}}{s^{1/4}}\right\}\\
&\quad + \min\left\{\frac{(mn)^{3\theta/4+1/16}(mn,s)^{1/16}}{s^{1/4}}, \frac{(mn)^{1/4}(mn,s)^{1/4}}{s}\right\}\Bigg)
\end{aligned} \tag{3.27}$$

for $\delta > 0$ small enough. The contribution from $\mathcal{I}_3$ is the same as in (3.25). Combining (3.26), (3.27), and (3.25) gives (3.15).

3.4. *Putting things together.* In order to show Theorem 5, we add up all the inequalities (3.3), (3.12), (3.13), (3.14) respectively (3.15), and make the choice $T = O(s^{2/3}C^{2/3})$, which is allowed since $s \ll \min\{(mn)^{1/4}, C^{1/2}\}$. One may note that we have

$$\frac{(m,n)^{1/2}}{\sqrt{C}} \leqslant X^{1/2} \leqslant 1 + X$$

and

$$\frac{T}{C}X^{-2\theta} \ll s^{2/3-4\theta}C^{2\theta-1/3}\cdot s^{4\theta}(mn)^{-\theta} \ll 1.$$

Theorem 1 follows now at once by estimating the range $c \leqslant \min\{(mn)^{1/2}, (1+|\alpha|^{2/3})s^{2/3}(mn)^{1/3}\}$ trivially using the Weil bound, which gives

$$\sum_{\substack{c\leqslant(1+|\alpha|^{2/3})s^{2/3}(mn)^{1/3}\\ c\equiv 0 \bmod (s)}} \frac{1}{c}|S(m,n;c)| \lesssim \frac{\left((1+|\alpha|^{2/3})s^{2/3}(mn)^{1/3}\right)^{1/2}}{s}$$

$$\lesssim (1+|\alpha|^{1/3})\frac{(mn)^{1/6}}{s^{2/3}}.$$

For the remaining range $\min\{(mn)^{1/2}, (1+|\alpha|^{2/3})s^{2/3}(mn)^{1/3}\} \leqslant c \leqslant C$, we use Theorem 5. Furthermore, note that

$$\int_1^\infty |Y_{2t}(x)|\frac{dx}{x} \ll \int_1^\infty x^{-3/2}\,dx \ll 1$$

uniformly for $t \leqslant \theta$ and hence we have

$$\sum_{t_h\in \mathrm{i}[0,\theta]} \frac{\sqrt{mn}\cdot|\rho_h(m)\rho_h(n)|}{\cos(\pi|t_h|)}\int_1^\infty |Y_{2|t_h|}(x)|\frac{dx}{x}$$
$$\lesssim \min\left\{(mn)^\theta, 1+\frac{m^{1/4}(m,s)^{1/4}+n^{1/4}(n,s)^{1/4}}{s^{1/2}}+\frac{(mn)^{1/4}(mn,s)^{1/4}}{s}\right\}, \tag{3.28}$$

which allows us to extend the integral in the main term to infinity. This proves Theorem 1. For Corollary 2, there is no need to extend the integral to infinity, which allows us to use the Weil bound for the whole range $c \leqslant (1+|\alpha|^{2/3})s^{2/3}(mn)^{1/3}$ and Theorem 5 for the complementary range. Upon recalling (3.28), it suffices to show that

$$\sum_{t_h\in \mathrm{i}[0,\theta]} \frac{\sqrt{mn}\cdot|\rho_h(m)\rho_h(n)|}{\cos(\pi|t_h|)}\int_X^1 |Y_{2|t_h|}(x)|\frac{dx}{x} \lesssim C^{2\theta}$$

when $C \geqslant \sqrt{mn}$. This follows from the two estimates

$$\int_X^1 |Y_{2t}(x)|\frac{dx}{x} \ll_\epsilon \int_X^1 x^{-2\theta-1-\epsilon}\,dx \ll_\epsilon X^{-2\theta-\epsilon}$$

and

$$\sum_{t_h\in \mathrm{i}[0,\theta]} \frac{\sqrt{mn}\cdot|\rho_h(m)\rho_h(n)|}{\cos(\pi|t_h|)} \ll_\epsilon (mn)^{\theta+\epsilon}.$$

Theorem 3 is proved analogously.

§4. *Transform estimates.* In this section, we prove the claimed upper bounds in Lemma 7 on the transforms of f. Since all the estimates are very different in nature, we split them up into multiple lemmas. We generally follow the arguments of [23] and [7], but tweak them to account for our introduced twist. First, we shall need two preliminary lemmas, which will be used frequently.

LEMMA 10. *Let $F, G \in C([A, B], \mathbb{C})$ with G having a continuous derivative. Then we have*

$$\left|\int_A^B F(x)G(x)\,dx\right| \ll (\|G\|_\infty + \|G'\|_1) \sup_{C\in[A,B]}\left|\int_A^C F(x)\,dx\right|.$$

Proof. We integrate by parts and find that

$$\int_A^B F(x)G(x)\,dx = \int_A^B F(x)\,dx \cdot G(B) - \int_A^B \int_A^y F(x)\,dx \cdot G'(y)\,dy,$$

from which the first statement is trivially deduced. □

LEMMA 11. *Let $G, H \in C^1([A, B], \mathbb{C})$ and assume that G has a zero and H' has at most K zeros. Then we have*

$$\|GH\|_\infty + \|(GH)'\|_1 \ll_K \|G'\|_1\|H\|_\infty.$$

Proof. We have $\|GH\|_\infty \leqslant \|G\|_\infty\|H\|_\infty$ and $\|G\|_\infty \leqslant \|G'\|_1$ since we have $G(b) = \int_a^b G'(x)\,dx$, where a is a zero of G. Furthermore, we have

$$\begin{aligned}\|(GH)'\|_1 &\leqslant \|G'H\|_1 + \|GH'\|_1 \leqslant \|G'\|_1\|H\|_\infty + \|G\|_\infty\|H'\|_1\\ &\leqslant \|G'\|_1(\|H\|_\infty + \|H'\|_1)\end{aligned}$$

and

$$\|H'\|_1 \leqslant 2(K+1)\|H\|_\infty$$

by splitting up the integral into intervals on which H' has a constant sign. □

LEMMA 12. *Let f be defined as it is immediately preceding (3.1) and $|\alpha| \leqslant 1$. Then we have*

$$\widetilde{f}(t) \ll \frac{1+|\log(X)|}{1+X^{1/2}+|1-|\alpha|^2|^{1/2}X} \quad \textit{for all } t \in \mathbb{R}.$$

Proof. We follow the proof of Lemma 7.1 in [7] and Proposition 5 in [23]. To prove the first statement, we use the Bessel representation

$$J_t(x) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}(x\sin\xi - t\xi)}\,d\xi,$$

which upon exchanging the order of integration yields

$$\widetilde{f}(t) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} \mathrm{e}^{\mathrm{i}x \sin\xi} \frac{f(x)}{x}\, dx\, \mathrm{e}^{-\mathrm{i}t\xi}\, d\xi.$$

Integration by parts yields

$$\begin{aligned}\int_0^{\infty} \mathrm{e}^{\mathrm{i}x \sin\xi} \frac{f(x)}{x}\, dx &= \int_0^{\infty} \mathrm{e}^{\mathrm{i}x(\sin\xi+\alpha)} \frac{g(x)}{x}\, dx \\ &= \frac{\mathrm{i}}{\sin\xi+\alpha} \int_0^{\infty} \mathrm{e}^{\mathrm{i}x(\sin\xi+\alpha)} \left(\frac{g(x)}{x}\right)' dx \\ &\ll \min\{1, X^{-1}|\sin\xi+\alpha|^{-1}\}.\end{aligned}$$

Thus, we find that

$$\widetilde{f}(t) \ll \int_0^{2\pi} \min\left\{1, X^{-1}|\sin\xi+\alpha|^{-1}\right\} d\xi.$$

Now, clearly, $\widetilde{f}(t) \ll 1$. For $X \geqslant 1$, we can do better though. We have $|\sin\xi + \alpha| \geqslant ||\sin\xi| - |\alpha||$; thus, we may assume that $\xi \in [0, \pi/2]$ and $\alpha \geqslant 0$. Set $\alpha = \sin\varphi$ with $\varphi \in [0, \pi/2]$. Then we have

$$\sin\xi - \alpha = 2\sin\left(\frac{\xi-\varphi}{2}\right)\sin\left(\frac{\pi-\xi-\varphi}{2}\right).$$

Now, for $x \in [-\pi/2, \pi/2]$, we have $|\sin(x)| \asymp |x|$; thus,

$$\begin{aligned}\widetilde{f}(t) &\ll \int_0^{\pi/2} \min\{1, X^{-1}|\xi-\varphi|^{-1}|\pi-\xi-\varphi|^{-1}\}\, d\xi \\ &\ll \int_0^{\pi/2} \min\left\{1, X^{-1}|\xi-\varphi|^{-1}\left|\frac{\pi}{2}-\varphi\right|^{-1}, X^{-1}|\xi-\varphi|^{-2}\right\} d\xi \\ &\ll \min\left\{\frac{1+\log(X)}{|\pi/2-\varphi|X}, X^{-1/2}\right\}.\end{aligned}$$

The first estimate follows from splitting the region of integration into $|\xi - \varphi| \leqslant X^{-1}|\pi/2 - \varphi|^{-1}$, for which we use the trivial bound of 1, and $|\xi - \varphi| \geqslant X^{-1}|\pi/2 - \varphi|^{-1}$, for which we use the bound $X^{-1}|\xi - \varphi|^{-1}|\pi/2 - \varphi|^{-1}$. In order to prove the second estimate, we split the integral into $|\xi - \varphi| \leqslant X^{-1/2}$, for which we use the trivial bound of 1, and $|\xi-\varphi| \geqslant X^{-1/2}$, for which we use the bound $X^{-1}|\xi - \varphi|^{-2}$. Now, we just have to note that $\pi/2-\varphi \asymp \sin(\pi/2-\varphi) = \sqrt{1 - |\alpha|^2}$. □

LEMMA 13. *Let f be defined as it is immediately preceding (3.1) and $|\alpha| \geqslant 1$. Then we have*

$$\widetilde{f}(t) \ll \frac{1+|\log(X)|}{1+X^{1/2}+||\alpha|^2-1|^{1/2}X} \quad \textit{for all } t \in \mathbb{R}.$$

Proof. As before, we find that $\widetilde{f}(t) \ll 1$ and for $X \geqslant 1$ we have

$$\begin{aligned}
\widetilde{f}(t) &\ll \int_0^{\pi/2} \min\left\{1, X^{-1}(|\alpha| - |\sin\xi|)^{-1}\right\} d\xi \\
&\ll \int_0^{\pi/2} \min\left\{1, X^{-1}\left(|\alpha| - 1 + \frac{1}{\pi}\left(\frac{\pi}{2} - \xi\right)^2\right)^{-1}\right\} d\xi \\
&\ll \int_0^{\pi/2} \min\left\{1, X^{-1}(|\alpha|-1)^{-1}, X^{-1}(|\alpha|-1)^{-1/2}\left(\frac{\pi}{2} - \xi\right)^{-1},\right. \\
&\qquad \left. X^{-1}\left(\frac{\pi}{2} - \xi\right)^2\right\} d\xi \\
&\ll \min\left\{\frac{1}{||\alpha| - 1|X}, \frac{1 + \log(X)}{||\alpha| - 1|^{1/2} X}, X^{-1/2}\right\}. \qquad \square
\end{aligned}$$

We also require some more refined estimates. For this, we consider the different regions of the J-Bessel function.

LEMMA 14. *Let f be defined as it is immediately preceding (3.1) and $|\alpha| \leqslant 1$. Then we have for $t \geqslant 8$,*

$$\begin{aligned}
\int_0^{t/2} J_t(y) f(y) \frac{dy}{y} &\ll \mathbb{1}_{[2X/3,\infty)}(t) \cdot t^{-1/2} e^{-(2/5)t}, \\
\int_{t/2}^{t - t^{1/3}} J_t(y) f(y) \frac{dy}{y} &\ll \mathbb{1}_{[1/4,\infty)}(X) \mathbb{1}_{[X/3,4X]}(t) \cdot t^{-1} (\log(t))^{2/3}, \\
\int_{t - t^{1/3}}^{t + t^{1/3}} J_t(y) f(y) \frac{dy}{y} &\ll \mathbb{1}_{[1/4,\infty)}(X) \mathbb{1}_{[3X/16,3X]}(t) \cdot t^{-1}, \\
\int_{t + t^{1/3}}^{\infty} J_t(y) f(y) \frac{dy}{y} &\ll \mathbb{1}_{[1/4,\infty)}(X) \mathbb{1}_{[0,2X]}(t) \cdot t^{-1} \\
&\qquad \times \min\left\{ |1 - |\alpha||^{-1/4}, \left(\frac{X}{t}\right)^{1/2} \right\}, \qquad (4.1)
\end{aligned}$$

where $\mathbb{1}_{\mathcal{I}}$ is the characteristic function of the interval $\mathcal{I}$.

Proof. We require some uniform estimates on the J-Bessel functions of real order. For small argument, we have exponential decay

$$0 \leqslant J_t(x) \leqslant \frac{e^{-tF(0,x/t)}}{(1 - (x/t)^2)^{1/4}\sqrt{2\pi t}} \quad \text{for all } x < t, \qquad (4.2)$$

where $F(0, x) = \log((1 + \sqrt{1 - x^2})/x) - \sqrt{1 - x^2}$. The left-hand side follows from the fact that the first zero of the Bessel function of order t is $> t$ [**25**, p. 254] and the right-hand side is [**25**, equation (9), p. 255]. We will also make use of Langer's formulas; see [**9**, pp. 30 and 89]. The first formula is

$$\begin{aligned}
J_t(x) = w^{-1/2}(w - \arctan(w))^{1/2}\left(\frac{\sqrt{3}}{2} J_{1/3}(z) - \frac{1}{2} Y_{1/3}(z)\right) + O(t^{-4/3}) \\
\text{for all } x > t, \qquad (4.3)
\end{aligned}$$

where

$$w = \sqrt{\frac{x^2}{t^2} - 1} \quad \text{and} \quad z = t(w - \arctan(w)).$$

The second one is

$$J_t(x) = \frac{1}{\pi} w^{-1/2} (\operatorname{artanh}(w) - w)^{1/2} K_{1/3}(z) + O(t^{-4/3}) \quad \text{for all } x < t, \tag{4.4}$$

where

$$w = \sqrt{1 - \frac{x^2}{t^2}} \quad \text{and} \quad z = t(\operatorname{artanh}(w) - w).$$

And finally for the transitional range $|x - t| \leqslant t^{1/3}$, we have

$$J_t(x) \ll t^{-1/3}, \tag{4.5}$$

by [25, equation (7), p. 247]. Although the required uniformity is not clearly stated, in this special case it does follow from their proof (see [25, pp. 244–247]).

The first inequality follows directly from (4.2):

$$\int_0^{t/2} J_t(y) f(y) \frac{dy}{y} \ll t^{-1/2} \mathrm{e}^{-(2/5)t} \cdot \frac{X}{X}.$$

Note that if $X \leqslant \frac{1}{2}$, then the other integrals are equal to zero as f vanishes identically on the range of integration. Thus, we may assume that $X \geqslant \frac{1}{2}$ from now on. For the range $[t/2, t - t^{1/3}]$, we use (4.4) and

$$\begin{aligned} z^{1/2} K_{1/3}(z) &= \left(\frac{\pi}{2}\right)^{1/2} \Gamma\left(\frac{5}{6}\right)^{-1} \mathrm{e}^{-z} \int_0^{\infty} \mathrm{e}^{-\xi} \left(\xi \left(1 + \frac{\xi}{2z}\right)\right)^{-1/6} d\xi \\ &\leqslant \left(\frac{\pi}{2}\right)^{1/2} \mathrm{e}^{-z} \quad \text{for all } z \geqslant 0; \end{aligned}$$

see [13, Appendix B] for the above integral representation. Thus, we find that

$$J_t(y) \ll (t^2 - y^2)^{-1/4} \mathrm{e}^{-z} + O(t^{-4/3}).$$

Now if $y \leqslant \min\{t - 9t^{1/3}(\log t)^{2/3}, t - t^{1/3}\}$, we have $z \geqslant \log t$ and thus $J_t(y) \ll t^{-4/3}$; otherwise, we have $J_t(y) \ll t^{-1/3}$. We conclude that

$$\int_{t/2}^{t - t^{1/3}} J_t(y) f(y) \frac{dy}{y} \ll t^{-4/3} \cdot \frac{X}{X} + t^{-1/3} \cdot \frac{t^{1/3} (\log(t))^{2/3}}{t}.$$

For the range $t - t^{1/3} \leqslant y \leqslant t + t^{1/3}$, we use (4.5) and get

$$\int_{t - t^{1/3}}^{t + t^{1/3}} J_t(y) f(y) \frac{dy}{y} \ll t^{-1/3} \cdot \frac{t^{1/3}}{t}.$$

We are left to deal with the range $t + t^{1/3} \leqslant y$. We make a change of variable $y \to ty$ and we are left to estimate

$$\int_{1+t^{-2/3}}^{\infty} J_t(ty)\mathrm{e}^{\mathrm{i}\alpha t y} g(ty) \frac{dy}{y}. \tag{4.6}$$

We make use of (4.3) and find that $z \gg 1$ in this range of y. By making use of Langer's formula (4.3), we introduce an error of the size

$$\ll t^{-4/3} \cdot \frac{X}{X},$$

which is acceptable. Since $z \gg 1$, we are able to make use of the classical estimates

$$\begin{aligned} J_{1/3}(z) &= \sqrt{\frac{2}{\pi z}} \left(\cos\left(z - \frac{\pi}{6} - \frac{\pi}{4} \right) + O(z^{-1}) \right), \\ Y_{1/3}(z) &= \sqrt{\frac{2}{\pi z}} \left(\sin\left(z - \frac{\pi}{6} - \frac{\pi}{4} \right) + O(z^{-1}) \right). \end{aligned} \tag{4.7}$$

Inserting (4.7) into (4.6) introduces another error of the size

$$t^{-1/2} \int_{1+t^{-2/3}}^{\infty} w^{-1/2} z^{-1} g(ty) \frac{dy}{y},$$

where $w = \sqrt{y^2 - 1}$ and $z = t(w - \arctan(w))$. We have $z \gg t \min\{w^3, w\}$ and thus we are able to estimate the above as

$$\begin{aligned} &\ll t^{-3/2} \int_{1+t^{-2/3}}^{2} \frac{g(ty)}{(y^2-1)^{7/4} y}\, dy + t^{-3/2} \int_{2}^{\infty} \frac{g(ty)}{(y^2-1)^{3/4} y}\, dy \\ &\ll t^{-3/2} \int_{1+t^{-2/3}}^{2} \frac{g(ty) y}{(y^2-1)^{7/4}}\, dy + t^{-3/2} \int_{2}^{\infty} \frac{g(ty)}{y^{5/2}}\, dy \\ &\ll t^{-1} + t^{-3/2} \end{aligned}$$

as $g(y) \ll 1$. This is again sufficient.

For the main term, we have to consider

$$t^{-1/2} \int_{1+t^{-2/3}}^{\infty} \mathrm{e}^{\mathrm{i}t(\pm\omega(y) + \alpha y)} \frac{g(ty)}{(y^2-1)^{1/4} y}\, dy, \tag{4.8}$$

where

$$\begin{aligned} \omega(y) &= \sqrt{y^2 - 1} - \arctan \sqrt{y^2 - 1}, \\ \omega'(y) &= \frac{\sqrt{y^2 - 1}}{y}. \end{aligned}$$

We would like to integrate $t(\pm\omega'(y) + \alpha)\mathrm{e}^{\mathrm{i}t(\pm\omega(y)+\alpha y)}$ by parts, but for the sign "$-\mathrm{sign}(\alpha)$" and $y_0 = (1 - \alpha^2)^{-1/2}$ we have $\omega'(y_0) = |\alpha|$ and we pick up a stationary phase. Let us first assume that α is close to 0 such that $y_0 < 1+t^{-2/3}$. For $|\alpha| \ll t^{-1/3}$ or the sign "sign(α)", we have $|\pm\omega'(1+t^{-2/3})+\alpha| \gg t^{-1/3}$ and we get by means of Lemmas 10 and 11 with $F(y) = (\pm\omega'(y)+\alpha)\mathrm{e}^{\mathrm{i}t(\pm\omega(y)+\alpha y)}$, $G(y) = g(ty)$, and $H(y) = [(\pm\omega'(y)+\alpha)(y^2-1)^{1/4}y]^{-1}$ a bound of $t^{-1/2}$ on the integral, which gives a satisfactory contribution of t^{-1} by recalling the extra factor $t^{-1/2}$ in front of the integral (4.8). So from now on we can assume that $\alpha > 0$, $\alpha \geqslant kt^{-1/3}$, for some small constant k, and the sign being "$-$". We treat first the case where $\alpha < 1$, where we make use of a Taylor expansion around y_0. We split up the integral (4.8) into three parts $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$ corresponding to the intervals $[1+t^{-2/3}, y_0-A]$, $[y_0-A, y_0+A]$, $[y_0+A, \infty]$, respectively, where A will be suitably chosen at a later stage. For $\mathcal{I}_1$ and $\mathcal{I}_3$, we again make use of Lemmas 10 and 11 with $F(y) = (\omega'(y)-\alpha)\mathrm{e}^{\mathrm{i}t(\omega(y)-\alpha y)}$, $G(y) = g(ty)$, and $H(y) = [(\omega'(y)-\alpha)(y^2-1)^{1/4}y]^{-1}$. For this, we require lower bounds on

$$R(x) = \sqrt{x^2-1} - \alpha x \quad \text{and} \quad (x^2-1)^{1/4}.$$

We have

$$R'(x) = \frac{x}{\sqrt{x^2-1}} - \alpha \quad \text{and} \quad R''(x) = -\frac{1}{(x^2-1)^{3/2}}.$$

We have that $R'(x)$ is decreasing and positive and hence $R(x)$ is increasing with a zero at y_0. Furthermore, we have that $R''(x)$ is increasing and negative. We conclude that

$$\begin{aligned} R(y_0+A) &\geqslant R(y_0) + R'(y_0)\cdot A + R''(y_0)\cdot\frac{A^2}{2} \\ &= \frac{1-\alpha^2}{\alpha}\cdot A - \left(\frac{1-\alpha^2}{\alpha^2}\right)^{3/2}\cdot\frac{A^2}{2} \\ &= \frac{1-\alpha^2}{\alpha}\cdot A\cdot\left(1-\frac{(1-\alpha^2)^{1/2}}{\alpha^2}\cdot\frac{A}{2}\right) \\ &\gg \frac{1-\alpha^2}{\alpha}\cdot A \end{aligned}$$

for $A \leqslant \alpha^2(1-\alpha^2)^{-1/2}$. We also have

$$\begin{aligned} -R(y_0-A) &\geqslant -R(y_0) + R'(y_0)A \\ &\gg \frac{1-\alpha^2}{\alpha}\cdot A. \end{aligned}$$

For the second factor, we have

$$((y_0+A)^2-1)^{1/4} \geqslant \left(\frac{\alpha^2}{1-\alpha^2}\right)^{1/4}$$

and

$$((y_0 - A)^2 - 1)^{1/4} \geqslant \left(\frac{\alpha^2}{1-\alpha^2} - \frac{2A}{(1-\alpha^2)^{1/2}}\right)^{1/4} \gg \left(\frac{\alpha^2}{1-\alpha^2}\right)^{1/4}$$

for $A \leqslant \frac{1}{4}\alpha^2(1-\alpha^2)^{-1/2}$. Thus, for $A \leqslant \frac{1}{4}\alpha^2(1-\alpha^2)^{-1/2}$, we find that the contribution from $\mathcal{I}_3$ is at most

$$t^{-3/2}\frac{1}{((1-\alpha^2)/\alpha)A \cdot (\alpha^2/(1-\alpha^2))^{1/4}} \ll t^{-3/2}\frac{\alpha^{1/2}}{(1-\alpha^2)^{3/4}A}$$

after recalling the extra factor of $t^{-1/2}$ in front of the integral (4.8). We claim that $-R(x)(x^2-1)^{1/4}$ increases first and then decreases in $[1, y_0]$. For this, it suffices to prove that its derivative has exactly one zero in that interval and is positive at $1+\epsilon$. Note that since our function is zero at the end points, we have by Rolle's theorem that there is at least a zero of the derivative. The derivative is

$$\frac{3\alpha x^2 - 3x(x^2-1)^{1/2} - 2\alpha}{2(x^2-1)^{3/4}},$$

which is clearly positive at $1+\epsilon$. Assume now that we have two zeros y_1, y_2 in $[1, y_0]$. They both satisfy the equation

$$3\alpha x^2 - 3x(x^2-1)^{1/2} - 2\alpha = 0 \Rightarrow 9(1-\alpha^2)x^4 + (12\alpha^2 - 9)x^2 - 4\alpha^2 = 0.$$

Now by Vieta's formula we have

$$2 \leqslant y_1^2 + y_2^2 = \frac{9 - 12\alpha^2}{9(1-\alpha^2)} = \frac{4}{3} - \frac{1}{3(1-\alpha^2)} \leqslant \frac{4}{3}$$

and thus a contradiction. With this information, we conclude that if $\alpha \geqslant Kt^{-1/3}$, for some large constant K, we have that the contribution from $\mathcal{I}_1$ to (4.8) is at most

$$\max\left\{t^{-1}, t^{-3/2}\frac{\alpha^{1/2}}{(1-\alpha^2)^{3/4}A}\right\}.$$

Furthermore, we estimate the integral over $\mathcal{I}_2$ trivially and get the bound

$$t^{-1/2}A\frac{(1-\alpha^2)^{3/4}}{\alpha^{1/2}}.$$

Choosing $A = t^{-1/2}\alpha^{1/2}(1-\alpha^2)^{-1/2}$, which we are allowed to do for K large enough, we get that (4.8) is bounded by

$$t^{-1}(1-|\alpha|)^{-1/4}.$$

We are left to deal with the case $\alpha \asymp t^{-1/3}$. In this case, we elongate the interval $\mathcal{I}_2$ to $[1+t^{-2/3}, y_0 + A]$ and estimate trivially again. Letting $A = \frac{1}{4}\alpha^2(1-\alpha^2)^{-1/2}$, we find that in this case one also has a bound of t^{-1} for $\mathcal{I}_2, \mathcal{I}_3$. This proves the first half of (4.1).

Let us assume now that $\alpha \geqslant 2\sqrt{2}/3$, so that α is close to 1 and $y_0 \geqslant 3$. Assume that $2X/t \leqslant y_0/2$, in which case the integrals over $\mathcal{I}_2$ and $\mathcal{I}_3$ vanish. We have

$$\min_{\substack{x\in[1+t^{-2/3},y_0/2]\\ x\in\frac{1}{t}\operatorname{Supp} g}} -R(x)(x^2-1)^{1/4} = \min_{\substack{x\in[1+t^{-2/3},y_0/2]\\ x\in\frac{1}{t}\operatorname{Supp} g}} \frac{1-(1-\alpha^2)x^2}{\alpha x+\sqrt{x^2-1}}(x^2-1)^{1/4}$$
$$\gg \min\left\{t^{-1/6}, \left(\frac{X}{t}\right)^{-1/2}\right\}$$

and hence the contribution from $\mathcal{I}_1$ is bounded by

$$t^{-3/2}\left(t^{1/6}+\left(\frac{X}{t}\right)^{1/2}\right).$$

Similarly for $(\frac{1}{3})X/t \geqslant 2y_0$, we have that the integrals over $\mathcal{I}_1$ and $\mathcal{I}_2$ are 0 and furthermore

$$\min_{\substack{x\in[2y_0,\infty)\\ x\in\frac{1}{t}\operatorname{Supp} g}} R(x)(x^2-1)^{1/4} = \min_{\substack{x\in[2y_0,\infty)\\ x\in\frac{1}{t}\operatorname{Supp} g}} \frac{(1-\alpha^2)x^2-1}{\alpha x+\sqrt{x^2-1}}(x^2-1)^{1/4}$$
$$\gg \left(\frac{X}{t}\right)^{-1/2}$$

and hence the contribution from $\mathcal{I}_3$ is bounded by

$$t^{-3/2}\left(\frac{X}{t}\right)^{1/2}.$$

Finally, when $X/t \asymp y_0$, we are able to replace $|1-|\alpha||^{-1/4}$ by $(X/t)^{1/2}$, which proves the last inequality in full for $|\alpha|<1$.

Now let us have a look at $\alpha = 1$. We proceed as before only that this time the stationary phase is at infinity; thus, we can directly apply Lemmas 10 and 11 with $F(y) = (\omega'(y)-1)\mathrm{e}^{\mathrm{i}t(\omega(y)-1y)}$, $G(y) = g(ty)$, and $H(y) = [(\omega'(y)-1)(y^2-1)^{1/4}y]^{-1}$. We need an upper bound on the quantity

$$\frac{1}{(y-\sqrt{y^2-1})(y^2-1)^{1/4}} \quad \text{for } y\in[1+t^{-2/3},\infty) \text{ and } ty\in \operatorname{Supp} g.$$

This function decreases and then increases; thus, it takes its maximum at the boundary. The values at the boundary are easily bounded by

$$\max\left\{t^{1/6}, \left(\frac{X}{t}\right)^{1/2}\right\}$$

and therefore we find that the same upper bound as for the case $|\alpha|<1$ holds for $|\alpha|=1$. $\square$

LEMMA 15. *Let f be defined as it is immediately preceding (3.1) and $|\alpha| \geqslant 1$. Then we have for $t \geqslant 8$,*

$$\int_0^{t/2} J_t(y) f(y) \frac{dy}{y} \ll \mathbb{1}_{[2X/3,\infty)}(t) \cdot t^{-1/2} \mathrm{e}^{-(2/5)t},$$
$$\int_{t/2}^{t-t^{1/3}} J_t(y) f(y) \frac{dy}{y} \ll \mathbb{1}_{[1/4,\infty)}(X) \mathbb{1}_{[X/3,4X]}(t) \cdot t^{-1} (\log(t))^{2/3},$$
$$\int_{t-t^{1/3}}^{t+t^{1/3}} J_t(y) f(y) \frac{dy}{y} \ll \mathbb{1}_{[1/4,\infty)}(X) \mathbb{1}_{[3X/16,3X]}(t) \cdot t^{-1},$$
$$\int_{t+t^{1/3}}^{\infty} J_t(y) f(y) \frac{dy}{y} \ll \mathbb{1}_{[1/4,\infty)}(X) \mathbb{1}_{[0,2X]}(t) \cdot t^{-1} \times \min\left\{1 + ||\alpha| - 1|^{-1/4}, \left(\frac{X}{t}\right)^{1/2}\right\},$$

where $\mathbb{1}_{\mathcal{I}}$ is the characteristic function of the interval $\mathcal{I}$.

Proof. We follow the argument as in the previous lemma. The first three inequalities follow immediately. For the last inequality, we need a lower bound on

$$\begin{aligned} &\min_{\substack{y \geqslant 1+t^{-2/3} \\ y \sim X/t}} l|(|\alpha| - \omega'(y))(y^2-1)^{1/4} y| \\ &\gg \min_{\substack{y \geqslant 1+t^{-2/3} \\ y \sim X/t}} \left(|\alpha| - 1 + \frac{y - \sqrt{y^2-1}}{y}\right)(y^2-1)^{1/4} y \\ &\gg \min_{\substack{y \geqslant 1+t^{-2/3} \\ y \sim X/t}} \left(|\alpha| - 1 + \frac{1}{y^2}\right)(y^2-1)^{1/4} y. \end{aligned}$$

If $X/t \asymp 1$, then the minimum is at least $|\alpha| t^{-1/6}$, which gives a contribution of $t^{-4/3}|\alpha|^{-1} \ll t^{-1}$; otherwise $X/t \gg 1$, in which case the minimum is at least

$$\max\left\{||\alpha| - 1|\left(\frac{X}{t}\right)^{3/2}, \left(\frac{X}{t}\right)^{-1/2}\right\} \gg \max\left\{||\alpha| - 1|^{1/4}, \left(\frac{X}{t}\right)^{-1/2}\right\}$$

giving a contribution of

$$t^{-3/2} \min\left\{||\alpha| - 1|^{-1/4}, \left(\frac{X}{t}\right)^{1/2}\right\}. \qquad \square$$

LEMMA 16. *Let f be defined as it is immediately preceding (3.1) and $|\alpha| \leqslant 1$. Then we have*

$$\widehat{f}(t) \ll \frac{1 + |\log(X)|}{1 + X^{1/2} + |1 - |\alpha|^2|^{1/2} X} \quad \text{for all } t \in \mathbb{R},$$

$$\widehat{f}(t) \ll |t|^{-3/2}\left(1+\min\left\{\left(\frac{X}{|t|}\right)^{1/2}, |1-|\alpha|^2|^{-1}\left(\frac{X}{|t|}\right)^{-3/2}\right\}\right)$$
for all $|t| \geqslant 1$,
$$\widehat{f}(t) \ll \frac{C}{T}|t|^{-5/2}\left(1+\min\left\{\left(\frac{X}{|t|}\right)^{3/2}, |1-|\alpha|^2|^{-2}\left(\frac{X}{|t|}\right)^{-5/2}\right\}\right)$$
for all $|t| \geqslant 1$.

Proof. We follow the proof of Lemma 7.1 in [7] and Proposition 5 in [23]. To prove the first inequality, we use the equation

$$J_{2\mathrm{i}t}(x) - J_{-2\mathrm{i}t}(x) = \frac{4\mathrm{i}}{\pi}\sinh \pi t \int_0^\infty \cos(x\cosh\xi)\cos(2t\xi)\,d\xi,$$

which follows from the integral representation found in [25, equation (12), p. 180]. We have by partial integration

$$\begin{aligned}\int_0^\infty \mathrm{e}^{\mathrm{i}(\pm x\cosh\xi)}\frac{f(x)}{x}\,dx &= \int_0^\infty \mathrm{e}^{\mathrm{i}x(\pm\cosh\xi+\alpha)}\frac{g(x)}{x}\,dx\\ &= \frac{\mathrm{i}}{\pm\cosh\xi+\alpha}\int_0^\infty \mathrm{e}^{\mathrm{i}x(\pm\cosh\xi+\alpha)}\left(\frac{g(x)}{x}\right)'dx\\ &\ll \min\{1, X^{-1}|\cosh\xi\pm\alpha|^{-1}\}.\end{aligned}$$

Thus, we find that

$$\widehat{f}(t) \ll \int_0^\infty \min\{1, X^{-1}|\cosh\xi\pm\alpha|^{-1}\}\,d\xi.$$

Hence, it suffices to bound the latter integral. It is bounded by

$$\begin{aligned}&\ll \int_0^1 \min\{1, X^{-1}(\xi^2+1-|\alpha|)^{-1}\}\,d\xi + \int_1^\infty \min\{1, X^{-1}\mathrm{e}^{-\xi}\}\,d\xi\\ &\ll \int_0^1 \min\{1, X^{-1}\xi^{-2}, X^{-1}\xi^{-1}|1-|\alpha||^{-1/2}, X^{-1}|1-|\alpha||^{-1}\}\,d\xi\\ &\quad + \int_1^\infty \min\{1, X^{-1}\mathrm{e}^{-\xi}\}\,d\xi.\end{aligned}$$

For $X \geqslant 1$, this is bounded by

$$\ll \min\left\{X^{-1/2}, \frac{1+\log(X)}{|1-|\alpha||^{1/2}X}, X^{-1}|1-|\alpha||^{-1}\right\} + X^{-1}$$

and for $X \leqslant 1$ it is bounded by

$$\ll_\epsilon 1 + |\log(X)|.$$

The first inequality follows immediately.

The final two inequalities require some more work. Note that $\widehat{f}(t)$ is even in t and thus we can restrict ourselves to $t \geqslant 1$. We make the substitution $x \to 2tx$ in the definition of $\widehat{f}(t)$

$$\widehat{f}(t) = \frac{\mathrm{i}}{\sinh \pi t} \int_0^\infty \frac{J_{2\mathrm{i}t}(2tx) - J_{-2\mathrm{i}t}(2tx)}{2} f(2tx) \frac{dx}{x}$$

and use the uniform asymptotic expansion of the function $G_{\mathrm{i}\nu}(\nu s)$ from [8, pp. 1008–1010][4]:

$$\begin{aligned} G_{2\mathrm{i}t}(2tx) &= \frac{1}{\sinh(\pi t)} \frac{J_{2\mathrm{i}t}(2tx) - J_{-2\mathrm{i}t}(2tx)}{2\mathrm{i}} \\ &= \left(\frac{1}{\pi t}\right)^{1/2} (1+x^2)^{-1/4} \left[\sin\left(2t\omega(x) - \frac{\pi}{4}\right) \right. \\ &\quad - \cos\left(2t\omega(x) - \frac{\pi}{4}\right) \frac{3(1+x^2)^{-1/2} - 5(1+x^2)^{-3/2}}{48t} \\ &\quad \left. + \frac{1}{2\mathrm{i}} (\mathrm{e}^{-\mathrm{i}(\pi/4)} \mathcal{E}_{2,1}(2t, \omega(x)) - \mathrm{e}^{\mathrm{i}(\pi/4)} \mathcal{E}_{2,2}(2t, \omega(x))) \right], \end{aligned}$$

where

$$\omega(x) = \sqrt{1+x^2} + \log\left(\frac{x}{1+\sqrt{1+x^2}}\right)$$

and the error terms satisfy

$$\mathcal{E}_{2,1}(2t, \omega(x)),\ \mathcal{E}_{2,2}(2t, \omega(x)) \ll |t|^{-2} \exp(O(|t|^{-1})).$$

Let us first deal with the error term. The contribution of the error term is bounded by

$$t^{-5/2} \int_0^\infty |f(2tx)| \frac{dx}{x} \ll t^{-5/2} \ll \min\left\{ |t|^{-3/2}, \frac{C}{T} |t|^{-5/2} \right\}.$$

For the remaining summands, we have to deal with integrals of the type

$$t^{-1/2} \int_0^\infty \frac{\mathrm{e}^{\pm 2\mathrm{i}t\omega(x)}}{(1+x^2)^{1/4+\beta}} f(2tx) \frac{dx}{x} = t^{-1/2} \int_0^\infty \frac{\mathrm{e}^{2\mathrm{i}t(\pm\omega(x)+\alpha x)}}{(1+x^2)^{1/4+\beta}} g(2tx) \frac{dx}{x}$$

with $\beta \in \{0, \frac{1}{2}, \frac{3}{2}\}$. We rewrite the above as

$$\frac{1}{2} t^{-3/2} \int_0^\infty (\mathrm{e}^{2\mathrm{i}t(\pm\omega(x)+\alpha x)} 2t(\pm\omega'(x) + \alpha)) \frac{g(2tx)}{x(\pm\omega'(x)+\alpha)(1+x^2)^{1/4+\beta}} \, dx. \tag{4.9}$$

Since

$$\omega'(x) = \frac{\sqrt{1+x^2}}{x} > 1,$$

[4] Unfortunately, [8, equation (5.16), p. 1010] only displays the odd expansions. The even expansions follow in the same way from the preceding discussions in the reference.

we have $\omega'(x) - |\alpha| > 0$. We apply Lemmas 10 and 11 with $F(x) = e^{2it(\pm\omega(x)+\alpha)}2t(\pm\omega'(x)+\alpha)$, $G(x) = g(2tx)$, and $H(x) = [x(\pm\omega'(x)+\alpha)(1+x^2)^{1/4+\beta}]^{-1}$. Moreover, we have

$$\begin{aligned}\min_{x\sim X/t}&\left|x(\pm\omega'(x)+\alpha)(1+x^2)^{1/4+\beta}\right|\\ &\gg \min_{x\sim X/t}\left|x\left(\frac{1}{x\sqrt{1+x^2}}+1-|\alpha|\right)(1+x^2)^{1/4}\right|\\ &\gg \min_{x\sim X/t}\max\left\{(1+x^2)^{-1/4},\,(1-|\alpha|)x(1+x^2)^{1/4}\right\}.\end{aligned}$$

For $x \ll 1$, we see that the minimum is bounded below by 1. If $x \gg 1$, then the minimum is bounded by below by

$$\max\left\{\left(\frac{X}{t}\right)^{-1/2}, |1-|\alpha||\left(\frac{X}{t}\right)^{3/2}\right\}.$$

Therefore, the integral (4.9) is bounded by

$$t^{-3/2}\left(1+\min\left\{\left(\frac{X}{t}\right)^{1/2}, |1-|\alpha||^{-1}\left(\frac{X}{t}\right)^{-3/2}\right\}\right).$$

This yields the second inequality. For the third inequality, we proceed from (4.9) with integration by parts. We have to deal with four new integrals

$$\begin{aligned}\mathcal{I}_1 &= t^{-5/2}\int_0^\infty (e^{2it(\pm\omega(x)+\alpha)}2t(\pm\omega'(x)+\alpha))\\ &\qquad\times\frac{g(2tx)}{x^2(\pm\omega'(x)+\alpha)^2(1+x^2)^{1/4+\beta}}\,dx,\\ \mathcal{I}_2 &= t^{-5/2}\int_0^\infty (e^{2it(\pm\omega(x)+\alpha)}2t(\pm\omega'(x)+\alpha))\\ &\qquad\times\frac{g(2tx)(\pm\omega''(x)x^2)}{x^3(\pm\omega'(x)+\alpha)^3(1+x^2)^{1/4+\beta}}\,dx,\\ \mathcal{I}_3 &= t^{-5/2}\int_0^\infty (e^{2it(\pm\omega(x)+\alpha)}2t(\pm\omega'(x)+\alpha))\\ &\qquad\times\frac{g(2tx)x^2}{x^2(\pm\omega'(x)+\alpha)^2(1+x^2)^{5/4+\beta}}\,dx,\\ \mathcal{I}_4 &= t^{-5/2}\int_0^\infty (e^{2it(\pm\omega(x)+\alpha)}2t(\pm\omega'(x)+\alpha))\\ &\qquad\times\frac{tx\cdot g'(2tx)}{x^2(\pm\omega'(x)+\alpha)^2(1+x^2)^{1/4+\beta}}\,dx.\end{aligned}$$

Proceeding as before, we find that

$$\mathcal{I}_1 \ll t^{-5/2}\left(1+\min\left\{\left(\frac{X}{t}\right)^{3/2}, |1-|\alpha||^{-2}\left(\frac{X}{t}\right)^{-5/2}\right\}\right),$$

$$\mathcal{I}_2 \ll t^{-5/2}\left(1+\min\left\{\left(\frac{X}{t}\right)^{3/2}, |1-|\alpha||^{-3}\left(\frac{X}{t}\right)^{-9/2}\right\}\right) \quad ,$$
$$\mathcal{I}_3 \ll t^{-5/2}\left(1+\min\left\{\left(\frac{X}{t}\right)^{3/2}, |1-|\alpha||^{-2}\left(\frac{X}{t}\right)^{-5/2}\right\}\right),$$
$$\mathcal{I}_4 \ll \frac{C}{T}t^{-5/2}\left(1+\min\left\{\left(\frac{X}{t}\right)^{3/2}, |1-|\alpha||^{-2}\left(\frac{X}{t}\right)^{-5/2}\right\}\right).$$

We conclude the third inequality from this. □

LEMMA 17. *Let f be defined as it is immediately preceding (3.1) and $|\alpha| \geqslant 1$. Then we have*

$$\widehat{f}(t) \ll \frac{1+|\log(X)|+\log(|\alpha|)}{1+X^{1/2}+||\alpha|^2-1|^{1/2}X} \quad \textit{for all } t \in \mathbb{R}.$$

When $|t| \notin [(\frac{1}{12})||\alpha|^2-1|^{1/2}X, 2||\alpha|^2-1|^{1/2}X]$ and $|t| \geqslant 1$, we can do better and find in that case that

$$\widehat{f}(t) \ll |t|^{-3/2}\left(1+\min\left\{\left(\frac{X}{|t|}\right)^{1/2}, ||\alpha|^2-1|^{-1}\left(\frac{X}{|t|}\right)^{-3/2}\right\}\right),$$
$$\widehat{f}(t) \ll \frac{C}{T}|t|^{-5/2}\left(1+\min\left\{\left(\frac{X}{|t|}\right)^{3/2}, ||\alpha|^2-1|^{-2}\left(\frac{X}{|t|}\right)^{-5/2}\right\}\right).$$

Proof. We follow the proof of the previous lemma, which leads us to estimate

$$\widehat{f}(t) \ll \int_0^\infty \min\left\{1, X^{-1}|\cosh\xi-|\alpha||^{-1}\right\} d\xi.$$

Set $\cosh(\varphi) = |\alpha|$ and note that we have $e^\varphi \asymp |\alpha|$ and $\log(|\alpha|) \leqslant \varphi \leqslant 1+\log(|\alpha|)$ for $|\alpha| \geqslant 1$. This leads to

$$\widehat{f}(t) \ll \int_0^\infty \min\left\{1, X^{-1}\sinh\left(\frac{\xi+\varphi}{2}\right)^{-1}\sinh\left(\frac{|\xi-\varphi|}{2}\right)^{-1}\right\} d\xi.$$

Thus, it suffices to bound the latter integral. We split up the region of integration into three parts $\mathcal{I}_1, \mathcal{I}_2$, and $\mathcal{I}_3$, where we restrict ourselves to $|\xi-\varphi| \geqslant 1$, $|\xi-\varphi| \leqslant 1 \wedge \xi+\varphi \geqslant 1$, and $|\xi-\varphi| \leqslant 1 \wedge \xi+\varphi \leqslant 1$, respectively. In particular, the last case may only occur when $\varphi \leqslant 1$. For $X \geqslant 1$, we have

$$\begin{aligned}\mathcal{I}_1 &\ll \int_0^\infty \min\{1, X^{-1}e^{-\max\{\varphi,\xi\}}\} d\xi \\ &\ll \int_0^\varphi \frac{e^{-\varphi}}{X} d\xi + \int_\varphi^\infty \frac{e^{-\xi}}{X} d\xi \\ &\ll_\epsilon \frac{1+\log(|\alpha|)}{|\alpha|X},\end{aligned}$$

$$
\begin{aligned}
\mathcal{I}_2 &\ll \int_{\max\{0,\varphi-1\}}^{\varphi+1} \min\{1, X^{-1}\mathrm{e}^{-(\xi+\varphi)/2}|\xi-\varphi|^{-1}\}\,d\xi \\
&\ll \int_{-1}^{1} \min\{1, X^{-1}\mathrm{e}^{-\varphi}|\psi|^{-1}\}\,d\psi \\
&\ll \int_0^{1/|\alpha|X} d\psi + \int_{1/|\alpha|X}^{1} \frac{1}{|\alpha|X\psi}\,d\psi \\
&\ll \frac{1+\log(|\alpha|X)}{|\alpha|X},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I}_3 &\ll \int_{\max\{0,\varphi-1\}}^{1-\varphi} \min\{1, X^{-1}|\xi^2-\varphi^2|^{-1}\}\,d\xi \\
&\ll \int_{\max\{-1,-\varphi\}}^{1-2\varphi} \min\{1, X^{-1}\varphi^{-1}|\psi|^{-1}, X^{-1}|\psi|^{-2}\}\,d\psi \\
&\ll \mathbb{1}_{[0,1]}(\varphi)\min\left\{1, \frac{1+\log^+(X\varphi)}{X\varphi}, X^{-1/2}\right\} \\
&\ll \mathbb{1}_{[0,1]}(\varphi)\min\left\{1, \frac{1+\log(X)}{||\alpha|-1|^{1/2}X}, X^{-1/2}\right\}.
\end{aligned}
$$

For $X \leqslant 1$, we have

$$
\begin{aligned}
\mathcal{I}_1 &\ll \int_0^{\infty} \min\{1, X^{-1}\mathrm{e}^{-\max\{\varphi,\xi\}}\}\,d\xi \\
&\ll \int_0^{\max\{\varphi,-\log(X)\}} \min\left\{1, \frac{\mathrm{e}^{-\varphi}}{X}\right\}d\xi + \int_{\max\{\varphi,-\log(X)\}}^{\infty} \frac{\mathrm{e}^{-\xi}}{X}\,d\xi \\
&\ll \frac{1+\log(|\alpha|)+|\log(X)|}{1+|\alpha|X} + \frac{1}{X}\min\{|\alpha|^{-1}, X\} \\
&\ll \frac{1+\log(|\alpha|)+|\log(X)|}{1+|\alpha|X},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I}_2 &\ll \int_{\max\{0,\varphi-1\}}^{\varphi+1} \min\{1, X^{-1}\mathrm{e}^{-(\xi+\varphi)/2}|\xi-\varphi|^{-1}\}\,d\xi \\
&\ll \int_{-1}^{1} \min\{1, X^{-1}\mathrm{e}^{-\varphi}|\psi|^{-1}\}\,d\psi \\
&\ll \min\left\{1, \frac{1+\log^+(|\alpha|X)}{|\alpha|X}\right\},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I}_3 &\ll \int_{\max\{0,\varphi-1\}}^{1-\varphi} \min\{1, X^{-1}|\xi^2-\varphi^2|^{-1}\}\,d\xi \\
&\ll \mathbb{1}_{[0,1]}(\varphi).
\end{aligned}
$$

This completes the case $X \leqslant 1$ of the first inequality.

For the second inequality, we proceed as in Lemma 16 and have to consider the integral

$$t^{-1/2}\int_0^\infty \frac{e^{2it(\pm\omega(x)+\alpha x)}}{(1+x^2)^{1/4+\beta}}g(2tx)\frac{dx}{x}.$$

We would pick up a stationary phase at $x_0=(\alpha^2-1)^{-1/2}$; however, we have $x\in[\frac{1}{6}(X/t), X/t]$, which does not intersect $[\frac{1}{2}x_0, 2x_0]$. Thus, we split up the integral into two parts $\mathcal{I}_1$ and $\mathcal{I}_2$ corresponding to the intervals $[0,\frac{1}{2}x_0]$ and $[2x_0,\infty)$. Without loss of generality, let $\alpha\geqslant 1$.

Assume first that $X/t\leqslant 1$. In this case, we have by Lemmas 10 and 11 with the choice $F(x)=(\pm\omega'(x)+\alpha)e^{2it(\pm\omega(x)+\alpha)}$, $G(x)=g(2tx)$, and $H(x)=[(1+x^2)^{1/4+\beta}(\sqrt{1+x^2}\pm\alpha x)]^{-1}$,

$$\mathcal{I}_1 \ll t^{-3/2}\frac{1}{\min\limits_{x\in[0,\frac{1}{2}x_0]\cap[\frac{1}{6}(X/t),X/t]}\sqrt{1+x^2}\pm\alpha x},$$

$$\mathcal{I}_2 \ll t^{-3/2}\frac{1}{\min\limits_{x\in[2x_0,\infty)\cap[\frac{1}{6}(X/t),X/t]}\alpha x\pm\sqrt{1+x^2}}.$$

In both estimates, the sign being "−" will give the larger contribution. The allowed range for t leaves us with two cases, either $x_0\geqslant 2X/t$ or $x_0\leqslant\frac{1}{12}(X/t)$. If $x_0\geqslant 2X/t$, then the integral over $\mathcal{I}_2$ is 0 and

$$\sqrt{1+x^2}-\alpha x=\frac{1-x^2(\alpha^2-1)}{\sqrt{1+x^2}+\alpha x}\gg 1 \quad \text{for } x\leqslant\frac{1}{2}x_0 \text{ and } x\leqslant 1.$$

Thus, we get a total bound of $t^{-3/2}$. Similarly, if $x_0\leqslant\frac{1}{12}(X/t)$, we have that the integral over $\mathcal{I}_1$ is 0 and

$$\alpha x-\sqrt{1+x^2}=\frac{x^2(\alpha^2-1)-1}{\sqrt{1+x^2}+\alpha x}\gg\frac{1}{\alpha x} \quad \text{for } x\geqslant 2x_0 \text{ and } x\leqslant 1.$$

Note that for $x\leqslant 1$ we also have $\alpha x-\sqrt{1+x^2}\geqslant\alpha x-\sqrt{2}$ and hence

$$\alpha x-\sqrt{1+x^2}\gg\alpha x+\frac{1}{\alpha x} \quad \text{for } x\geqslant 2x_0 \text{ and } x\leqslant 1.$$

This yields a total bound of $t^{-3/2}$.

Assume now that $X/t\geqslant 1$. In this case, we have

$$\mathcal{I}_1 \ll t^{-3/2}\frac{1}{\min\limits_{x\in[0,\frac{1}{2}x_0]\cap[\frac{1}{6}(X/t),X/t]}\big(\sqrt{1+x^2}-\alpha x\big)x^{1/2}},$$

$$\mathcal{I}_2 \ll t^{-3/2}\frac{1}{\min\limits_{x\in[2x_0,\infty)\cap[\frac{1}{6}(X/t),X/t]}\big(\alpha x-\sqrt{1+x^2}\big)x^{1/2}}.$$

If $x_0 \geqslant 2X/t$, then we have that the integral over $\mathcal{I}_2$ is 0 and

$$\left(\sqrt{1+x^2}-\alpha x\right)x^{1/2} = \frac{1-x^2(\alpha^2-1)}{\sqrt{1+x^2}+\alpha x}x^{1/2} \gg x^{-1/2}$$
$$\text{for } x \leqslant \frac{1}{2}x_0 \text{ and } x \geqslant \frac{1}{12}.$$

Thus, we get a total bound of $t^{-3/2}(X/t)^{1/2} \ll t^{-3/2}|\alpha^2-1|^{-1}(X/t)^{-3/2}$. Similarly, if $x_0 \leqslant \frac{1}{12}(X/t)$, we have that the integral over $\mathcal{I}_1$ is 0 and

$$(\alpha x - \sqrt{1+x^2}) = \frac{x^2(\alpha^2-1)-1}{\sqrt{1+x^2}+\alpha x} \geqslant \frac{3}{8}\frac{x^2(\alpha^2-1)}{\alpha x} \gg \frac{1}{\alpha x}$$
$$\text{for } x \geqslant 2x_0 \text{ and } x \geqslant \frac{1}{6}.$$

This yields a total bound of

$$t^{-3/2} \cdot \min\left\{\alpha\left(\frac{X}{t}\right)^{1/2}, \frac{\alpha}{\alpha^2-1}\left(\frac{X}{t}\right)^{-3/2}\right\}$$
$$\ll t^{-3/2}\left(1+\min\left\{\left(\frac{X}{t}\right)^{1/2}, \frac{1}{\alpha^2-1}\left(\frac{X}{t}\right)^{-3/2}\right\}\right)$$

since $X/t \geqslant 1$. This proves the second inequality.

For the third inequality, we integrate once by parts. We then have to consider the integrals

$$\mathcal{I}_4 = t^{-5/2}\int_0^\infty (\mathrm{e}^{2\mathrm{i}t(\pm\omega(x)+\alpha)}2t(\pm\omega'(x)+\alpha))$$
$$\times \frac{g(2tx)}{x^2(\pm\omega'(x)+\alpha)^2(1+x^2)^{1/4+\beta}}\,dx,$$
$$\mathcal{I}_5 = t^{-5/2}\int_0^\infty (\mathrm{e}^{2\mathrm{i}t(\pm\omega(x)+\alpha)}2t(\pm\omega'(x)+\alpha))$$
$$\times \frac{g(2tx)(\pm\omega''(x)x^2)}{x^3(\pm\omega'(x)+\alpha)^3(1+x^2)^{1/4+\beta}}\,dx,$$
$$\mathcal{I}_6 = t^{-5/2}\int_0^\infty (\mathrm{e}^{2\mathrm{i}t(\pm\omega(x)+\alpha)}2t(\pm\omega'(x)+\alpha))$$
$$\times \frac{g(2tx)x^2}{x^2(\pm\omega'(x)+\alpha)^2(1+x^2)^{5/4+\beta}}\,dx,$$
$$\mathcal{I}_7 = t^{-5/2}\int_0^\infty (\mathrm{e}^{2\mathrm{i}t(\pm\omega(x)+\alpha)}2t(\pm\omega'(x)+\alpha))$$
$$\times \frac{tx\cdot g'(2tx)}{x^2(\pm\omega'(x)+\alpha)^2(1+x^2)^{1/4+\beta}}\,dx.$$

By similar means as before, we have that

$$\mathcal{I}_4 \ll t^{-5/2}\left(1+\min\left\{\left(\frac{X}{t}\right)^{3/2}, ||\alpha|^2-1|^{-2}\left(\frac{X}{t}\right)^{-5/2}\right\}\right),$$

$$\mathcal{I}_5 \ll t^{-5/2}\left(1+\min\left\{\left(\frac{X}{t}\right)^{3/2}, ||\alpha|^2-1|^{-3}\left(\frac{X}{t}\right)^{-9/2}\right\}\right) \quad ,$$
$$\mathcal{I}_6 \ll t^{-5/2}\left(1+\min\left\{\left(\frac{X}{t}\right)^{3/2}, ||\alpha|^2-1|^{-2}\left(\frac{X}{t}\right)^{-5/2}\right\}\right),$$
$$\mathcal{I}_7 \ll \frac{C}{T}t^{-5/2}\left(1+\min\left\{\left(\frac{X}{t}\right)^{3/2}, ||\alpha|^2-1|^{-2}\left(\frac{X}{t}\right)^{-5/2}\right\}\right).$$

We conclude the last inequality from this. □

LEMMA 18. *Let f be defined as it is immediately preceding (3.1). For $0 \leqslant t \leqslant \frac{1}{4}-\delta$, we have the following expansion:*

$$\widehat{f}(\mathrm{i}t) = -\frac{1}{2}\int_{X/2}^{X} Y_{2t}(x)\mathrm{e}^{\mathrm{i}\alpha x}\frac{dx}{x} + O_{\epsilon,\delta}\left(1+\frac{T}{C}X^{-2t-\epsilon}\right).$$

Proof. We have

$$\begin{aligned}\widehat{f}(\mathrm{i}t) &= \frac{1}{\sin(2\pi t)}\int_0^\infty \frac{J_{-2t}(x)-J_{2t}(x)}{2}f(x)\frac{dx}{x}\\ &= -\frac{1}{2}\int_0^\infty\left[\frac{J_{2t}(x)\cos(2\pi t)-J_{-2t}(x)}{\sin(2\pi t)}+\frac{J_{2t}(x)-J_{2t}(x)\cos(2\pi t)}{\sin(2\pi t)}\right]\\ &\quad\times f(x)\frac{dx}{x}\\ &= -\frac{1}{2}\int_0^\infty [Y_{2t}(x)+J_{2t}(x)\tan(\pi t)]f(x)\frac{dx}{x}.\end{aligned}$$

Now, we have

$$\int_0^\infty J_{2t}(x)\tan(\pi t)f(x)\frac{dx}{x} \ll \int_0^\infty \min\{x^{2t}, x^{-1/2}\}\frac{g(x)}{x}\,dx \ll 1$$

and

$$\left(\int_{2\pi\sqrt{mn}/(C+T)}^{X/2} + \int_X^{4\pi\sqrt{mn}/(C-T)}\right)Y_{2t}(x)f(x)\frac{dx}{x} \ll \frac{T}{C}\sup_{x\sim X}|Y_{2t}(x)|.$$

The following inequality will imply the result:

$$|Y_{2t}(x)| \ll_\epsilon \begin{cases} x^{-2t-\epsilon} & \text{if } x \leqslant 1,\\ x^{-1/2} & \text{if } x \geqslant 1.\end{cases}$$

The range $x \geqslant 1$ can be found in [**13**, Appendix B.35] and for the range $x \leqslant 1$ we make use of the following integral representation [**25**, p. 170]:

$$Y_{2t}(x) = -\frac{2(x/2)^{-2t}}{\sqrt{\pi}\,\Gamma(1/2-2t)}\int_1^\infty \frac{\cos(xy)}{(y^2-1)^{2t+1/2}}\,dy.$$

The integral from 1 to $1/x$ is bounded by

$$\int_1^2 \frac{1}{(y-1)^{1-2\delta}}\,dy + \int_2^{\max\{2,1/x\}} \frac{1}{(y^2-1)^{1/2}}\,dy$$
$$= \frac{1}{2\delta}(y-1)^{2\delta}\bigg|_{y=1}^{2} + \log\left(\sqrt{y^2-1}+y\right)\bigg|_{y=2}^{\max\{2,1/x\}} \ll_{\epsilon,\delta} x^{-\epsilon}$$

and the remaining integral is bounded by $O(1)$, by Lemma 10 with $F(y) = \cos(xy)$ and $G(y) = (y^2-1)^{2t+1/2}$. □

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Raphael S. Steiner,
FH-317,
Institute for Advanced Study,
1 Einstein Drive,
Princeton,
NJ 08540,
U.S.A.
E-mail: raphael.steiner.academic@gmail.com