

C^1 MAPPINGS IN $\mathbb{R}^5$ WITH DERIVATIVE OF RANK AT MOST 3 CANNOT BE UNIFORMLY APPROXIMATED BY C^2 MAPPINGS WITH DERIVATIVE OF RANK AT MOST 3

PAWEŁ GOLDSTEIN AND PIOTR HAJŁASZ

ABSTRACT. We find a counterexample to a conjecture of Gałęski [1] by constructing for some positive integers $m < n$ a mapping $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ satisfying $\text{rank } Df \leq m$ that, even locally, cannot be uniformly approximated by C^2 mappings f_ε satisfying the same rank constraint: $\text{rank } Df_\varepsilon \leq m$.

1. INTRODUCTION

In the context of geometric measure theory Jacek Gałęski [1, Conjecture 1.1 and Section 3.3] formulated the following conjecture.

Conjecture 1. *Let $1 \leq m < n$ be integers and let $\Omega \subset \mathbb{R}^n$ be open. If $f \in C^1(\Omega, \mathbb{R}^n)$ satisfies $\text{rank } Df \leq m$ everywhere in Ω , then f can be uniformly approximated by smooth mappings $g \in C^\infty(\Omega, \mathbb{R}^n)$ such that $\text{rank } Dg \leq m$ everywhere in Ω .*

A weaker form of the conjecture is whether any mapping as in Conjecture 1 can be approximated locally.

Conjecture 2. *Let $1 \leq m < n$ be integers and let $\Omega \subset \mathbb{R}^n$ be open. If $f \in C^1(\Omega, \mathbb{R}^n)$ satisfies $\text{rank } Df \leq m$ everywhere in Ω , then for every point $x \in \Omega$ there is a neighborhood $\mathbb{B}^n(x, \varepsilon) \subset \Omega$ and a sequence $f_i \in C^\infty(\mathbb{B}^n(x, \varepsilon), \mathbb{R}^n)$ such that $\text{rank } Df_i \leq m$ and f_i converges to f uniformly on $\mathbb{B}^n(x, \varepsilon)$.*

In fact Gałęski formulated a stronger conjecture than Conjecture 1, expecting that f can be approximated by $g_i \in C^\infty(\Omega, \mathbb{R}^n)$, $\text{rank } Dg_i \leq m$, not only uniformly, but in the C^1 norm, i.e. $\|f - g_i\|_\infty + \|Df - Dg_i\|_\infty \rightarrow 0$ as $i \rightarrow \infty$. Our main result, Theorem 4, answers Conjecture 1 (and hence the original conjecture of Gałęski) in the negative. However, in a local form the original conjecture of Gałęski is true on an open and dense subset of Ω . The following result is an easy consequence of the Rank Theorem.

Theorem 3. *Let $1 \leq m < n$ be integers and let $\Omega \subset \mathbb{R}^n$ be open. If $f \in C^1(\Omega, \mathbb{R}^n)$ satisfies $\text{rank } Df \leq m$ everywhere in Ω , then there is an open and dense set $G \subset \Omega$ such that for every point $x \in G$ there is a neighborhood $\mathbb{B}^n(x, \varepsilon) \subset G$ and a sequence $f_i \in C^\infty(\mathbb{B}^n(x, \varepsilon), \mathbb{R}^n)$ such that $\text{rank } Df_i \leq m$ and f_i converge to f in $C^1(\mathbb{B}^n(x, \varepsilon), \mathbb{R}^n)$ (i.e. both f_i and their first derivatives converge uniformly).*

2010 *Mathematics Subject Classification.* Primary: 41A29; Secondary: 26B05, 26B10, 57R12.

Key words and phrases. approximation by smooth mappings, rank of the derivative, Sard's theorem.

P.G. was partially supported by National Science Center grant no 2012/05/E/ST1/03232.

P.H. was supported by NSF grant DMS-1800457.

The main result of the paper shows that, in general Conjecture 2 (and hence Conjecture 1) is false by providing a family counterexamples for certain ranges of n and m .

Theorem 4. *Suppose that $m + 1 \leq k < 2m - 1$, $\ell \geq k + 1$, $r \geq m + 1$, and the homotopy group $\pi_k(\mathbb{S}^m)$ is non-trivial. Then there is a map $f \in C^1(\mathbb{R}^\ell, \mathbb{R}^r)$ with $\text{rank } Df \leq m$ in $\mathbb{R}^\ell$ and a Cantor set $E \subset \mathbb{R}^\ell$ with the following property:*

For every $x_o \in E$ and $\varepsilon > 0$ there is $\delta > 0$ such that if $g \in C^{k-m+1}(\mathbb{B}^\ell(x_o, \varepsilon), \mathbb{R}^r)$ and $|f(x) - g(x)| < \delta$ for all $x \in \mathbb{B}^\ell(x_o, \varepsilon)$, then $\text{rank } Dg \geq m + 1$ on a non-empty open set in $\mathbb{B}^\ell(x_o, \varepsilon)$.

(Here by a Cantor set we mean a set that is homeomorphic to the ternary Cantor set.)

Therefore the mapping f cannot be approximated in the supremum norm by C^{k-m+1} mappings with rank of the derivative $\leq m$ in any neighborhood of any point of the set E .

Remark 5. In fact, the mapping f constructed in the proof of Theorem 4 is C^∞ smooth on $\mathbb{R}^\ell \setminus E$, so $G = \mathbb{R}^\ell \setminus E$ is an open and dense set where we can approximate f smoothly, cf. Theorem 3.

Since the assumptions of the theorem are quite complicated, let us show explicit situations when the approximation cannot hold.

Example 1. If $n \geq 3$, $\ell \geq n+2$ and $r \geq n+1$, then there is $f \in C^1(\mathbb{R}^\ell, \mathbb{R}^r)$ with $\text{rank } Df \leq n$ in $\mathbb{R}^\ell$ that cannot be locally approximated in the supremum norm by mappings $g \in C^2(\mathbb{R}^\ell, \mathbb{R}^r)$ satisfying $\text{rank } Dg \leq n$.

Indeed, if $n \geq 3$, $k = n+1$ and $m = n$, then $\pi_k(\mathbb{S}^m) = \mathbb{Z}_2$ (see [3]) and $m+1 \leq k < 2m-1$.

In particular, there is $f \in C^1(\mathbb{R}^5, \mathbb{R}^5)$ with $\text{rank } Df \leq 3$ that cannot be locally approximated in the supremum norm by mappings $g \in C^2(\mathbb{R}^5, \mathbb{R}^5)$ satisfying $\text{rank } Dg \leq 3$.

Example 2. $\pi_6(\mathbb{S}^4) = \mathbb{Z}_2$, $k = 6$, $m = 4$, $m + 1 \leq k < 2m - 1$, so there is $f \in C^1(\mathbb{R}^7, \mathbb{R}^7)$, $\text{rank } Df \leq 4$, that cannot be locally approximated by mappings $g \in C^3(\mathbb{R}^7, \mathbb{R}^7)$ satisfying $\text{rank } Dg \leq 4$.

Example 3. $\pi_8(\mathbb{S}^5) = \mathbb{Z}_{24}$, $k = 8$, $m = 5$, $m + 1 \leq k < 2m - 1$, so there is $f \in C^1(\mathbb{R}^9, \mathbb{R}^9)$, $\text{rank } Df \leq 5$, that cannot be locally approximated by mappings $g \in C^4(\mathbb{R}^9, \mathbb{R}^9)$ satisfying $\text{rank } Dg \leq 5$.

Infinitely many essentially different situations when the assumptions of Theorem 4 are satisfied can be easily obtained by examining the catalogue of homotopy groups of spheres.

While, in general, Gałęski's conjecture is not true, Theorem 4 covers only a certain range of dimensions and ranks, leaving other cases unsolved. We believe that the following special case of the conjecture is true.

Conjecture 6. *If $f \in C^1(\mathbb{R}^n, \mathbb{R}^k)$, $n, k \geq 2$, satisfies $\text{rank } Df \leq 1$, then f can be uniformly approximated (at least locally) by mappings $g \in C^\infty(\mathbb{R}^n, \mathbb{R}^k)$ satisfying $\text{rank } Dg \leq 1$.*

Our belief is based on the fact that in that case the structure of the mapping f is particularly simple: on the open set where $\text{rank } Df = 1$, it is a C^1 curve that branches on the set where $\text{rank } Df = 0$.

2. PROOF OF THEOREM 3

Let $G \subset \Omega$ be the set of points where the function $x \mapsto \text{rank } Df(x)$ attains a local maximum i.e.,

$$G = \{x \in \Omega : \exists \varepsilon > 0 \forall y \in \mathbb{B}^n(x, \varepsilon) \text{ rank } Df(y) \leq \text{rank } Df(x)\}.$$

We claim that the set G is open, and that $\text{rank } Df$ is locally constant in G . Indeed, the set $\{\text{rank } Df \geq k\}$ is open so if $x \in G$ and $\text{rank } Df(x) = k$, then $\text{rank } Df(y) \geq k$ in a neighborhood $\mathbb{B}^n(x, \varepsilon)$ of x , but $\text{rank } Df$ attains a local maximum at x , so $\text{rank } Df(y) = k$ in $\mathbb{B}^n(x, \varepsilon)$. Clearly, $\mathbb{B}^n(x, \varepsilon) \subset G$ and $\text{rank } Df$ is constant in the neighborhood $\mathbb{B}^n(x, \varepsilon) \subset G$.

We also claim that the set $G \subset \Omega$ is dense. Let $x \in \Omega$ and $\mathbb{B}^n(x, \varepsilon) \subset \Omega$. Since $\text{rank } Df$ can attain only a finite number of values, it attains a local maximum at some point $y \in \mathbb{B}^n(x, \varepsilon)$. Clearly, $y \in G$. That proves density of G .

It remains to prove now that f can be locally approximated in G . Let $x \in G$. Then $\text{rank } Df(x) = k \leq m$. Since $\text{rank } Df$ is constant in a neighborhood of x , it follows from the Rank Theorem [6, Theorem 8.6.2/2] that there are C^1 diffeomorphisms Φ and Ψ , defined in neighborhoods of x and $f(x)$, respectively, such that $\Phi(x) = 0$, $\Psi(f(x)) = 0$, and

$$\Psi \circ f \circ \Phi^{-1}(x_1, \dots, x_n) = (x_1, \dots, x_k, 0, \dots, 0) \quad \text{in a neighborhood of } 0 \in \mathbb{R}^n.$$

Let $\pi_k : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\pi_k(x_1, \dots, x_n) = (x_1, \dots, x_k, 0, \dots, 0)$. Then $\Psi \circ f \circ \Phi^{-1} = \pi_k$, so $f = \Psi^{-1} \circ \pi_k \circ \Phi$ in a neighborhood of x . If $\Phi_\varepsilon = \Phi * \varphi_\varepsilon$ and $(\Psi^{-1})_\varepsilon = (\Psi^{-1}) * \varphi_\varepsilon$ are smooth approximations by mollification, then $f_\varepsilon = (\Psi^{-1})_\varepsilon \circ \pi_k \circ \Phi_\varepsilon$ is C^∞ smooth and it converges in C^1 to f in a neighborhood of x as $\varepsilon \rightarrow 0$. Indeed, uniform convergence of f_ε to f is obvious and the uniform convergence of the derivatives follows from the chain rule and the fact that $D\Phi_\varepsilon = (D\Phi) * \varphi_\varepsilon$ and $D(\Psi^{-1})_\varepsilon = (D(\Psi^{-1})) * \varphi_\varepsilon$.

Clearly, $\text{rank } Df_\varepsilon \leq k \leq m$ by the chain rule, since $\text{rank } D\pi_k = k$. $\square$

Remark 7. It is easy to see that in fact $\text{rank } Df_\varepsilon = k$ in a neighborhood of x , provided ε is sufficiently small. Indeed, $\Phi_\varepsilon = \Phi * \varphi_\varepsilon$ (approximation by mollification) so $D\Phi_\varepsilon = (D\Phi) * \varphi_\varepsilon$. Since $\det(D\Phi(x)) \neq 0$, for small $\varepsilon > 0$ we also have that $\det(D\Phi_\varepsilon)(x) \neq 0$ and hence Φ_ε is a diffeomorphism near x . Similarly, $(\Psi^{-1})_\varepsilon$ is a diffeomorphism near 0.

3. PROOF OF THEOREM 4

In the first step of the proof we shall construct a mapping $F : \mathbb{B}^{k+1} \rightarrow \mathbb{R}^{m+1}$ defined on the unit ball $\mathbb{B}^{k+1} = \mathbb{B}^{k+1}(0, 1)$, with the properties announced by Theorem 4.

Lemma 8. *Suppose that $m + 1 \leq k < 2m - 1$ and $\pi_k(\mathbb{S}^m) \neq \emptyset$. Then there exists a map $F \in C^1(\mathbb{B}^{k+1}, \mathbb{R}^{m+1})$ with $\text{rank } DF \leq m$ in $\mathbb{B}^{k+1}$ and a Cantor set $E_F \subset \mathbb{B}^{k+1}$ such that for every $x_o \in E_F$ and $1 - |x_o| > \varepsilon > 0$ there is $\delta > 0$ with the following property: if $G \in C^{k-m+1}(\mathbb{B}^{k+1}(x_o, \varepsilon), \mathbb{R}^{m+1})$ satisfies $|F(x) - G(x)| < \delta$ at all points $x \in \mathbb{B}^{k+1}(x_o, \varepsilon)$, then $\text{rank } DG \geq m + 1$ on an open, non-empty set in $\mathbb{B}^{k+1}(x_o, \varepsilon)$.*

Before we prove Lemma 8, let us show how Theorem 4 follows from it. To this end, let $\tilde{\mathbb{B}}^{k+1} \subsetneq \mathbb{B}^{k+1}$ be a ball concentric with $\mathbb{B}^{k+1}$, containing the Cantor set E_F and let

$\Phi : \mathbb{B}^{k+1} \rightarrow \mathbb{R}^{k+1}$ be a diffeomorphism onto $\mathbb{R}^{k+1}$ that is identity on $\tilde{\mathbb{B}}^{k+1}$, so $F \circ \Phi^{-1} : \mathbb{R}^{k+1} \rightarrow \mathbb{R}^{m+1}$ coincides with F on $\tilde{\mathbb{B}}^{k+1}$ and hence in a neighborhood of the set E_F . Denote the points in $\mathbb{R}^\ell$ and $\mathbb{R}^r$ by

$$(x, y) \in \mathbb{R}^{k+1} \times \mathbb{R}^{\ell-k-1} = \mathbb{R}^\ell \quad \text{and} \quad (z, v) \in \mathbb{R}^{m+1} \times \mathbb{R}^{r-m-1} = \mathbb{R}^r$$

and let $\pi : \mathbb{R}^r \rightarrow \mathbb{R}^{m+1}$, $\pi(z, v) = z$ be the orthogonal projection.

It easily follows that the mapping

$$\mathbb{R}^\ell \ni (x, y) \mapsto f(x, y) := (F \circ \Phi^{-1}(x), 0) \in \mathbb{R}^r$$

satisfies the claim of Theorem 4 with $E = E_F \times \{0\} \subset \mathbb{R}^{k+1} \times \mathbb{R}^{\ell-k-1} = \mathbb{R}^\ell$.

Indeed, in a neighborhood of $x_o \in E_F$, $f(x, y) = (F(x), 0)$.

Suppose that $g \in C^{k-m+1}(\mathbb{B}^\ell((x_o, 0), \varepsilon), \mathbb{R}^r)$ is such that

$$|f(x, y) - g(x, y)| < \delta \quad \text{for all } (x, y) \in \mathbb{B}^\ell((x_o, 0), \varepsilon).$$

Then $G(x) = \pi(g(x, 0)) \in C^{k-m+1}(\mathbb{B}^{k+1}(x_o, \varepsilon), \mathbb{R}^{m+1})$ satisfies

$$|F(x) - G(x)| < \delta \quad \text{for all } x \in \mathbb{B}^{k+1}(x_o, \varepsilon)$$

provided $\varepsilon > 0$ is so small that $f(x, y) = (F(x), 0)$ for all $x \in \mathbb{B}^{k+1}(x_o, \varepsilon)$.

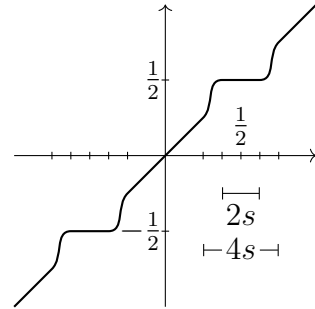
Hence $\text{rank } DG \geq m + 1$ on an open, non-empty set in $\mathbb{B}^{k+1}(x_o, \varepsilon)$ by Lemma 8. Since $\text{rank } Dg(x, 0) \geq \text{rank } DG(x)$ and the set $\{\text{rank } Dg \geq m + 1\}$ is open, $\text{rank } Dg \geq m + 1$ on an open, non-empty subset of $\mathbb{B}^\ell((x_o, 0), \varepsilon)$, which completes the proof of Theorem 4. Therefore it remains to prove Lemma 8.

Proof of Lemma 8. Let $\mathbb{I}$ denote the unit cube $[-\frac{1}{2}, \frac{1}{2}]^{m+1}$ in $\mathbb{R}^{m+1}$. Since, by assumption, $\pi_k(\mathbb{S}^m) \neq 0$ and $\partial\mathbb{I}$ is homeomorphic to $\mathbb{S}^m$, there is a continuous mapping $\hat{\phi} : \mathbb{S}^k \rightarrow \partial\mathbb{I}$ that is not homotopic to a constant map. Approximating $\hat{\phi}$ by standard mollification, we obtain a smooth mapping from $\mathbb{S}^k$ to $\mathbb{R}^{m+1}$, uniformly close to $\hat{\phi}$, with the image lying in a small neighborhood of $\partial\mathbb{I}$. Then, composing it with a C^∞ smooth mapping R that is homotopic to the identity and maps a neighborhood of $\partial\mathbb{I}$ onto $\partial\mathbb{I}$ we obtain a mapping $\phi : \mathbb{S}^k \rightarrow \partial\mathbb{I}$ that is not homotopic to a constant map and is C^∞ smooth as a mapping to $\mathbb{R}^{m+1}$.

A smooth mapping $R : \mathbb{R}^{m+1} \rightarrow \mathbb{R}^{m+1}$ homotopic to the identity, that maps a neighborhood of $\partial\mathbb{I}$ onto $\partial\mathbb{I}$ can be defined by a formula

$$R(x_1, x_2, \dots, x_{m+1}) = (\lambda_s(x_1), \lambda_s(x_2), \dots, \lambda_s(x_{m+1})),$$

where for $s \in (0, \frac{1}{4})$ the function $\lambda_s : \mathbb{R} \rightarrow \mathbb{R}$ is smooth, odd, non-decreasing and such that $\lambda_s(t) = t$ when $||t| - \frac{1}{2}| > 2s$ and $\lambda(t) = 1$ when $||t| - \frac{1}{2}| < s$, see the graph on the right. Taking $s \rightarrow 0$ gives a homotopy between R and the identity.



Lemma 8 is a simple consequence of the following result proved in [2, Lemma 5.1]. (Note that in the statement of Lemma 5.1 in [2], k plays the role of m and m plays the role of k .) The self-similarity property of the mapping F in Lemma 9 is explicitly stated in the proof of Lemma 5.1 in [2].

Lemma 9. *Suppose that $m + 1 \leq k < 2m - 1$ and $\pi_k(\mathbb{S}^m) \neq 0$. Then there is a mapping $F \in C^1(\mathbb{B}^{k+1}, \mathbb{I})$ satisfying $\text{rank } DF \leq m$ everywhere, such that F maps the boundary $\partial \mathbb{B}^{k+1} = \mathbb{S}^k$ to $\partial \mathbb{I}$ and $F|_{\partial \mathbb{B}^{k+1}} = \phi$, where ϕ has been defined above.*

Moreover, F is self-similar in the following sense. There is a Cantor set $E_F \subset \mathbb{B}^{k+1}$ such that for every $x_o \in E_F$ there is a sequence of balls $\mathbb{D}_i \subset \mathbb{B}^{k+1}$, $x_o \in \mathbb{D}_i$, with radii convergent to zero, and similarity transformations

$$\Sigma_i : \mathbb{B}^{k+1} \rightarrow \mathbb{D}_i, \quad \Sigma_i(\mathbb{B}^{k+1}) = \mathbb{D}_i, \quad T_i : \mathbb{R}^{m+1} \rightarrow \mathbb{R}^{m+1},$$

each being a composition of a translation and scaling, such that

$$T_i^{-1} \circ F|_{\mathbb{D}_i} \circ \Sigma_i = F.$$

Here the C^1 regularity of F means that it is C^1 as a mapping into $\mathbb{R}^{m+1}$, with the image being the cube $\mathbb{I}$.

The mappings T_i and Σ_i are compositions $T_i = \tau_{j_1} \circ \dots \circ \tau_{j_i}$ and $\Sigma_i = \sigma_{j_1} \circ \dots \circ \sigma_{j_i}$ of similarity transformations τ_j and σ_j that are used at the very end of the proof of Lemma 5.1 in [2]. The Cantor set E_F is the same as the Cantor set C in the proof of Lemma 5.1 in [2].

In other words, F restricted to an arbitrarily small ball $\mathbb{D}_i$ that contains x_o is a scaled copy of $F : \mathbb{B}^{k+1} \rightarrow \mathbb{I}$.

The mapping F is obtained through an iterative construction, described in detail in [2]. We shall present here a sketch of that construction.

Sketch of the construction of the mapping F .

By assumption, $\pi_k(\mathbb{S}^m) \neq 0$. By Freudenthal's theorem ([3, Corollary 4.24]), also $\pi_{k-1}(\mathbb{S}^{m-1}) \neq 0$; let $h : \mathbb{S}^{k-1} \rightarrow \mathbb{S}^{m-1}$ be a mapping that is not homotopic to a constant.

We begin by choosing in the ball $\mathbb{B}^{k+1}$ disjoint, closed balls $\mathbb{B}_i$, $i = 1, 2, \dots, N = n^{m+1}$, of radius $\frac{2}{n}$, all inside $\frac{1}{2}\mathbb{B}^{k+1}$. This is possible, if n is chosen sufficiently large, since, for n large, the $(k+1)$ -dimensional volume of $\frac{1}{2}\mathbb{B}^{k+1}$ is much larger than the sum of volumes of $\mathbb{B}_i$, $2^{-(k+1)} \gg n^{m+1} 2^{k+1} n^{-(k+1)}$.

We define a C^∞ -mapping F in $\mathbb{B}^{k+1} \setminus \bigcup_{i=1}^N \mathbb{B}_i$; then, the same mapping is iterated inside each of the balls $\mathbb{B}_i = \mathbb{B}_{i,1}$, which defines F outside a family of N^2 second generation balls $\mathbb{B}_{i,2}$, and so on – in this way we obtain a mapping which is C^∞ outside a Cantor set. Finally, we extend F continuously to the Cantor set C defined by the subsequent generations of balls $\mathbb{B}_{i,j}$, as the intersection $C = \bigcap_{j=1}^\infty \bigcup_{i=1}^{N^j} \mathbb{B}_{i,j}$.

The mapping F in $\mathbb{B}^{k+1} \setminus \bigcup_{i=1}^N \mathbb{B}_i$ is (in principle – see comments below) defined as a composition of four steps (see Figure 1):

- (1) First, we realign all the balls $\mathbb{B}_i$ inside $\mathbb{B}^{k+1}$, by a diffeomorphism G_1 equal to the identity near $\partial \mathbb{B}^{k+1}$, so that the images of $\mathbb{B}_i$ are identical, disjoint, closed balls

lying along the vertical axis of $\mathbb{B}^{k+1}$. Obviously, this diffeomorphism has to shrink the balls $\mathbb{B}_i$ somewhat.

- (2) The next step, the mapping $H : \mathbb{B}^{k+1} \rightarrow \mathbb{B}^{m+1}$, is defined in the following way: it maps $(k-1)$ -dimensional spheres centered at the vertical axis of $\mathbb{B}^{k+1}$, lying in the hyperplane orthogonal to that axis, to $(m-1)$ -dimensional spheres of the same radius, centered at analogous points on the vertical axis of $\mathbb{B}^{m+1}$. On each such sphere, H is an appropriately scaled copy of the mapping h . This way, H restricted to any k -sphere centered on the axis (in particular to $\partial\mathbb{B}_{k+1}$ and to $\partial(G_1(\mathbb{B}_i))$) equals (up to scaling) to the suspension of h .
- (3) Next, we define the diffeomorphism G_2 : we inflate the ball $\mathbb{B}^{m+1}$ to $\frac{1}{2}\sqrt{m+1}\mathbb{B}^{m+1}$, so that we can inscribe the unit cube $[-\frac{1}{2}, \frac{1}{2}]^{m+1}$ in it, and inside that ball, we rearrange the N balls $H(G_1(\mathbb{B}_i))$, so that each of them is almost inscribed in one of the cubes of the grid obtained by partitioning the unit cube into $N = n^{m+1}$ cubes of edge length $\frac{1}{n}$.
- (4) Finally, we project $\frac{1}{2}\sqrt{m+1}\mathbb{B}^{m+1} \setminus \bigcup_{i=1}^N G_2(H(G_1(\mathbb{B}_i)))$ onto the m -dimensional skeleton of the grid: first, we project the outside of the unit cube onto the boundary of the cube using the nearest point projection π , then in each of the N closed cubes of the grid we use the mapping R defined in the proof of Lemma 8. Even though π is not smooth, this composition turns out to be smooth (see [2, Lemma 5.3]).

In fact, this construction of F outside $\bigcup_i \mathbb{B}_i$ is almost correct – the resulting mapping is not C^∞ , but Lipschitz: it is not differentiable at the points of the vertical axis, and some technical modifications are necessary to make it C^∞ . Similarly, some additional work is necessary to glue F with scaled copies of F in each of the balls $\mathbb{B}_i$ in a differentiable way. These are purely technical difficulties, the details are provided in [2].

The third iteration of that construction is depicted in Figure 2.

One easily checks that the derivative of F tends to 0 as we approach the points of the Cantor set C , thus the limit mapping, extended to the whole $\mathbb{B}^{k+1}$, is C^1 . For each point of $\mathbb{B}^{k+1} \setminus C$, the image of its small neighborhood is mapped to the m -dimensional skeleton of the grid, thus $\text{rank } DF \leq m$ at all these points, and since $DF = 0$ at the points of C , the condition $\text{rank } DF \leq m$ holds everywhere in $\mathbb{B}^{k+1}$.

□

Lemma 9 allows us to complete the proof of Lemma 8 as follows. Let $x_o \in E_F$ and $1 - |x_o| > \varepsilon > 0$ be given. Suppose to the contrary, that there is a sequence $G_j \in C^{k-m+1}(\mathbb{B}^{k+1}(x_o, \varepsilon), \mathbb{R}^{m+1})$ with $\text{rank } DG_j \leq m$, that is uniformly convergent to F on $\mathbb{B}^{k+1}(x_o, \varepsilon)$.

Let $\mathbb{D}_i$ be a sequence of balls convergent to x_o as in the statement of Lemma 9. If i is sufficiently large, then $\mathbb{D}_i \subset \mathbb{B}^{k+1}(x_o, \varepsilon)$ and the sequence G_j converges uniformly to F on $\mathbb{D}_i$. Hence

$$\tilde{G}_j := T_i^{-1} \circ G_j|_{\mathbb{D}_i} \circ \Sigma_i : \mathbb{B}^{k+1} \rightarrow \mathbb{R}^{m+1}$$

converges uniformly to

$$T_i^{-1} \circ F|_{\mathbb{D}_i} \circ \Sigma_i = F : \mathbb{B}^{k+1} \rightarrow \mathbb{I}.$$

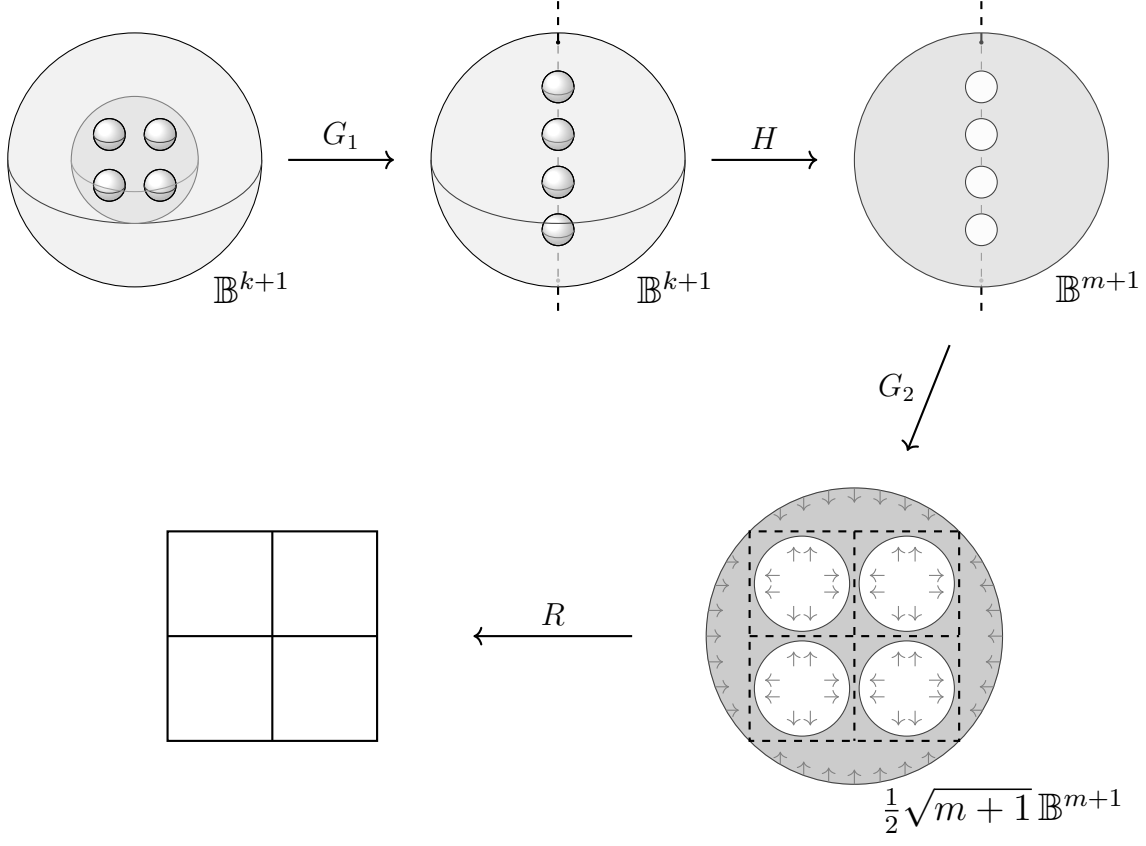


FIGURE 1. The construction of F in $\mathbb{B}^{m+1} \setminus \bigcup_{i=1}^N \mathbb{B}_i$.

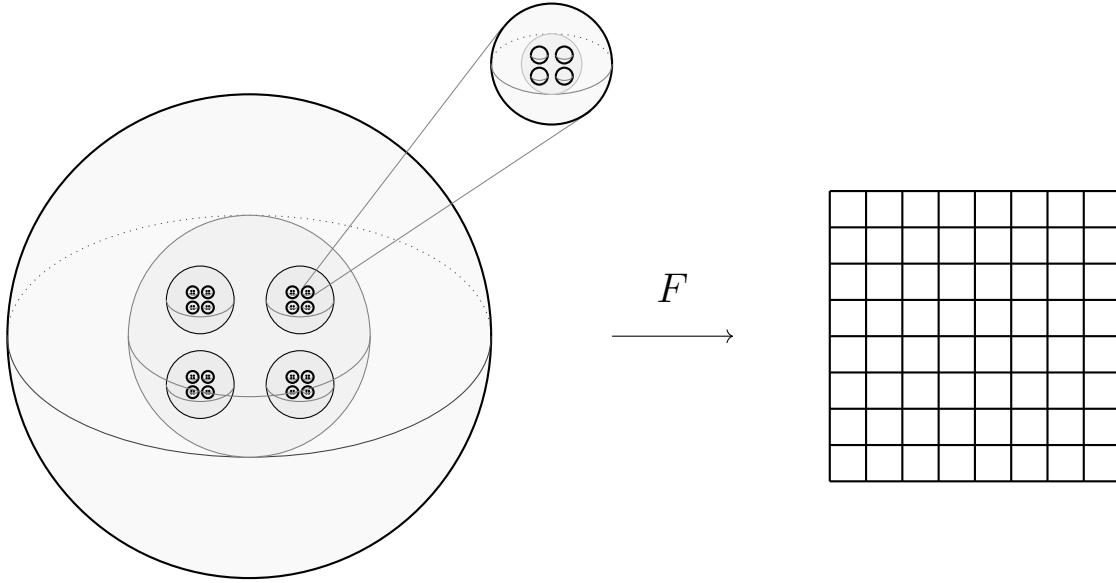


FIGURE 2. The third iteration: F outside the third generation of balls $\bigcup_i \mathbb{B}_{3,i}$.

Obviously, $\text{rank } D\tilde{G}_j \leq m$. Since $\tilde{G}_j$ is uniformly close to F on $\partial\mathbb{B}^{k+1}$ and $F|_{\partial\mathbb{B}^{k+1}} : \mathbb{S}^k \rightarrow \partial\mathbb{I}$ is not homotopic to a constant map, it easily follows that for j sufficiently large the image $\tilde{G}_j(\mathbb{B}^{k+1})$ contains the cube $\frac{1}{2}\mathbb{I}$ that is concentric with $\mathbb{I}$ and has half the diameter.

Indeed, assume that $\sup_{x \in \partial\mathbb{B}^{k+1}} |\tilde{G}_j(x) - F(x)| < \frac{1}{4}$ – this ensures that $\tilde{G}_j(\partial\mathbb{B}^{k+1})$ does not intersect $\frac{1}{2}\mathbb{I}$. We will show that $\frac{1}{2}\mathbb{I} \subset \tilde{G}_j(\mathbb{B}^{k+1})$. Assume to the contrary, that there is $y \in \frac{1}{2}\mathbb{I} \setminus \tilde{G}_j(\mathbb{B}^{k+1})$. Let P denote the ‘radial’ projection from $\mathbb{I} \setminus \{y\}$ to $\partial\mathbb{I}$, centered at y (we set $P(z)$, for $z \in \mathbb{I} \setminus \{y\}$, to be the unique intersection of the half-line $\{y + t(z - y) : t > 0\}$ and $\partial\mathbb{I}$). Then P is continuous and $P|_{\partial\mathbb{I}} = \text{id}$. Note that $P \circ \tilde{G}_j$ is continuous on $\mathbb{B}^{k+1}$, because y does not belong to the image of $\tilde{G}_j$. Also, for large j , $P \circ \tilde{G}_j|_{\partial\mathbb{B}^{k+1}}$ is uniformly close to $P \circ F = F$, in particular the two mappings are homotopic so $P \circ \tilde{G}_j|_{\partial\mathbb{B}^{k+1}}$ is not homotopic to a constant map. However, $H : [0, 1] \times \partial\mathbb{B}^{k+1} \rightarrow \partial\mathbb{I}$ given by the formula

$$H(t, x) = P \circ \tilde{G}_j(tx)$$

is continuous homotopy between $H(1, x) = P \circ \tilde{G}_j(x)$, and the constant map $H(0, x) = P \circ \tilde{G}_j(0)$, which is a contradiction.

Recall that according to Sard’s theorem [4, 5], the map $\tilde{G}_j \in C^{k-m+1}$ maps the set of its critical points to a set of measure zero. Since $\text{rank } D\tilde{G}_j \leq m$, all points in $\mathbb{B}^{k+1}$ are critical, so the set $\tilde{G}_j(\mathbb{B}^{k+1})$ has measure zero, which contradicts the fact that it contains the cube $\frac{1}{2}\mathbb{I}$. The proof is complete. $\square$

REFERENCES

- [1] GAŁĘSKI, J.: Besicovitch–Federer projection theorem for continuously differentiable mappings having constant rank of the Jacobian matrix. *Math. Z.* 289 (2018), 995–1010.
- [2] GOLDSTEIN, P., HAJŁASZ, P., PANKKA, P.: Topologically nontrivial counterexamples to Sard’s theorem. *Int. Math. Res. Not. IMRN* (to appear).
- [3] HATCHER, A.: *Algebraic topology*. Cambridge University Press, Cambridge, 2002.
- [4] SARD, A.: The measure of the critical values of differentiable maps. *Bull. Amer. Math. Soc.* 48 (1942), 883–890.
- [5] STERNBERG, S.: *Lectures on differential geometry*. Chelsea Publishing Co., New York, second edition, 1983. With an appendix by Sternberg and Victor W. Guillemin.
- [6] ZORICH, V. A.: *Mathematical analysis. I*. Second edition. Universitext. Springer-Verlag, Berlin, 2015.

PAWEŁ GOLDSTEIN, INSTITUTE OF MATHEMATICS, FACULTY OF MATHEMATICS, INFORMATICS AND MECHANICS, UNIVERSITY OF WARSAW, BANACHA 2, 02-097 WARSAW, POLAND

E-mail address: P.Goldstein@mimuw.edu.pl

PIOTR HAJŁASZ, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PITTSBURGH, PITTSBURGH, PA 15260, USA

E-mail address: hajlasz@pitt.edu