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UNIQUE DETERMINATION OF A TRANSVERSELY ISOTROPIC PERTURBATION IN A LINEARIZED INVERSE BOUNDARY VALUE PROBLEM FOR ELASTICITY

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Abstract. We consider a linearized inverse boundary value problem for the elasticity system. From the linearized Dirichlet-to-Neumann map at zero frequency, we show that a transversely isotropic perturbation of a homogeneous isotropic elastic tensor can be uniquely determined. From the linearized Dirichlet-to-Neumann map at two distinct positive frequencies, we show that a transversely isotropic perturbation of a homogeneous isotropic density can be identified at the same time.

1. **Introduction and main result.** In this paper, we investigate the problem of determining interior material property of an elastic body from boundary measurements. We will consider the linearized inverse boundary value problem for the equation

$$\mathrm{div}(\mathbf{C}\nabla u) + \omega^2 \rho u = 0,$$

which reads in components as

$$\partial_j C_{ijkl}(x)\partial_k u_l(x) + \omega^2 \rho_{ik}(x) u_k(x) = 0, \qquad i = 1, 2, 3. \tag{1}$$

Here $\omega \geq 0$ is the frequency, u is the displacement vector, $\rho = (\rho_{ik})$ is a symmetric matrix representing the density of mass; $\mathbf{C} = (C_{ijkl})$ is the elastic tensor whose components obey the symmetry conditions

$$C_{ijkl} = C_{jikl} = C_{klij}. \tag{2}$$

We have used Einstein's summation convention in (1) such that repeated indices are summed up over $\{1, 2, 3\}$. Note that $\mathbf{C}$ with the above symmetry has a total number of 21 linearly independent components. For a fixed ω, the case $\omega = 0$ corresponds to the governing equations for linear elasticity in equilibrium, while the case $\omega > 0$ represents the time-harmonic elastic wave with frequency ω.

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Let Ω be an open bounded domain in $\mathbb{R}^3$ with $C^{1,1}$ boundary $\partial\Omega$. Suppose the density of mass ρ and the elastic tensor $\mathbf{C}$ are both bounded, in the sense that $\rho_{ik}, C_{ijkl} \in L^\infty(\Omega)$ for all $i, j, k, l \in \{1, 2, 3\}$. We further assume that the density of mass ρ and the elasticity tensor $\mathbf{C}$ satisfy the following positivity conditions: there exists $\delta > 0$ such that for any real-valued 3-vector $\sigma = (\sigma_1, \sigma_2, \sigma_3)$,

$$\sum_{i,k=1}^{3} \rho_{ik}\sigma_i\sigma_k \geq \delta \sum_{i=1}^{3} \sigma_i^2;$$

and for any 3×3 real-valued symmetric matrix (ε_{ij}),

$$\sum_{i,j,k,l=1}^{3} C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \geq \delta \sum_{i,j=1}^{3} \varepsilon_{ij}^2.$$

If ω^2 is not a Dirichlet eigenvalue of the operator $-\rho^{-1}\mathrm{div}(\mathbf{C}\nabla\cdot)$, then for any $f \in H^{1/2}(\partial\Omega)$, standard elliptic theory ensures a unique solution $u^f \in H^1(\Omega)$ to the boundary value problem

$$\begin{cases} \partial_j C_{ijkl}(x)\partial_k u_l^f(x) + \omega^2 \rho_{ik}(x) u_k^f(x) = 0 \text{ in } \Omega, & i = 1, 2, 3 \\ u^f|_{\partial\Omega} = f. \end{cases}$$

We define the Dirichlet-to-Neumann map (DN map) $\Lambda_{\mathbf{C},\rho,\omega}$ by

$$\Lambda_{\mathbf{C},\rho,\omega} : f \mapsto C_{ijkl}\nu_j\partial_k u_l^f|_{\partial\Omega}$$

where $\nu = (\nu_1, \nu_2, \nu_3)$ denotes the outer unit normal vector to $\partial\Omega$. It follows that $\Lambda_{\mathbf{C},\rho,\omega} : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ is a bounded linear operator, and the equivalent weak formulation is

$$\langle \Lambda_{\mathbf{C},\rho,\omega} f, g\rangle = \int_\Omega C_{ijkl}\partial_i u_j^f \partial_k u_l^g - \omega^2 \rho_{ik} u_i^f u_k^g \, dx$$

for any $f, g \in H^{1/2}(\partial\Omega)$. We are interested in determining $\mathbf{C}, \rho$ from $\Lambda_{\mathbf{C},\rho,\omega}$ This is related to the invertibility of the non-linear map $(\mathbf{C}, \rho) \mapsto \Lambda_{\mathbf{C},\rho,\omega}$. The question is difficult in the general setting, so it is commonly studied under additional a-priori information.

The case $\boldsymbol{\omega = 0}$. Note that when $\omega = 0$, the density ρ does not appear in the equation (1), thus one can only expect to recover information on $\mathbf{C}$. We henceforth write $\Lambda_{\mathbf{C},\rho,0}$ as $\Lambda_{\mathbf{C}}$ for the ease of notation.

We say the elastic tensor $\mathbf{C}$ (or the medium) is *homogeneous* if it is a constant tensor (that is, independent of x); it is *isotropic* if it can be written as

$$C_{ijkl}(x) = \lambda(x)\delta_{ij}\delta_{kl} + \mu(x)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

where the two functions $\lambda(x)$ and $\mu(x)$ are known as *Lamé parameters*; and it is *fully anisotropic* if the components C_{ijkl} are subject to no other relations other than (2). For isotropic $\mathbf{C}$, a global uniqueness result can be found [20] in dimension two. The problem remains open in dimension three, yet some special cases have been tackled. Among them, Nakamura and Uhlmann [27] proved uniqueness when the Lamé parameters are smooth and $\mu(x)$ is close to a positive constant, see [13] for a similar result by Eskin and Ralston and [19] for a partial data result; uniqueness for recovering piecewise constant Lamé parameters was proved in [6, 7]; and some boundary determination results were shown in [22, 25, 26]. For fully anisotropic $\mathbf{C}$, uniqueness was proved in [10] for piecewise homogeneous medium.

It is widely believed that a fully anisotropic $\mathbf{C}$ without additional assumption cannot be uniquely recovered. For the inverse conductivity problem, that is the problem to determine the coefficients $\gamma = (\gamma_{ij}(x))$ in the equation

$$\partial_i(\gamma_{ij}(x)\partial_j u(x)) = 0$$

from the associated Dirichlet-to-Neumann map, it is known that an anisotropic $\gamma(x)$ can at best be determined up to boundary-fixing diffeomorphisms [14]. In contrast, many anisotropic elastic materials have extra structural symmetries which cannot be preserved under diffeomorphisms. It is therefore important to study the uniqueness of elasticity parameters with extra symmetries in anisotropy. We list some frequently considered anisotropies with symmetries in the table below, see [4, Chapter 2.6] [30, Chapter 3.4] for detailed description. It is worth mentioning that these concepts of anisotropy are purely Cartesian (in a prescribed coordinate system (x_1, x_2, x_3)).

Type of anisotropy	Symmetry	Number of independent components
isotropic	radial symmetry	2
cubic	three mutually orthogonal planes of reflection symmetry plus $\frac{\pi}{2}$ rotation symmetry with respect to those planes	3
transversely isotropic	three mutually orthogonal planes of reflection symmetry and one symmetry axis perpendicular to one symmetry plane	5
orthotropic (orthorhombic)	three mutually orthogonal planes of reflection symmetry	9
monoclinic	one plane of reflection symmetry	13
fully anisotropic	no symmetry	21

In this article, we investigate the linearization of the map $\mathbf{C} \mapsto \Lambda_{\mathbf{C}}$ at a homogeneous isotropic elastic tensor. More specifically, suppose

$$\mathbf{C}(x) = \mathbf{C}^0 + \delta\mathbf{C}(x)$$

where $\mathbf{C}^0 = \lambda^0\delta_{ij}\delta_{kl} + \mu^0(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ is a homogeneous, isotropic background tensor with Lamé parameters (λ^0, μ^0) satisfying

$$\mu^0 > 0, \qquad 3\lambda^0 + 2\mu^0 > 0, \tag{3}$$

and $\delta\mathbf{C}(x)$ is viewed as a perturbation term with components $\delta C_{ijkl}(x)$. It is routine to verify that the map $\mathbf{C} \mapsto \Lambda_{\mathbf{C}}$ is Frechét differentiable at $\mathbf{C}^0$ (we refer to [15] for more details), and the Frechét derivative

$$\dot{\Lambda}_{\mathbf{C}^0} : L^\infty(\Omega) \ni \delta\mathbf{C} \mapsto \dot{\Lambda}_{\mathbf{C}^0}(\delta\mathbf{C}) \in \mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))$$

is characterized by

$$\langle \dot{\Lambda}_{\mathbf{C}^0}(\delta\mathbf{C})f, g\rangle = \int_\Omega \delta C_{ijkl}(x)\,\partial_i u_j(x)\,\partial_k v_l(x)\,dx \tag{4}$$

where u (resp. v) solves

$$\left\{\begin{array}{rcll} \mu^0\Delta u + (\lambda^0 + \mu^0)\nabla\nabla\cdot u &=& 0 & \text{in } \Omega \\ u|_{\partial\Omega} &=& f. & \end{array}\right. \tag{5}$$

$$\left(\text{resp.} \begin{cases} \mu^0 \Delta v + (\lambda^0 + \mu^0)\nabla\nabla\cdot v &= 0 \quad \text{in } \Omega \\ v|_{\partial\Omega} &= g. \end{cases} \right)$$

The question we are interested in is whether the linearized map $\dot{\Lambda}_{\mathbf{C}^0}$ is injective on anisotropic perturbations with certain symmetry. It was proved in [15] that the linearization $\dot{\Lambda}_{\mathbf{C}^0}$ is injective on isotropic perturbations. Our main theorem (see Theorem 1 below) generalizes this injectivity result from isotropic perturbtions to transversely isotropic perturbations.

A *transversely isotropic* material is one with physical properties that are symmetric about an axis that is normal to a plane of isotropy. It is also known as "polar anisotropic" since the material properties are the same in all directions within the transverse plane. Examples of transversely isotropic materials include some piezoelectric materials and fiber-reinforced composites where all fibers are in parallel. Geological layers of rocks are often interpreted as being transversely isotropic as well in terms of their effective properties. Transversely isotropic materials have been extensively studied in geophysical literature, see [1, 2, 21, 23] and the references therein. For mathematical treatment, unique determination of transversely isotropic parameters from boundary measurements is studied in [11, 12].

As is indicated in the above table, a transverse isotropic material has three mutually orthogonal planes of micro-structural reflection symmetry and one symmetry axis perpendicular to one of the three symmetry planes. Assume the symmetry axis is x_3, then a transversely isotropic $\delta\mathbf{C}$ obeys the invariance

$$Q_{ip}Q_{jq}Q_{kr}Q_{ls}\delta C_{pqrs} = \delta C_{ijkl},$$

where Q can take any of the following reflection and rotation matrices.

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$
$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; 0 \le \theta \le 2\pi. \tag{6}$$

Writing the above invariance component-wisely results in 9 non-zero components in $\delta\mathbf{C}$

$$\delta C_{1111},\ \delta C_{2222},\ \delta C_{3333},\ \delta C_{1122},\ \delta C_{1133},\ \delta C_{2233},\ \delta C_{1212},\ \delta C_{1313},\ \delta C_{2323}$$

subject to 4 linear relations

$$\begin{aligned} \delta C_{1111} &= \delta C_{2222}, & \delta C_{1133} &= \delta C_{2233}, \\ \delta C_{1313} &= \delta C_{2323}, & \delta C_{1212} &= \frac{1}{2}(\delta C_{1111} - \delta C_{1122}). \end{aligned} \tag{7}$$

Hence a traversally isotropic $\delta\mathbf{C}$ has only 5 linearly independent components. We will prove that these independent components are uniquely determined by the linearized map $\dot{\Lambda}_{\mathbf{C}^0}$. More precisely, we show

Theorem 1. *Let $\mathbf{C}^0 = \lambda^0\delta_{ij}\delta_{kl} + \mu^0(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ be homogeneous and isotropic with Lamé parameters (λ^0, μ^0) satisfying (3). If $\dot{\Lambda}_{\mathbf{C}^0}(\delta\mathbf{C}) = 0$ and $\delta\mathbf{C} \in L^\infty(\Omega)$ is transversely isotropic with known axis of symmetry, then $\delta\mathbf{C} = 0$.*

The 5 linearly independent components of $\delta\mathbf{C}$ we will determine are δC_{1111}, δC_{1122}, δC_{1133}, δC_{1313}, and δC_{3333}.

Injectivity of the linearized map $\dot{\Lambda}_{\mathbf{C}^0}$ has been studied in previous literature. In dimension two or higher, it is known that $\dot{\Lambda}_{\mathbf{C}^0}$ is injective on isotropic $\delta\mathbf{C}$ [15]. Theorem 1 can be viewed as generalization of such injectivity result from isotropic perturbations to transversely isotropic perturbations in dimension three. Note that Theorem 1 has greatly increased the number of independent parameters that can be simultaneously identified by $\dot{\Lambda}_{\mathbf{C}_0}$ – from 2 Lamé parameters in the isotropic case [15] to 5 independent parameters in the transversely isotropic case. In dimension two, Ikehata [16, 18, 17] characterized the injectivity with general anisotropic $\mathbf{C}^0$.

The case $\boldsymbol{\omega > 0}$. The time-harmonic case has important application in (reflection) seismology, where one hopes to recover the material parameters of the Earth's subsurface areas from vibroseis data. Unique determination of piecewise homogeneous isotropic parameters from $\Lambda_{\mathbf{C},\rho,\omega}$ was established in [5]; unique determination of an anisotropic density with homogeneous isotropic elastic tensor was proved in [3]. On the other hand, inverse boundary value problems for the dynamic elasticity system has been considered in [28, 29, 31, 8, 11].

We still consider the linearization of the map $(\mathbf{C},\rho)\mapsto\Lambda_{\mathbf{C},\rho,\omega}$ at a homogeneous and isotropic $(\mathbf{C}^0,\rho^0)$. Assume ω^2 is not a Dirichlet eigenvalue of $-(\rho^0)^{-1}\mathrm{div}(\mathbf{C}^0\nabla\cdot)$. Then ω^2 is also not a Dirichlet eigenvalue of $-(\rho)^{-1}\mathrm{div}(\mathbf{C}\nabla\cdot)$ for $(\mathbf{C},\rho)$ close enough to $(\mathbf{C}_0,\rho_0)$, and thus the Frechét derivative

$$\dot{\Lambda}_{\mathbf{C}^0,\rho^0,\omega}: L^\infty(\Omega)\ni(\delta\mathbf{C},\delta\rho)\mapsto\dot{\Lambda}_{\mathbf{C}^0,\rho^0,\omega}(\delta\mathbf{C},\delta\rho)\in\mathcal{L}(H^{1/2}(\partial\Omega),H^{-1/2}(\partial\Omega))$$

is well defined and given by

$$\langle\dot{\Lambda}_{\mathbf{C}^0,\rho^0,\omega}(\delta\mathbf{C},\delta\rho)f,g\rangle=\int_\Omega\delta C_{ijkl}(x)\,\partial_i u_j(x)\,\partial_k v_l(x)-\omega^2\delta\rho_{ik}u_iv_k\,dx, \tag{8}$$

where u and v solve respectively

$$\begin{cases}\mu^0\Delta u+(\lambda^0+\mu^0)\nabla\nabla\cdot u+\omega^2\rho^0u &= 0 \quad\text{in }\Omega\\ u|_{\partial\Omega} &= f,\end{cases}$$

$$\begin{cases}\mu^0\Delta v+(\lambda^0+\mu^0)\nabla\nabla\cdot v+\omega^2\rho^0v &= 0 \quad\text{in }\Omega\\ v|_{\partial\Omega} &= g.\end{cases} \tag{9}$$

By definition, a transversely isotropic $\delta\rho$ with symmetry axis x_3 has the property

$$Q_{ip}Q_{jq}\delta\rho_{pq}=\delta\rho_{ij}$$

for any Q of the forms (6). Then it can be written as

$$\delta\rho=\begin{pmatrix}\delta\rho_{11}&&\\&\delta\rho_{11}&\\&&\delta\rho_{33}\end{pmatrix}.$$

For the inverse boundary value problem for time-harmonic acoustic wave equation

$$\nabla\cdot\gamma\nabla u+\omega^2qu=0,$$

it is known that the simultaneous recovery of γ and q requires two frequency data $\Lambda_{\gamma,q,\omega_1}$, $\Lambda_{\gamma,q,\omega_2}$ [24]. Therefore it is also natural to conjecture that we also need two frequency data for our problem. More precisely, we will prove

Theorem 2. *Let $\mathbf{C}^0=\lambda^0\delta_{ij}\delta_{kl}+\mu^0(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})$ be homogeneous and isotropic with Lamé parameters (λ^0,μ^0) satisfying (3), and $\rho^0_{ik}=\rho^0\delta_{ik}$ be a homogeneous isotropic density. If $\dot{\Lambda}_{\mathbf{C}^0,\rho^0,\omega_i}(\delta\mathbf{C},\delta\rho)=0$ for two distinct positive frequencies ω_1,ω_2*

and $(\delta\mathbf{C}, \delta\rho) \in L^\infty(\Omega)$ *are transversely isotropic with known axis of symmetry, then* $(\delta\mathbf{C}, \delta\rho) = 0$.

Although single frequency data is enough for recovering piecewise homogeneous parameters [5], we do believe a second frequency is necessary without piecewise homogeneity assumptions.

The proofs of Theorem 1 and Theorem 2 are based on construction of the *Complex Geometric Optics* (CGO) solutions for the system (1) and (9), respectively. CGO solutions were initiated by Sylvester and Uhlmann [32] in their solving Calderón's inverse conductivity problem [9]. Solutions of this type with $\omega = 0$ were introduced in [15] for the elasticity system with constant coefficients, and in [13, 27] for variable coefficients. Solutions with $\omega > 0$ were utilized in [3].

We remark here that one can follow the steps in the proofs of Theorem 1 and Theorem 2 to obtain an explicit reconstruction procedure as in [15] , using identities (4) and (8).

There are notable differences in the construction of CGO solutions for $\omega = 0$ and $\omega > 0$. To see this, consider the solution ψ to the scalar wave equation

$$(\lambda^0 + 2\mu^0)\Delta\psi + \rho^0\omega^2\psi = 0.$$

For $\omega = 0$, a CGO solution can be taken as $\psi = e^{\zeta\cdot x}$ with $\zeta \in \mathbf{C}^3$, $\zeta\cdot\zeta = 0$, then $u := \nabla\psi$ is a solution to the elasticity equation (1) that is divergence free, which is a property for S-wave. For $\omega > 0$, a similar CGO solution can be constructed as $\psi = e^{\zeta'\cdot x}$ but with $\zeta' \in \mathbf{C}^3$, $\zeta'\cdot\zeta' = \frac{\omega^2\rho^0}{\lambda^0+2\mu^0}$. Then $u := \nabla\psi$ remains a solution to the elasticity equation in (9), but is not divergence free any more. In fact, this solution u corresponds to a P-wave. Of course, this argument is just heuristic as the equation with $\omega = 0$ does not really describe waves; but it demonstrates the difference between the cases $\omega = 0$ and $\omega > 0$. The proofs of Theorem 1 and Theorem 2 are therefore presented separately due to some essential differences in the construction of CGO solutions. The rest of the paper is devoted to these proofs.

2. Zero frequency case: Proof of Theorem 1.

We prove Theorem 1 in this section. In view of (4), $\dot\Lambda_{\mathbf{C}^0}(\delta\mathbf{C}) = 0$ implies

$$\int_\Omega \delta C_{ijkl}\partial_i u_j \partial_k v_l\, dx = 0, \tag{10}$$

for any u, v satisfying (1). The key ingredient of our proof is constructing CGO solutions to (1) and inserting them into (10) to obtain sufficiently many linearly independent equations in the 5 independent components of $\delta\mathbf{C}$. For the ease of notation, we abbreviate $\mathbf{C}$ for $\delta\mathbf{C}(x)$ and C_{ijkl} for $\delta C_{ijkl}(x)$ from now on. We reserve the letter i for an index and write the bold face $\mathbf{i}$ for the imaginary unit.

Step 1. Set

$$\zeta^{(1)} := \mathbf{i}(s,0,t) + (-t,0,s), \qquad \zeta^{(2)} := \mathbf{i}(s,0,t) - (-t,0,s), \tag{11}$$

and

$$a^{(1)} = a^{(2)} = a = (0,1,0).$$

We take

$$u = a^{(1)}e^{\zeta^{(1)}\cdot x}, \qquad v = a^{(2)}e^{\zeta^{(2)}\cdot x}.$$

The choice of $\zeta^{(i)} \in \mathbb{C}^3$ ensures $\zeta^{(1)} \cdot \zeta^{(1)} = \zeta^{(2)} \cdot \zeta^{(2)} = 0$, hence $\Delta u = \Delta v = 0$. The choice of a ensures $a \perp \Re\zeta^{(i)}$, $a \perp \Im\zeta^{(i)}$ for $i = 1, 2$, hence $\nabla \cdot u = \nabla \cdot v = 0$. This verifies that u, v defined in this manner satisfy the equations (1).

Substituting u and v into (10), we have

$$\begin{aligned} 0 &= \int_\Omega C_{ijkl} a_i \zeta_j^{(1)} \, a_k \zeta_l^{(2)} e^{(\zeta^{(1)}+\zeta^{(2)})\cdot x} \, dx \\ &= \int_\Omega [C_{1212}(\mathbf{i}s - t)(\mathbf{i}s + t) + C_{2323}(\mathbf{i}t + s)(\mathbf{i}t - s)] e^{2\mathbf{i}(s,0,t)\cdot x} \, dx \\ &= \int_\Omega (s^2 + t^2)(-C_{1212} - C_{1313}) e^{2\mathbf{i}(s,0,t)\cdot x} \, dx. \end{aligned}$$

This implies the Fourier transform $\mathcal{F}[\chi_\Omega(C_{1212} + C_{1313})](-2(s, 0, t)) = 0$ for any s, t with $s^2 + t^2 \neq 0$. Here χ_Ω is the characteristic function of the domain Ω. Since s, t can be any real number, this Fourier transform vanishes on the punctured x_1x_3-plane. The axial symmetry with respect to x_3-axis in the definition of transversal isotropy allows one to obtain similar vanishing result in any plane containing x_3-axis. We conclude $\mathcal{F}[\chi_\Omega(C_{1212} + C_{1313})](\xi) = 0$ for any $\xi \neq 0$. This forces

$$C_{1212} + C_{1313} = 0 \qquad \text{in } \Omega.$$

Using the relation $C_{1212} = \frac{1}{2}(C_{1111} - C_{1122})$ in (7) we have

$$C_{1111} - C_{1122} + 2C_{1313} = 0 \qquad \text{in } \Omega. \tag{12}$$

Step 2. Take

$$u = \zeta^{(1)} e^{\zeta^{(1)}\cdot x}, \qquad v = \zeta^{(2)} e^{\zeta^{(2)}\cdot x},$$

with $\zeta^{(1)}$, $\zeta^{(2)}$ defined in (11). One still has $\Delta u = \Delta v = 0$ as before. On the other hand, the i-th component of u (resp. v) is $u_i = \zeta_i^{(1)} e^{\zeta^{(1)}\cdot x}$ (resp. $v_i = \zeta_i^{(2)} e^{\zeta^{(2)}\cdot x}$), $i = 1, 2, 3$. The derivatives of these components are

$$\partial_i u_j = \zeta_i^{(1)} \zeta_j^{(1)} e^{\zeta^{(1)}\cdot x}, \qquad \partial_k v_l = \zeta_k^{(2)} \zeta_l^{(2)} e^{\zeta^{(2)}\cdot x}.$$

Then $\nabla \cdot u = \zeta^{(1)} \cdot \zeta^{(1)} e^{\zeta^{(1)}\cdot x} = 0$ and likewise $\nabla \cdot v = 0$. We see that u, v solve (1).

Inserting u, v into the integral identity (10), we obtain

$$\begin{aligned} 0 = &\int_\Omega C_{ijkl} \zeta_i^{(1)} \zeta_j^{(1)} e^{\zeta^{(1)}\cdot x} \, \zeta_k^{(2)} \zeta_l^{(2)} e^{\zeta^{(2)}\cdot x} \, dx \\ = &\int_\Omega \Big[C_{1111}(\mathbf{i}s - t)^2(\mathbf{i}s + t)^2 + C_{1133}(\mathbf{i}s - t)^2(\mathbf{i}t - s)^2 \\ &+ C_{1313}(\mathbf{i}s - t)(\mathbf{i}t + s)(\mathbf{i}s + t)(\mathbf{i}t - s) + C_{1331}(\mathbf{i}s - t)(\mathbf{i}t + s)(\mathbf{i}t - s)(\mathbf{i}s + t) \\ &+ C_{3113}(\mathbf{i}t + s)(\mathbf{i}s - t)(\mathbf{i}s + t)(\mathbf{i}t - s) + C_{3131}(\mathbf{i}t + s)(\mathbf{i}s - t)(\mathbf{i}t - s)(\mathbf{i}s + t) \\ &+ C_{3311}(\mathbf{i}t + s)^2(\mathbf{i}s + t)^2 + C_{3333}(\mathbf{i}t + s)^2(\mathbf{i}t - s)^2 \Big] e^{2\mathbf{i}(s,0,t)\cdot x} \, dx \end{aligned}$$

Combining the terms, one has

$$0 = \int_\Omega (t - \mathbf{i}s)^2(t + \mathbf{i}s)^2 [C_{1111} - 2C_{1133} + 4C_{1313} + C_{3333}] e^{2\mathbf{i}(s,0,t)\cdot x} \, dx.$$

This means $\mathcal{F}\left[\chi_\Omega(C_{1111} - 2C_{1133} + 4C_{1313} + C_{3333})\right](-2(s, 0, t))(t^2 + s^2)^2 = 0$. A similar argument as in Step 1 shows $\mathcal{F}\left[\chi_\Omega(C_{1111} - 2C_{1133} + 4C_{1313} + C_{3333})\right](\xi) = 0$ for any $\xi \neq 0$, hence

$$C_{1111} - 2C_{1133} + 4C_{1313} + C_{3333} = 0, \qquad \text{in } \Omega. \tag{13}$$

Step 3. We still take u, v of the form

$$u = \zeta^{(1)} e^{\zeta^{(1)}\cdot x}, \qquad v = \zeta^{(2)} e^{\zeta^{(2)}\cdot x},$$

but with different phases $\zeta^{(1)}$, $\zeta^{(2)}$. Set $d := \sqrt{s^2+t^2}$ and $\beta := \sqrt{\frac{r^2}{d^2}-1}$. The new phases to be used are

$$\begin{aligned}\zeta^{(1)} &= \mathbf{i}(s,0,t) + \mathbf{i}\beta(-t,0,s) + r(0,1,0) = (\mathbf{i}s - \mathbf{i}\beta t, r, \mathbf{i}t + \mathbf{i}\beta s),\\ \zeta^{(2)} &= \mathbf{i}(s,0,t) - \mathbf{i}\beta(-t,0,s) - r(0,1,0) = (\mathbf{i}s + \mathbf{i}\beta t, -r, \mathbf{i}t - \mathbf{i}\beta s).\end{aligned}$$

It is easy to verify $\zeta^{(1)}\cdot\zeta^{(1)} = \zeta^{(2)}\cdot\zeta^{(2)} = 0$. This property again makes $\Delta u = \Delta v = 0$ and $\nabla\cdot u = \nabla\cdot v = 0$. Note that these new phases include the old ones: they coincide with (11) if one takes $r = 0$ and $\beta = -\mathbf{i}$.

Using such u, v in (10), we have

$$0 = \int_\Omega C_{ijkl}\zeta_i^{(1)}\zeta_j^{(1)} e^{\zeta^{(1)}\cdot x}\, \zeta_k^{(2)}\zeta_l^{(2)} e^{\zeta^{(2)}\cdot x}\, dx =: G_1 + G_2 + G_3,$$

where

$$\begin{aligned}G_1 :=& \int_\Omega \Big[C_{1111}(\mathbf{i}s - \mathbf{i}\beta t)^2(\mathbf{i}s + \mathbf{i}\beta t)^2 + C_{2222}(r)^2(-r)^2\\ &\qquad + C_{3333}(\mathbf{i}t + \mathbf{i}\beta s)^2(\mathbf{i}t - \mathbf{i}\beta s)^2\Big] e^{2\mathbf{i}(s,0,t)\cdot x}\, dx\\ =& \int_\Omega \Big[C_{1111}(s-\beta t)^2(s+\beta t)^2 + C_{2222}r^4\\ &\qquad + C_{3333}(t+\beta s)^2(t-\beta s)^2\Big] e^{2\mathbf{i}(s,0,t)\cdot x}\, dx,\end{aligned}$$

$$\begin{aligned}G_2 :=& \int_\Omega \big[C_{1122}(\mathbf{i}s - \mathbf{i}\beta t)^2(-r)^2 + C_{2211}(r)^2(\mathbf{i}s + \mathbf{i}\beta t)^2\\ &+ C_{1133}(\mathbf{i}s - \mathbf{i}\beta t)^2(\mathbf{i}t - \mathbf{i}\beta s)^2 + C_{3311}(\mathbf{i}t + \mathbf{i}\beta s)^2(\mathbf{i}s + \mathbf{i}\beta t)^2\\ &+ C_{2233}(r)^2(\mathbf{i}t - \mathbf{i}\beta s)^2 + C_{3322}(\mathbf{i}t + \mathbf{i}\beta s)^2(-r)^2\big]\, e^{2\mathbf{i}(s,0,t)\cdot x}\, dx\\ =& \int_\Omega \big[-2C_{1122}(s^2 + \beta^2 t^2)r^2\\ &+ C_{1133}(s-\beta t)^2(t-\beta s)^2 + C_{3311}(t+\beta s)^2(s+\beta t)^2\\ &-2C_{2233}(t^2 + \beta^2 s^2)r^2\big]\, e^{2\mathbf{i}(s,0,t)\cdot x}\, dx,\end{aligned}$$

$$\begin{aligned}G_3 :=& \int_\Omega \Big[4C_{1212}(\mathbf{i}s - \mathbf{i}\beta t)(r)(\mathbf{i}s + \mathbf{i}\beta t)(-r)\\ &\qquad + 4C_{1313}(\mathbf{i}s - \mathbf{i}\beta t)(\mathbf{i}t + \mathbf{i}\beta s)(\mathbf{i}s + \mathbf{i}\beta t)(\mathbf{i}t - \mathbf{i}\beta s)\\ &\qquad + 4C_{2323}(r)(\mathbf{i}t + \mathbf{i}\beta s)(-r)(\mathbf{i}t - \mathbf{i}\beta s)\Big] e^{2\mathbf{i}(s,0,t)\cdot x}\, dx\\ =& \int_\Omega \Big[4C_{1212}(s^2 - \beta^2 t^2)r^2 + 4C_{1313}(s^2 - \beta^2 t^2)(t^2 - \beta^2 s^2)\\ &\qquad + 4C_{2323}(t^2 - \beta^2 s^2)r^2\Big] e^{2\mathbf{i}(s,0,t)\cdot x}\, dx.\end{aligned}$$

We will analyze the asymptotic behavior as $r \to \infty$. Direct calculation (though tedious) shows

$$G_1 = \int_\Omega \left(C_{1111}\frac{t^4}{d^4} + C_{2222} + C_{3333}\frac{s^4}{d^4}\right) r^4 e^{2\mathbf{i}(s,0,t)\cdot x}\, dx + O(r^3),$$

$$G_2 = \int_\Omega \left(-2C_{1122}\frac{t^2}{d^2} + 2C_{1133}\frac{t^2s^2}{d^4} - 2C_{2233}\frac{s^2}{d^2}\right) r^4 e^{2\mathbf{i}(s,0,t)\cdot x}\, dx + O(r^3),$$
$$G_3 = \int_\Omega \left(-4C_{1212}\frac{t^2}{d^2} + 4C_{1313}\frac{t^2s^2}{d^4} - 4C_{2323}\frac{s^2}{d^2}\right) r^4 e^{2\mathbf{i}(s,0,t)\cdot x}\, dx + O(r^3).$$

Equating the terms of order $O(r^4)$ yields

$$\int_\Omega \Big[\frac{t^4}{d^4}C_{1111} + C_{2222} + \frac{s^4}{d^4}C_{3333} - \frac{2t^2}{d^2}C_{1122} + \frac{2t^2s^2}{d^4}C_{1133} - \frac{2s^2}{d^2}C_{2233} - \frac{4t^2}{d^2}C_{1212} + \frac{4t^2s^2}{d^4}C_{1313} - \frac{4s^2}{d^2}C_{2323}\Big] e^{2\mathbf{i}(s,0,t)\cdot x}\, dx = 0.$$

Using the linear relations in (7) and $d^2 = t^2 + s^2$, one can eliminate C_{2222}, C_{2233}, C_{1212}, C_{2323} and get

$$\int_\Omega \Big[\frac{s^4}{d^4}C_{1111} - \frac{2s^4}{d^4}C_{1133} - \frac{4s^4}{d^4}C_{1313} + \frac{s^4}{d^4}C_{3333}\Big] e^{2\mathbf{i}(s,0,t)\cdot x}\, dx = 0.$$

In other words, $\mathcal{F}[\chi_\Omega(C_{1111} - 2C_{1133} - 4C_{1313} + C_{3333})](-2(s,0,t)) = 0$ when $s \neq 0$. Using the definition of transverse isotropy, one sees that the Fourier transform vanishes in the entire $\mathbb{R}^3$ except on the x_3 axis. Moreover, the Fourier transform is actually an analytic function since $\chi_\Omega(C_{1111} - 2C_{1133} - 4C_{1313} + C_{3333})$ is compactly supported. This forces

$$C_{1111} - 2C_{1133} - 4C_{1313} + C_{3333} = 0 \qquad \text{in } \Omega. \tag{14}$$

Let us put the three pieces of information (12)(13)(14) together

$$\begin{aligned} &C_{1111} - C_{1122} + 0\cdot C_{1133} + 2C_{1313} + 0\cdot C_{3333} = 0;\\ &C_{1111} + 0\cdot C_{1122} - 2C_{1133} + 4C_{1313} + C_{3333} = 0;\\ &C_{1111} + 0\cdot C_{1122} - 2C_{1133} - 4C_{1313} + C_{3333} = 0. \end{aligned}$$

We observe that these combinations are linearly independent and thus can be used to eliminate 3 independent components of C. In fact, solving this linear system yields

$$C_{1313} = C_{1212} = \frac{1}{2}(C_{1111} - C_{1122}) = 0; \tag{15}$$
$$2C_{1133} = C_{1122} + C_{3333}. \tag{16}$$

We are therefore left with only 2 independent components, say C_{1111} and C_{1133}.

The need for different solutions. The previous CGOs are not enough to determine the remaining independent components. To see this, we employ the known relations (15)(16) to simplify the integral identity (10), then

$$\begin{aligned} 0 = &\int_\Omega C_{1111}(\partial_1 u_1 \partial_1 v_1 + \partial_1 u_1 \partial_2 v_2 + \partial_2 u_2 \partial_1 v_1 + \partial_2 u_2 \partial_2 v_2)\\ &+ C_{1133}(\partial_1 u_1 \partial_3 v_3 + \partial_3 u_3 \partial_1 v_1 + \partial_2 u_2 \partial_3 v_3 + \partial_3 u_3 \partial_2 v_2)\\ &+ C_{3333}\partial_3 u_3 \partial_3 v_3\, dx\\ = &\int_\Omega C_{1111}\nabla\cdot u \nabla\cdot v\\ &+ (C_{1133} - C_{1111})(\partial_1 u_1 \partial_3 v_3 + \partial_3 u_3 \partial_1 v_1 + \partial_2 u_2 \partial_3 v_3 + \partial_3 u_3 \partial_2 v_2)\\ &+ (C_{3333} - C_{1111})\partial_3 u_3 \partial_3 v_3\, dx. \end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega C_{1111}\nabla\cdot u\nabla\cdot v \\
&\quad +(C_{1133}-C_{1111})(\partial_1u_1\partial_3v_3+\partial_3u_3\partial_1v_1+\partial_2u_2\partial_3v_3+\partial_3u_3\partial_2v_2) \\
&\quad +2(C_{1133}-C_{1111})\partial_3u_3\partial_3v_3\,dx \\
&= \int_\Omega C_{1111}\nabla\cdot u\nabla\cdot v \\
&\quad +(C_{1133}-C_{1111})(\partial_1u_1\partial_3v_3+\partial_3u_3\partial_1v_1 \\
&\qquad\qquad +\partial_2u_2\partial_3v_3+\partial_3u_3\partial_2v_2+2\partial_3u_3\partial_3v_3)\,dx \qquad (17)\\
&= \int_\Omega C_{1111}\nabla\cdot u\nabla\cdot v+(C_{1133}-C_{1111})(\nabla\cdot u\partial_3v_3+\partial_3u_3\nabla\cdot v)\,dx.
\end{aligned}
$$

All the solutions we have constructed have divergence zero, so they cannot give new information about the tensor $\mathbf{C}$.

Remark 1. *With only CGO solutions of divergence zero, one cannot even determine an isotropic perturbation from* $\dot{\Lambda}_{\mathbf{C}^0}$ *(cf.* [15]*). To see this, suppose* $C_{ijkl} = \lambda\delta_{ij}\delta_{kl}+\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})$*, then* (10) *reduces (with* δC_{ijkl} *abbreviated as* C_{ijkl}*) to*

$$
\int_\Omega 2\mu\mathrm{Sym}(\nabla u):\mathrm{Sym}(\nabla v)+\lambda(\nabla\cdot u)(\nabla\cdot v)dx=0. \qquad (18)
$$

Here $\mathrm{Sym}(\nabla u):=\frac{1}{2}(\nabla u+(\nabla u)^T)$ *and* $A:B=\sum_{i,j=1}^3 A_{ij}B_{ij}$ *for any* 3×3 *matrices* A,B*. It is obvious that solutions with divergence zero cannot provide information about* λ*.*

Different type of solutions. This above analysis suggests the necessity to construct solutions with non-vanishing divergence. We proceed to construct different CGO-type solutions with this property. They are of the form

$$
u=[(b\cdot x)\hat{\zeta}+c]e^{\zeta\cdot x}
$$

where $\zeta\in\mathbb{C}^3$ satisfies $\zeta\cdot\zeta=0$, $\hat{\zeta}$ denotes $\frac{\zeta}{|\zeta|}$, and b,c are constant vectors to be determined. This type of solutions can be constructed as in [15]. The divergence of u is

$$
\nabla\cdot u=\nabla\cdot\left([(b\cdot x)\hat{\zeta}+c]e^{\zeta\cdot x}\right)=[(b\cdot x)\hat{\zeta}\cdot\zeta+b\cdot\hat{\zeta}+c\cdot\zeta]e^{\zeta\cdot x}=(b\cdot\hat{\zeta}+c\cdot\zeta)e^{\zeta\cdot x},
$$

hence

$$
\nabla\nabla\cdot u=(b\cdot\hat{\zeta}+c\cdot\zeta)\zeta e^{\zeta\cdot x}.
$$

On the other hand, the gradient of u is

$$
\nabla u=(b\otimes\hat{\zeta}+\zeta\otimes c+(b\cdot x)\zeta\otimes\hat{\zeta})e^{\zeta\cdot x}
$$

so Δu can be computed:

$$
\Delta u=\nabla\cdot\nabla u=[(\hat{\zeta}\cdot b)\zeta+(\zeta\cdot\zeta)c+(b\cdot x)(\zeta\cdot\zeta)\hat{\zeta}+(b\cdot\hat{\zeta})\zeta]e^{\zeta\cdot x}=2(\hat{\zeta}\cdot b)\zeta e^{\zeta\cdot x}.
$$

We then have

$$
\begin{aligned}
\mu^0\Delta u+(\lambda^0+\mu^0)\nabla\nabla\cdot u =&[2\mu^0(\hat{\zeta}\cdot b)\zeta+(\lambda^0+\mu^0)(b\cdot\hat{\zeta}+c\cdot\zeta)\zeta]e^{\zeta\cdot x} \\
=&[(\lambda^0+\mu^0)c\cdot\zeta+(\lambda^0+3\mu^0)b\cdot\hat{\zeta}]\zeta e^{\zeta\cdot x}.
\end{aligned}
$$

Taking $b=(\lambda^0+\mu^0)\Re\hat{\zeta}$ and $c=-\frac{\lambda^0+3\mu^0}{|\zeta|}\Re\hat{\zeta}$ guarantees the right hand side is zero, making u a solution to (1). Notice that with such b,c, the divergence of u is

$$
\nabla\cdot u=(b\cdot\hat{\zeta}+c\cdot\zeta)e^{\zeta\cdot x}=-2\mu^0\Re\hat{\zeta}\cdot\hat{\zeta}\,e^{\zeta\cdot x}=-\mu^0\,e^{\zeta\cdot x},
$$

which is non-vanishing since $\mu^0 > 0$.

Step 4. We take

$$u = \zeta^{(1)} e^{\zeta^{(1)}\cdot x}, \qquad v = [(b\cdot x)\widehat{\zeta^{(2)}} + c]e^{\zeta^{(2)}\cdot x},$$

with

$$\zeta^{(1)} := \mathbf{i}(s,0,t) + \mathbf{i}\beta(-t,0,s) + r(0,1,0) = (\mathbf{i}s - \mathbf{i}\beta t, r, \mathbf{i}t + \mathbf{i}\beta s),$$
$$\zeta^{(2)} := \mathbf{i}(s,0,t) - \mathbf{i}\beta(-t,0,s) - r(0,1,0) = (\mathbf{i}s + \mathbf{i}\beta t, -r, \mathbf{i}t - \mathbf{i}\beta s).$$

It has been verified that $\zeta^{(1)}\cdot\zeta^{(1)} = \zeta^{(2)}\cdot\zeta^{(2)} = 0$; moreover, $|\zeta^{(1)}| = |\zeta^{(2)}| = \sqrt{2}r$. Correspondingly, we take

$$b = (\lambda^0 + \mu^0)\Re\widehat{\zeta^{(2)}} = (0, -\frac{\lambda^0+\mu^0}{\sqrt{2}}, 0), \qquad c = \frac{\lambda^0 + 3\mu^0}{\sqrt{2}r}(0, \frac{1}{\sqrt{2}}, 0).$$

Substitute u, v into (17) and notice $\nabla\cdot v = -\mu^0\, e^{\zeta^{(2)}\cdot x}$, $\nabla\cdot u = 0$. we have

$$\begin{aligned} 0 =& -\int_\Omega (C_{1133} - C_{1111})\mu^0 \zeta_3^{(1)}\zeta_3^{(1)} e^{(\zeta^{(1)}+\zeta^{(2)})\cdot x}\,dx \\ =&\mu^0 \int_\Omega (C_{1133} - C_{1111})(t+\beta s)^2 e^{2\mathbf{i}(s,0,t)\cdot x}\,dx \\ =&\mu^0 \left[\int_\Omega (C_{1133} - C_{1111})\frac{s^2}{d^2} e^{2\mathbf{i}(s,0,t)\cdot x}\,dx\right] r^2 + O(r) \end{aligned}$$

where the asymptotics is again when $r\to\infty$. This implies

$$\frac{s^2}{t^2+s^2}\int_\Omega (C_{1133} - C_{1111}) e^{2\mathbf{i}(s,0,t)\cdot x}\,dx = 0. \tag{19}$$

Therfore $C_{1133} = C_{1111}$ in Ω.

Step 5. Now we have $C_{1111} = C_{1133} = C_{3333}$, and (17) becomes

$$\int_\Omega C_{1111}\nabla\cdot u \nabla\cdot v\,dx = 0. \tag{20}$$

Take

$$u = [(b^{(1)}\cdot x)\widehat{\zeta^{(1)}} + c^{(1)}]e^{\zeta^{(1)}\cdot x}, \qquad v = [(b^{(2)}\cdot x)\widehat{\zeta^{(2)}} + c^{(2)}]e^{\zeta^{(2)}\cdot x},$$

with

$$\zeta^{(1)} := \mathbf{i}(s,0,t) + (-t,0,s),$$
$$\zeta^{(2)} := \mathbf{i}(s,0,t) - (-t,0,s),$$
$$b^{(1)} = -b^{(2)} = \left(-\frac{t(\lambda^0+\mu^0)}{\sqrt{2}d}, 0, \frac{s(\lambda^0+\mu^0)}{\sqrt{2}d}\right),$$
$$c^{(1)} = -c^{(2)} = (\lambda^0 + 3\mu^0)\left(\frac{t}{2d^2}, 0, -\frac{s}{2d^2}\right).$$

Substitue u and v into (20)

$$\int_\Omega C_{1111}\left(\mu^0\right)^2 e^{2\mathbf{i}(s,0,t)\cdot x}\,dx = 0.$$

Then we get $C_{1111} = 0$. This completes the proof of the uniqueness of all parameters in $\mathbf{C}$.

3. **Non-zero frequency case: Proof of Theorem 2.** We present the proof of Theorem 2 in this section. Notice that $\dot{\Lambda}_{\mathbf{C}^0,\rho^0,\omega}(\delta\mathbf{C},\delta\rho)=0$ implies

$$\int_\Omega \delta C_{ijkl}(x)\,\partial_i u_j(x)\,\partial_k v_l(x) - \omega^2\delta\rho_{ik}u_i v_k\,dx = 0, \tag{21}$$

where u, v satisfies (9).

Step 1. For any $s^2+t^2 > k_{\mathbf{s}}^2 := \omega^2 c_{\mathbf{s}}^{-2}$, set

$$\zeta^{(1)} := \mathbf{i}(s,0,t)+(-t,0,s)\sqrt{1-\frac{k_{\mathbf{s}}^2}{s^2+t^2}}, \quad \zeta^{(2)} := \mathbf{i}(s,0,t)-(-t,0,s)\sqrt{1-\frac{k_{\mathbf{s}}^2}{s^2+t^2}}, \tag{22}$$

where $c_{\mathbf{s}}^2 := \frac{\mu^0}{\rho^0}$ is the speed of S-wave, and

$$a^{(1)} = a^{(2)} = a = (0,1,0).$$

Take

$$u = a^{(1)}e^{\zeta^{(1)}\cdot x}, \qquad v = a^{(2)}e^{\zeta^{(2)}\cdot x}.$$

It is easy to verify that $\zeta^{(1)}\cdot\zeta^{(1)} = \zeta^{(2)}\cdot\zeta^{(2)} = -k_{\mathbf{s}}^2$, $a\cdot\zeta^{(j)}=0$, $j=1,2$ and u, v solve the equations (9). Denote $\mathfrak{K} := \sqrt{1-\frac{k_{\mathbf{s}}^2}{s^2+t^2}}$, substitute u, v into (21), we have

$$\begin{aligned} 0 &= \int_\Omega (C_{ijkl}a_i\zeta_j^{(1)}\,a_k\zeta_l^{(2)} - \omega^2\rho_{ik}a_ia_k)e^{(\zeta^{(1)}+\zeta^{(2)})\cdot x}\,dx \\ &= \int_\Omega [C_{1212}(\mathbf{i}s-\mathfrak{K}t)(\mathbf{i}s+\mathfrak{K}t) + C_{1313}(\mathbf{i}t+\mathfrak{K}s)(\mathbf{i}t-\mathfrak{K}s)]e^{2\mathbf{i}(s,0,t)\cdot x} \\ &\qquad\qquad - \omega^2\rho_{11}e^{2\mathbf{i}(s,0,t)\cdot x}\,dx \\ &= \int_\Omega [(s^2+t^2)(-C_{1212}-C_{1313}) \\ &\qquad\qquad + \frac{\omega^2 c_{\mathbf{s}}^{-2}}{s^2+t^2}\left(t^2C_{1212}+s^2C_{1313}\right) - \omega^2\rho_{11}]e^{2\mathbf{i}(s,0,t)\cdot x}\,dx. \end{aligned} \tag{23}$$

If we have above identity at two different frequencies $\omega=\omega_1,\omega_2$, we can separate the two parts to obtain

$$\mathcal{F}[\chi_\Omega(C_{1212}+C_{1313})](2s,0,2t) = 0 \tag{24}$$

and

$$\frac{c_{\mathbf{s}}^{-2}}{s^2+t^2}\left(t^2\mathcal{F}[\chi_\Omega C_{1212}](2s,0,2t)+s^2\mathcal{F}[\chi_\Omega C_{1313}](2s,0,2t)\right) - \mathcal{F}[\chi_\Omega\rho_{11}](2s,0,2t) = 0. \tag{25}$$

By the transverse isotropy assumption, (24) implies $\mathcal{F}[\chi_\Omega(C_{1212}+C_{1313})](\xi)=0$ for any $|\xi|\geq\sqrt{2}k_{\mathbf{s}}$. Then we use the analyticity of the Fourier transform of compactly supported functions to obtain

$$C_{1212}+C_{1313} = 0 \qquad \text{in } \Omega. \tag{26}$$

Alternatively, we can allow complex square root in (22) to include the situation $s^2+t^2<k_{\mathbf{s}}^2$. Notice this is exactly the same identity as what we got in Step 1 for the zero-frequency case.

Step 2. The proof will be quite different from the zero-freqeuncy case from now on. Denote $k_{\mathbf{p}}^2 := \omega^2 c_{\mathbf{p}}^{-2}$, where $c_{\mathbf{p}}^2 = \frac{\lambda^0+2\mu^0}{\rho^0}$, take u, v of the form

$$u = \zeta^{(1)}e^{\zeta^{(1)}\cdot x}, \qquad v = \zeta^{(2)}e^{\zeta^{(2)}\cdot x},$$

with

$$\begin{aligned} \zeta^{(1)} &= \mathbf{i}(s,0,t) + \mathbf{i}\beta(-t,0,s) + r(0,1,0) = (\mathbf{i}s - \mathbf{i}\beta t, r, \mathbf{i}t + \mathbf{i}\beta s), \\ \zeta^{(2)} &= \mathbf{i}(s,0,t) - \mathbf{i}\beta(-t,0,s) - r(0,1,0) = (\mathbf{i}s + \mathbf{i}\beta t, -r, \mathbf{i}t - \mathbf{i}\beta s), \end{aligned} \tag{27}$$

where $d := \sqrt{s^2+t^2}$ and $\beta := \sqrt{\frac{r^2}{d^2} - 1 + \frac{k_{\mathbf{p}}^2}{d^2}}$. It is easy to verify $\zeta^{(1)} \cdot \zeta^{(1)} = \zeta^{(2)} \cdot \zeta^{(2)} = -k_{\mathbf{p}}^2$. Therefore the solutions u, v constructed here is not divergence free. Substitute u, v into (21), we have
(28)

$$0 = \int_\Omega C_{ijkl}\zeta_i^{(1)}\zeta_j^{(1)} e^{\zeta^{(1)}\cdot x}\, \zeta_k^{(2)}\zeta_l^{(2)} e^{\zeta^{(2)}\cdot x}\, dx =: \int_\Omega (\sum_{j=1}^{9} I_j + \omega^2 \sum_{k=1}^{3} J_k) e^{2\mathbf{i}(s,0,t)\cdot x}\, dx.$$

Here

$$\begin{aligned} I_1 &= C_{1111}(s-\beta t)^2(s+\beta t)^2, \\ I_2 &= C_{2222}r^4, \\ I_3 &= C_{3333}(t+\beta s)^2(t-\beta s)^2, \\ I_4 &= -2C_{1122}(s^2+\beta^2 t^2)r^2, \\ I_5 &= C_{1133}(s-\beta t)^2(t-\beta s)^2 + C_{3311}(t+\beta s)^2(s+\beta t)^2, \\ I_6 &= -2C_{2233}(t^2+\beta^2 s^2)r^2, \\ I_7 &= 4C_{1212}(s^2-\beta^2 t^2)r^2, \\ I_8 &= 4C_{1313}(s^2-\beta^2 t^2)(t^2-\beta^2 s^2), \\ I_9 &= 4C_{2323}(t^2-\beta^2 s^2)r^2, \end{aligned}$$

and

$$\begin{aligned} J_1 &= \rho_{11}(s-\beta t)(s+\beta t), \\ J_2 &= \rho_{11}r^2, \\ J_3 &= \rho_{33}(t-\beta s)(t+\beta s). \end{aligned}$$

We use the following asymptotics of β and its powers in large r:

$$\begin{aligned} \beta &= \frac{r}{d} + \frac{(k_{\mathbf{p}}^2 - d^2)r^{-1}}{2d} + O(r^{-3}), \\ \beta^2 &= \frac{r^2}{d^2} + \frac{k_{\mathbf{p}}^2 - d^2}{d^2} + O(r^{-2}), \\ \beta^4 &= \frac{r^4}{d^4} + \frac{2r^2(k_{\mathbf{p}}^2 - d^2)}{d^4} + \frac{(k_{\mathbf{p}}^2 - d^2)^2}{d^4} + O(r^{-2}), \end{aligned}$$

and do some tedious calculation to obtain

$$\begin{aligned} I_1 &\sim r^4 C_{1111}\frac{t^4}{d^4} + r^2 C_{1111}\left(\frac{2(k_{\mathbf{p}}^2 - d^2)t^4}{d^4} - \frac{2}{d^2}t^2 s^2\right) \\ &\qquad + C_{1111}\left(2\frac{d^2 - k_{\mathbf{p}}^2}{d^2}t^2 s^2 + \frac{(d^2 - k_{\mathbf{p}}^2)^2}{d^4}t^4 + s^4\right) \\ I_2 &\sim r^4 C_{1111} \\ I_3 &\sim r^4 C_{3333}\frac{s^4}{d^4} + r^2 C_{3333}\left(\frac{2(k_{\mathbf{p}}^2 - d^2)s^4}{d^4} - \frac{2}{d^2}t^2 s^2\right) \end{aligned}$$

$$+ C_{3333}\left(2\frac{d^2-k_{\mathbf{p}}^2}{d^2}t^2s^2+\frac{(d^2-k_{\mathbf{p}}^2)^2}{d^4}s^4+t^4\right)$$

$$I_4 \sim r^4(-2C_{1122}\frac{t^2}{d^2}) + r^2 2C_{1122}\left(\frac{d^2-k_{\mathbf{p}}^2}{d^2}t^2-s^2\right)$$

$$I_5 \sim r^4(2C_{1133}\frac{t^2s^2}{d^4}) + r^2\frac{2C_{1133}}{d^2}\left(s^4+t^4+2t^2s^2+2t^2s^2\frac{k_{\mathbf{p}}^2}{d^2}\right)$$

$$+ 2C_{1133}\left(t^2s^2\frac{(d^2-k_{\mathbf{p}}^2)^2}{d^4}-(s^4+t^4+4t^2s^2)\frac{d^2-k_{\mathbf{p}}^2}{d^2}+t^2s^2\right)$$

$$I_6 \sim r^4\left(-2C_{2233}\frac{s^2}{d^2}\right) + r^2 2C_{2233}\left(\frac{d^2-k_{\mathbf{p}}^2}{d^2}s^2-t^2\right)$$

$$I_7 \sim r^4\left(-4C_{1212}\frac{t^2}{d^2}\right) + r^2 4C_{1212}\left(\frac{d^2-k_{\mathbf{p}}^2}{d^2}t^2+s^2\right);$$

$$I_8 \sim r^4 4C_{1313}\frac{t^2s^2}{d^4} + r^2 4C_{1313}\left(-2t^2s^2\frac{d^2-k_{\mathbf{p}}^2}{d^4}-\frac{t^4+s^4}{d^2}\right)$$

$$+ 4C_{1313}\left(t^2s^2\frac{(d^2-k_{\mathbf{p}}^2)^2}{d^4}+\frac{d^2-k_{\mathbf{p}}^2}{d^2}(t^4+s^4)+s^2t^2\right);$$

$$I_9 \sim r^4\left(-4C_{2323}\frac{s^2}{d^2}\right) + r^2 4C_{2323}\left(\frac{d^2-k_{\mathbf{p}}^2}{d^2}s^2+t^2\right).$$

We also have

$$J_1 \sim r^2(-\rho_{11})\frac{t^2}{d^2} + \rho_{11}\left(s^2+\frac{d^2-k_{\mathbf{p}}^2}{d^2}t^2\right);$$

$$J_2 \sim r^2\rho_{11};$$

$$J_3 \sim r^2(-\rho_{33})\frac{s^2}{d^2} + \rho_{33}\left(t^2+\frac{d^2-k_{\mathbf{p}}^2}{d^2}s^2\right).$$

The $O(r^4)$ terms in (28) are exactly the same as in Step 3 for the zero-frequency case, which give

$$C_{1111} - 2C_{1133} - 4C_{1313} + C_{3333} = 0 \qquad \text{in } \Omega. \tag{29}$$

The coefficient of r^2 in the integrand is

$$\omega^2\Bigg[c_{\mathbf{p}}^{-2}\Bigg(\frac{2C_{1111}t^4}{d^4}+\frac{2C_{3333}s^4}{d^4}-2C_{1122}\frac{t^2}{d^2}+4\frac{C_{1133}}{d^2}t^2s^2-2C_{2233}\frac{s^2}{d^2}$$

$$-4C_{1212}\frac{t^2}{d^2}+8C_{1313}\frac{t^2s^2}{d^4}-4C_{2323}\frac{s^2}{d^2}\Bigg)-\rho_{11}\frac{t^2}{d^2}+\rho_{11}-\rho_{33}\frac{s^2}{d^2}\Bigg]$$

$$+\Bigg[-C_{1111}\frac{2t^4}{d^2}-C_{1111}\frac{2t^2s^2}{d^2}-2C_{3333}\frac{s^4}{d^2}-C_{3333}\frac{2t^2s^2}{d^2}$$

$$+2C_{1122}(t^2-s^2)+2C_{1133}\frac{(s^2+t^2)^2}{d^2}+2C_{2233}(s^2-t^2)+4C_{1212}(t^2+s^2)$$

$$+ 4C_{1313}(-\frac{2t^2s^2}{d^2} - \frac{t^4+s^4}{d^2}) + 4C_{2323}(s^2+t^2)\Big]$$

Write this expression as $\omega^2 A(x,s,t) + B(x,s,t)$, then

$$\int_\Omega \left(\omega^2 A(x,s,t) + B(x,s,t)\right) e^{2\mathbf{i}(s,0,t)\cdot x}\, dx = 0.$$

Evaluating above identity at two different frequencies gives

$$\int_\Omega A(x,s,t)e^{2\mathbf{i}(s,0,t)\cdot x}\, dx = \int_\Omega B(x,s,t)e^{2\mathbf{i}(s,0,t)\cdot x}\, dx = 0. \tag{30}$$

Using the relation (7), we get

$$B(x,s,t) = \left(2s^2C_{1111} - 4s^2C_{1122} + 4s^2C_{1133} - 2s^2C_{3333}\right).$$

Then (30) implies

$$C_{1111} - 2C_{1122} + 2C_{1133} - C_{3333} = 0 \qquad \text{in } \Omega. \tag{31}$$

We summarize identities (26)(29)(31) as

$$\begin{aligned}
1\cdot C_{1212} + 0\cdot(C_{1122} - 2C_{1133} + C_{3333}) + 1\cdot C_{1313} = 0,\\
2\cdot C_{1212} + 1\cdot(C_{1122} - 2C_{1133} + C_{3333}) - 4\cdot C_{1313} = 0,\\
2\cdot C_{1212} - 1\cdot(C_{1122} - 2C_{1133} + C_{3333}) + 0\cdot C_{1313} = 0.
\end{aligned}$$

By solving the above linear equations, we have $C_{1212} = C_{1313} = 0$ and $2C_{1133} = C_{1122} + C_{3333}$. Then (23) becomes $\mathcal{F}[\chi_\Omega\rho_{11}](2s,0,2t) = 0$. Then we can get

$$\rho_{11} = 0 \qquad \text{in } \Omega.$$

Step 3. Take

$$u = \vartheta^{(1)}e^{\zeta^{(1)}\cdot x}, \qquad v = \vartheta^{(2)}e^{\zeta^{(2)}\cdot x},$$

with $\zeta^{(j)}$ defined in (22) and

$$\vartheta^{(1)} := \mathbf{i}(s,0,t)\sqrt{1-\frac{k_{\mathbf{s}}^2}{s^2+t^2}} + (-t,0,s), \quad \vartheta^{(2)} := \mathbf{i}(s,0,t)\sqrt{1-\frac{k_{\mathbf{s}}^2}{s^2+t^2}} - (-t,0,s). \tag{32}$$

Notice that $\vartheta^{(1)}\cdot\zeta^{(1)} = \vartheta^{(2)}\cdot\zeta^{(2)} = 0$. The solution u,v used here is divergence free.

Inserting u,v into the integral identity (21) and use the fact $C_{1313} = 0$, $2C_{1133} = C_{1122} + C_{3333}$ and $\rho_{11} = 0$, we obtain

$$\begin{aligned}
0 &= \int_\Omega C_{ijkl}\vartheta_i^{(1)}\zeta_j^{(1)}e^{\zeta^{(1)}\cdot x}\ \vartheta_k^{(2)}\zeta_l^{(2)}e^{\zeta^{(2)}\cdot x} - \omega^2\rho_{ik}\vartheta_i^{(1)}e^{\zeta^{(1)}\cdot x}\vartheta_k^{(2)}e^{\zeta^{(2)}\cdot x}, dx\\
&= \int_\Omega \Big[C_{1111}(\mathfrak{K}^2s^2+t^2)(\mathfrak{K}^2t^2+s^2) + C_{1133}(\mathbf{i}\mathfrak{K}s-t)(\mathbf{i}s-\mathfrak{K}t)(\mathbf{i}\mathfrak{K}t-s)(\mathbf{i}t-\mathfrak{K}s)\\
&\quad +C_{3311}(\mathbf{i}\mathfrak{K}t+s)(\mathbf{i}t+\mathfrak{K}s)(\mathbf{i}\mathfrak{K}s+t)(\mathbf{i}s+\mathfrak{K}t) + C_{3333}(\mathfrak{K}^2s^2+t^2)(\mathfrak{K}^2t^2+s^2)\\
&\quad +\omega^2\rho_{33}(\mathfrak{K}^2t^2+s^2)\Big]e^{2\mathbf{i}(s,0,t)\cdot x}\, dx\\
&= \int_\Omega \Big[(C_{1111} - 2C_{1133} + C_{3333})(s^2+\mathfrak{K}^2t^2)(t^2+\mathfrak{K}^2s^2)\\
&\qquad\qquad + \omega^2\rho_{33}(\mathfrak{K}^2t^2+s^2)\Big]e^{2\mathbf{i}(s,0,t)\cdot x}\, dx
\end{aligned}$$

$$= \int_\Omega \omega^2[\rho_{33}(\mathfrak{K}^2 t^2 + s^2)] e^{2\mathbf{i}(s,0,t)\cdot x}\, dx$$
$$= \int_\Omega \omega^2[\rho_{33}(s^2+t^2) - \omega^2 c_{\mathbf{s}}^{-2}\rho_{33}\frac{t^2}{s^2+t^2}] e^{2\mathbf{i}(s,0,t)\cdot x}\, dx.$$

Again using the above identity at two different frequencies, we end up with

$$\rho_{33} = 0 \qquad \text{in } \Omega.$$

At this point we have recovered the density $\delta\rho$ and the same quantities for the elastic tensor $\delta\mathbf{C}$ as the zero-frequency case. Similar to (17) we have

$$\int_\Omega C_{1111}\nabla\cdot u\nabla\cdot v + (C_{1133} - C_{1111})(\nabla\cdot u\partial_3 v_3 + \partial_3 u_3 \nabla\cdot v)\, dx = 0. \tag{33}$$

Step 4. Take

$$u = \zeta^{(1)} e^{\zeta^{(1)}\cdot x}, \qquad v = \vartheta^{(2)} e^{\zeta^{(2)}\cdot x},$$

with $\zeta^{(1)}, \zeta^{(2)}$ defined as (27) and

$$\vartheta^{(2)} = (\mathbf{i}s + \mathbf{i}\beta t, -\frac{r^2 + k_{\mathbf{p}}^2}{r}, \mathbf{i}t - \mathbf{i}\beta s).$$

Substitute into (33), and notice $\nabla\cdot u = -k_{\mathbf{p}}^2 u$ and $\nabla\cdot v = 0$. Then we obtain the following identity similar to (19) from the leading order terms in r:

$$\frac{s^2}{t^2+s^2}\int_\Omega (C_{1133} - C_{1111}) e^{2\mathbf{i}(s,0,t)\cdot x}\, dx = 0,$$

from which we get $C_{1133} = C_{1111}$.

Step 5. For the last step, simply take

$$u = \zeta^{(1)} e^{\zeta^{(1)}\cdot x}, \qquad v = \zeta^{(2)} e^{\zeta^{(2)}\cdot x},$$

with $\zeta^{(1)}, \zeta^{(2)}$ defined as (27). Substitute into

$$\int_\Omega C_{1111}\nabla\cdot u\nabla\cdot v\, dx = 0, \tag{34}$$

and notice $\nabla\cdot u = -k_{\mathbf{p}}^2 u$, $\nabla\cdot v = -k_{\mathbf{p}}^2 v$. We can easily obtain $C_{1111} = 0$, and conclude the proof of Theorem 2.

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