



Virtual enriching operators

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Abstract

We construct bounded linear operators that map H^1 conforming Lagrange finite element spaces to H^2 conforming virtual element spaces in two and three dimensions. These operators are useful for the analysis of nonstandard finite element methods.

Keywords Virtual elements · Fourth order elliptic boundary value problems · Enriching operator

Mathematics Subject Classification 65N30 · 35J40

1 Introduction

Let $\Omega \in \mathbb{R}^d$ ($d = 2, 3$) be a bounded polygonal/polyhedral domain, $\mathcal{T}_h$ be a simplicial triangulation of Ω and $V_h \subset H^1(\Omega)$ be the Lagrange P_k finite element space [9, 15] with $k \geq 3$. The mesh dependent semi-norm $\|\cdot\|_h$ is defined by

$$\|v\|_h^2 = \|D_h^2 v\|_{L_2(\Omega)}^2 + J(v, v) \quad \forall v \in V_h, \quad (1.1)$$

where $D_h^2 v$ is the piecewise Hessian of v with respect to $\mathcal{T}_h$, and

$$J(w, v) = \sum_{e \in \mathcal{E}_h^i} h_e^{-1} \int_e \llbracket \partial w / \partial n \rrbracket \llbracket \partial v / \partial n \rrbracket ds \quad \text{for } d = 2, \quad (1.2)$$

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$$J(w, v) = \sum_{F \in \mathcal{F}_h^i} h_F^{-1} \int_F \llbracket \partial w / \partial n \rrbracket \llbracket \partial v / \partial n \rrbracket dS \quad \text{for } d = 3. \quad (1.3)$$

Here $\mathcal{E}_h^i$ (resp., $\mathcal{F}_h^i$) is the set of interior edges (resp., faces) of $\mathcal{T}_h$, h_e (resp., h_F) is the diameter of the edge e (resp., face F), $\llbracket \partial v / \partial n \rrbracket$ is the jump of the normal derivative across an edge e (resp., face F), and ds (resp., dS) is the infinitesimal length (resp., area).

Our goal is to construct a linear operator $E_h : V_h \rightarrow H^2(\Omega)$ such that

$$\|v - E_h v\|_h \leq C_\sharp \sqrt{J(v, v)} \quad \forall v \in V_h, \quad (1.4)$$

$$\sum_{\ell=0}^2 h^\ell |\zeta - E_h \Pi_h \zeta|_{H^\ell(\Omega)} \leq C_b h^{k+1} |\zeta|_{H^{k+1}(\Omega)} \quad \forall \zeta \in H^{k+1}(\Omega), \quad (1.5)$$

where $\Pi_h : C(\bar{\Omega}) \rightarrow V_h$ is the Lagrange nodal interpolation operator, and the positive constants $C_\sharp$ and C_b only depend on the shape regularity of $\mathcal{T}_h$ and k . Moreover, the operator E_h maps $V_h \cap H_0^1(\Omega)$ into $H^2(\Omega) \cap H_0^1(\Omega)$.

Remark 1.1 Throughout the paper we will follow the standard notation for differential operators, functions spaces and norms that can be found for example in [1,9,15].

Enriching operators that satisfy (1.4) and (1.5) are useful for a priori and a posteriori error analyses for fourth order elliptic problems [6–8,10,19], and they also play an important role in fast solvers for fourth order problems [4,5,11]. A recent application to Hamilton–Jacobi–Bellman equations can be found in [23].

In the two dimensional case, one can construct E_h through the C^1 macro finite elements in [14,16,17,24,25]. This was carried out in [8] for the quadratic element and in [19] for higher order elements. However macro elements of order higher than 3 are not available in three dimensions and therefore this approach can only be carried out for quadratic and cubic Lagrange elements (cf. [23]) using the three dimensional cubic Clough–Tocher macro element from [27].

We take a different approach in this paper by connecting the k -th order Lagrange finite element space to a k -th order H^2 conforming virtual element space. In two dimensions such spaces are already in the literature [12,13], and we will develop a version of three dimensional H^2 conforming virtual element spaces that are sufficient for the construction of E_h .

Remark 1.2 The assumption that the order of the Lagrange finite element space is at least 3 allows a uniform construction of E_h . For the quadratic Lagrange finite element space we can simply take E_h to be the restriction of the cubic enriching operator.

The rest of the paper is organized as follows. The construction of E_h in two dimensions is carried out in Sect. 2, followed by the construction in three dimensions in Sect. 3 and some concluding remarks in Sect. 4. Appendix A contains some technical results concerning inverse trace theorems that are needed for the construction of H^2 conforming virtual elements.

A list of notations and conventions that will be used throughout the paper is provided here for convenience.

- A polygon/polyhedron is an open subset in $\mathbb{R}^2/\mathbb{R}^3$, an edge of a polygon/polyhedron does not include the endpoints and a face of a polyhedron does not include the vertices and edges. These conventions apply in particular to triangles and tetrahedra.
- Let G be an open line segment, a triangle or a tetrahedron, and k be an integer. $P_k(G)$ is the space of polynomials of total degree $\leq k$ restricted to G if $k \geq 0$ and $P_k(G) = \{0\}$ if $k < 0$. We say that two functions u and v defined on G have identical moments up to order ℓ if the integral of $(u - v)q$ on G vanishes for all $q \in P_\ell(G)$. The orthogonal projection from $L_2(G)$ onto $P_k(G)$ is denoted by $Q_{G,k}$.
- $\mathcal{V}_h$ is the set of all the vertices of the triangles/tetrahedra in $\mathcal{T}_h$, $\mathcal{V}_h^i$ is the set of vertices in Ω and $\mathcal{V}_h^b$ is the set of vertices on $\partial\Omega$.
- $\mathcal{E}_h$ is the set of all the edges of the triangles/tetrahedra in $\mathcal{T}_h$, $\mathcal{E}_h^i$ is the set of edges in Ω and $\mathcal{E}_h^b$ is the set of edges that are subsets of $\partial\Omega$.
- $\mathcal{F}_h$ is the set of all the faces of the tetrahedra in $\mathcal{T}_h$, $\mathcal{F}_h^i$ is the set of faces in Ω and $\mathcal{F}_h^b$ is the set of faces that are subsets of $\partial\Omega$.
- $\mathcal{T}_p$ is the set of all the triangles/tetrahedra in $\mathcal{T}_h$ that share p as a common vertex.
- $\mathcal{T}_e$ is the set of all the triangles/tetrahedra in $\mathcal{T}_h$ that share e as a common edge.
- $\mathcal{T}_F$ is the set of all the tetrahedra in $\mathcal{T}_h$ that share F as a common face.
- $\mathcal{F}_e$ is the set of the faces of the tetrahedra in $\mathcal{T}_h$ that share e as a common edge.
- If v is a function defined on a triangle/tetrahedron, then v_e (resp., v_F) is the restriction of v to an edge e (resp., a face F).
- If v is a function defined on a triangle/tetrahedron, then $\partial v / \partial n$ denotes the outward normal derivative of v along ∂T . In the case of a triangle (resp. tetrahedron), $\partial v / \partial n$ is double-valued at the vertices (resp., edges) of T .
- If e is an edge of the triangle T , then $\mathbf{n}_{e,T}$ is the unit vector orthogonal to e and pointing towards the outside of T . If e is an edge of a face F of a tetrahedron T , then $\mathbf{n}_{e,F}$ is the unit vector tangential to F , orthogonal to e and pointing towards the outside of F .
- If F is a face of the tetrahedron T , then $\mathbf{n}_{F,T}$ is the unit vector orthogonal to F and pointing towards the outside of T .

2 The two dimensional case

The construction of E_h is based on the characterizations of trace spaces associated with a triangle and the construction of polynomial data on the skeleton of $\mathcal{T}_h$ that satisfy these characterizations on all the triangles of $\mathcal{T}_h$.

2.1 Trace spaces for a triangle

Let T be a triangle with vertices p_1 , p_2 and p_3 , e_i be the edge of T opposite p_i , $\mathbf{n}_i$ be the unit outer normal along e_i , and $\mathbf{t}_i$ be the counterclockwise unit tangent of e_i . Let ℓ be a nonnegative number. A function u belongs to the piecewise Sobolev space $H^\ell(\partial T)$ (resp., piecewise polynomial space $P_\ell(\partial T)$) if and only if u_i , the restriction

of u to e_i , belongs to $H^\ell(e_i)$ (resp., $P_\ell(e_i)$) for $1 \leq i \leq 3$. It follows from the Sobolev Embedding Theorem [1, Theorem 4.12] that we can define a linear operator $\text{Tr} : H^2(T) \longrightarrow H^{\frac{3}{2}}(\partial T) \times H^{\frac{1}{2}}(\partial T)$ by

$$\text{Tr } \zeta = (\zeta, \partial \zeta / \partial n)|_{\partial T}, \quad (2.1)$$

where the restrictions of ζ and $\partial \zeta / \partial n$ are in the sense of trace and defined piecewise with respect to the edges. For the subspace $H^{\frac{5}{2}}(T)$ of $H^2(T)$, we have $\text{Tr } H^{\frac{5}{2}}(T) \subset H^2(\partial T) \times H^1(\partial T)$. Our first task is to identify the image of $H^{\frac{5}{2}}(T)$.

Definition 2.1 A pair of functions $(f, g) \in H^2(\partial T) \times H^1(\partial T)$ belongs to the space $(H^2 \times H^1)(\partial T)$ if and only if the following conditions are satisfied:

$$f_j(p_i) = f_k(p_i) \quad \text{for } 1 \leq i \leq 3 \text{ and } j, k \in \{1, 2, 3\} \setminus \{i\}, \quad (2.2)$$

and there exist $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \in \mathbb{R}^2$ (which depend on (f, g)) such that

$$(\partial f_j / \partial t_j)(p_i) = \mathbf{w}_i \cdot \mathbf{t}_j \quad \text{for } 1 \leq i \leq 3 \text{ and } j \in \{1, 2, 3\} \setminus \{i\}, \quad (2.3)$$

$$g_j(p_i) = \mathbf{w}_i \cdot \mathbf{n}_j \quad \text{for } 1 \leq i \leq 3 \text{ and } j \in \{1, 2, 3\} \setminus \{i\}. \quad (2.4)$$

Note that the compatibility conditions (2.3)–(2.4) are equivalent to

$$(\partial f_j / \partial t_j) \mathbf{t}_j + g_j \mathbf{n}_j = (\partial f_k / \partial t_k) \mathbf{t}_k + g_k \mathbf{n}_k \quad \text{at } p_i \quad (2.5)$$

for $1 \leq i \leq 3$ and $j, k \in \{1, 2, 3\} \setminus \{i\}$.

It follows from the Sobolev Embedding Theorem that $\text{Tr } \zeta \in (H^2 \times H^1)(\partial T)$ for $\zeta \in H^{\frac{5}{2}}(T)$, where $\mathbf{w}_i = \nabla \zeta(p_i)$, and we can recover $\nabla \zeta$ on ∂T from $(f, g) = \text{Tr } \zeta$ by

$$\nabla \zeta = (\partial f_i / \partial t_i) \mathbf{t}_i + g_i \mathbf{n}_i \quad \text{on } e_i \text{ for } 1 \leq i \leq 3. \quad (2.6)$$

We want to show that in fact $\text{Tr } H^{\frac{5}{2}}(T) = (H^2 \times H^1)(\partial T)$. For this purpose it is useful to construct a linear isomorphism

$$\Phi^* : (H^2 \times H^1)(\partial T) \longrightarrow (H^2 \times H^1)(\partial \tilde{T})$$

such that

$$\text{Tr } (\zeta \circ \Phi) = \Phi^*(\text{Tr } \zeta) \quad \forall \zeta \in H^{\frac{5}{2}}(T), \quad (2.7)$$

where Φ is an orientation preserving affine transformation that maps the triangle $\tilde{T}$ onto T . We assume that Φ maps the vertex $\tilde{p}_i$ of $\tilde{T}$ to the vertex p_i of T and hence it also maps the edge $\tilde{e}_i$ of $\tilde{T}$ to the edge e_i of T .

First we note that, by the chain rule,

$$\nabla(\zeta \circ \Phi) = J_\Phi^t(\nabla \zeta \circ \Phi) \quad \forall \zeta \in H^{\frac{5}{2}}(T), \quad (2.8)$$

where J_Φ (a constant 2×2 matrix with a positive determinant) is the Jacobian of Φ .

Let $(f, g) \in (H^2 \times H^1)(\partial T)$. Motivated by (2.6)–(2.8), we define $\Phi^*(f, g) = (\tilde{f}, \tilde{g})$, where

$$\tilde{f} = f \circ \Phi, \quad (2.9)$$

and

$$\tilde{g} = J_\Phi^t(g \circ \Phi) \cdot \tilde{n}_i \quad \text{on } \tilde{e}_i \quad \text{for } 1 \leq i \leq 3, \quad (2.10)$$

where $\tilde{n}_i$ is the outward pointing unit normal along the edge $\tilde{e}_i$ and the vector field g on ∂T is given by

$$g = (\partial f_i / \partial t_i) t_i + g_i n_i \quad \text{on } e_i \quad \text{for } 1 \leq i \leq 3. \quad (2.11)$$

It is straightforward to check that $(\tilde{f}, \tilde{g}) \in (H^2 \times H^1)(\partial \tilde{T})$, $\Phi^* : (H^2 \times H^1)(\partial T) \rightarrow (H^2 \times H^1)(\partial \tilde{T})$ is a bijection, and that (2.7) follows from (2.6) and (2.8)–(2.11).

We are now ready to characterize $\text{Tr } H^{\frac{5}{2}}(T)$.

Lemma 2.1 *The image of $H^{\frac{5}{2}}(T)$ under Tr is the space $(H^2 \times H^1)(\partial T)$.*

Proof We already know that $\text{Tr } H^{\frac{5}{2}}(T) \subset (H^2 \times H^1)(\partial T)$. In the other direction, we want to construct $\zeta \in H^{\frac{5}{2}}(T)$ that satisfies (2.1) for a given $(f, g) \in (H^2 \times H^1)(\partial T)$.

If the functions f and g vanish near the vertices, we can use the operator L_1 in Lemma A.1 and cut-off functions to obtain ζ . Therefore, by using a partition of unity, we can reduce the construction to a neighborhood of a vertex and, by an affine transformation [cf. (2.7)], we can further assume that the angle at the vertex is a right angle. The existence of ζ near such a vertex then follows from Lemma A.3. $\square$

2.2 Affine invariant H^2 virtual element spaces

The construction of the virtual element spaces involves polynomial subspaces of $(H^2 \times H^1)(\partial T)$.

Definition 2.2 Let T be a triangle. We will denote the intersection of $(H^2 \times H^1)(\partial T)$ and $P_k(\partial T) \times P_{k-1}(\partial T)$ by $(H^2 \times H^1)_{k,k-1}(\partial T)$.

Remark 2.1 It follows from the compatibility conditions (2.2)–(2.4) that the function pair $(f, g) \in (H^2 \times H^1)_{k,k-1}(\partial T)$ is determined by (i) the values of f at the vertices, (ii) the tangential derivatives of f at the vertices, (iii) moments of f on e_i up to order $k-4$ that together with (i) and (ii) determine $f_i \in P_k(e_i)$, and (iv) moments of g up to order $k-3$ that together with (ii) [through (2.4)] determine $g_i \in P_{k-1}(e_i)$. These degrees of freedom (dofs) are depicted in Fig. 1 for $k=3$ and 4, where (i) the values of f at the vertices and the moments of f on the edges are represented by solid dots, and (ii) the tangential derivatives of f at the vertices and the moments of g on the edges are represented by arrows. Altogether we have $\dim(H^2 \times H^1)_{k,k-1}(\partial T) = 6(k-1)$.

Remark 2.2 Since polynomial spaces are preserved by an affine transformation, the map $\Phi^* : (H^2 \times H^1)(\partial T) \rightarrow (H^2 \times H^1)(\partial \tilde{T})$ defined by (2.9)–(2.10) maps $(H^2 \times H^1)_{k,k-1}(\partial T)$ one-to-one and onto $(H^2 \times H^1)_{k,k-1}(\partial \tilde{T})$.

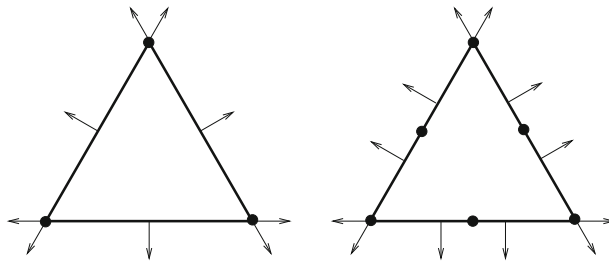


Fig. 1 Degrees of freedom for $(H^2 \times H^1)_{3,2}(\partial T)$ and $(H^2 \times H^1)_{4,3}(\partial T)$

2.2.1 Virtual element spaces on the reference triangle

We begin with a simple well-posedness result for the biharmonic problem.

Lemma 2.2 *Given any $(f, g) \in (H^2 \times H^1)(\partial T)$ and $\rho \in L_2(T)$, there exists a unique $\xi \in H^2(T)$ such that*

$$(\Delta \xi, \Delta z)_{L_2(T)} = (\rho, z)_{L_2(T)} \quad \forall z \in H_0^2(T) \quad \text{and} \quad \text{Tr } \xi = (f, g). \quad (2.12)$$

Proof Let $\zeta \in H^{\frac{5}{2}}(T)$ satisfy (2.1) and $\eta \in H_0^2(T)$ be defined by

$$(\Delta \eta, \Delta z)_{L_2(T)} = (\rho - \Delta \zeta, z)_{L_2(T)} \quad \forall z \in H_0^2(T).$$

Then $\xi = \eta + \zeta$ is the unique solution of (2.6). $\square$

Let $\hat{T}$ be the reference triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$. In view of Lemma 2.2 and the fact that $(H^2 \times H^1)_{k,k-1}(\partial \hat{T})$ is a subspace of $(H^2 \times H^1)(\partial \hat{T})$, we can now define the reference virtual element spaces $\mathcal{V}^k(\hat{T})$, which are identical to the virtual element spaces in [12] for the special case of the reference triangle.

Definition 2.3 A function $\hat{\xi} \in H^2(\hat{T})$ belongs to the virtual element space $\mathcal{V}^k(\hat{T})$ if and only if $\text{Tr } \hat{\xi} \in (H^2 \times H^1)_{k,k-1}(\partial \hat{T})$ and the distributional derivative $\Delta^2 \hat{\xi}$ belongs to $P_{k-4}(\hat{T})$, i.e., there exists $\hat{\rho} \in P_{k-4}(\hat{T})$ such that

$$(\Delta \hat{\xi}, \Delta \hat{z})_{L_2(\hat{T})} = (\hat{\rho}, \hat{z})_{L_2(\hat{T})} \quad \forall \hat{z} \in H_0^2(\hat{T}). \quad (2.13)$$

Remark 2.3 According to Remark 2.1 and Lemma 2.2, we have

$$\dim \mathcal{V}^k(\hat{T}) = \dim (H^2 \times H^1)_{k,k-1}(\partial \hat{T}) + \dim P_{k-4}(\hat{T}) = \frac{k^2 + 7k - 6}{2}. \quad (2.14)$$

The following result is well-known (cf. [12]). We provide a proof here for self-containedness.

Lemma 2.3 *A function $\hat{\xi}$ in $\mathcal{V}^k(\hat{T})$ is uniquely determined by $\text{Tr } \hat{\xi} \in (H^2 \times H^1)_{k,k-1}(\partial \hat{T})$ and $Q_{\hat{T},k-4} \hat{\xi} \in P_{k-4}(\hat{T})$.*

Proof In view of Remark 2.3, it suffices to show that $\hat{\xi} = 0$ is the only function in $\mathcal{V}^k(\hat{T})$ with the properties that $\text{Tr } \hat{\xi} = (0, 0)$ and $Q_{\hat{T}, k-4} \hat{\xi} = 0$. Indeed, using integration by parts and the fact that the distributional derivative $\Delta^2 \hat{\xi} \in P_{k-4}(\hat{T})$, we have

$$(\Delta \hat{\xi}, \Delta \hat{\xi})_{L_2(\hat{T})} = (\Delta^2 \hat{\xi}, \hat{\xi})_{L_2(\hat{T})} = 0.$$

Therefore $\hat{\xi} \in H^2(\hat{T})$ is a harmonic function that vanishes on $\partial \hat{T}$ and hence $\hat{\xi} = 0$. $\square$

Remark 2.4 The definition of the virtual element space $\mathcal{V}^k(\hat{T})$ relies on the fact that $(H^2 \times H^1)_{k, k-1}(\partial \hat{T})$ is a subspace of $\text{Tr } H^{\frac{5}{2}}(\hat{T}) \subset \text{Tr } H^2(\hat{T})$. One can show by using macro elements of order k that a pair $(\hat{f}, \hat{g}) \in P_k(\partial \hat{T}) \times P_{k-1}(\partial \hat{T})$ satisfying the compatibility conditions (2.2)–(2.4) automatically belongs to $\text{Tr } H^2(\hat{T})$. Hence Lemma 2.1 is not necessary for the definition of the virtual element space $\mathcal{V}^k(\hat{T})$ in two dimensions. However, the definition of the virtual element spaces in three dimensions requires the characterization of the trace of $H^{\frac{5}{2}}(\hat{T})$ for the reference tetrahedron $\hat{T}$, since macro elements of arbitrary order are not available. The approach here provides a preview of the three dimensional case.

2.2.2 Virtual element spaces for a general triangle

We now define $\mathcal{V}^k(T)$ for an arbitrary triangle T in terms of $\mathcal{V}^k(\hat{T})$.

Definition 2.4 Let T be an arbitrary triangle and Φ be an orientation preserving affine transformation that maps $\hat{T}$ onto T . Then $\xi \in \mathcal{V}^k(T)$ if and only if $\xi \circ \Phi \in \mathcal{V}^k(\hat{T})$.

Remark 2.5 The definition of $\mathcal{V}^k(T)$ is independent of the choice of Φ . The polynomial space $P_k(T)$ is a subspace of $\mathcal{V}^k(T)$ since $P_k(\hat{T})$ is obviously a subspace of $\mathcal{V}^k(\hat{T})$. The dimension of $\mathcal{V}^k(T)$ is also given by the formula in (2.14).

We have an analog of Lemma 2.3.

Lemma 2.4 A function ξ in $\mathcal{V}^k(T)$ is uniquely determined by $\text{Tr } \xi \in (H^2 \times H^1)_{k, k-1}(\partial T)$ and $Q_{T, k-4} \xi \in P_{k-4}(T)$.

Proof This is a direct consequence of (2.7), Remark 2.2, Lemma 2.3 and the relation $Q_{\hat{T}, k-4}(\xi \circ \Phi) = (Q_{T, k-4} \xi) \circ \Phi$. $\square$

Remark 2.6 Our definition of $\mathcal{V}^k(T)$, which is invariant under affine transformations, differs from the one in [12] for a general triangle. The affine invariance simplifies the proofs of (1.4) and (1.5) in Sect. 2.4. We note that it is also possible to use the virtual finite element spaces from [12] in the construction of E_h . But then the proofs of (1.4) and (1.5) will become more involved.

Remark 2.7 The definition of H^2 virtual element spaces on polygons and their applications to the plate bending problem can be found in [12, 13].

2.3 Construction on the skeleton $\bigcup_{T \in \mathcal{T}_h} \partial T$

Given $v \in V_h$, our goal is to define $f_{v,T}$ representing (the desired) $E_h v|_{\partial T}$ and $g_{v,T}$ representing (the desired) $(\partial E_h v / \partial n)|_{\partial T}$ for all $T \in \mathcal{T}_h$, such that

$$(f_{v,T}, g_{v,T}) \in (H^2 \times H^1)_{k,k-1}(\partial T) \text{ for all } T \in \mathcal{T}_h, \quad (2.15)$$

and the following conditions are satisfied:

$$\begin{aligned} &\text{if } T_1, T_2 \in \mathcal{T}_p \text{ (resp., } T_1, T_2 \in \mathcal{T}_e), \\ &\quad \text{then } f_{v,T_1}(p) = f_{v,T_2}(p) \text{ (resp., } f_{v,T_1} = f_{v,T_2} \text{ on } e), \end{aligned} \quad (2.16)$$

$$\text{if two distinct } T_1 \text{ and } T_2 \text{ belong to } \mathcal{T}_e, \text{ then } g_{v,T_1} + g_{v,T_2} = 0 \text{ on } e, \quad (2.17)$$

$$\text{if } e \in \mathcal{E}_h^b \text{ is an edge of } T \text{ and } v \in H_0^1(\Omega), \text{ then } f_{v,T} = 0 \text{ on } e. \quad (2.18)$$

Note that (2.16) and (2.17) imply any piecewise H^2 function ξ satisfying $(\xi_T, \partial \xi_T / \partial n)|_{\partial T} = (f_{v,T}, g_{v,T})$ for all $T \in \mathcal{T}_h$ will belong to $H^2(\Omega)$, and (2.18) implies that $\xi \in H^2(\Omega) \cap H_0^1(\Omega)$ if $v \in H_0^1(\Omega)$.

2.3.1 Construction at the vertices

In view of the compatibility conditions (2.3) and (2.4), we need to define vectors $\mathbf{w}_p \in \mathbb{R}^2$ associated with the vertices p of $\mathcal{T}_h$. There are three cases: (i) p is an interior vertex, (ii) p is boundary vertex that is not a corner of Ω and (iii) p is a corner of Ω . **Case (i)** For an interior vertex p , we define $\mathbf{w}_p$ to be ∇v_T , where T is any triangle in $\mathcal{T}_p$.

Case (ii) For a boundary vertex p that is not a corner of Ω , we define $\mathbf{w}_p$ to be ∇v_T , where T is one of the triangles in $\mathcal{T}_p$ that has an edge on $\partial\Omega$. This choice ensures that $\mathbf{w}_p \cdot \mathbf{t} = 0$ if $v \in H_0^1(\Omega)$, where $\mathbf{t}$ is any vector tangential to $\partial\Omega$ at p .

Case (iii) At a corner p of Ω , we define $\mathbf{w}_p$ by

$$\mathbf{w}_p \cdot \mathbf{t}_i = (\partial v / \partial t_i)(p) \quad \text{for } i = 1, 2, \quad (2.19)$$

where $e_1, e_2 \in \mathcal{E}_h^b$ are the two edges emanating from p and $\partial / \partial t_i$ is the derivative in the direction of the unit tangent $\mathbf{t}_i$ of e_i . Note that $\mathbf{w}_p = 0$ at a corner p of Ω if $v \in H_0^1(\Omega)$.

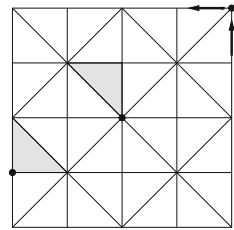
The choices of the triangles and tangent vectors in Case (i)–Case (iii) are illustrated in Fig. 2.

Remark 2.8 If the condition

$$\nabla v_{T_1}(p) = \nabla v_{T_2}(p) \quad \forall T_1, T_2 \in \mathcal{T}_p \quad (2.20)$$

is satisfied at a vertex p , then obviously $\mathbf{w}_p = \nabla v_T(p)$ for all $T \in \mathcal{T}_p$.

Fig. 2 Triangles and tangent vectors in the definition of $\mathbf{w}_p$



2.3.2 Construction on the edges

On any edge $e \in \mathcal{E}_h$, we define a polynomial $g_e \in P_{k-1}(e)$ as follows: First we choose $T \in \mathcal{T}_e$ and then we specify that

$$g_e(p) = \mathbf{w}_p \cdot \mathbf{n}_{e,T} \text{ at an endpoint } p \text{ of } e. \quad (2.21)$$

$$g_e \text{ and } \partial v_T / \partial n \text{ have the same moments up to order } k - 3 \text{ on } e. \quad (2.22)$$

2.3.3 Construction on the triangles

We are now ready to define $(f_{v,T}, g_{v,T}) \in H^2(\partial T) \times H^1(\partial T)$ for any T as follows. Given any edge e of T , the function $f_{v,T}$ on e is the unique polynomial in $P_k(e)$ with the following properties:

$$\begin{aligned} f_{v,T} \text{ agrees with } v \text{ at the two endpoints of } e \text{ and shares the same moments} \\ \text{up to order } k - 4 \text{ with } v, \end{aligned} \quad (2.23)$$

$$\begin{aligned} \text{the directional derivative of } f_{v,T} \text{ at an endpoint } p \text{ of } e \text{ in the direction} \\ \text{of the tangent } \mathbf{t}_e \text{ of } e \text{ is given by } \mathbf{w}_p \cdot \mathbf{t}_e. \end{aligned} \quad (2.24)$$

Remark 2.9 If the condition (2.20) is satisfied at both endpoints of e , then $\mathbf{w}_p = \nabla v_T(p)$ at the two endpoints p of e by Remark 2.8 and then conditions (2.23) and (2.24) imply $f_{v,T} = v$ on e .

Given any edge e of T , we define

$$\begin{aligned} g_{v,T} = g_e \text{ if } T \text{ is the triangle chosen in the definition of } g_e \text{ (cf. Sect. 2.3.2),} \\ \text{otherwise } g_{v,T} = -g_e. \end{aligned} \quad (2.25)$$

Remark 2.10 If the condition (2.20) is satisfied at both endpoints of e and v is C^1 across e , then Remark 2.8 and (2.21)–(2.22) imply that $g_{v,T} = \partial v_T / \partial n$ on e .

By construction, the condition (2.15) is satisfied because the compatibility conditions (2.2)–(2.4) follow from (2.21) and (2.23)–(2.24). The condition (2.16) follows from (2.23)–(2.24) and the condition (2.17) follows from (2.25). The choices we make in the definition of $\mathbf{w}_p$ for $p \in \partial\Omega$ [cf. Case (ii) and Case (iii) in Sect. 2.3.1 and (2.23)–(2.24)] also implies (2.18).

2.4 The operator E_h

Let $v \in V_h$ and $T \in \mathcal{T}_h$ be arbitrary, and $(f_{v,T}, g_{v,T}) \in (H^2 \times H^1)_{k,k-1}(\partial T)$ be the function pair constructed in Sect. 2.3. We define $E_h v \in \mathcal{V}^k(T)$ by the following conditions (cf. Lemma 2.4):

$$\begin{aligned} (E_h v, \partial E_h v / \partial n) &= (f_{v,T}, g_{v,T}) \quad \text{on } \partial T \\ \text{and } Q_{T,k-4}(E_h v) &= Q_{T,k-4}(v). \end{aligned} \quad (2.26)$$

It follows from (2.16)–(2.17) that the piecewise H^2 function $E_h v$ belongs to $H^2(\Omega)$, and (2.18) implies $E_h v \in H_0^1(\Omega)$ if $v \in H_0^1(\Omega)$. It only remains to establish the estimates (1.4) and (1.5).

Note that Remarks 2.9 and 2.10 imply

$$(f_{v,T}, g_{v,T}) = (v_T, \partial v_T / \partial n) \quad \text{on } \partial T \quad \text{if } v \text{ is } C^1 \text{ on } \partial T, \quad (2.27)$$

and hence $v = E_h v$ if v is C^1 on ∂T , which is the rationale behind (1.4) and (1.5).

Theorem 2.1 *The estimate (1.4) holds with a positive constant $C_{\sharp}$ that only depends on k and the shape regularity of $\mathcal{T}_h$.*

Proof All the constants (explicit or hidden) that appear below will only depend on the minimum angle of $\mathcal{T}_h$.

Let $T \in \mathcal{T}_h$ be arbitrary. In view of Remark 2.1, Lemma 2.4 and the equivalence of norms on finite dimensional vector spaces, we have, by scaling,

$$\begin{aligned} \|\xi\|_{L_2(T)}^2 &\approx \|Q_{T,k-4}\xi\|_{L_2(T)}^2 + \sum_{e \in \mathcal{E}_T} [h_T \|Q_{e,k-4}\xi\|_{L_2(e)}^2 + h_T^3 \|Q_{e,k-3}(\partial \xi / \partial n)\|_{L_2(e)}^2] \\ &\quad + \sum_{p \in \mathcal{V}_T} [h_T^2 \xi^2(p) + h_T^4 |\nabla \xi(p)|^2] \quad \forall \xi \in \mathcal{V}^k(T), \end{aligned} \quad (2.28)$$

where h_T is the diameter of T and $\mathcal{V}_T$ (resp., $\mathcal{E}_T$) is the set of the three vertices (resp., edges) of T . Moreover the affine invariance of $\mathcal{V}^k(T)$ (cf. Definition 2.4) together with (2.10) and (2.11) implies that the hidden constants in (2.28) only depend on the shape regularity of T .

It follows from (2.23), (2.26) and (2.28) that

$$\begin{aligned} \|v - E_h v\|_{L_2(T)}^2 &\approx \sum_{p \in \mathcal{V}_T} h_T^4 |\nabla(v - E_h v)(p)|^2 \\ &\quad + \sum_{e \in \mathcal{E}_T} h_T^3 \|Q_{e,k-3} \partial(v - E_h v) / \partial n\|_{L_2(e)}^2, \end{aligned} \quad (2.29)$$

and we also have, by the construction of $\mathbf{w}_p$ in Sect. 2.3.1, (2.22), (2.24), and (2.25),

$$|\nabla(v - E_h v)(p)|^2 \leq C_1 \sum_{e \in \mathcal{E}_p} h_e^{-1} \|\llbracket \partial v / \partial n \rrbracket\|_{L_2(e)}^2,$$

$$\|Q_{e,k-3} \partial(v - E_h v) / \partial n\|_{L_2(e)}^2 \leq \|\llbracket \partial v / \partial n \rrbracket\|_{L_2(e)}^2,$$

where $\mathcal{E}_p$ is the set of all the edges in $\mathcal{E}_h$ that share p as a common vertex, and hence

$$\|v - E_h v\|_{L_2(T)}^2 \leq C_2 h_T^3 \sum_{p \in \mathcal{V}_T} \sum_{e \in \mathcal{E}_p} \|\llbracket \partial v / \partial n \rrbracket\|_{L_2(e)}^2. \quad (2.30)$$

We then deduce from (2.30) and scaling that

$$\|D^2(v - E_h v)\|_{L_2(T)}^2 \leq C_3 h_T^{-1} \sum_{p \in \mathcal{V}_T} \sum_{e \in \mathcal{E}_p} \|\llbracket \partial v / \partial n \rrbracket\|_{L_2(e)}^2. \quad (2.31)$$

Note that, because of the affine invariance of $\mathcal{V}^k(T)$, the scaling constants behind (2.31) only depend on the shape regularity of T .

The estimate (1.4) follows immediately from (1.1) and (2.31). $\square$

Theorem 2.2 *The estimate (1.5) holds with a positive constant C_b that only depends on k and the shape regularity of $\mathcal{T}_h$.*

Proof Let $T \in \mathcal{T}_h$ be arbitrary and S_T (the star of T) be the interior of the union of the closures of all the triangles in $\mathcal{T}_h$ that share a common vertex with T . If $\zeta \in H^{k+1}(\Omega)$ belongs to $P_k(S_T)$, then $\Pi_h \zeta = \zeta$ in S_T and hence $\zeta - E_h \Pi_h \zeta = \Pi_h \zeta - E_h \Pi_h \zeta = 0$ on T by (2.30). The estimate (1.5) can then be established through the Bramble–Hilbert lemma [3, 18]. $\square$

3 The three dimensional case

The construction of E_h in three dimensions follows the same strategy as in Sect. 2, and our treatment will be brief regarding the results and arguments that are (almost) identical with the two dimensional case.

3.1 Trace spaces for a tetrahedron

Let T be a tetrahedron with vertices p_1, p_2, p_3, p_4 , and F_i be the face of T opposite p_i . Let ℓ be a nonnegative number. A function u belongs to the piecewise Sobolev space $H^\ell(\partial T)$ (resp. piecewise polynomial space $P_\ell(\partial T)$) if and only if u_i , the restriction of u to F_i , belongs to $H^\ell(F_i)$ (resp., $P_\ell(F_i)$) for $1 \leq i \leq 4$.

For a function ϕ defined on a face F of the tetrahedron T , the planar gradient $\nabla_{F_j} \phi$ is defined by

$$\nabla_{F_j} \phi = \nabla \tilde{\phi} - (\nabla \tilde{\phi} \cdot \mathbf{n}_{F,T}) \mathbf{n}_{F,T},$$

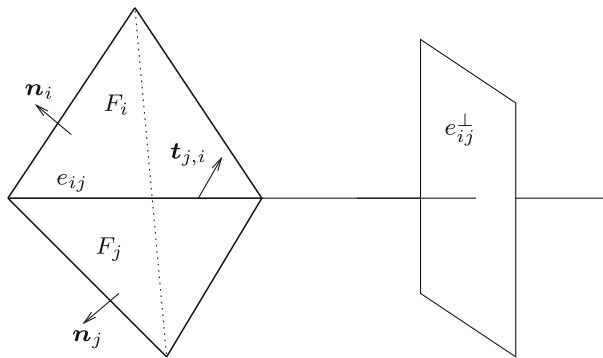


Fig. 3 Faces, normals, edge and orthogonal subspace

where $\tilde{\phi}$ is any extension of ϕ to a neighborhood of F in $\mathbb{R}^3$.

The operator $\text{Tr} : H^2(T) \longrightarrow H^{\frac{3}{2}}(\partial T) \times H^{\frac{1}{2}}(\partial T)$ is again defined by (2.1) in a piecewise sense. We want to characterize the image of $H^{\frac{5}{2}}(T)$ in $(H^2 \times H^1)(\partial T)$ under the operator Tr , for which we will need more notations and definitions.

The common edge of F_i and F_j is denoted by $e_{ij}(=e_{ji})$ and $e_{ij}^\perp$ denotes the two dimensional subspace of $\mathbb{R}^3$ perpendicular to e_{ij} . The outward unit normal on F_j is denoted by $\mathbf{n}_j$, and we denote by $\mathbf{t}_{j,i}$ the unit vector tangential to F_j , perpendicular to e_{ij} and pointing outside F_j (cf. Fig. 3).

Definition 3.1 The space $H^{\frac{1}{2}}(e_{ij}, e_{ij}^\perp)$ consists of all vector functions $\mathbf{w}$ defined on e_{ij} with image in $e_{ij}^\perp$ such that $\mathbf{w} \cdot \mathbf{z} \in H^{\frac{1}{2}}(e_{ij})$ for all $\mathbf{z} \in e_{ij}^\perp$.

Definition 3.2 A pair of functions $(f, g) \in H^2(\partial T) \times H^1(\partial T)$ belongs to the space $(H^2 \times H^1)(\partial T)$ if and only if the following conditions are satisfied :

$$f_i = f_j \quad \text{on } e_{ij} \text{ for } 1 \leq i \neq j \leq 4, \quad (3.1)$$

and there exist $\mathbf{w}_{ij} = \mathbf{w}_{ji} \in H^{\frac{1}{2}}(e_{ij}, e_{ij}^\perp)$ such that

$$\nabla_{F_j} f_j \cdot \mathbf{t}_{j,i} = \mathbf{w}_{ij} \cdot \mathbf{t}_{j,i} \quad \text{on } e_{ij} \text{ for } 1 \leq i \neq j \leq 4, \quad (3.2)$$

$$g_j = \mathbf{w}_{ij} \cdot \mathbf{n}_j \quad \text{on } e_{ij} \text{ for } 1 \leq i \neq j \leq 4. \quad (3.3)$$

Note that we can replace the compatibility conditions (3.2)–(3.3) by the condition

$$\nabla_{F_i} f_i + g_i \mathbf{n}_i = \nabla_{F_j} f_j + g_j \mathbf{n}_j \quad \text{on } e_{ij} \text{ for } 1 \leq i \neq j \leq 4. \quad (3.4)$$

It follows from the Sobolev Embedding Theorem that $\text{Tr } \zeta \in (H^2 \times H^1)(\partial T)$ for $\zeta \in H^{\frac{5}{2}}(T)$, where $\mathbf{w}_{ij}$ is the orthogonal projection of $\nabla \zeta$ along e_{ij} onto the subspace $e_{ij}^\perp$, and we can recover $\nabla \zeta$ on F_i from $(f, g) = \text{Tr } \zeta$ through the relation

$$\nabla \zeta = \nabla_{F_i} f_i + g_i \mathbf{n}_i \quad \text{on } F_i \text{ for } 1 \leq i \leq 4. \quad (3.5)$$

We want to show that $\text{Tr } H^{\frac{5}{2}}(T) = (H^2 \times H^1)(\partial T)$.

Again we construct a linear bijection

$$\Phi^* : (H^2 \times H^1)(\partial T) \longrightarrow (H^2 \times H^1)(\partial \tilde{T})$$

so that (2.7) is valid, where Φ is an orientation preserving affine transformation that maps the tetrahedron $\tilde{T}$ onto T . Let $(f, g) \in (H^2 \times H^1)(\partial T)$. Motivated by (2.7), (2.8) and (3.4), we define $\Phi^*(f, g) = (\tilde{f}, \tilde{g})$, where $\tilde{f}$ is given by (2.9), $\tilde{g}$ is given by (2.10) (where $\tilde{n}_i$ is the outward pointing unit normal along the face $\tilde{F}_i$) and the vector field $\mathbf{g}$ on ∂T is given by

$$\mathbf{g} = \nabla_{F_i} f_i + g_i \mathbf{n}_i \quad \text{on } F_i \text{ for } 1 \leq i \leq 4. \quad (3.6)$$

It is straightforward to check that $(\tilde{f}, \tilde{g}) \in (H^2 \times H^1)(\partial \tilde{T})$, $\Phi^* : (H^2 \times H^1)(\partial T) \longrightarrow (H^2 \times H^1)(\partial \tilde{T})$ is a bijection, and that (2.7) follows from (2.8)–(2.10), (3.5) and (3.6).

We can now establish the following analog of Lemma 2.1.

Lemma 3.1 *The image of $H^{\frac{5}{2}}(T)$ under Tr is the space $(H^2 \times H^1)(\partial T)$.*

Proof Given $(f, g) \in H^2(\partial T) \times H^1(\partial T)$ that satisfies (3.1)–(3.3), we can reduce the construction of ζ to the following three cases by a partition of unity. (i) f and g vanish near the vertices of T and the edges of T , in which case we can use the operator L_2 in Lemma A.2 to obtain ζ . (ii) f and g are supported in a neighborhood of an edge and vanish near the vertices of T , in which case we can assume through an affine transformation [cf. (2.7)] that the dihedral angle at the edge is a right angle and obtain ζ through Lemma A.5. (iii) f and g are supported near a vertex of T , in which case we can assume through an affine transformation that the angle at the vertex is a solid right angle and obtain ζ through Lemma A.4. $\square$

3.2 Affine invariant H^2 virtual element spaces

We will use the same notation $(H^2 \times H^1)(\partial T)$ to denote $\text{Tr } H^{\frac{5}{2}}(T)$ for a tetrahedron T . But the definition of $(H^2 \times H^1)_{k,k-1}(\partial T)$ is different.

Definition 3.3 Let T be a tetrahedron. A pair $(f, g) \in (H^2 \times H^1)(\partial T)$ belongs to $(H^2 \times H^1)_{k,k-1}(\partial T)$ if and only if $(f_i, g_i) \in \mathcal{V}^k(F_i) \times P_{k-1}(F_i)$ for $1 \leq i \leq 4$.

Remark 3.1 It follows from Remark 2.1, Lemma 2.3 and the constraints (3.1)–(3.3) that we need the following dofs for $(H^2 \times H^1)_{k,k-1}(\partial T)$: (i) The value of v at each vertex p together with the values of the three directional derivatives along the three edges emanating from p , which requires 4×4 dofs. (ii) The moments of v up to order $k - 4$ on each edge, which together with (i) ensure the constraint (3.1). This requires $6 \times (k - 3)$ dofs. (iii) The moments of order up to $k - 3$ on each edge in order to define, together with (i), a polynomial (vector) function of order $\leq k - 1$ on e with images in $e^\perp$, which requires $6 \times 2(k - 2)$ dofs. We can then use this polynomial (vector) function

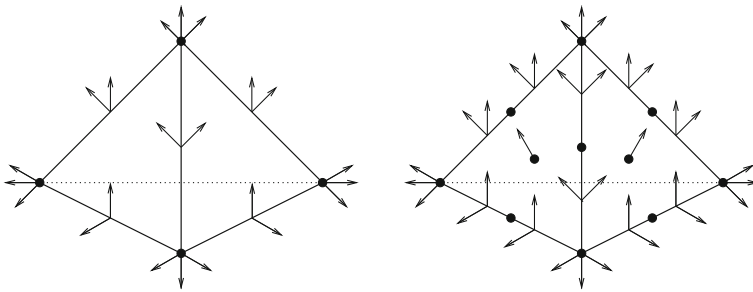


Fig. 4 Visible degrees of freedom for $(H^2 \times H^1)_{3,2}(\partial T)$ and $(H^2 \times H^1)_{4,3}(\partial T)$

to define $\nabla_F v_F \cdot \mathbf{n}_{e,F}$ on any edge e of F through (3.2) and $\partial v / \partial n$ on ∂F through (3.3). (iv) On each face F we need to specify the moments of v and $\partial v / \partial n$ up to order $k-4$ in order to complete the definition of $v_F \in \mathcal{V}^k(F)$ and $\partial v / \partial n \in P_{k-1}(F)$, which requires $4 \times 2 \times \frac{(k-3)(k-2)}{2}$ dofs. Altogether we have

$$\begin{aligned} \dim (H^2 \times H^1)_{k,k-1}(\partial T) \\ = 16 + 6(k-3) + 12(k-2) + 4(k-3)(k-2)2(k-1)(2k+1). \end{aligned} \quad (3.7)$$

The (visible) dofs of $(H^2 \times H^1)_{k,k-1}(\partial T)$ for $k = 3$ and 4 are depicted in Fig. 4, where (i) the values of f at the vertices and the moments of f on the edges and faces are represented by solid dots, and (ii) the directional derivatives of f at the vertices and the moments of g on the edges and faces are represented by arrows.

The well-posedness result in Lemma 2.2 remains valid for a tetrahedron T and the definition of the virtual element space $\mathcal{V}^k(\hat{T})$ on the reference tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ is identical to the one in Definition 2.3 for the reference triangle. The virtual element space $\mathcal{V}^k(T)$ for an arbitrary tetrahedron is then defined as in Definition 2.4 through an orientation preserving affine transformation Φ that maps $\hat{T}$ onto T , and Lemma 2.4 also holds for a general tetrahedron.

The dimension of $\mathcal{V}^k(T)$ is now given by

$$\begin{aligned} \dim \mathcal{V}^k(T) &= \dim(H^2 \times H^1)_{k,k-1}(\partial T) + \dim P_{k-4}(T) \\ &= 2(k-1)(2k+1) + \frac{1}{6}(k-3)(k-2)(k-1) \\ &= \frac{(k-1)(k+1)(k+18)}{6}. \end{aligned} \quad (3.8)$$

Remark 3.2 The definition of $\mathcal{V}^k(T)$ for a tetrahedron relies crucially on the fact that boundary data satisfying the compatibility condition (3.1)–(3.3) will belong to $\text{Tr } H^2(T)$. Unlike the two dimensional case (cf. Remark 2.4), this cannot be taken for granted since macro elements of arbitrary order that share the same boundary data are yet to be developed.

Remark 3.3 Three dimensional H^2 virtual elements on arbitrary polyhedron have recently been proposed in [2].

3.3 Construction on the skeleton $\bigcup_{T \in \mathcal{T}_h} \partial T$

Given any $v \in V_h$, we want to define $f_{v,T}$ representing (the desired) $E_h v|_{\partial T}$ and $g_{v,T}$ representing (the desired) $(\partial E_h v / \partial n)|_{\partial T}$ for all $T \in \mathcal{T}_h$, such that

$$(f_{v,T}, g_{v,T}) \in (H^2 \times H^1)_{k,k-1}(\partial T) \quad \forall T \in \mathcal{T}_h \quad (3.9)$$

and the following conditions are satisfied:

$$\begin{aligned} &\text{if } T_1 \text{ and } T_2 \text{ belong to } \mathcal{T}_p \text{ (resp., } \mathcal{T}_e \text{ or } \mathcal{T}_F), \text{ then } f_{v,T_1}(p) = f_{v,T_2}(p) \\ &\quad (\text{resp., } f_{v,T_1} = f_{v,T_2} \text{ on } e \text{ or } f_{v,T_1} = f_{v,T_2} \text{ on } F), \end{aligned} \quad (3.10)$$

$$\begin{aligned} &\text{if } T_1 \text{ and } T_2 \text{ are two distinct tetrahedra in } \mathcal{T}_F, \\ &\quad \text{then } g_{v,T_1} + g_{v,T_2} = 0 \text{ on } F, \end{aligned} \quad (3.11)$$

$$\text{if } F \in \mathcal{F}_h^b \text{ is a face of } T \text{ and } v \in H_0^1(\Omega), \text{ then } f_{v,T} = 0 \text{ on } F. \quad (3.12)$$

Note that (3.10) and (3.11) imply any piecewise H^2 function ξ satisfying $(\xi_T, \partial \xi_T / \partial n)|_{\partial T} = (f_{v,T}, g_{v,T})$ for all $T \in \mathcal{T}_h$ will belong to $H^2(\Omega)$, and (3.12) implies that $\xi \in H^2(\Omega) \cap H_0^1(\Omega)$ if $v \in H_0^1(\Omega)$.

3.3.1 Construction at the vertices

As in Sect. 2.3, we first define the vectors $\mathbf{w}_p$ associated with the vertices p of $\mathcal{T}_h$. There are three cases: (i) p is an interior vertex, (ii) p is a boundary vertex that belongs to a face of Ω , (iii) p is a boundary vertex that does not belong to any face of Ω .

Case (i) For an interior vertex p , we choose a tetrahedron T in $\mathcal{T}_p$ and define $\mathbf{w}_p$ to be ∇v_T .

Case (ii) For a boundary vertex p that belongs to a face F of Ω , we define $\mathbf{w}_p$ to be ∇v_T , where T is a tetrahedron in $\mathcal{T}_p$ that has a face on F . This choice ensures that $\mathbf{w}_p \cdot \mathbf{t} = 0$ if $v \in H_0^1(\Omega)$, where $\mathbf{t}$ is any vector tangential to $\partial\Omega$ at p .

Case (iii) In this case p is either a corner of Ω or p belongs to an edge of Ω . We define $\mathbf{w}_p$ implicitly by

$$\mathbf{w}_p \cdot \mathbf{t}_i = \frac{\partial v}{\partial t_i}(p) \quad \text{for } i = 1, 2, 3, \quad (3.13)$$

where $\partial/\partial t_1$, $\partial/\partial t_2$ and $\partial/\partial t_3$ are the tangential derivatives along three edges $e_1, e_2, e_3 \in \mathcal{E}_h^b$ emanating from p that are not coplanar. This choice of e_1, e_2, e_3 implies $\mathbf{w}_p = \mathbf{0}$ if $v \in H_0^1(\Omega)$.

Remark 3.4 Note that Remark 2.8 is also valid here, i.e., $\mathbf{w}_p = \nabla v_T(p)$ for all $T \in \mathcal{T}_p$ if v is C^1 at the vertex p .

3.3.2 Construction on the edges

In view of (3.2) and (3.3), we also need to define polynomial vector functions $\mathbf{w}_e : e \rightarrow e^\perp$ on the edges $e \in \mathcal{E}_h$. There are three cases: (i) e is an interior edge of $\mathcal{T}_h$, (ii) e is a subset of a face of Ω and (iii) e is a subset of an edge of Ω .

Case (i) Let e belong to $\mathcal{E}_h^i$. We choose $T \in \mathcal{T}_e$, and then define $\mathbf{w}_e$ by the following conditions:

$$\text{at an endpoint } p \text{ of } e, \mathbf{w}_e(p) \text{ is the projection of } \mathbf{w}_p \text{ on } e^\perp. \quad (3.14)$$

$$\mathbf{w}_e \text{ and the projection of } \nabla v_T \text{ on } e^\perp \text{ have the same moments along } e \text{ up to order } k - 3. \quad (3.15)$$

Case (ii) Let e be an edge of $\mathcal{T}_h$ that is a subset of a face F of Ω . We define $\mathbf{w}_e$ again by (3.14)–(3.15), but with the stipulation that one of the faces of T is a subset of F . This additional condition (together with the choices made in Cases (ii) and (iii) in Sect. 3.3.1) implies that $\mathbf{w}_e \cdot \mathbf{t} = 0$ on e if $v \in H_0^1(\Omega)$, where $\mathbf{t}$ is any vector tangential to F .

Case (iii) Let e be an edge of $\mathcal{T}_h$ that is a subset of an edge of Ω . Then there are two distinct faces $F_1, F_2 \in \mathcal{F}_h^b \cap \mathcal{F}_e$ and we define $\mathbf{w}_e$ by (3.14) together with the condition that

$$\begin{aligned} \mathbf{w}_e \cdot \mathbf{n}_{e,F_j} \text{ and } \nabla_{F_j} v_{F_j} \cdot \mathbf{n}_{e,F_j} \text{ have identical moments up to order } k - 3 \\ \text{for } j = 1, 2. \end{aligned} \quad (3.16)$$

Our choice of F_1 and F_2 (together with the choices made in Cases (ii) and (iii) in Sect. 3.3.1) ensures that $\mathbf{w}_e = 0$ on e if $v \in H_0^1(\Omega)$.

Remark 3.5 In the case where $v \in V_h$ is C^1 across an edge $e \in \mathcal{E}_h$ and at the endpoints of e , it follows from Remark 3.4 and (3.14)–(3.16) that the vector field $\mathbf{w}_e$ is the projection of ∇v_T on $e^\perp$ for all $T \in \mathcal{T}_e$.

3.3.3 Construction on the faces

We define $g_{v,F}$ on a face $F \in \mathcal{F}_h$ as follows. We choose $T \in \mathcal{T}_F$ and stipulate that

$$\text{on an edge } e \text{ of } F, g_{v,F} \in P_{k-1}(e) \text{ is given by } \mathbf{w}_e \cdot \mathbf{n}_{F,T}, \quad (3.17)$$

$$g_{v,F} \text{ and } \partial v_T / \partial n \text{ have the same moments up to order } k - 4 \text{ on } F. \quad (3.18)$$

Remark 3.6 If v is C^1 across $e \in \mathcal{E}_h$ and at the endpoints of e , then we have $g_{v,F} = \partial v_T / \partial n$ on e for all $F \in \mathcal{F}_e$ and $T \in \mathcal{T}_F$ by Remark 3.5 and (3.17).

3.3.4 Construction on the tetrahedra

We are now ready to define $(f_{v,T}, g_{v,T}) \in H^2(\partial T) \times H^1(\partial T)$ for any $T \in \mathcal{T}_h$ as follows. On any edge e of a face F of T , $f_{v,\partial F}$ is the unique polynomial in $P_k(e)$ with

the following properties:

$f_{v,\partial F}$ agrees with v at the two endpoints of e and shares the same moments up to order $k - 4$,

(3.19)

the directional derivative of $f_{v,\partial F}$ at an endpoint p of e in the direction

of the tangent $\mathbf{t}_e$ of e is given by $\mathbf{w}_p \cdot \mathbf{t}_e$.

(3.20)

Remark 3.7 Remark 2.9 is also valid here, i.e., $f_{v,\partial F} = v$ on e if v is C^1 at the endpoints of e .

Let F be a face of T and e be an edge of F , we define $q_{v,\partial F} \in P_{k-1}(e)$ by

$$q_{v,\partial F} = \mathbf{w}_e \cdot \mathbf{n}_{e,F}. \quad (3.21)$$

On each face F of T , the pair $(f_{v,\partial F}, q_{v,\partial F})$ belongs to $(H^2 \times H^1)_{k,k-1}(\partial F)$ by (3.14) and (3.19)–(3.21). Hence we can define $f_{v,F} \in \mathcal{V}^k(F)$ to be the virtual element function (cf. Lemma 2.4) that satisfies the following conditions:

$$\text{Tr } f_{v,F} = (f_{v,\partial F}, q_{v,\partial F}) \text{ on } \partial F \text{ and } Q_{F,k-4} f_{v,F} = Q_{F,k-4} v. \quad (3.22)$$

Remark 3.8 If v is C^1 on ∂F , then $q_{v,\partial F} = \partial v_F / \partial n$ on ∂F by Remark 3.5 and (3.21). It then follows from Remarks 2.5, 3.7 and (3.22) that $f_{v,F} = v$ on F .

Given any face F of T , we define

$$g_{v,T} = g_{v,F} \text{ if } T \text{ is the tetrahedron chosen in the definition of } g_{v,F} \\ (\text{cf. Sect. 3.3.3}), \text{ otherwise } g_{v,T} = -g_{v,F}. \quad (3.23)$$

Remark 3.9 If v is C^1 on ∂T , then (3.18), Remark 3.6 and (3.23) imply $g_{v,T} = \partial v_T / \partial n$ on ∂T .

At the end of this process, we have constructed $(f_{v,T}, g_{v,T}) \in H^2(\partial T) \times H^1(\partial T)$ for every polyhedron $T \in \mathcal{T}_h$. The pair $(f_{v,T}, g_{v,T})$ belongs to $(H^2 \times H^1)_{k,k-1}(\partial T)$ because (i) the condition (3.1) is implied by (3.19)–(3.20), (ii) the condition (3.2) is implied by (3.21)–(3.22), and (iii) the condition (3.3) is implied by (3.17).

It follows from (3.19)–(3.22) that (3.10) is satisfied, and the condition (3.11) follows from (3.23). The choices we make in Sects. 3.3.1 and 3.3.2 ensure that $f_{v,\partial F}$ defined by (3.19)–(3.20) and $q_{v,\partial F}$ defined by (3.21) both vanish on ∂F if the face F of T is a subset of $\partial \Omega$ and $v \in H_0^1(\Omega)$. The condition (3.12) then follows from (3.22).

In view of Remark 3.8 and Remark 3.9 the relation (2.27) remains valid, i.e., $E_h v = v$ if $v \in V_h$ is C^1 on ∂T , which is the basis for the estimates (1.4) and (1.5).

3.4 The operator E_h

We proceed as in Sect. 2.4. Let $v \in V_h$ and $T \in \mathcal{T}_h$ be arbitrary, and $(f_{v,T}, g_{v,T}) \in (H^2 \times H^1)_{k,k-1}(\partial T)$ be the function pair constructed in Sect. 3.3. We define $E_h v \in$

$\mathcal{V}^k(T)$ again by the conditions in (2.26), i.e.,

$$\begin{aligned} (E_h v, \partial E_h v / \partial n) &= (f_{v,T}, g_{v,T}) \quad \text{on } \partial T \\ \text{and } Q_{T,k-4}(E_h v) &= Q_{T,k-4}(v). \end{aligned} \quad (3.24)$$

It follows from (3.10)–(3.11) that $E_h v \in H^2(\Omega)$, and (3.12) implies that $E_h v \in H_0^1(\Omega)$ if $v \in H_0^1(\Omega)$.

The estimates (1.4) and (1.5) are established by similar arguments as in Sect. 2.4, where the analog of (2.28) for a tetrahedron T (cf. Lemma 2.4 and Remark 3.1) is given by

$$\begin{aligned} &\|\xi\|_{L_2(T)}^2 \\ &\approx \|Q_{T,k-4}\xi\|_{L_2(T)}^2 + \sum_{F \in \mathcal{F}_T} h_T \|Q_{F,k-4}\xi\|_{L_2(F)}^2 + \sum_{F \in \mathcal{F}_T} h_T^3 \|Q_{F,k-4}(\partial\xi/\partial n)\|_{L_2(F)}^2 \\ &\quad + \sum_{e \in \mathcal{E}_T} h_T^2 \|Q_{e,k-4}\xi\|_{L_2(e)}^2 + \sum_{e \in \mathcal{E}_T} h_T^4 \|Q_{e,k-3}(\nabla\xi)_{e^\perp}\|_{L_2(e)}^2 \\ &\quad + \sum_{p \in \mathcal{V}_T} [h_T^3 \xi^2(p) + h_T^5 |\nabla\xi(p)|^2] \end{aligned} \quad (3.25)$$

for all $\xi \in \mathcal{V}^k(T)$, where $\mathcal{F}_T$ (resp., $\mathcal{E}_T$ and $\mathcal{V}_T$) is the set of the four faces (resp., six edges and four vertices) of T and $(\nabla\xi)_{e^\perp}$ is the orthogonal projection of $\nabla\xi$ onto the subspace of $\mathbb{R}^3$ perpendicular to e . The hidden constants in (3.25) only depend on the shape regularity of $\mathcal{T}_h$ because of the affine invariance of the virtual element spaces.

It follows from (3.19), (3.22), (3.24) and (3.25) that we have the following analog of (2.29):

$$\begin{aligned} &\|v - E_h v\|_{L_2(T)}^2 \\ &\approx \sum_{p \in \mathcal{V}_T} h_T^5 |\nabla(v - E_h v)(p)|^2 + \sum_{e \in \mathcal{E}_T} h_T^4 \|Q_{e,k-3}(\nabla(v - E_h v))_{e^\perp}\|_{L_2(e)}^2 \\ &\quad + \sum_{F \in \mathcal{F}_T} h_T^3 \|Q_{F,k-4}[\partial v / \partial n]\|_{L_2(F)}^2. \end{aligned} \quad (3.26)$$

We can then establish the three-dimensional analogs of Theorems 2.1 and 2.2 as in Sect. 2.4.

4 Concluding remarks

Following the approach of this paper (and with more patience and persistence), one can construct enriching operators E_h that maps the totally discontinuous P_k finite element space into $H^2(\Omega)$, where (1.4) and (1.5) are valid for $J(w, v)$ given by

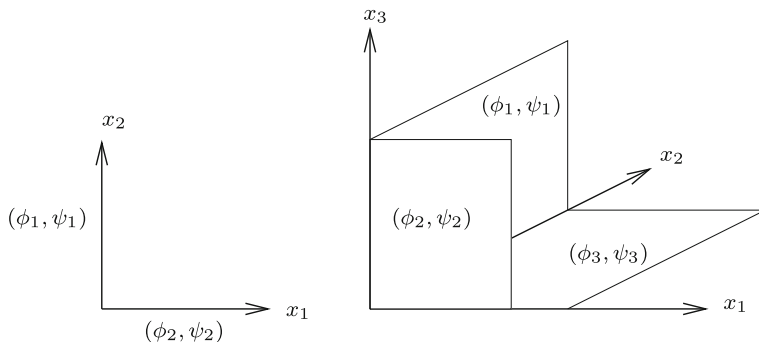


Fig. 5 Boundary data for $\mathbb{R}_+^2$ and $\mathbb{R}_+^3$

$$J(w, v) = \sum_{e \in \mathcal{E}_h^i} \left[h_e^{-3} \int_e \llbracket w \rrbracket \llbracket v \rrbracket ds + h_e^{-1} \int_e \llbracket \partial w / \partial n \rrbracket \llbracket \partial v / \partial n \rrbracket ds \right] \quad \text{for } d = 2,$$

$$J(w, v) = \sum_{F \in \mathcal{F}_h^i} \left[h_F^{-3} \int_F \llbracket w \rrbracket \llbracket v \rrbracket dS + h_F^{-1} \int_F \llbracket \partial w / \partial n \rrbracket \llbracket \partial v / \partial n \rrbracket dS \right] \quad \text{for } d = 3.$$

One can also construct $E_h : V_h \cap H_0^1(\Omega) \longrightarrow H_0^2(\Omega)$ such that (1.4) and (1.5) are valid, provided the sum in (1.2) [resp., (1.3)] is taken over $\mathcal{E}_h$ (resp., $\mathcal{F}_h$). This can also be carried out for the totally discontinuous P_k finite element space.

Lemma 3.1 is also of independent interest, since inverse trace theorems for polyhedral domains in $\mathbb{R}^3$ do not appear to be readily available in the literature.

A Inverse trace theorems for $\mathbb{R}_+^2$ and $\mathbb{R}_+^3$

We consider inverse trace theorems for $\mathbb{R}_+^2$ and $\mathbb{R}_+^3$ with data on the boundaries of these domains (cf. Fig. 5). We will rely on the results in Lemmas A.1 and A.2 that follow from the construction of inverse trace operators through the Fourier transform [22, 26] and the Paley–Wiener theorem [21].

Lemma A.1 *There exists a bounded linear operator*

$$L_1 : H^2(\mathbb{R}) \times H^1(\mathbb{R}) \longrightarrow H^{\frac{5}{2}}(\mathbb{R}^2)$$

such that (i) $[L_1(\phi, \psi)](t, 0) = \phi(t)$, (ii) $[\partial L_1(\phi, \psi) / \partial x_2](t, 0) = \psi(t)$, and (iii) $L_1(\phi, \psi)(x_1, x_2)$ vanishes on the half plane $x_1 < 0$ if $\phi(t)$ and $\psi(t)$ vanish on the half line $t < 0$.

Lemma A.2 *There exists a bounded linear operator*

$$L_2 : H^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \longrightarrow H^{\frac{5}{2}}(\mathbb{R}^3)$$

such that (i) $[L_2(\phi, \psi)](x_1, x_2, 0) = \phi(x_1, x_2)$, (ii) $[\partial L_2(\phi, \psi)/\partial x_3](x_1, x_2, 0) = \psi(x_1, x_2)$, and (iii) $L_2(\phi, \psi)(x_1, x_2, x_3)$ vanishes on the half space $x_1 < 0$ (resp., $x_2 < 0$) if $\phi(x_1, x_2)$ and $\psi(x_1, x_2)$ vanish on the half plane $x_1 < 0$ (resp., $x_2 < 0$).

We begin with a two-dimensional inverse trace theorem. We note that similar results for $H^2(\mathbb{R}_+^2)$ can be found in [20, Section 1.5.2]. Our approach is simpler (since we are considering $H^{\frac{5}{2}}(\mathbb{R}_+^2)$) and therefore its extension to three dimensions is easier.

Lemma A.3 *Let (ϕ_1, ψ_1) and (ϕ_2, ψ_2) belong to $H^2(\mathbb{R}_+) \times H^1(\mathbb{R}_+)$ such that*

$$\phi_1(0) = \phi_2(0), \quad (\text{A.1})$$

$$\psi_1(0) = \phi_2'(0), \quad (\text{A.2})$$

$$\psi_2(0) = \phi_1'(0). \quad (\text{A.3})$$

Then there exists $\zeta \in H^{\frac{5}{2}}(\mathbb{R}_+^2)$ such that

$$(\zeta, \partial\zeta/\partial x_i) = (\phi_i, \psi_i) \quad \text{if } x_i = 0, \quad 1 \leq i \leq 2. \quad (\text{A.4})$$

Proof First we extend ϕ_1 and ψ_1 to $\mathbb{R}$, so that the extensions (still denoted by ϕ_1 and ψ_1) satisfy $\phi_1 \in H^2(\mathbb{R})$ and $\psi_1 \in H^1(\mathbb{R})$. This can be achieved by reflection (cf. [22, Theorem 2.3.9] and [1, Theorem 5.19]). Let L_1 be the lifting operator in Lemma A.1 and $\zeta_1 = L_1(\phi_1, \psi_1) \in H^{\frac{5}{2}}(\mathbb{R}^2)$ so that

$$\zeta_1(0, x_2) = \phi_1(x_2) \quad \text{and} \quad (\partial\zeta_1/\partial x_1)(0, x_2) = \psi_1(x_2). \quad (\text{A.5})$$

Then we define $\tilde{\phi}_2(x_1) = \phi_2(x_1) - \zeta_1(x_1, 0)$, $\tilde{\psi}_2(x_1) = \psi_2(x_1) - (\partial\zeta_1/\partial x_2)(x_1, 0)$ for $x_1 > 0$.

Note that $\tilde{\phi}_2 \in H^2(\mathbb{R}_+)$, and

$$\tilde{\phi}_2(0) = \phi_2(0) - \zeta_1(0, 0) = \phi_2(0) - \phi_1(0) = 0$$

by (A.1) and (A.5), and

$$\tilde{\phi}_2'(0) = \phi_2'(0) - (\partial\zeta_1/\partial x_1)(0, 0) = \phi_2'(0) - \psi_1(0) = 0$$

by (A.2) and (A.5). Moreover we have $\tilde{\psi}_2 \in H^1(\mathbb{R}_+)$, and

$$\tilde{\psi}_2(0) = \psi_2(0) - (\partial\zeta_1/\partial x_2)(0, 0) = \psi_2(0) - \phi_1'(0) = 0$$

by (A.3) and (A.5). Hence their trivial extensions (still denoted by $\tilde{\phi}_2$ and $\tilde{\psi}_2$) satisfy $\tilde{\phi}_2 \in H^2(\mathbb{R})$ and $\tilde{\psi}_2 \in H^1(\mathbb{R})$.

Let $\zeta_2 = L_1(\tilde{\phi}_2, \tilde{\psi}_2) \in H^{\frac{5}{2}}(\mathbb{R}^2)$ such that $\zeta_2(x_1, 0) = \tilde{\psi}_2(x_1, 0)$ and

$$(\partial\zeta_2/\partial x_2)(x_1, 0) = \tilde{\psi}_1(x_1).$$

Then $\zeta_2 = 0$ on the half plane $x_1 < 0$ by Lemma A.3, which implies

$$\zeta_2(0, x_2) = (\partial \zeta_2 / \partial x_1)(0, x_2) = 0 \quad \forall x_2 > 0.$$

We can now take ζ to be the restriction of $\zeta_1 + \zeta_2$ to $\mathbb{R}_+^2$. □

Next we consider the three dimensional analog of Lemma A.3.

Lemma A.4 *Let (ϕ_1, ψ_1) , (ϕ_2, ψ_2) and (ϕ_3, ψ_3) belong to $H^2(\mathbb{R}_+^2) \times H^1(\mathbb{R}_+^2)$ such that the following conditions are satisfied:*

$$\phi_1(0, x_3) = \phi_2(0, x_3) \quad \forall x_3 > 0, \quad (\text{A.6})$$

$$\phi_2(x_1, 0) = \phi_3(x_1, 0) \quad \forall x_1 > 0, \quad (\text{A.7})$$

$$\phi_3(0, x_2) = \phi_1(x_2, 0) \quad \forall x_2 > 0, \quad (\text{A.8})$$

$$\psi_1(0, x_3) = \frac{\partial \phi_2}{\partial x_1}(0, x_3) \quad \forall x_3 > 0, \quad (\text{A.9})$$

$$\psi_1(x_2, 0) = \frac{\partial \phi_3}{\partial x_1}(0, x_2) \quad \forall x_2 > 0, \quad (\text{A.10})$$

$$\psi_2(x_1, 0) = \frac{\partial \phi_3}{\partial x_2}(x_1, 0) \quad \forall x_1 > 0, \quad (\text{A.11})$$

$$\psi_2(0, x_3) = \frac{\partial \phi_1}{\partial x_2}(0, x_3) \quad \forall x_3 > 0, \quad (\text{A.12})$$

$$\psi_3(x_1, 0) = \frac{\partial \phi_2}{\partial x_3}(x_1, 0) \quad \forall x_1 > 0, \quad (\text{A.13})$$

$$\psi_3(0, x_2) = \frac{\partial \phi_1}{\partial x_3}(x_2, 0) \quad \forall x_2 > 0. \quad (\text{A.14})$$

Then there exists $\zeta \in H^{\frac{5}{2}}(\mathbb{R}_+^3)$ such that

$$(\zeta, \partial \zeta / \partial x_i) = (\phi_i, \psi_i) \quad \text{if } x_i = 0, \quad 1 \leq i \leq 3. \quad (\text{A.15})$$

Proof First we extend ϕ_1 and ψ_1 to $\mathbb{R}^2$ by reflection (twice) so that the extensions (still denoted by ϕ_1 and ψ_1) satisfy $\phi_1 \in H^2(\mathbb{R}^2)$ and $\psi_1 \in H^1(\mathbb{R}^2)$. Let L_2 be the lifting operator in Lemma A.2 and $\zeta_1 = L_2(\phi_1, \psi_1)$ so that

$$\zeta_1(0, x_2, x_3) = \phi_1(x_2, x_3), \quad (\text{A.16})$$

$$(\partial \zeta_1 / \partial x_1)(0, x_2, x_3) = \psi_1(x_2, x_3). \quad (\text{A.17})$$

Then we define, for $(x_1, x_3) \in \mathbb{R}_+^2$,

$$\tilde{\phi}_2(x_1, x_3) = \phi_2(x_1, x_3) - \zeta_1(x_1, 0, x_3), \quad (\text{A.18})$$

$$\tilde{\psi}_2(x_1, x_3) = \psi_2(x_1, x_3) - (\partial \zeta_1 / \partial x_2)(x_1, 0, x_3). \quad (\text{A.19})$$

Note that $\tilde{\phi}_2$ belongs to $H^2(\mathbb{R}_+^2)$ and

$$\tilde{\phi}_2(0, x_3) = \phi_2(0, x_3) - \zeta_1(0, 0, x_3) = \phi_2(0, x_3) - \phi_1(0, x_3) = 0 \quad \text{for } x_3 > 0$$

by (A.6), (A.16) and (A.18), and

$$\frac{\partial \tilde{\phi}_2}{\partial x_1}(0, x_3) = \frac{\partial \phi_2}{\partial x_1}(0, x_3) - \frac{\partial \zeta_1}{\partial x_1}(0, 0, x_3) = \frac{\partial \phi_2}{\partial x_1}(0, x_3) - \psi_1(0, x_3) = 0$$

by (A.9), (A.17) and (A.18). Furthermore $\tilde{\psi}_2$ belongs to $H^1(\mathbb{R}_+^2)$ and

$$\begin{aligned} \tilde{\psi}_2(0, x_3) &= \psi_2(0, x_3) - \frac{\partial \zeta_1}{\partial x_2}(0, 0, x_3) \\ &= \psi_2(0, x_3) - \frac{\partial \phi_1}{\partial x_2}(0, x_3) = 0 \quad \text{for } x_3 > 0 \end{aligned}$$

by (A.12), (A.16) and (A.19).

Hence we can extend $\tilde{\phi}_2$ and $\tilde{\psi}_2$ to $\mathbb{R}_+ \times \mathbb{R}$ by reflection across $x_3 = 0$ (still denoted by $\tilde{\phi}_2$ and $\tilde{\psi}_2$) so that $\tilde{\phi}_2 \in H^2(\mathbb{R}_+ \times \mathbb{R})$, $\tilde{\psi}_2 \in H^2(\mathbb{R}_+ \times \mathbb{R})$, $\tilde{\phi}_2(0, x_3) = (\partial \tilde{\phi}_2 / \partial x_1)(0, x_3) = 0$ for $x_3 \in \mathbb{R}$ and $\tilde{\psi}_2(0, x_3) = 0$ for $x_3 \in \mathbb{R}$. Therefore the trivial extensions of $\tilde{\phi}_2$ and $\tilde{\psi}_2$ to $\mathbb{R}^2$ (still denoted by $\tilde{\phi}_2$ and $\tilde{\psi}_2$) belong to $H^2(\mathbb{R}^2)$ and $H^1(\mathbb{R}^2)$ respectively.

Let $\zeta_2 = L_2(\tilde{\phi}_2, \tilde{\psi}_2)$. Then we have, by Lemma A.2,

$$\zeta_2(x_1, 0, x_3) = \tilde{\phi}_2(x_1, x_3), \quad (\text{A.20})$$

$$(\partial \zeta_2 / \partial x_2)(x_1, 0, x_3) = \tilde{\psi}_2(x_1, x_3), \quad (\text{A.21})$$

and

$$\zeta_2(x_1, x_2, x_3) = 0 \quad \text{if } x_1 < 0,$$

which implies

$$\zeta_2 = \partial \zeta_2 / \partial x_1 = 0 \quad \text{if } x_1 = 0. \quad (\text{A.22})$$

We now define, for $(x_1, x_2) \in \mathbb{R}_+^2$,

$$\tilde{\phi}_3(x_1, x_2) = \phi_3(x_1, x_2) - \zeta_1(x_1, x_2, 0) - \zeta_2(x_1, x_2, 0), \quad (\text{A.23})$$

$$\tilde{\psi}_3(x_1, x_2) = \psi_3(x_1, x_2) - (\partial \zeta_1 / \partial x_3)(x_1, x_2, 0) - (\partial \zeta_2 / \partial x_3)(x_1, x_2, 0). \quad (\text{A.24})$$

Then $\tilde{\phi}_3$ (resp., $\tilde{\psi}_3$) belongs to $H^2(\mathbb{R}_+^2)$ (resp., $H^1(\mathbb{R}_+^2)$).

Moreover, it follows from (A.8), (A.16), (A.22) and (A.23) that

$$\tilde{\phi}_3(0, x_2) = \phi_3(0, x_2) - \zeta_1(0, x_2, 0) = \phi_3(0, x_2) - \phi_1(x_2, 0) = 0 \quad \text{for } x_2 > 0,$$

and (A.10), (A.17), (A.22) and (A.23) imply

$$\begin{aligned}\frac{\partial \tilde{\phi}_3}{\partial x_1}(0, x_2) &= \frac{\partial \phi_3}{\partial x_1}(0, x_2) - \frac{\partial \zeta_1}{\partial x_1}(0, x_2, 0) \\ &= \frac{\partial \phi_3}{\partial x_1}(0, x_2) - \psi_1(x_2, 0) = 0 \quad \text{for } x_2 > 0.\end{aligned}$$

From (A.14), (A.16), (A.22) and (A.24) we also have

$$\tilde{\psi}_3(0, x_2) = \psi_3(0, x_2) - \frac{\partial \zeta_1}{\partial x_3}(0, x_2, 0) = \psi_3(0, x_2) - \frac{\partial \phi_1}{\partial x_3}(x_2, 0) \quad \text{for } x_2 > 0.$$

Next we check the behavior of $\tilde{\phi}_3$ and $\tilde{\psi}_3$ at $x_2 = 0$. We have

$$\begin{aligned}\tilde{\phi}_3(x_1, 0) &= \phi_3(x_1, 0) - \zeta_1(x_1, 0, 0) - \zeta_2(x_1, 0, 0) \\ &= \phi_3(x_1, 0) - \phi_2(x_1, 0) = 0 \quad \text{for } x_1 > 0\end{aligned}$$

by (A.7), (A.18), (A.20) and (A.23);

$$\begin{aligned}\frac{\partial \tilde{\phi}_3}{\partial x_2}(x_1, 0) &= \frac{\partial \phi_3}{\partial x_2}(x_1, 0) - \frac{\partial \zeta_1}{\partial x_2}(x_1, 0, 0) - \frac{\partial \zeta_2}{\partial x_2}(x_1, 0, 0) \\ &= \frac{\partial \phi_3}{\partial x_2}(x_1, 0) - \psi_2(x_1, 0) = 0 \quad \text{for } x_1 > 0\end{aligned}$$

by (A.11), (A.18), (A.21) and (A.23);

$$\begin{aligned}\tilde{\psi}_3(x_1, 0) &= \psi_3(x_1, 0) - \frac{\partial \zeta_1}{\partial x_3}(x_1, 0, 0) - \frac{\partial \zeta_2}{\partial x_3}(x_1, 0, 0) \\ &= \psi_3(x_1, 0) - \frac{\partial \phi_2}{\partial x_3}(x_1, 0) = 0 \quad \text{for } x_1 > 0\end{aligned}$$

by (A.13), (A.18), (A.20) and (A.24).

The calculations above show that $\tilde{\phi}_3 = \partial \tilde{\phi}_3 / \partial n = \tilde{\psi}_3 = 0$ on the boundary of $\mathbb{R}_+^2$. Hence their trivial extensions to $\mathbb{R}^2$ (still denoted by $\tilde{\phi}_3$ and $\tilde{\psi}_3$) belongs to $H^2(\mathbb{R}^2)$ and $H^1(\mathbb{R}^2)$.

Let $\zeta_3 = L_2(\tilde{\phi}_1, \tilde{\psi}_1)$. Then we have, by Lemma A.2, $\zeta_3 \in H^3(\mathbb{R}^3)$,

$$\zeta_3(x_1, x_2, 0) = \tilde{\phi}_3(x_1, x_2), \quad (\text{A.25})$$

$$(\partial \zeta_3 / \partial x_3)(x_1, x_2, 0) = \tilde{\psi}_3(x_1, x_2), \quad (\text{A.26})$$

and

$$\zeta_3(x_1, x_2, x_3) = 0 \quad \text{if } x_1 < 0 \text{ or } x_2 < 0,$$

which implies

$$\zeta_3 = \frac{\partial \zeta_3}{\partial x_1} = 0 \quad \text{if } x_1 = 0 \quad \text{and} \quad \zeta_3 = \frac{\partial \zeta_3}{\partial x_2} = 0 \quad \text{if } x_2 = 0. \quad (\text{A.27})$$

We can now take ζ to be the restriction of $\zeta_1 + \zeta_2 + \zeta_3$ to $\mathbb{R}_+^3$, and (A.15) follows from (A.16)–(A.27). $\square$

Finally we have a three-dimensional result that is two-dimensional in nature and which can be derived by using the arguments in the proof of either Lemma A.3 or Lemma A.4.

Lemma A.5 *Let (ϕ_1, ψ_1) and (ϕ_2, ψ_2) belong to $H^2(\mathbb{R}_+ \times \mathbb{R}) \times H^1(\mathbb{R}_+ \times \mathbb{R})$ such that*

$$\begin{aligned} \phi_1(0, x_3) &= \phi_2(0, x_3) \quad \forall x_3 \in \mathbb{R}, \\ \psi_1(0, x_3) &= \frac{\partial \phi_2}{\partial x_1}(0, x_3) \quad \forall x_3 \in \mathbb{R}, \\ \psi_2(0, x_3) &= \frac{\partial \phi_1}{\partial x_2}(0, x_3) \quad \forall x_3 \in \mathbb{R}. \end{aligned}$$

Then there exists $\zeta \in H^{\frac{5}{2}}(\mathbb{R}_+^3)$ such that

$$(\zeta, \partial \zeta / \partial x_i) = (\phi_i, \psi_i) \quad \text{if } x_i = 0, \quad 1 \leq i \leq 2.$$

References

1. Adams, R., Fournier, J.: Sobolev Spaces, 2nd edn. Academic Press, Amsterdam (2003)
2. Beirão da Veiga, L., Dassi, F., Russo, A.: A C^1 virtual element method on polyhedral meshes. [arXiv:1808.01105v2](https://arxiv.org/abs/1808.01105v2) [math.NA] (to appear in Comput. Math. Appl.)
3. Bramble, J., Hilbert, S.: Estimation of linear functionals on Sobolev spaces with applications to Fourier transforms and spline interpolation. SIAM J. Numer. Anal. **7**, 113–124 (1970)
4. Brenner, S.: Two-level additive Schwarz preconditioners for nonconforming finite element methods. Math. Comput. **65**, 897–921 (1996)
5. Brenner, S.: Convergence of nonconforming multigrid methods without full elliptic regularity. Math. Comput. **68**, 25–53 (1999)
6. Brenner, S.: C^0 interior penalty methods. In: Blowey, J., Jensen, M. (eds.) Frontiers in Numerical Analysis-Durham 2010, Lecture Notes in Computational Science and Engineering, vol. 85, pp. 79–147. Springer, Berlin (2012)
7. Brenner, S., Gedicke, J., Sung, L.Y., Zhang, Y.: An a posteriori analysis of C^0 interior penalty methods for the obstacle problem of clamped Kirchhoff plates. SIAM J. Numer. Anal. **55**, 87–108 (2017)
8. Brenner, S., Gudi, T., Sung, L.Y.: An a posteriori error estimator for a quadratic C^0 interior penalty method for the biharmonic problem. IMA J. Numer. Anal. **30**, 777–798 (2010)
9. Brenner, S., Scott, L.: The Mathematical Theory of Finite Element Methods, 3rd edn. Springer, New York (2008)
10. Brenner, S., Sung, L.Y.: A new convergence analysis of finite element methods for elliptic distributed optimal control problems with pointwise state constraints. SIAM J. Control Optim. **55**, 2289–2304 (2017)
11. Brenner, S., Wang, K.: Two-level additive Schwarz preconditioners for C^0 interior penalty methods. Numer. Math. **102**, 231–255 (2005)

12. Brezzi, F., Marini, L.: Virtual element methods for plate bending problems. *Comput. Methods Appl. Mech. Eng.* **253**, 455–462 (2013)
13. Chinosi, C., Marini, L.: Virtual element method for fourth order problems: L^2 -estimates. *Comput. Math. Appl.* **72**, 1959–1967 (2016)
14. Ciarlet, P.: Sur l'élément de Clough et Tocher. *RAIRO Anal. Numér.* **8**, 19–27 (1974)
15. Ciarlet, P.: The Finite Element Method for Elliptic Problems. North-Holland, Amsterdam (1978)
16. Clough, R., Tocher, J.: Finite element stiffness matrices for analysis of plate bending. In: *Proceedings of Conference on Matrix Methods in Structural Mechanics*, pp. 515–545. Wright-Patterson Air Force Base (1965)
17. Douglas Jr., J., Dupont, T., Percell, P., Scott, R.: A family of C^1 finite elements with optimal approximation properties for various Galerkin methods for 2nd and 4th order problems. *R.A.I.R.O. Modél. Math. Anal. Numér.* **13**, 227–255 (1979)
18. Dupont, T., Scott, R.: Polynomial approximation of functions in Sobolev spaces. *Math. Comput.* **34**, 441–463 (1980)
19. Georgoulis, E., Houston, P., Virtanen, J.: An a posteriori error indicator for discontinuous Galerkin approximations of fourth-order elliptic problems. *IMA J. Numer. Anal.* **31**, 281–298 (2011)
20. Grisvard, P.: *Elliptic Problems in Nonsmooth Domains*, *Classics in Applied Mathematics*, vol. 69. Society for Industrial and Applied Mathematics (SIAM), Philadelphia (2011)
21. Lax, P.: *Functional Analysis*. Wiley-Interscience, New York (2002)
22. Nečas, J.: *Direct Methods in the Theory of Elliptic Equations*. Springer, Heidelberg (2012)
23. Neilan, M., Wu, M.: Discrete Miranda–Talenti estimates and applications to linear and nonlinear PDEs. *J. Comput. Appl. Math.* **356**, 358–376 (2019)
24. Peano, A.: Hierarchies of conforming finite elements for plane elasticity and plate bending. *Comput. Math. Appl.* **2**, 211–224 (1976)
25. Percell, P.: On cubic and quartic Clough–Tocher finite elements. *SIAM J. Numer. Anal.* **13**, 100–103 (1976)
26. Wloka, J.: *Partial Differential Equations*. Cambridge University Press, Cambridge (1987)
27. Worsey, A., Farin, G.: An n -dimensional Clough–Tocher interpolant. *Constr. Approx.* **3**, 99–110 (1987)

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