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A domain decomposition method for the time-dependent Navier-Stokes-Darcy model with Beavers-Joseph interface condition and defective boundary condition [☆]

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ABSTRACT

In this article a domain decomposition method is proposed to solve a time-dependent Navier-Stokes-Darcy model with Beavers-Joseph interface condition and defective boundary condition. Robin boundary conditions between the Navier-Stokes domain and Darcy domain are constructed by directly re-organizing the terms in the three interface conditions, including the Beavers-Joseph condition. In order to avoid the traditional iteration for the domain decomposition method at each time step, the interface information, which is needed for the Robin type transmission conditions at the current time step, is directly predicted based on the numerical solution of the previous time steps. Backward Euler scheme is first utilized for the temporal discretization while finite elements are used for the spatial discretization. The convergences of this domain decomposition method are rigorously analyzed for the time-dependent Navier-Stokes-Darcy model with Beavers-Joseph interface condition. The major difficulties in the analysis arise from nonlinear terms and Beavers-Joseph interface condition, including a series of technical treatments and the final special norm used in the discrete Gronwall's inequality for the analysis of full discretization. Based on the above preparation, we further develop a Lagrange multiplier method under the framework of the domain decomposition method to overcome the difficulty of non-unique solutions arising from the defective boundary condition. One interesting finding of this paper is that the Lagrange multipliers are time dependent functions instead of constants. In order to improve the accuracy order for the temporal discretization, a three-step backward differentiation scheme is used to replace the backward Euler scheme. Compared with the first scheme, the second one allows us to use the relative larger time step to reduce the computational cost while keeping the same accuracy. Numerical examples are provided to illustrate the features of the proposed method.

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1. Introduction

In the past decade many scientists and engineers have investigated the Stokes-Darcy or Navier-Stokes-Darcy type models for the coupling of fluid flows and porous media flows. The Stokes-Darcy interface model has attracted significant attention from scientists and engineers due to its wide range of applications, such as interaction between surface and subsurface flows [1–4], industrial filtrations [5,6], groundwater system in karst aquifers [7–10], and petroleum extraction [11–15]. Therefore it is not surprising that many different numerical methods have been proposed and analyzed for the Stokes-Darcy model, including domain decomposition methods [16–19,2,20–23], Lagrange multiplier methods [24–27,4], discontinuous Galerkin methods [28–31], multigrid methods [32,33], partitioned time stepping methods [34–37], coupled finite element methods [7, 38–40], and many others [41–51]. Particularly, a parallel, non-iterative, multi-physics domain decomposition method (DDM) was proposed for the time-dependent Stokes-Darcy model with Beavers-Joseph-Saffman-Jones (BJSJ) interface condition in [52].

Recently, the Navier-Stokes-Darcy model has attracted scientists' attention, including the steady state problem [53–60] and the unsteady problem [61–63]. Compared with the extensively studied Stokes-Darcy model, the more difficult time-dependent Navier-Stokes-Darcy model is still in great need of continued efforts for developing and analyzing stable, accurate, and efficient numerical methods, especially for the model with more realistic and difficult boundary/interface conditions. In fact, it is difficult or expensive in many applications to measure the fluid flow velocity for the boundary conditions but much easier and more cost-efficient to obtain flow rates on the boundary [64,65]. Therefore, the corresponding defective boundary conditions were considered for the Navier-Stokes equation [66]. More recent development for defective boundary problems can be found in [67–69].

Furthermore, there are two choices for the interface condition in the tangential direction: the original Beavers-Joseph (BJ) interface condition [70] and the simplified Beavers-Joseph-Saffman-Jones (BJSJ) interface condition [71–73]. It is true for some cases that the contribution of the Darcy flow in tangential direction is heuristically much smaller than that of Stokes flow on the interface and hence the BJSJ simplification can be used. There are related theoretical works in [74] using the Brinkman-Stokes model as the starting point and periodicity in the horizontal (along the interface) direction. They demonstrated that the BJ interface condition is more accurate than the BJSJ interface condition or its further simplifications. The error is not necessarily small for all parameters and it could be of order 1 for the lower values of the hydraulic conductivity/permeability/porosity.

Based on the key ideas of [52] which was a fundamental development for the simple Stokes-Darcy model with BJSJ interface condition, in this article we first develop a parallel, non-iterative, multi-physics domain decomposition method to solve the sophisticated time-dependent Navier-Stokes-Darcy system with BJ interface condition. In order to avoid the traditional iteration for the domain decomposition at each time step, the interface information at the current time step is directly predicted based on the numerical solution of the previous time steps. Therefore, the method is non-iterative for the domain decomposition even though iterations are still needed for the temporal discretization and Newton's method to handle time-dependence and nonlinearity. Beavers-Joseph interface condition needs special treatments in both the analysis and the construction of the Robin boundary conditions for the domain decomposition. The nonlinear advection also increases the difficulty of the analysis. Therefore, the analysis for the proposed method in this article is much more difficult than that of [52], thus needs significant extra efforts, which will be illustrated in details in the analysis section. Finite elements are used for the spatial discretization. Backward Euler and three-step backward differentiation schemes are used for the temporal discretization. Based on the solid foundation built for the domain decomposition method of the Navier-Stokes-Darcy system with BJ interface condition, we further propose the Lagrange multipliers to deal with this model with defective boundary condition under the same framework of the domain decomposition method. One interesting finding of this paper is that the Lagrange multipliers are time dependent functions instead of constants.

The rest of paper is organized as follows. In section 2, we introduce the time-dependent Navier-Stokes-Darcy model with BJ interface condition and defective boundary condition. In section 3, we propose the parallel, non-iterative, multi-physics domain decomposition method, which uses backward Euler scheme in temporal discretization and finite elements for the spatial discretization, for the time-dependent Navier-Stokes-Darcy system with BJ interface condition (but without considering the defective boundary condition). In section 4, we analyze the stability and convergence for the method proposed in section 3. In section 5, we propose the Lagrange multiplier method in the framework of the proposed domain decomposition method with three-step backward differentiation scheme for the time-dependent Navier-Stokes-Darcy system with defective boundary condition and BJ interface condition. In section 6, we provide numerical experiments to validate the proposed method and illustrate its features. In section 7, we draw the conclusions.

2. The Navier-Stokes-Darcy model

In this section, we briefly introduce the time-dependent Navier-Stokes-Darcy model with Beavers-Joseph interface condition and defective boundary condition.

2.1. The Navier-Stokes-Darcy system with the Beavers-Joseph interface condition

We consider a coupled Navier-Stokes-Darcy system on a bounded domain $\Omega = \Omega_D \bigcup \Omega_S \subset \mathbb{R}^d$, $(d = 2, 3)$, see Fig. 1.

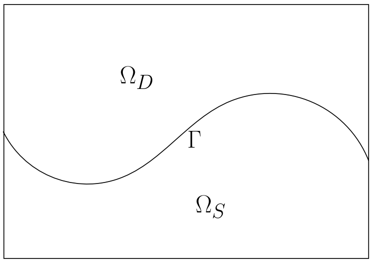

Fig. 1. A sketch of the porous median domain Ω_D, fluid domain Ω_S, and the interface Γ.

In the porous media region Ω_D, let $\vec{u}_D$ denote the fluid discharge rate in the porous media, $\mathbb{K}$ denotes the hydraulic conductivity tensor, f_D denotes the sink/source term and ϕ_D denote the hydraulic head. Specifically, $\phi_D = z + \frac{p_D}{\rho g}$ where p_D is the dynamic pressure, z is the height, ρ is the density and g is the gravity constant. S denotes the mass storativity coefficient. Then the porous media flow is assumed to satisfy the following Darcy equation:

$$\vec{u}_D = -\mathbb{K}\nabla\phi_D, \tag{1}$$

$$S\frac{\partial \phi_D}{\partial t} + \nabla\cdot\vec{u}_D = f_D,\ t\in[0,T]. \tag{2}$$

Plugging (1) into (2), we obtain

$$S\frac{\partial \phi_D}{\partial t} - \nabla\cdot(\mathbb{K}\nabla\phi_D) = f_D,\ t\in[0,T]. \tag{3}$$

In this article, we assume the media in Ω_D is isotropic and $\mathbb{K} = \frac{\prod g}{\nu}$, $\prod(\vec{x}) = k(\vec{x})\mathbb{I}$, and $\mathbb{I}$ is the identity matrix [8]. In order to simplify the expression of formulation and the process of proof in this article, we set $S = 1$.

In the fluid region Ω_S, let $\vec{u}_S$ denote the fluid velocity, p_S denote the kinematic pressure, $\vec{f}_S$ denote the external body force, and ν denote the kinematic viscosity of the fluid. Then the fluid flow is assumed to satisfy the Navier-Stokes equation:

$$\frac{\partial \vec{u}_S}{\partial t} + (\vec{u}_S\cdot\nabla)\vec{u}_S - \nabla\cdot\mathbb{T}(\vec{u}_S, p_S) = \vec{f}_S,\ t\in[0,T], \tag{4}$$

$$\nabla\cdot\vec{u}_S = 0, \tag{5}$$

where $\mathbb{T}(\vec{u}_S, p_S) = 2\nu\mathbb{D}(\vec{u}_S) - p_S\mathbb{I}$ is the stress tensor and $\mathbb{D}(\vec{u}_S) = 1/2(\nabla\vec{u}_S + \nabla^T\vec{u}_S)$ is the deformation tensor.

Let $\overline{\Gamma} = \overline{\Omega}_D \cap \overline{\Omega}_S$ denote the interface between the fluid and porous media regions. Along the interface Γ, we first impose the following two well-accepted interface conditions:

$$\vec{u}_S\cdot\vec{n}_S = -\vec{u}_D\cdot\vec{n}_D, \quad -\vec{n}_S\cdot(\mathbb{T}(\vec{u}_S, p_S)\cdot\vec{n}_S) = g\phi_D, \tag{6}$$

where $\vec{n}_S$ and $\vec{n}_D$ denote the unit outer normal to the fluid and the porous media regions at the interface Γ respectively. These two interface conditions are for the continuity of normal velocity and the balance of force normal to the interface. Then the following Beavers-Joseph (BJ) interface condition [70] is imposed in the tangential direction on the interface

$$-\boldsymbol{\tau}_j\cdot(\mathbb{T}(\vec{u}_S, p_S)\cdot\vec{n}_S) = \frac{\alpha\nu\sqrt{d}}{\sqrt{\mathrm{trace}(\prod)}}\boldsymbol{\tau}_j\cdot(\vec{u}_S - \vec{u}_D), \tag{7}$$

where $\boldsymbol{\tau}_j$ $(j = 1, \cdots, d-1)$ denote mutually orthogonal unit vectors tangential to the interface Γ.

Assume that the hydraulic head ϕ_D and the fluid velocity $\vec{u}_S$ satisfies homogeneous Dirichlet boundary condition except on Γ, i.e., $\phi_D = 0$ on the boundary $\partial\Omega_D\backslash\Gamma$ and $\vec{u}_S = 0$ on the boundary $\partial\Omega_S\backslash\Gamma$. Assume that the hydraulic head ϕ_D and the fluid velocity $\vec{u}_S$ satisfy the following initial conditions

$$\phi_D(0,x,y) = \phi_0(x,y) \quad \textit{and} \quad \vec{u}_S(0,x,y) = \vec{u}_0(x,y). \tag{8}$$

2.2. The Navier-Stokes-Darcy system with the defective boundary and the Beavers-Joseph interface conditions

Fig. 2 illustrates a simple system with defective boundary conditions. For example, in a simplified typical karst aquifer system, the free flow is confined in the underground conduit while porous media is surrounding the conduit. The region occupied by the conduit and porous media are denoted by Ω_S and Ω_D, respectively. On the boundary of Ω_S, we particularly consider $\Gamma_S = \partial\Omega_S\backslash\Gamma = \bigcup_{i=0}^{m} S_i$ for the defective boundary condition.

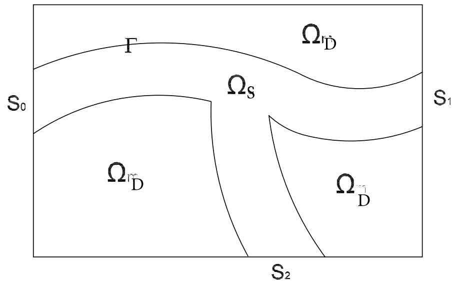

Fig. 2. Typical components of a karst aquifer.

The governing equations and interface conditions are still (3)-(7). However, it is often difficult to obtain the velocity data on Γ_S for different applications, but easier to obtain flow rates Q_i on the boundary S_i [66,65]. Hence we consider the following prescribed flow rate condition on $\Gamma_S = \partial\Omega_S \backslash \Gamma = \bigcup_{i=0}^{m} S_i$:

$$\int_{S_i} \vec{u}_S \cdot \vec{n}_S \, ds = Q_i, \quad \text{for } i = 0, 1, \cdots, m, \tag{9}$$

where the flow rates Q_i (also called velocity fluxes) are functions of time.

The other boundary and initial conditions we consider for the model are the same as in section 2.1.

3. The parallel, non-iterative, multiphysics domain decomposition method

In this section, we consider the time-dependent Navier-Stokes-Darcy with BJ interface condition in section 2.1. We will first present the coupled weak formulation and introduce Robin boundary conditions of the Darcy and Navier-Stokes systems on the interface Γ for the domain decomposition. Then we will present the parallel, non-iterative, multi-physics domain decomposition method with backward Euler scheme in temporal discretization, whose stability and convergence will be analyzed in section 4.

3.1. Coupled weak formulation for the Navier-Stokes-Darcy with BJ interface condition

First, we define the following function spaces

$$\begin{aligned}
X_S &= \{\vec{v} \in [H^1(\Omega_S)]^d \mid \vec{v} = 0 \text{ on } \partial\Omega_S \backslash \Gamma\} \\
Q_S &= L^2(\Omega_S) \\
X_D &= \{\psi \in H^1(\Omega_D) \mid \psi = 0 \text{ on } \partial\Omega_D \backslash \Gamma\} \\
L^2(0,T;Q_S) &= \{\phi : \phi(t,\cdot) \in Q_S, \ \forall t \in [0,T]\} \\
H^1(0,T;X_D,X_D') &= \{\phi : \phi \in L^2(0,T;X_D) \text{ and } \frac{\partial \phi}{\partial t} \in L^2(0,T;X_D')\} \\
H^1(0,T;X_S,X_S') &= \{\phi : \phi \in L^2(0,T;X_S) \text{ and } \frac{\partial \phi}{\partial t} \in L^2(0,T;X_S')\}.
\end{aligned}$$

Here X_D' and X_S' are the dual spaces of X_D and X_S. For the domain D ($D = \Omega_S$ or Ω_D), $(\cdot,\cdot)_D$ denotes the L^2 inner product on the domain D, and $\langle\cdot,\cdot\rangle$ denotes the L^2 inner product on the interface Γ or the duality pairing between $(H_{00}^{1/2}(\Gamma))'$ and $H_{00}^{1/2}(\Gamma)$. P_τ denoted the projection onto the tangent space on Γ, i.e. $P_\tau \vec{u} = \sum_{j=1}^{d-1} (\vec{u} \cdot \boldsymbol{\tau}_j)\boldsymbol{\tau}_j$.

We also define the following bilinear forms

$$a_D(\phi_D, \psi) = (\mathbb{K}\nabla\phi_D, \nabla\psi)_{\Omega_D}, \ \ a_S(\vec{u}_S, \vec{v}) = 2\nu(\mathbb{D}(\vec{u}_S), \mathbb{D}(\vec{v}))_{\Omega_S}, \ \ b_S(\vec{v}, q) = -(\nabla \cdot \vec{v}, q)_{\Omega_S},$$

and the trilinear form

$$C_s(\vec{u}_S, \vec{u}_S, \vec{v}) = ((\vec{u}_S \cdot \nabla)\vec{u}_S, \vec{v}).$$

With these notations, the weak formulation of the coupled Navier-Stokes-Darcy model with Beavers-Joseph interface condition is defined as follows: find $(\vec{u}_S, p_S) \in H^1(0,T;X_S,X_S') \times L^2(0,T;Q_S)$ and $\phi_D \in H^1(0,T;X_D,X_D')$ such that

$$
\begin{aligned}
&(\frac{\partial \vec{u}_S}{\partial t}, \vec{v})_{\Omega_S} + \eta(\frac{\partial \phi_D}{\partial t}, \psi)_{\Omega_D} + C_S(\vec{u}_S, \vec{u}_S, \vec{v}) + a_S(\vec{u}_S, \vec{v}) + b_S(\vec{v}, p_S) + \frac{\eta}{S} a_D(\phi_D, \psi) \\
&+ \langle g\phi_D, \vec{v} \cdot \vec{n}_S \rangle - \frac{\eta}{S} \langle \vec{u}_S \cdot \vec{n}_S, \psi \rangle + \frac{\alpha \nu \sqrt{d}}{\sqrt{\text{trace}(\prod)}} \langle P_\tau(\vec{u}_S + \mathbb{K}\nabla\phi_D), P_\tau \vec{v} \rangle \\
&= \frac{\eta}{S}(f_D, \psi)_{\Omega_D} + (\vec{f}_S, \vec{v})_{\Omega_S}, \ \forall \vec{v} \in X_S, \ \ \psi \in X_D, \qquad (10) \\
&b_S(\vec{u}_S, q) = 0, \ \forall \ q \in Q_S. \qquad (11)
\end{aligned}
$$

Here η is a scaling parameter. This weak formulation is similar to that of [52], but takes the nonlinear advection and Beavers-Joseph interface condition into account.

Remark 1. Essentially the well-posedness of the non-stationary Navier-Stokes-Darcy model with Beavers-Joseph (BJ) interface condition can be obtained by combining the ideas and techniques in [8] (for non-stationary Stokes-Darcy model with BJ condition) and [60] (for stationary Navier-Stokes-Darcy model with BJ condition). Additionally, there exist other well-posedness analysis results in [61,63] for the non-stationary Navier-Stokes-Darcy model with the easier Beavers-Joseph-Saffman (BJS) interface condition. By utilizing the rescaling technique in [8] for the Beavers-Joseph interface condition in the non-stationary Stokes-Darcy system, the well-posedness analysis results in [61,63], which are based on the BJS condition, can be also improved to show the well-posedness of the non-stationary Navier-Stokes-Darcy model with BJ condition.

3.2. Robin boundary conditions and the decoupled system

In order to solve the coupled Navier-Stokes-Darcy problem utilizing the domain decomposition idea, we naturally consider Robin boundary conditions for the Darcy and the Navier-Stokes equations by following the idea in [52]. But the Robin boundary conditions need to be modified according to the Beavers-Joseph interface condition.

First we consider the following two Robin type conditions for the Navier-Stokes equations

$$
\begin{aligned}
&-P_\tau(\mathbb{T}(\vec{u}_S, p_S) \cdot \vec{n}_S) - \frac{\alpha \nu \sqrt{d}}{\sqrt{\text{trace}(\prod)}} P_\tau \vec{u}_S = \xi_{S\tau} \ \text{ on } \Gamma, \qquad (12) \\
&\vec{n}_S \cdot (\mathbb{T}(\vec{u}_S, p_S) \cdot \vec{n}_S) + \vec{u}_S \cdot \vec{n}_S = \xi_S \qquad \text{on } \Gamma, \qquad (13)
\end{aligned}
$$

for two given functions $\xi_{S\tau}, \ \xi_S \in L^2(0, T; L^2(\Gamma))$. Then, the corresponding weak formulation for the Navier-Stokes system is to find $\vec{u}_S \in H^1(0, T; X_S, X_S')$ and $p_S \in L^2(0, T; Q_S)$ such that

$$
\begin{aligned}
&(\frac{\partial \vec{u}_S}{\partial t}, \vec{v})_{\Omega_S} + C_S(\vec{u}_S, \vec{u}_S, \vec{v}) + a_S(\vec{u}_S, \vec{v}) + b_S(\vec{v}, p_S) + \langle \vec{u}_S \cdot \vec{n}_S, \vec{v} \cdot \vec{n}_S \rangle \\
&+ \frac{\alpha \nu \sqrt{d}}{\sqrt{\text{trace}(\prod)}} \langle P_\tau \vec{u}_S, P_\tau \vec{v} \rangle \ = \ (\vec{f}_S, \vec{v})_{\Omega_S} + \langle \xi_S, \vec{v} \cdot \vec{n}_S \rangle - \langle \xi_{S\tau}, P_\tau \vec{v} \rangle, \qquad (14) \\
&b_S(\vec{u}_S, q) \ = \ 0, \qquad (15)
\end{aligned}
$$

where $\forall \ \vec{v} \in X_S, \ \forall \ q \in Q_S$.

On the other hand, we consider the following Robin condition for the Darcy system

$$
\mathbb{K}\nabla\phi_D \cdot \vec{n}_D + g\phi_D = \xi_D \qquad \text{on } \Gamma, \qquad (16)
$$

for a given function $\xi_D \in L^2(0, T; L^2(\Gamma))$. Hence, the corresponding weak formulation for the Darcy system is to find $\phi_D \in H^1(0, T; X_D, X_D')$ such that

$$
\eta(\frac{\partial \phi_D}{\partial t}, \psi)_{\Omega_D} + \frac{\eta}{S} a_D(\phi_D, \psi) + \frac{\eta}{S} \langle g\phi_D, \psi \rangle = \frac{\eta}{S}(f_D, \psi)_{\Omega_D} + \frac{\eta}{S} \langle \xi_D, \psi \rangle, \forall \psi \in X_D. \qquad (17)
$$

The Navier-Stokes and Darcy systems with Robin boundary conditions can be combined into one system. Indeed, recall (5.4)-(5.5) in [60], (3.16)-(3.17) in [17] and (2.14) in [8], it is easy to see that if ξ_D, ξ_S and $\xi_{S\tau}$ are given, then, there exists a unique solution $(\phi_D, \vec{u}_S, p_S) \in H^1(0, T; X_D, X_D') \times H^1(0, T; X_S, X_S') \times L^2(0, T; Q_S)$ such that

$$
\begin{aligned}
&(\frac{\partial \vec{u}_S}{\partial t}, \vec{v})_{\Omega_S} + \eta(\frac{\partial \phi_D}{\partial t}, \psi)_{\Omega_D} + C_S(\vec{u}_S, \vec{u}_S, \vec{v}) + a_S(\vec{u}_S, \vec{v}) + b_S(\vec{v}, p_S) + \frac{\eta}{S} a_D(\phi_D, \psi) \\
&+ \langle \vec{u}_S \cdot \vec{n}_S, \vec{v} \cdot \vec{n}_S \rangle + \frac{\eta}{S} \langle \phi_D, \psi \rangle + \frac{\alpha \nu \sqrt{d}}{\sqrt{\text{trace}(\prod)}} \langle P_\tau \vec{u}_S, P_\tau \vec{v} \rangle = \frac{\eta}{S}(f_D, \psi)_{\Omega_D}
\end{aligned}
$$

$$+(\vec{f}_S,\vec{v})_{\Omega_S}+\langle\xi_S,\vec{v}\cdot\vec{n}_S\rangle+\frac{\eta}{S}\langle\xi_D,\psi\rangle-\langle\xi_{S\tau},P_\tau\vec{v}\rangle,\ \forall\psi\in Q_S,\ \vec{v}\in X_S, \tag{18}$$

$$b_S(\vec{u}_S,q)=0,\ \forall q\in Q_S, \tag{19}$$

$$\phi_D(0)\ =\ \phi_0,\ \ \vec{u}_S(0)\ =\ \vec{u}_0. \tag{20}$$

Similar to Proposition 3.1 in [52], it is easy to show that the solutions of the coupled Navier-Stokes-Darcy system are equivalent to solutions of the decoupled system if the following compatibility conditions are satisfied:

$$\xi_D=\vec{u}_S\cdot\vec{n}_S+g\phi_D,\ \xi_S=\vec{u}_S\cdot\vec{n}_S-g\phi_D,\ \xi_{S\tau}=\frac{\alpha\nu\sqrt{d}}{\sqrt{\mathrm{trace}(\prod)}}P_\tau(\mathbb{K}\nabla\phi_D). \tag{21}$$

These compatibility conditions provide the key tool to predict ξ_D,ξ_S and $\xi_{S\tau}$ on the interface at each time step based on the results from the previous time steps.

3.3. The non-iterative domain decomposition method

First we present the semi-discretization for (17) and (14)-(15). Suppose we have finite element spaces $X_{Dh}\subset X_D$, $X_{Sh}\subset X_S$ and $Q_{Sh}\subset Q_S$. Here we assume that $X_{Sh}\subset X_S$ and $Q_{Sh}\subset Q_S$ satisfy the following inf-sup condition: there exists a constant $\gamma>0$ such that

$$\inf_{0\neq q\in Q_{sh}}\ \sup_{0\neq\vec{v}\in X_{Sh}}\frac{b_S(\vec{v},q)}{\|\vec{v}\|_1\|q\|_0}>\gamma. \tag{22}$$

Define $P_h:X_D\to X_{Dh}$ and $\mathbb{P}_h:X_S\to X_{Sh}$ to be the regular orthogonal projections. We have the following regular approximation capability for them.

$$\|P_h\phi-\phi\|_0\ \le\ Ch^r\|\phi\|_r,\ \forall\phi\in H^r(\Omega_D), \tag{23}$$

$$\|\mathbb{P}_h\vec{u}-\vec{u}\|_0\ \le\ Ch^r\|\vec{u}\|_r,\ \forall\vec{u}\in[H^r(\Omega_S)]^d. \tag{24}$$

Then the semi-discretization of the decoupled system (17) and (14)-(15) is to find $\phi_h\in H^1(0,T;X_{Dh})$, $\vec{u}_h\in H^1(0,T;X_{Sh})$ and $p_h\in L^2(0,T;Q_{Sh})$ such that

$$\eta(\frac{\partial\phi_h}{\partial t},\psi_h)_{\Omega_D}+\frac{\eta}{S}a_D(\phi_h,\psi_h)+\frac{\eta}{S}\langle g\phi_h,\psi_h\rangle=\frac{\eta}{S}(f_D,\psi_h)_{\Omega_D}+\frac{\eta}{S}\langle\xi_{Dh},\psi_h\rangle, \tag{25}$$

$$(\frac{\partial\vec{u}_h}{\partial t},\vec{v}_h)_{\Omega_S}+C_S(\vec{u}_h,\vec{u}_h,\vec{v})+\langle\vec{u}_h\cdot\vec{n}_S,\vec{v}_h\cdot\vec{n}_S\rangle+a_S(\vec{u}_h,\vec{v}_h)+b_S(\vec{v}_h,p_h)$$

$$+\frac{\alpha\nu\sqrt{d}}{\sqrt{\mathrm{trace}(\prod)}}\langle P_\tau\vec{u}_h,P_\tau\vec{v}_h\rangle=(\vec{f}_S,\vec{v}_h)_{\Omega_S}+\langle\xi_{Sh},\vec{v}_h\cdot\vec{n}_S\rangle-\langle\xi_{S\tau h},P_\tau\vec{v}_h\rangle, \tag{26}$$

$$b_S(\vec{u}_h,q_h)=0, \tag{27}$$

$$\phi_h(0)\ =\ P_h\phi_0,\ \ \vec{u}_h(0)\ =\ \mathbb{P}_h\vec{u}_0, \tag{28}$$

where $\forall\ \psi_h\in X_{Dh}$, $\forall\ \vec{v}_h\in X_{Sh}$, $\forall\ q_h\in Q_{Sh}$ and

$$\xi_{Dh}=\vec{u}_h\cdot\vec{n}_S+g\phi_h,\quad \xi_{Sh}=\vec{u}_h\cdot\vec{n}_S-g\phi_h,\quad \xi_{S\tau h}=\frac{\alpha\nu\sqrt{d}}{\sqrt{\mathrm{trace}(\prod)}}P_\tau(\mathbb{K}\nabla\phi_h).$$

Based on the compatibility conditions above and the backward Euler scheme in temporal discretization, now we present the following full discretization for the parallel, non-iterative, multi-physics domain decomposition method: at the nth ($n=0,1,2,\cdots,N-1$) time iteration step:

1. compute

$$\xi_D^n=\vec{u}_h^n\cdot\vec{n}_S+g\phi_h^n,\quad \xi_S^n=\vec{u}_h^n\cdot\vec{n}_S-g\phi_h^n,\quad \xi_{S\tau}^n=\frac{\alpha\nu\sqrt{d}}{\sqrt{\mathrm{trace}(\prod)}}P_\tau(\mathbb{K}\nabla\phi_h^n) \tag{29}$$

   by using the initial conditions $\phi_h^0=P_h\phi_0$ and $\vec{u}_h^0=\mathbb{P}_h\vec{u}_0$, and the numerical solutions ϕ_h^n and $\vec{u}_h^n$ at t_n.

2. independently solve

$$\eta(\frac{\phi_h^{n+1}-\phi_h^n}{\triangle t},\psi_h)_{\Omega_D}+\frac{\eta}{S}a_D(\phi_h^{n+1},\psi_h)+\frac{\eta}{S}\langle g\phi_h^{n+1},\psi_h\rangle$$
$$=\frac{\eta}{S}(f_D^{n+1},\psi_h)_{\Omega_D}+\frac{\eta}{S}\langle\xi_D^n,\psi_h\rangle,\ \forall\,\psi_h\in X_{Dh} \tag{30}$$
$$(\frac{\vec{u}_h^{n+1}-\vec{u}_h^{n}}{\triangle t},\vec{v}_h)_{\Omega_S}+C_S(\vec{u}_h^{n+1},\vec{u}_h^{n+1},\vec{v}_h)+a_S(\vec{u}_h^{n+1},\vec{v}_h)+b_S(\vec{v}_h,p_h^{n+1})$$
$$+\langle\vec{u}_h^{n+1}\cdot\vec{n}_S,\vec{v}_h\cdot\vec{n}_S\rangle+\frac{\alpha\nu\sqrt{d}}{\sqrt{\mathrm{trace}(\prod)}}\langle P_\tau\vec{u}_h^{n+1},P_\tau\vec{v}_h\rangle$$
$$=(\vec{f}_S^{n+1},\vec{v}_h)_{\Omega_S}+\langle\xi_S^n,\vec{v}_h\cdot\vec{n}_S\rangle-\langle\xi_{S\tau}^n,P_\tau\vec{v}_h\rangle,\ \forall\,\vec{v}_h\in X_{Sh} \tag{31}$$
$$b_S(\vec{u}_h^{n+1},q_h)=0,\ \forall\,q_h\in Q_{Sh}, \tag{32}$$

for ϕ_h^{n+1}, $\vec{u}_h^{n+1}$ and p_h^{n+1}.

Again, this method is non-iterative for the domain decomposition even though iterations are still needed for the temporal discretization and Newton's method to handle time-dependence and nonlinearity.

4. Stability and convergence analysis for the decoupled system

In this section, we will analyze the convergence for the parallel, non-iterative, multi-physics domain decomposition method proposed above. The major difficulty is to bound the terms arising from the nonlinear advection and the BJ interface condition.

In order to deal with the nonlinear terms, we recall the following inequalities [10,63]: there exists constants C_1 and C_2 depending only on Ω_S, such that

$$|v|\le C_1\|\nabla v\|_0,\qquad \|v\|_{L^4}\le C_2|v|, \tag{33}$$

where $|\cdot|$ denote the semi-norm of space $H^1(0,T;X_S,X_S')$ and $v\in X_S$. Based on the work in [10,63], we have the following lemma.

Lemma 1. *Assume that both $\vec{u}_S$ and $\vec{u}_h$ satisfy the following smallness condition*

$$\|\nabla\vec{u}\|_{L^2}<\frac{\nu}{8C_1^3C_2^2}\ \forall\,t\in[0,T]. \tag{34}$$

Then, we have the estimate

$$|((\vec{u}\cdot\nabla)v,\omega)|\le\frac{\nu}{8}\|\nabla v\|_0\|\nabla\omega\|_0\ \forall\,v,w\in X_S. \tag{35}$$

Proof. By using *Hölder's* inequality and (33), we have

$$|((\vec{u}\cdot\nabla)v,\omega)|\le\|\vec{u}\|_{L^4}|v|\|\omega\|_{L^4}\le C_2^2|\vec{u}||v||\omega|\le C_1^3C_2^2\|\nabla\vec{u}\|_0\|\nabla v\|_0\|\nabla\omega\|_0\le\frac{\nu}{8}\|\nabla v\|_0\|\nabla\omega\|_0.$$

4.1. Stability analysis for the decoupled system

In this section, we will analyze the stability for the parallel, non-iterative, multi-physics domain decomposition method proposed above. Let C_p, C_k and C_t denote the constants from Poincaré inequality, Korn's inequality and trace inequality, respectively. We have the stability theorem as follows.

Theorem 1. *Assume that α is small enough such that $C_t^4\alpha^2g^2k\le C_k^2\nu^2$ which implies $(\frac{C_t^4\alpha^2g^2k}{C_k\nu}+\frac{C_k\nu}{2})\frac{8\nu}{3gk}\le\frac{4C_k\nu^2-2\nu^2}{gk}$, the scaling parameter η is selected to satisfy $(\frac{C_t^4\alpha^2g^2k}{C_k\nu}+\frac{C_k\nu}{2})\frac{8\nu}{3gk}\le\eta\le\frac{4C_k\nu^2-2\nu^2}{gk}$ and the time step size satisfies the following condition:*

$$[\frac{(2g^4+4g^2+4+2\eta)\nu C_t^4}{gk}+\frac{(3\eta+g^4)C_t^4}{\eta C_k\nu}]\Delta t<1.$$

Then we have

$$
\begin{aligned}
&\|\overrightarrow{u}_h^N\|_0^2+\eta\|\phi_h^N\|_0^2+\sum_{n=0}^{N-1}(\|\overrightarrow{u}_h^{n+1}-\overrightarrow{u}_h^n\|_0^2+\eta\|\phi_h^{n+1}-\phi_h^n\|_0^2)\\
&+\frac{(4C_k-1)\nu}{4}\Delta t\|\nabla\overrightarrow{u}_h^N\|_0^2+\frac{\eta gk}{2\nu}\Delta t\|\nabla\phi_h^N\|_0^2\\
&\le Ce^{CT}(\|\overrightarrow{u}_h^0\|_0^2+\|\phi_h^0\|_0^2+\frac{(4C_k-1)\nu}{4}\Delta t\|\nabla\overrightarrow{u}_h^0\|_0^2+\frac{\eta gk}{2\nu}\Delta t\|\nabla\phi_h^0\|_0^2\\
&+\frac{C_p^2\Delta t}{C_k\nu}\sum_{n=0}^{N-1}\|\overrightarrow{f}_S^{n+1}\|_0^2+\frac{C_p^2\eta\nu\Delta t}{gk}\sum_{n=0}^{N-1}\|f_D^{n+1}\|_0^2).
\end{aligned}
$$

Proof. First, setting $\psi_h=2\Delta t\phi_h^{n+1}$ in (30) and substituting ξ_D^n into (30), we have

$$
\begin{aligned}
&2\eta(\phi_h^{n+1}-\phi_h^n,\phi_h^{n+1})_{\Omega_D}+2\eta\Delta t a_D(\phi_h^{n+1},\phi_h^{n+1})+2\eta\Delta t\langle g\phi_h^{n+1},\phi_h^{n+1}\rangle\\
&=2\eta\Delta t(f_D^{n+1},\phi_h^{n+1})_{\Omega_D}+2\eta\Delta t\langle\overrightarrow{u}_h^n\cdot\overrightarrow{n}_S+g\phi_h^n,\phi_h^{n+1}\rangle,
\end{aligned}\tag{36}
$$

where $S=1$ for simplifying the process. By using the fact $2(a-b,a)=|a|^2-|b|^2+|a-b|^2$, the relation $\mathbb{K}=\frac{\prod g}{\nu}$ and $\prod(\vec{x})=k(\vec{x})\mathbb{I}$ in [8,36], (36) can be rewritten as

$$
\begin{aligned}
&\eta(\|\phi_h^{n+1}\|_0^2-\|\phi_h^n\|_0^2+\|\phi_h^{n+1}-\phi_h^n\|_0^2)+\frac{2\eta gk\Delta t}{\nu}\|\nabla\phi_h^{n+1}\|_0^2+2\eta g\Delta t\|\phi_h^{n+1}\|_{0,\Gamma}^2.\\
&=2\eta\Delta t(f_D^{n+1},\phi_h^{n+1})_{\Omega_D}+2\eta\Delta t\langle\overrightarrow{u}_h^n\cdot\overrightarrow{n}_S+g\phi_h^n,\phi_h^{n+1}\rangle.
\end{aligned}\tag{37}
$$

Second, setting $\overrightarrow{v}_h=2\Delta t\overrightarrow{u}_h^{n+1}$, $q_h=2\Delta t p_h$ in (31)-(32), substituting ξ_S and $\xi_{S\tau}$ into (38) and adding the resulting equations together gives

$$
\begin{aligned}
&2(\overrightarrow{u}_h^{n+1}-\overrightarrow{u}_h^n,\overrightarrow{u}_h^{n+1})+2\Delta t C_S(\overrightarrow{u}_h^{n+1},\overrightarrow{u}_h^{n+1},\overrightarrow{u}_h^{n+1})+2\Delta t a_S(\overrightarrow{u}_h^{n+1},\overrightarrow{u}_h^{n+1})\\
&+2\Delta t\langle\overrightarrow{u}_h^{n+1}\cdot\overrightarrow{n}_S,\overrightarrow{u}_h^{n+1}\cdot\overrightarrow{n}_S\rangle+2\Delta t\beta\langle P_\tau\overrightarrow{u}_h^{n+1},P_\tau\overrightarrow{u}_h^{n+1}\rangle\\
&=2\Delta t(\overrightarrow{f}_S^{n+1},\overrightarrow{u}_h^{n+1})+2\Delta t\langle\overrightarrow{u}_h^n\cdot\overrightarrow{n}_S-g\phi_h^n,\overrightarrow{u}_h^{n+1}\cdot\overrightarrow{n}_S\rangle\\
&-2\Delta t\langle\beta P_\tau(\mathbb{K}\nabla\phi_h^n),P_\tau\overrightarrow{u}_h^{n+1}\rangle,
\end{aligned}\tag{38}
$$

where $\beta=\frac{\alpha\nu\sqrt{d}}{\sqrt{\mathrm{trace}(\prod)}}$. By using $2(a-b,a)=|a|^2-|b|^2+|a-b|^2$, we have

$$
\begin{aligned}
&\|\overrightarrow{u}_h^{n+1}\|_0^2-\|\overrightarrow{u}_h^n\|_0^2+\|\overrightarrow{u}_h^{n+1}-\overrightarrow{u}_h^n\|_0^2+2\Delta t C_S(\overrightarrow{u}_h^{n+1},\overrightarrow{u}_h^{n+1},\overrightarrow{u}_h^{n+1})\\
&+2\Delta t a_S(\overrightarrow{u}_h^{n+1},\overrightarrow{u}_h^{n+1})+2\Delta t\|\overrightarrow{u}_h^{n+1}\cdot\overrightarrow{n}_S\|_{0,\Gamma}^2+2\Delta t\beta\|P_\tau\overrightarrow{u}_h^{n+1}\|_{0,\Gamma}^2\\
&=2\Delta t(\overrightarrow{f}_S^{n+1},\overrightarrow{u}_h^{n+1})+2\Delta t\langle\overrightarrow{u}_h^n\cdot\overrightarrow{n}_S-g\phi_h^n,\overrightarrow{u}_h^{n+1}\cdot\overrightarrow{n}_S\rangle-2\Delta t\langle\beta P_\tau(\mathbb{K}\nabla\phi_h^n),P_\tau\overrightarrow{u}_h^{n+1}\rangle,
\end{aligned}\tag{39}
$$

Adding (37) with (39) and using Korn's inequality, we have

$$
\begin{aligned}
&\|\overrightarrow{u}_h^{n+1}\|_0^2-\|\overrightarrow{u}_h^n\|_0^2+\|\overrightarrow{u}_h^{n+1}-\overrightarrow{u}_h^n\|_0^2+\eta(\|\phi_h^{n+1}\|_0^2-\|\phi_h^n\|_0^2+\|\phi_h^{n+1}-\phi_h^n\|_0^2)\\
&+4C_k\nu\Delta t\|\nabla\overrightarrow{u}_h^{n+1}\|_0^2+\frac{2\eta gk\Delta t}{\nu}\|\nabla\phi_h^{n+1}\|_0^2\\
&\le-2\Delta t C_S(\overrightarrow{u}_h^{n+1},\overrightarrow{u}_h^{n+1},\overrightarrow{u}_h^{n+1})-2\Delta t\|\overrightarrow{u}_h^{n+1}\cdot\overrightarrow{n}_S\|_{0,\Gamma}^2-2\eta g\Delta t\|\phi_h^{n+1}\|_{0,\Gamma}^2\\
&-\frac{2\nu\alpha\Delta t}{\sqrt{k}}\|P_\tau\overrightarrow{u}_h^{n+1}\|_{0,\Gamma}^2+2\Delta t(\overrightarrow{f}_S^{n+1},\overrightarrow{u}_h^{n+1})+2\eta\Delta t(f_D^{n+1},\phi_h^{n+1})\\
&+2\eta\Delta t\langle\overrightarrow{u}_h^n\cdot\overrightarrow{n}_S+g\phi_h^n,\phi_h^{n+1}\rangle+2\Delta t\langle\overrightarrow{u}_h^n\cdot\overrightarrow{n}_S-g\phi_h^n,\overrightarrow{u}_h^{n+1}\cdot\overrightarrow{n}_S\rangle\\
&-2\Delta t\langle\beta P_\tau(\mathbb{K}\nabla\phi_h^n),P_\tau\overrightarrow{u}_h^{n+1}\rangle,
\end{aligned}\tag{40}
$$

where C_k is the constant from Korn's inequality.

For the force terms on the right-hand side, using Hölder, Poincaré and Young inequalities, we have

$$
\begin{aligned}
2\Delta t(\overrightarrow{f}_S^{n+1},\overrightarrow{u}_h^{n+1})+2\eta\Delta t(f_D^{n+1},\phi_h^{n+1})&\le C_k\nu\Delta t\|\nabla\overrightarrow{u}_h^{n+1}\|_0^2+\tfrac{C_p^2\Delta t}{C_k\nu}\|\overrightarrow{f}_S^{n+1}\|_0^2\\
&+\tfrac{\eta gk\Delta t}{\nu}\|\nabla\phi_h^{n+1}\|_0^2+\tfrac{C_p^2\eta\nu\Delta t}{gk}\|f_D^{n+1}\|_0^2,
\end{aligned}\tag{41}
$$

where C_p is the constant from Poincaré inequality.

For the nonlinear term, by using the Lemma 1, we have

$$2\Delta t C_S(\overrightarrow{u}_h^{n+1}, \overrightarrow{u}_h^{n+1}, \overrightarrow{u}_h^{n+1}) \le \frac{\nu\Delta t}{4}\|\nabla \overrightarrow{u}_h^{n+1}\|_0^2 \tag{42}$$

For the interface terms, by using Young's, Hölder inequalities and trace inequality, we have

$$\begin{aligned} 2\Delta t\|\overrightarrow{u}_h^{n+1}\cdot\overrightarrow{n}_S\|_{0,\Gamma}^2 &\le 2\Delta t C_t^2\|\overrightarrow{u}_h^{n+1}\|_0\|\nabla\overrightarrow{u}_h^{n+1}\|_0 \\ &\le C_k\nu\Delta t\|\nabla\overrightarrow{u}_h^{n+1}\|_0^2 + \frac{C_t^4}{C_k\nu}\Delta t\|\overrightarrow{u}_h^{n+1}\|_0^2, \end{aligned} \tag{43}$$

$$\begin{aligned} 2\eta g\Delta t\|\phi_h^{n+1}\|_{0,\Gamma}^2 &\le 2\eta g C_t^2\Delta t\|\phi_h^{n+1}\|_0\|\nabla\phi_h^{n+1}\|_0 \\ &\le \frac{\eta g k\Delta t}{4\nu}\|\nabla\phi_h^{n+1}\|_0^2 + \frac{4\eta g\nu C_t^4}{k}\|\phi_h^{n+1}\|_0^2, \end{aligned} \tag{44}$$

$$\begin{aligned} &2\Delta t\langle\overrightarrow{u}_h^n\cdot\overrightarrow{n}_S - g\phi_h^n, \overrightarrow{u}_h^{n+1}\cdot\overrightarrow{n}_S\rangle \le \Delta t\big(\frac{1}{\epsilon_1}\|\overrightarrow{u}_h^n\|_{0,\Gamma}^2 + 2\epsilon_1\|\overrightarrow{u}_h^{n+1}\|_{0,\Gamma}^2 + \frac{g^2}{\epsilon_1}\|\phi_h^n\|_{0,\Gamma}^2\big) \\ &\le \Delta t\big(\frac{C_t^2}{\epsilon_1}\|\overrightarrow{u}_h^n\|_0\|\nabla\overrightarrow{u}_h^n\|_0 + 2\epsilon_1 C_t^2\|\overrightarrow{u}_h^{n+1}\|_0\|\nabla\overrightarrow{u}_h^{n+1}\|_0 + \frac{g^2C_t^2}{\epsilon_1}\|\phi_h^n\|_0\|\nabla\phi_h^n\|_0\big) \\ &\le \Delta t\big(\frac{\epsilon_2}{2\epsilon_1}\|\nabla\overrightarrow{u}_h^n\|_0^2 + \frac{C_t^4}{2\epsilon_1\epsilon_2}\|\overrightarrow{u}_h^n\|_0^2\big) + \Delta t\big(\epsilon_1\epsilon_2\|\nabla\overrightarrow{u}_h^{n+1}\|_0^2 + \frac{\epsilon_1 C_t^4}{\epsilon_2}\|\overrightarrow{u}_h^{n+1}\|_0^2\big) \\ &+\Delta t\big(\frac{\epsilon_2}{2\epsilon_1}\|\nabla\phi_h^n\|_0^2 + \frac{g^4C_t^4}{2\epsilon_1\epsilon_2}\|\phi_h^n\|_0^2\big) \end{aligned} \tag{45}$$

where C_t is the constant from trace inequality. Similarly, we have

$$\begin{aligned} &2\eta\Delta t\langle\overrightarrow{u}_h^n\cdot\overrightarrow{n}_S + g\phi_h^n, \phi_h^{n+1}\rangle \\ &\le \eta\Delta t\big(\frac{\epsilon_4}{2\epsilon_3}\|\nabla\overrightarrow{u}_h^n\|_0^2 + \frac{C_t^4}{2\epsilon_3\epsilon_4}\|\overrightarrow{u}_h^n\|_0^2\big) + \eta\Delta t\big(\epsilon_3\epsilon_4\|\nabla\phi_h^{n+1}\|_0^2 + \frac{\epsilon_3 C_t^4}{\epsilon_4}\|\phi_h^{n+1}\|_0^2\big) \\ &+\eta\Delta t\big(\frac{\epsilon_4}{2\epsilon_3}\|\nabla\phi_h^n\|_0^2 + \frac{g^4C_t^4}{2\epsilon_3\epsilon_4}\|\phi_h^n\|_0^2\big) \end{aligned} \tag{46}$$

For the last term in the right-hand side, by recalling [36], we have

$$\begin{aligned} 2\Delta t\langle\beta P_\tau(\mathbb{K}\nabla\phi_h^n), P_\tau\overrightarrow{u}_h^{n+1}\rangle &\le 2\alpha g\sqrt{k}\Delta t C_t^2\|\nabla\phi_h^n\|_0\|\nabla\overrightarrow{u}_h^{n+1}\|_0 \\ &\le C_k\nu\Delta t\|\nabla\overrightarrow{u}_h^{n+1}\|_0^2 + \frac{C_t^4\alpha^2g^2k}{C_k\nu}\Delta t\|\nabla\phi_h^n\|_0^2. \end{aligned} \tag{47}$$

Combining (41)-(47) with (40), we obtain

$$\begin{aligned} &\|\overrightarrow{u}_h^{n+1}\|_0^2 - \|\overrightarrow{u}_h^n\|_0^2 + \|\overrightarrow{u}_h^{n+1} - \overrightarrow{u}_h^n\|_0^2 + \eta(\|\phi_h^{n+1}\|_0^2 - \|\phi_h^n\|_0^2 + \|\phi_h^{n+1} - \phi_h^n\|_0^2) \\ &+(\frac{3}{2}C_k\nu - \frac{\nu}{4} - \epsilon_1\epsilon_2)\Delta t\|\nabla\overrightarrow{u}_h^{n+1}\|_0^2 - (\frac{\epsilon_2}{2\epsilon_1} + \eta\frac{\epsilon_4}{2\epsilon_3})\Delta t\|\nabla\overrightarrow{u}_h^n\|_0^2 \\ &+(\frac{3\eta gk}{4\nu} - \eta\epsilon_3\epsilon_4)\Delta t\|\nabla\phi_h^{n+1}\|_0^2 - (\frac{\epsilon_2}{2\epsilon_1} + \eta\frac{\epsilon_4}{2\epsilon_3} + \frac{C_t^4\alpha^2g^2k}{C_k\nu})\Delta t\|\nabla\phi_h^n\|_0^2 \\ &\le \frac{C_p^2\Delta t}{C_k\nu}\|\overrightarrow{f}_S^{n+1}\|_0^2 + \frac{C_p^2\eta\nu\Delta t}{gk}\|f_D^{n+1}\|_0^2 + (\frac{C_t^4}{C_k\nu} + \frac{\epsilon_1 C_t^4}{\epsilon_2})\Delta t\|\overrightarrow{u}_h^{n+1}\|_0^2 \\ &+(\frac{4\eta g\nu C_t^4}{k} + \frac{\eta\epsilon_3 C_t^4}{\epsilon_4})\Delta t\|\phi_h^{n+1}\|_0^2 + (\frac{C_t^4}{2\epsilon_1\epsilon_2} + \frac{\eta C_t^4}{2\epsilon_3\epsilon_4})\Delta t\|\overrightarrow{u}_h^n\|_0^2 + (\frac{g^4C_t^4}{2\epsilon_1\epsilon_2} + \frac{\eta g^4C_t^4}{2\epsilon_3\epsilon_4})\Delta t\|\phi_h^n\|_0^2, \end{aligned} \tag{48}$$

Then, by choosing appropriate ϵ_i ($\epsilon_1 = 1/\sqrt{2}, \epsilon_2 = C_k\nu/\sqrt{2}, \epsilon_3 = 1$, and $\epsilon_4 = gk/4\nu$), we can obtain

$$\begin{aligned} &\|\overrightarrow{u}_h^{n+1}\|_0^2 - \|\overrightarrow{u}_h^n\|_0^2 + \|\overrightarrow{u}_h^{n+1} - \overrightarrow{u}_h^n\|_0^2 + \eta(\|\phi_h^{n+1}\|_0^2 - \|\phi_h^n\|_0^2 + \|\phi_h^{n+1} - \phi_h^n\|_0^2) \\ &+(C_k\nu - \frac{\nu}{4})\Delta t\|\nabla\overrightarrow{u}_h^{n+1}\|_0^2 - (\frac{C_k\nu}{2} + \eta\frac{gk}{8\nu})\Delta t\|\nabla\overrightarrow{u}_h^n\|_0^2 + (\frac{3\eta gk}{4\nu} - \eta\frac{gk}{4\nu})\Delta t\|\nabla\phi_h^{n+1}\|_0^2 \\ &-(\frac{C_k\nu}{2} + \eta\frac{gk}{8\nu} + \frac{C_t^4\alpha^2g^2k}{C_k\nu})\Delta t\|\nabla\phi_h^n\|_0^2 \end{aligned}$$

$$\leq \frac{C_p^2\Delta t}{C_k\nu}\|\overrightarrow{f}_S^{n+1}\|_0^2+\frac{C_p^2\eta\nu\Delta t}{gk}\|f_D^{n+1}\|_0^2+(\frac{C_t^4}{C_k\nu}+\frac{C_t^4}{C_k\nu})\Delta t\|\overrightarrow{u}_h^{n+1}\|_0^2$$
$$+(\frac{4\eta g\nu C_t^4}{k}+\frac{4\eta\nu C_t^4}{gk})\Delta t\|\phi_h^{n+1}\|_0^2+(\frac{C_t^4}{C_k\nu}+\frac{2\eta\nu C_t^4}{gk})\Delta t\|\overrightarrow{u}_h^n\|_0^2$$
$$+(\frac{g^4C_t^4}{C_k\nu}+\frac{2\eta g^3\nu C_t^4}{k})\Delta t\|\phi_h^n\|_0^2. \tag{49}$$

In order to use discrete Gronwall's inequality, we need $C_k\nu-\frac{\nu}{4}\geq\frac{C_k\nu}{2}+\eta\frac{gk}{8\nu}$ and $\frac{3\eta gk}{4\nu}-\eta\frac{gk}{4\nu}\geq\frac{C_k\nu}{2}+\eta\frac{gk}{8\nu}+\frac{C_t^4\alpha^2g^2k}{C_k\nu}$ which lead to

$$(\frac{C_t^4\alpha^2g^2k}{C_k\nu}+\frac{C_k\nu}{2})\frac{8\nu}{3gk}\leq\eta\leq\frac{4C_k\nu^2-2\nu^2}{gk}. \tag{50}$$

In order to make sure the feasibility of (50), we assume α is small enough such that $C_t^4\alpha^2g^2k\leq C_k^2\nu^2$ which implies $(\frac{C_t^4\alpha^2g^2k}{C_k\nu}+\frac{C_k\nu}{2})\frac{8\nu}{3gk}\leq\frac{4C_k\nu^2-2\nu^2}{gk}$. Then η is selected to satisfy (50). We also denote $\tilde{C}=\frac{(2g^4+4g^2+4+2\eta)\nu C_t^4}{gk}+\frac{(3\eta+g^4)C_t^4}{\eta C_k\nu}$, then sum (49) from $n=0$ to $N-1$ such that

$$\|\overrightarrow{u}_h^N\|_0^2+\eta\|\phi_h^N\|_0^2+\sum_{n=0}^{N-1}(\|\overrightarrow{u}_h^{n+1}-\overrightarrow{u}_h^n\|_0^2+\eta\|\phi_h^{n+1}-\phi_h^n\|_0^2)$$
$$+(C_k\nu-\frac{\nu}{4})\Delta t\|\nabla\overrightarrow{u}_h^N\|_0^2+\frac{\eta gk}{2\nu}\Delta t\|\nabla\phi_h^N\|_0^2$$
$$\leq\tilde{C}\Delta t\sum_{n=0}^{N}(\|\overrightarrow{u}_h^n\|_0^2+\eta\|\phi_h^n\|_0^2)+\|\overrightarrow{u}_h^0\|_0^2+\eta\|\phi_h^0\|_0^2+(C_k\nu-\frac{\nu}{4})\Delta t\|\nabla\overrightarrow{u}_h^0\|_0^2$$
$$+\frac{\eta gk}{2\nu}\Delta t\|\nabla\phi_h^0\|_0^2+\frac{C_p^2\Delta t}{C_k\nu}\sum_{n=0}^{N-1}\|\overrightarrow{f}_S^{n+1}\|_0^2+\frac{C_p^2\eta\nu\Delta t}{gk}\sum_{n=0}^{N-1}\|f_D^{n+1}\|_0^2 \tag{51}$$

It follows from discrete Gronwall's inequality that when $\tilde{C}\Delta t<1$,

$$\|\overrightarrow{u}_h^N\|_0^2+\eta\|\phi_h^N\|_0^2+\sum_{n=0}^{N-1}(\|\overrightarrow{u}_h^{n+1}-\overrightarrow{u}_h^n\|_0^2+\eta\|\phi_h^{n+1}-\phi_h^n\|_0^2)$$
$$+\frac{(4C_k-1)\nu}{4}\Delta t\|\nabla\overrightarrow{u}_h^N\|_0^2+\frac{\eta gk}{2\nu}\Delta t\|\nabla\phi_h^N\|_0^2$$
$$\leq Ce^{CT}\Big[\|\overrightarrow{u}_h^0\|_0^2+\eta\|\phi_h^0\|_0^2+\frac{(4C_k-1)\nu}{4}\Delta t\|\nabla\overrightarrow{u}_h^0\|_0^2+\frac{\eta gk}{2\nu}\Delta t\|\nabla\phi_h^0\|_0^2$$
$$+\frac{C_p^2\Delta t}{C_k\nu}\sum_{n=0}^{N-1}\|\overrightarrow{f}_S^{n+1}\|_0^2+\frac{C_p^2\eta\nu\Delta t}{gk}\sum_{n=0}^{N-1}\|f_D^{n+1}\|_0^2\Big]. \tag{52}$$

4.2. Convergence analysis for the semi-discrete solution

We will follow the well-known framework of energy method to analyze the convergence of the semi-discrete solution [75–77]. The major difficulties in the analysis are caused by the nonlinear advection and the Beavers-Joseph interface condition. Let C be a generic constant independent of h and $\triangle t$, whose value might be different from line to line.

Assume X_{Dh} and X_{Sh} consist of piecewise polynomial of degree k and Q_{Sh} consists of piecewise polynomial of degree $k-1$. For the analysis in the Navier-Stokes-Darcy system, we introduce the projection operator $\mathbb{P}=(\mathbb{P}_1,\mathbb{P}_2,\mathbb{P}_3):X_D\times X_S\times Q_S\to X_{Dh}\times X_{Sh}\times Q_{Sh}$ such that for any $\phi\in X_D$, $\overrightarrow{u}\in X_S$, $p\in Q_S$, and a rescaling constant η, for $\forall\ \psi_h\in X_{Dh}, \forall\ \overrightarrow{v}_h\in X_{Sh}$, and $\forall\ q_h\in Q_{Sh}$, the projection satisfies

$$\eta a_D(\mathbb{P}_1\phi-\phi,\psi_h)+a_S(\mathbb{P}_2\overrightarrow{u}-\overrightarrow{u},\overrightarrow{v}_h)+\langle g(\mathbb{P}_1\phi-\phi),\overrightarrow{v}_h\cdot\overrightarrow{n}_S\rangle-\eta\langle(\mathbb{P}_2\overrightarrow{u}-\overrightarrow{u})\cdot\overrightarrow{n}_S,\psi_h\rangle$$
$$+\beta\langle P_\tau((\mathbb{P}_2\overrightarrow{u}-\overrightarrow{u})+\mathbb{K}\nabla(\mathbb{P}_1\phi-\phi)),P_\tau\overrightarrow{v}_h\rangle+b_S(\overrightarrow{v}_h,\mathbb{P}_3p-p)=0, \tag{53}$$
$$b_S(\mathbb{P}_2\overrightarrow{u}-\overrightarrow{u},q_h)\ =\ 0, \tag{54}$$

where $\beta=\frac{\alpha\nu\sqrt{d}}{\sqrt{\mathrm{trace}(\prod)}}$. Similar to Proposition 4.1 and Proposition 4.3 in [7], we have the following properties for the projection operator $\mathbb{P}$.

Lemma 2. *For any $\overrightarrow{u} \in X_S,\ p \in Q_S$, we have*

$$\|\mathbb{P}_1\phi - \phi\|_{L^q(0,T;H^1)} + \|\mathbb{P}_2\overrightarrow{u} - \overrightarrow{u}\|_{L^q(0,T;H^1)} + \|\mathbb{P}_3 p - p\|_{L^q(0,T;L^2)}$$
$$\le Ch^{r-1}(\|\phi\|_{L^q(0,T;H^r)} + \|\overrightarrow{u}\|_{L^q(0,T;H^r)} + \|p\|_{L^q(0,T;H^{r-1})}),\ q \ge 1 \tag{55}$$
$$\|\mathbb{P}_1\phi - \phi\|_{L^q(0,T;L^2)} + \|\mathbb{P}_2\overrightarrow{u} - \overrightarrow{u}\|_{L^q(0,T;L^2)} + h\|\mathbb{P}_3 p - p\|_{L^q(0,T;L^2)}$$
$$\le Ch^{r}(\|\phi\|_{L^q(0,T;H^r)} + \|\overrightarrow{u}\|_{L^q(0,T;H^r)} + \|p\|_{L^q(0,T;H^{r-1})}),\ q \ge 1 \tag{56}$$
$$\|\mathbb{P}_1\frac{\partial^m\phi}{\partial t^m} - \frac{\partial^m\phi}{\partial t^m}\|_{L^2(0,T;H^1)} + \|\mathbb{P}_2\frac{\partial^m\overrightarrow{u}}{\partial t^m} - \frac{\partial^m\overrightarrow{u}}{\partial t^m}\|_{L^2(0,T;H^1)} + \|\mathbb{P}_3\frac{\partial^m p}{\partial t^m} - \frac{\partial^m p}{\partial t^m}\|_{L^2(0,T;L^2)}$$
$$\le Ch^{r-1}\left(\|\phi\|_{H^m(0,T;H^r)} + \|\overrightarrow{u}\|_{H^m(0,T;H^r)} + \|p\|_{H^m(0,T;H^{r-1})}\right),\ m \ge 0. \tag{57}$$
$$\|\mathbb{P}_1\frac{\partial^m\phi}{\partial t^m} - \frac{\partial^m\phi}{\partial t^m}\|_{L^2(0,T;L^2)} + \|\mathbb{P}_2\frac{\partial^m\overrightarrow{u}}{\partial t^m} - \frac{\partial^m\overrightarrow{u}}{\partial t^m}\|_{L^2(0,T;L^2)} + h\|\mathbb{P}_3\frac{\partial^m p}{\partial t^m} - \frac{\partial^m p}{\partial t^m}\|_{L^2(0,T;L^2)}$$
$$\le Ch^{r}\left(\|\phi\|_{H^m(0,T;H^r)} + \|\overrightarrow{u}\|_{H^m(0,T;H^r)} + \|p\|_{H^m(0,T;H^{r-1})}\right),\ m \ge 0 \tag{58}$$

Then, the error estimates for the semi-discrete approximations are given as follows.

Theorem 2. *Assume that $\phi_D \in H^1(0,T;H^{r+1}(\Omega_D))$ and $\overrightarrow{u}_S \in H^1(0,T;[H^{r+1}(\Omega_S)]^d)$. Then*

$$\|\phi_h - \phi_D\|_0 + \|\overrightarrow{u}_h - \overrightarrow{u}_S\|_0 \le Ch^r\left(\|\phi_D\|_{H^1(0,T;H^{r+1}(\Omega_D))} + \|\overrightarrow{u}_S\|_{H^1(0,T;[H^{r+1}(\Omega_S)]^d)}\right),$$

where $0 < r \le k+1$. Here k is the degree of piecewise polynomial of X_{Dh} and X_{Sh}.

Proof. Taking $\psi = \psi_h \in X_{Dh}$ in (17), plugging ξ_D into (17) and subtracting (25) from (17) and setting $S = 1$ for simplification, we have

$$\eta(\frac{\partial\phi_D - \partial\phi_h}{\partial t}, \psi_h)_{\Omega_D} + \eta a_D(\phi_D - \phi_h, \psi_h) + \eta\langle g(\phi_D - \phi_h), \psi_h\rangle$$
$$= \eta\langle(\overrightarrow{u}_S - \overrightarrow{u}_h)\cdot\overrightarrow{n}_S + g(\phi_D - \phi_h), \psi_h\rangle,\ \forall\,\psi_h \in X_{Dh}. \tag{59}$$

Taking $\overrightarrow{v} = \overrightarrow{v}_h \in X_{Sh}$ and $q = q_h \in Q_{Sh}$ in (14) and (15), plugging ξ_S and $\xi_{S\tau}$ into (14), and subtracting (26)-(27) from (14)-(15), we have

$$(\frac{\partial(\overrightarrow{u}_S - \overrightarrow{u}_h)}{\partial t}, \overrightarrow{v}_h)_{\Omega_S} + C_S(\overrightarrow{u}_S - \overrightarrow{u}_h, \overrightarrow{u}_S, \overrightarrow{v}_h) + C_S(\overrightarrow{u}_h, \overrightarrow{u}_S - \overrightarrow{u}_h, \overrightarrow{v}_h)$$
$$+a_S(\overrightarrow{u}_S - \overrightarrow{u}_h, \overrightarrow{v}_h) + b_S(\overrightarrow{v}_h, p_S - p_h) - b_S(\overrightarrow{u}_S - \overrightarrow{u}_h, q_h) + \langle(\overrightarrow{u}_S - \overrightarrow{u}_h)\cdot\overrightarrow{n}_S, \overrightarrow{v}_h\cdot\overrightarrow{n}_S\rangle$$
$$+\beta\langle P_\tau\left(\overrightarrow{u}_S - \overrightarrow{u}_h\right), P_\tau\overrightarrow{v}_h\rangle$$
$$= \langle(\overrightarrow{u}_S - \overrightarrow{u}_h)\cdot\overrightarrow{n}_S - g(\phi_D - \phi_h), \overrightarrow{v}_h\cdot\overrightarrow{n}_S\rangle - \langle\beta P_\tau(\mathbb{K}\nabla(\phi_D - \phi_h)), P_\tau\overrightarrow{v}_h\rangle. \tag{60}$$

Define

$$\theta = \mathbb{P}_1\phi_D - \phi_h,\ \ \rho = \phi_D - \mathbb{P}_1\phi_D. \tag{61}$$

Then we can split the error $\phi_D - \phi_h = \theta + \rho$. Define

$$\overrightarrow{\theta}_1 = \mathbb{P}_2\overrightarrow{u}_S - \overrightarrow{u}_h,\ \ \overrightarrow{\rho}_1 = \overrightarrow{u}_S - \mathbb{P}_2\overrightarrow{u}_S,\ \ \theta_2 = \mathbb{P}_3 p_S - p_h,\ \ \rho_2 = p_S - \mathbb{P}_3 p_S. \tag{62}$$

Then $\overrightarrow{u}_S - \overrightarrow{u}_h = \overrightarrow{\theta}_1 + \overrightarrow{\rho}_1$ and $p_S - p_h = \theta_2 + \rho_2$.

Plugging (61) and (62) into (59) and (60), we have

$$(\frac{\partial(\overrightarrow{\theta}_1 + \overrightarrow{\rho}_1)}{\partial t}, \overrightarrow{v}_h)_{\Omega_S} + \eta(\frac{\partial(\theta+\rho)}{\partial t}, \psi_h)_{\Omega_D} + a_S(\overrightarrow{\theta}_1 + \overrightarrow{\rho}_1, \overrightarrow{v}_h) + \eta a_D(\theta+\rho, \psi_h)$$
$$+C_S(\overrightarrow{\theta}_1 + \overrightarrow{\rho}_1, \overrightarrow{u}_S, \overrightarrow{v}_h) + C_S(\overrightarrow{u}_h, \overrightarrow{\theta}_1 + \overrightarrow{\rho}_1, \overrightarrow{v}_h) + \eta\langle g(\theta+\rho), \psi_h\rangle$$
$$+\langle(\overrightarrow{\theta}_1 + \overrightarrow{\rho}_1)\cdot\overrightarrow{n}_S, \overrightarrow{v}_h\cdot\overrightarrow{n}_S\rangle + \beta\langle P_\tau\left(\overrightarrow{\theta}_1 + \overrightarrow{\rho}_1\right), P_\tau\overrightarrow{v}_h\rangle + b_S(\overrightarrow{v}_h, \theta_2 + \rho_2) \tag{63}$$
$$-b_S(\overrightarrow{\theta}_1 + \overrightarrow{\rho}_1, q_h)$$
$$= \eta\langle(\overrightarrow{\theta}_1 + \overrightarrow{\rho}_1)\cdot\overrightarrow{n}_S + g(\theta+\rho), \psi_h\rangle + \langle(\overrightarrow{\theta}_1 + \overrightarrow{\rho}_1)\cdot\overrightarrow{n}_S - g(\theta+\rho), \overrightarrow{v}_h\cdot\overrightarrow{n}_S\rangle$$
$$-\langle\beta P_\tau(\mathbb{K}\nabla(\theta+\rho)), P_\tau\overrightarrow{v}_h\rangle.$$

where η is the rescaling parameter. By using (53) and (54), we obtain

$$\begin{aligned}
&(\frac{\partial(\overrightarrow{\theta}_1+\overrightarrow{\rho}_1)}{\partial t},\overrightarrow{v}_h)_{\Omega_S}+\eta(\frac{\partial(\theta+\rho)}{\partial t},\psi_h)_{\Omega_D}+a_S(\overrightarrow{\theta}_1,\overrightarrow{v}_h)+\eta a_D(\theta,\psi_h)\\
&+C_S(\overrightarrow{\theta}_1+\overrightarrow{\rho}_1,\overrightarrow{u}_S,\overrightarrow{v}_h)+C_S(\overrightarrow{u}_h,\overrightarrow{\theta}_1+\overrightarrow{\rho}_1,\overrightarrow{v}_h)+\langle g\theta,\overrightarrow{v}_h\cdot\overrightarrow{n}_S\rangle\\
&-\eta\langle\overrightarrow{\theta}_1\cdot\overrightarrow{n}_S,\psi_h\rangle+\beta\langle P_\tau\overrightarrow{\theta}_1,P_\tau\overrightarrow{v}_h\rangle+b_S(\overrightarrow{v}_h,\theta_2)-b_S(\overrightarrow{\theta}_1,q_h)\\
&=-\langle\beta P_\tau(\mathbb{K}\nabla\theta,P_\tau\overrightarrow{v}_h\rangle.
\end{aligned}\tag{64}$$

Choosing $\psi_h=\theta$, $\overrightarrow{v}_h=\overrightarrow{\theta}_1$ and $q_h=\theta_2$ in (64), we can get

$$\begin{aligned}
&(\frac{\partial\overrightarrow{\theta}_1}{\partial t},\overrightarrow{\theta}_1)_{\Omega_S}+\eta(\frac{\partial\theta}{\partial t},\theta)_{\Omega_D}+a_S(\overrightarrow{\theta}_1,\overrightarrow{\theta}_1)+\eta a_D(\theta,\theta)+C_S(\overrightarrow{\theta}_1+\overrightarrow{\rho}_1,\overrightarrow{u}_S,\overrightarrow{\theta}_1)\\
&+C_S(\overrightarrow{u}_h,\overrightarrow{\theta}_1+\overrightarrow{\rho}_1,\overrightarrow{\theta}_1)+\langle g\theta,\overrightarrow{\theta}_1\cdot\overrightarrow{n}_S\rangle-\eta\langle\overrightarrow{\theta}_1\cdot\overrightarrow{n}_S,\theta\rangle+\beta\langle P_\tau(\overrightarrow{\theta}_1+\mathbb{K}\nabla\theta),P_\tau\overrightarrow{\theta}_1\rangle\\
&=-(\frac{\partial\overrightarrow{\rho}_1}{\partial t},\overrightarrow{\theta}_1)_{\Omega_S}-\eta(\frac{\partial\rho}{\partial t},\theta)_{\Omega_D}.
\end{aligned}$$

Since the estimates of ρ and $\overrightarrow{\rho}_1$ are given by Lemma 2, θ and $\overrightarrow{\theta}_1$ are the main objects of the analysis. Hence,

$$\begin{aligned}
&\frac{\eta}{2}\frac{d\|\theta\|_0^2}{dt}+\frac{1}{2}\frac{d\|\overrightarrow{\theta}_1\|_0^2}{dt}+\eta a_D(\theta,\theta)+a_S(\overrightarrow{\theta}_1,\overrightarrow{\theta}_1)-\eta\langle\overrightarrow{\theta}_1\cdot\overrightarrow{n}_S,\theta\rangle+\langle g\theta,\overrightarrow{\theta}_1\cdot\overrightarrow{n}_S\rangle\\
&+\beta\langle P_\tau(\overrightarrow{\theta}_1+\mathbb{K}\nabla\theta),P_\tau\overrightarrow{\theta}_1\rangle\\
&=-C_S(\overrightarrow{\theta}_1+\overrightarrow{\rho}_1,\overrightarrow{u}_S,\overrightarrow{\theta}_1)-C_S(\overrightarrow{u}_h,\overrightarrow{\theta}_1+\overrightarrow{\rho}_1,\overrightarrow{\theta}_1)-\eta(\frac{\partial\rho}{\partial t},\theta)_{\Omega_D}-(\frac{\partial\overrightarrow{\rho}_1}{\partial t},\overrightarrow{\theta}_1)_{\Omega_S}\\
&\le|C_S(\overrightarrow{\theta}_1+\overrightarrow{\rho}_1,\overrightarrow{u}_S,\overrightarrow{\theta}_1)|+|C_S(\overrightarrow{u}_h,\overrightarrow{\theta}_1+\overrightarrow{\rho}_1,\overrightarrow{\theta}_1)|+\eta\|\frac{\partial\rho}{\partial t}\|_0\|\theta\|_0+\|\frac{\partial\overrightarrow{\rho}_1}{\partial t}\|_0\|\overrightarrow{\theta}_1\|_0
\end{aligned}\tag{65}$$

By using the (34) and (35), the Holder inequality and the Young's inequality, we can obtain the following conclusions for the nonlinear terms:

$$\begin{aligned}
|C_S(\overrightarrow{\rho}_1,\overrightarrow{u}_S,\overrightarrow{\theta}_1)+C_S(\overrightarrow{u}_h,\overrightarrow{\rho}_1,\overrightarrow{\theta}_1)|&\le\frac{\nu}{8}\|\nabla\overrightarrow{\rho}_1\|_0\|\nabla\overrightarrow{\theta}_1\|_0+\frac{\nu}{8}\|\nabla\overrightarrow{\rho}_1\|_0\|\nabla\overrightarrow{\theta}_1\|_0\\
&=\frac{\nu}{4}\|\nabla\overrightarrow{\rho}_1\|_0\|\nabla\overrightarrow{\theta}_1\|_0\\
&\le\frac{\nu}{8}\|\nabla\overrightarrow{\theta}_1\|_0^2+\frac{\nu}{8}\|\nabla\overrightarrow{\rho}_1\|_0^2,
\end{aligned}\tag{66}$$

$$\begin{aligned}
|C_S(\overrightarrow{\theta}_1,\overrightarrow{u}_S,\overrightarrow{\theta}_1)+C_S(\overrightarrow{u}_h,\overrightarrow{\theta}_1,\overrightarrow{\theta}_1)|&\le\frac{\nu}{8}\|\nabla\overrightarrow{\theta}_1\|_0^2+\frac{\nu}{8}\|\nabla\overrightarrow{\theta}_1\|_0^2\\
&=\frac{\nu}{4}\|\nabla\overrightarrow{\theta}_1\|_0^2.
\end{aligned}\tag{67}$$

By plugging (66)-(67) into (65) and using the Young's inequality and Poincaré inequality, we have

$$\begin{aligned}
&\eta\frac{d\|\theta\|_0^2}{dt}+\frac{d\|\overrightarrow{\theta}_1\|_0^2}{dt}+2\eta a_D(\theta,\theta)+2a_S(\overrightarrow{\theta}_1,\overrightarrow{\theta}_1)-2\eta\langle\overrightarrow{\theta}_1\cdot\overrightarrow{n}_S,\theta\rangle+2\langle g\theta,\overrightarrow{\theta}_1\cdot\overrightarrow{n}_S\rangle\\
&+2\beta\langle P_\tau(\overrightarrow{\theta}_1+\mathbb{K}\nabla\theta),P_\tau\overrightarrow{\theta}_1\rangle\\
&\le\frac{3\nu}{4}\|\nabla\overrightarrow{\theta}_1\|_0^2+\eta\|\theta\|_0^2+\|\overrightarrow{\theta}_1\|_0^2+\frac{\nu}{4}\|\nabla\overrightarrow{\rho}_1\|_0^2+\eta\|\frac{\partial\rho}{\partial t}\|_0^2+\|\frac{\partial\overrightarrow{\rho}_1}{\partial t}\|_0^2
\end{aligned}$$

By (4.2) in [8], we can obtain the following conclusion to treat the BJ interface condition for small enough ν and large enough η:

$$\begin{aligned}
&2\eta a_D(\theta,\theta)+2a_S(\overrightarrow{\theta}_1,\overrightarrow{\theta}_1)-2\eta\langle\overrightarrow{\theta}_1\cdot\overrightarrow{n}_S,\theta\rangle+2\langle g\theta,\overrightarrow{\theta}_1\cdot\overrightarrow{n}_S\rangle\\
&+2\beta\langle P_\tau(\overrightarrow{\theta}_1+\mathbb{K}\nabla\theta),P_\tau\overrightarrow{\theta}_1\rangle\\
&\ge\frac{3\nu}{4}\|\nabla\overrightarrow{\theta}_1\|_0^2-C_3\|\theta\|_0^2-C_3\|\overrightarrow{\theta}_1\|_0^2
\end{aligned}$$

where $C_3>0$ is a constant. Hence,

$$\eta\frac{d\|\theta\|_0^2}{dt}+\frac{d\|\overrightarrow{\theta}_1\|_0^2}{dt}$$
$$\leq C(\eta\|\theta\|_0^2+\|\overrightarrow{\theta}_1\|_0^2+\|\frac{\partial\rho}{\partial t}\|_0^2+\|\frac{\partial\overrightarrow{\rho}_1}{\partial t}\|_0^2+\|\nabla\overrightarrow{\rho}_1\|_0^2). \tag{68}$$

Integrating (68) from 0 to t and apply Gronwall's inequality, we get

$$\eta\|\theta(t)\|_0^2+\|\overrightarrow{\theta}_1(t)\|_0^2$$
$$\leq C\Big[\eta\|\theta(0)\|_0^2+\|\overrightarrow{\theta}_1(0)\|_0^2+\int_0^t\left(\|\frac{\partial\rho}{\partial t}\|_0^2+\|\frac{\partial\overrightarrow{\rho}_1}{\partial t}\|_0^2+\|\nabla\overrightarrow{\rho}_1\|_0^2\right)ds\Big]$$

Then by Lemma 2, we finish the proof.

4.3. Convergence analysis of the fully discrete approximate solution

The following theorem states that the first parallel non-iterative domain decomposition method is unconditionally stable and has optimal rates of convergence.

Theorem 3. *If $\phi_D\in H^1(0,T;H^2(\Omega_D))\cap L^\infty(0,T;H^2(\Omega_D))\cap H^2(0,T;L^2(\Omega_D))$, $\overrightarrow{u}_S\in H^1(0,T;H^2(\Omega_S))\cap L^\infty(0,T;H^2(\Omega_S))\cap H^2(0,T;L^2(\Omega_S))$, $\xi_D\in H^1(0,T;L^2(\Gamma))$, and $\xi_S\in H^1(0,T;L^2(\Gamma))$, then*

$$\|\phi_h^n-\phi_D(t_n)\|_0+\|\overrightarrow{u}_h^n-\overrightarrow{u}_S(t_n)\|_0$$
$$\leq Ce^{CT}\triangle t\Big[\int_0^{t_n}\|\frac{\partial^2\phi_D}{\partial t^2}\|_0\,dt+\int_0^{t_n}\|\frac{\partial\xi_D}{\partial t}\|_{0,\Gamma}\,dt$$
$$+\int_0^{t_n}\|\frac{\partial^2\overrightarrow{u}_S}{\partial t^2}\|_0\,dt+\int_0^{t_n}\|\frac{\partial\xi_S}{\partial t}\|_{0,\Gamma}\,dt+\int_0^{t_n}\|\frac{\partial\phi_D}{\partial t}\|_1\,dt\Big]$$
$$+Ce^{CT}h^2\Big[\int_0^{t_n}\|\frac{\partial\phi_D}{\partial t}\|_2\,dt+\int_0^{t_n}\|\frac{\partial\overrightarrow{u}_S}{\partial t}\|_2\,dt$$
$$+\max_{0\leq s\leq t_n}\|\phi_D(s)\|_2+\max_{0\leq s\leq t_n}\left(\|\overrightarrow{u}_S(s)\|_2+\|p_S(s)\|_1\right)\Big]. \tag{69}$$

Proof. We follow the standard energy method framework [76,75,77] to analyze the error of fully discrete approximations. For the Darcy part, taking $\psi=\psi_h\in X_{Dh}$ in (17) and subtracting (17) from (30), we have

$$\eta(\frac{\phi_h^{n+1}-\phi_h^n}{\triangle t}-\frac{\partial\phi_D(t_{n+1})}{\partial t},\psi_h)_{\Omega_D}+\eta a_D(\phi_h^{n+1}-\phi_D(t_{n+1}),\psi_h)$$
$$+\eta\langle g(\phi_h^{n+1}-\phi_D(t_{n+1})),\psi_h\rangle\ =\ \eta\langle\xi_D^n-\xi_D(t_{n+1}),\psi_h\rangle\quad\forall\,\psi_h\in X_{Dh}. \tag{70}$$

We can define θ^n and ρ^n as follows,

$$\theta^n=\phi_h^n-\mathbb{P}_1\phi_D(t_n)\qquad\text{and}\qquad\rho^n=\mathbb{P}_1\phi_D(t_n)-\phi_D(t_n), \tag{71}$$

so we can get $\phi_h^n-\phi_D(t_n)=\theta^n+\rho^n$. Here, ρ^n is bounded because of Lemma 2 and we can get the similar estimates like (6.29) in [52].

$$\|\rho^n\|_0\leq Ch^2\|\phi_D(t_n)\|_2. \tag{72}$$

Define

$$\overrightarrow{\theta}_1^n=\overrightarrow{u}_h^n-\mathbb{P}_2\overrightarrow{u}_S(t_n),\qquad\overrightarrow{\rho}_1^n=\mathbb{P}_2\overrightarrow{u}_S(t_n)-\overrightarrow{u}_S(t_n), \tag{73}$$
$$\theta_2^n=p_h^n-\mathbb{P}_3p_S(t_n),\qquad\rho_2^n=\mathbb{P}_3p_S(t_n)-p_S(t_n), \tag{74}$$

then $\overrightarrow{u}_h^n-\overrightarrow{u}_S(t_n)=\overrightarrow{\theta}_1^n+\overrightarrow{\rho}_1^n$ and $p_h^n-p_S(t_n)=\theta_2^n+\rho_2^n$. From (6.35)-(6.36) in [52], we can get the estimates about $\overrightarrow{\rho}_1^n$ and ρ_2^n.

$$\|\vec{\rho}_1^{\,n}\|_0 + h\|\vec{\rho}_1^{\,n}\|_1 \le Ch^2(\|\vec{u}_S(t_n)\|_2 + \|p_S(t_n)\|_1) \tag{75}$$

$$\|\rho_2^n\|_0 \le Ch^2(\|\vec{u}_S(t_n)\|_2 + \|p_S(t_n)\|_1). \tag{76}$$

Also, we have the following relations for the approximations of the coupling functions. Subtracting ξ_D in (21) from (29), we have

$$\xi_D^n - \xi_D(t_n) = \left(\vec{\theta}_1^{\,n} + \vec{\rho}_1^{\,n}\right)\cdot \vec{n}_S + g\left(\theta^n + \rho^n\right). \tag{77}$$

Define

$$w_1^{n+1} = \mathbb{P}_1\left(\frac{\phi_D(t_{n+1}) - \phi_D(t_n)}{\triangle t}\right) - \frac{\partial\phi_D(t_{n+1})}{\partial t},\ \ w_2^{n+1} = \xi_D(t_{n+1}) - \xi_D(t_n) \tag{78}$$

Then, (70) becomes

$$\begin{aligned}&\eta(\frac{\theta^{n+1}-\theta^n}{\triangle t},\psi_h)_{\Omega_D} + \eta a_D(\theta^{n+1}+\rho^{n+1},\psi_h) + \eta\langle g(\theta^{n+1}+\rho^{n+1}),\psi_h\rangle\\&= -\eta(w_1^{n+1},\psi_h) + \eta\langle(\vec{\theta}_1^{\,n} + \vec{\rho}_1^{\,n})\cdot\vec{n}_S + g(\theta^n+\rho^n),\psi_h\rangle - \eta\langle w_2^{n+1},\psi_h\rangle,\ \ \forall\psi_h\in X_{Dh}.\end{aligned} \tag{79}$$

For the Navier-Stokes part, choosing $\vec{v} = \vec{v}_h \in X_{Sh}$ in (14) and $q = q_h \in Q_{Sh}$ in (15), then subtracting (14) and (15) from (31) and (32) separately, we obtain

$$\begin{aligned}&(\frac{\vec{u}_h^{\,n+1}-\vec{u}_h^{\,n}}{\triangle t} - \frac{\partial\vec{u}_S(t_{n+1})}{\partial t},\vec{v}_h)_{\Omega_S} + C_S(\vec{u}_h^{\,n+1} - \vec{u}_S(t_{n+1}),\vec{u}_h^{\,n+1} - \vec{u}_S(t_{n+1}),\vec{v}_h)\\&+a_S(\vec{u}_h^{\,n+1} - \vec{u}_S(t_{n+1}),\vec{v}_h) + b_S(\vec{v}_h, p_h^{n+1} - p_S(t_{n+1}))\\&+\langle\left(\vec{u}_h^{\,n+1} - \vec{u}_S(t_{n+1})\right)\cdot\vec{n}_S,\vec{v}_h\cdot\vec{n}_S\rangle + \beta\langle P_\tau\left(\vec{u}_h^{\,n+1} - \vec{u}_S(t_{n+1})\right),P_\tau\vec{v}_h\rangle\\&= \langle\xi_S^n - \xi_S(t_{n+1}),\vec{v}_h\cdot\vec{n}_S\rangle - \langle\xi_{S\tau}^n - \xi_{S\tau}(t_{n+1}),P_\tau\vec{v}_h\rangle\end{aligned} \tag{80}$$

$$b_S(\vec{u}_h^{\,n+1} - \vec{u}_S(t_{n+1}),q_h) = 0. \tag{81}$$

Subtracting ξ_S in (21) from (29), we have

$$\xi_{Sh}^n - \xi_S(t_n) = \left(\vec{\theta}_1^{\,n} + \vec{\rho}_1^{\,n}\right)\cdot\vec{n}_S - g\left(\theta^n+\rho^n\right). \tag{82}$$

For the special treatment for BJ interface condition, which is one of the major difficulties, we have

$$\begin{aligned}&\xi_{S\tau h}^n - \xi_{S\tau}(t_{n+1})\\&= \beta P_\tau\left(\mathbb{K}\nabla\phi_h^n\right) - \beta P_\tau\left(\mathbb{K}\nabla\phi_D(t_{n+1})\right)\\&= \beta P_\tau\left(\mathbb{K}\nabla\left(\phi_h^n - \phi_D(t_{n+1})\right)\right)\\&= \beta P_\tau\left(\mathbb{K}\nabla\left(\phi_h^n - \mathbb{P}_1\phi_D(t_n) + \mathbb{P}_1\phi_D(t_{n+1}) - \phi_D(t_{n+1}) - \mathbb{P}_1\phi_D(t_{n+1}) + \mathbb{P}_1\phi_D(t_n)\right)\right)\\&= \beta P_\tau\left(\mathbb{K}\nabla(\theta^n + \rho^{n+1} - (\mathbb{P}_1\phi_D(t_{n+1}) - \mathbb{P}_1\phi_D(t_n))\right).\end{aligned} \tag{83}$$

We can define the $\vec{w}_3^{\,n+1}$, w_4^{n+1} as follows

$$\vec{w}_3^{\,n+1} = \mathbb{P}_2\left(\frac{\vec{u}_S(t_{n+1}) - \vec{u}_S(t_n)}{\triangle t}\right) - \frac{\partial\vec{u}_S(t_{n+1})}{\partial t} \tag{84}$$

$$w_4^{n+1} = \xi_S(t_{n+1}) - \xi_S(t_n). \tag{85}$$

Then, (80) and (81) becomes

$$\begin{aligned}&(\frac{\vec{\theta}_1^{\,n+1} - \vec{\theta}_1^{\,n}}{\triangle t},\vec{v}_h)_{\Omega_S} + C_S(\vec{\theta}_1^{\,n+1} + \vec{\rho}_1^{\,n+1},\vec{u}_h^{\,n+1},\vec{\theta}_1^{\,n+1})\\&+C_S(\vec{u}_S(t_{n+1}),\vec{\theta}_1^{\,n+1} + \vec{\rho}_1^{\,n+1},\vec{\theta}_1^{\,n+1}) + a_S(\vec{\theta}_1^{\,n+1} + \vec{\rho}_1^{\,n+1},\vec{v}_h) + b_S(\vec{v}_h,\theta_2^{n+1} + \rho_2^{n+1})\\&+\langle\left(\vec{\theta}_1^{\,n+1} + \vec{\rho}_1^{\,n+1}\right)\cdot\vec{n}_S,\vec{v}_h\cdot\vec{n}_S\rangle + \beta\langle P_\tau\left(\vec{\theta}_1^{\,n+1} + \vec{\rho}_1^{\,n+1}\right),P_\tau\vec{v}_h\rangle\\&= -(\vec{w}_3^{\,n+1},\vec{v}_h) - \langle w_4^{n+1},\vec{v}_h\cdot\vec{n}_S\rangle + \langle\left(\vec{\theta}_1^{\,n} + \vec{\rho}_1^{\,n}\right)\cdot\vec{n}_S - g\left(\theta^n+\rho^n\right),\vec{v}_h\cdot\vec{n}_S\rangle\\&-\langle\beta P_\tau\left(\mathbb{K}\nabla(\theta^n + \rho^{n+1} - (\mathbb{P}_1\phi_D(t_{n+1}) - \mathbb{P}_1\phi_D(t_n))\right),P_\tau\vec{v}_h\rangle,\end{aligned} \tag{86}$$

$$b_S(\vec{\theta}_1^{\,n+1} + \vec{\rho}_1^{\,n+1},q_h) = 0. \tag{87}$$

Combining (79) and (86)-(87), we can get

$$
\begin{aligned}
&\eta(\frac{\theta^{n+1}-\theta^{n}}{\Delta t},\psi_h)_{\Omega_D}+(\frac{\vec{\theta}_1^{n+1}-\vec{\theta}_1^{n}}{\Delta t},\vec{v}_h)_{\Omega_S}+C_S(\vec{\theta}_1^{n+1}+\vec{\rho}_1^{n+1},\vec{u}_h^{n+1},\vec{\theta}_1^{n+1})\\
&+C_S(\vec{u}_S(t_{n+1}),\vec{\theta}_1^{n+1}+\vec{\rho}_1^{n+1},\vec{\theta}_1^{n+1})+\eta a_D(\theta^{n+1}+\rho^{n+1},\psi_h)+a_S(\vec{\theta}_1^{n+1}+\vec{\rho}_1^{n+1},\vec{v}_h)\\
&+b_S(\vec{v}_h,\theta_2^{n+1}+\rho_2^{n+1})+b_S(\vec{\theta}_1^{n+1}+\vec{\rho}_1^{n+1},q_h)+\eta\langle g(\theta^{n+1}+\rho^{n+1}),\psi_h\rangle\\
&+\langle\left(\vec{\theta}_1^{n+1}+\vec{\rho}_1^{n+1}\right)\cdot\vec{n}_S,\vec{v}_h\cdot\vec{n}_S\rangle+\beta\langle P_\tau\left(\vec{\theta}_1^{n+1}+\vec{\rho}_1^{n+1}\right),P_\tau\vec{v}_h\rangle\\
&=-\eta(w_1^{n+1},\psi_h)-(\vec{w}_3^{n+1},\vec{v}_h)-\langle w_4^{n+1},\vec{v}_h\cdot\vec{n}_S\rangle-\eta\langle w_2^{n+1},\psi_h\rangle\\
&+\eta\langle(\vec{\theta}_1^{n}+\vec{\rho}_1^{n})\cdot\vec{n}_S+g(\theta^n+\rho^n),\psi_h\rangle+\langle\left(\vec{\theta}_1^{n}+\vec{\rho}_1^{n}\right)\cdot\vec{n}_S-g\left(\theta^n+\rho^n\right),\vec{v}_h\cdot\vec{n}_S\rangle\\
&-\langle\beta P_\tau\mathbb{K}\nabla(\theta^n+\rho^{n+1}),P_\tau\vec{v}_h\rangle+\langle\beta P_\tau\mathbb{K}\nabla(\mathbb{P}_1\phi_D(t_{n+1})-\mathbb{P}_1\phi_D(t_n)),P_\tau\vec{v}_h\rangle.
\end{aligned}
\tag{88}
$$

Before using (53)-(54), we need to add some terms on both sides in (88) as follows.

$$
\begin{aligned}
&\eta(\frac{\theta^{n+1}-\theta^{n}}{\Delta t},\psi_h)_{\Omega_D}+(\frac{\vec{\theta}_1^{n+1}-\vec{\theta}_1^{n}}{\Delta t},\vec{v}_h)_{\Omega_S}+C_S(\vec{\theta}_1^{n+1}+\vec{\rho}_1^{n+1},\vec{u}_h^{n+1},\vec{\theta}_1^{n+1})\\
&+C_S(\vec{u}_S(t_{n+1}),\vec{\theta}_1^{n+1}+\vec{\rho}_1^{n+1},\vec{\theta}_1^{n+1})+\eta a_D(\theta^{n+1}+\rho^{n+1},\psi_h)+a_S(\vec{\theta}_1^{n+1}+\vec{\rho}_1^{n+1},\vec{v}_h)\\
&+b_S(\vec{v}_h,\theta_2^{n+1}+\rho_2^{n+1})+b_S(\vec{\theta}_1^{n+1}+\vec{\rho}_1^{n+1},q_h)+\langle g\rho^{n+1},\vec{v}_h\cdot\vec{n}_S\rangle\\
&-\eta\langle\vec{\rho}_1^{n+1}\cdot\vec{n}_S,\psi_h\rangle+\eta\langle\vec{\rho}_1^{n+1}\cdot\vec{n}_S+g(\theta^{n+1}+\rho^{n+1}),\psi_h\rangle\\
&+\langle\left(\vec{\theta}_1^{n+1}+\vec{\rho}_1^{n+1}\right)\cdot\vec{n}_S-g\rho^{n+1},\vec{v}_h\cdot\vec{n}_S\rangle+\beta\langle P_\tau\left(\vec{\theta}_1^{n+1}+\vec{\rho}_1^{n+1}\right),P_\tau\vec{v}_h\rangle\\
&=-\eta(w_1^{n+1},\psi_h)-(\vec{w}_3^{n+1},\vec{v}_h)-\langle w_4^{n+1},\vec{v}_h\cdot\vec{n}_S\rangle-\eta\langle w_2^{n+1},\psi_h\rangle\\
&+\eta\langle(\vec{\theta}_1^{n}+\vec{\rho}_1^{n})\cdot\vec{n}_S+g(\theta^n+\rho^n),\psi_h\rangle+\langle\left(\vec{\theta}_1^{n}+\vec{\rho}_1^{n}\right)\cdot\vec{n}_S-g\left(\theta^n+\rho^n\right),\vec{v}_h\cdot\vec{n}_S\rangle\\
&-\langle\beta P_\tau\mathbb{K}\nabla(\theta^n+\rho^{n+1}),P_\tau\vec{v}_h\rangle+\langle\beta P_\tau\mathbb{K}\nabla(\mathbb{P}_1\phi_D(t_{n+1})-\mathbb{P}_1\phi_D(t_n)),P_\tau\vec{v}_h\rangle.
\end{aligned}
$$

By using (53)-(54) and setting $\psi_h=\theta^{n+1}$, $\vec{v}_h=\vec{\theta}_1^{n+1}$ and $q_h=\theta_2^{n+1}$, we have

$$
\begin{aligned}
&\eta(\frac{\theta^{n+1}-\theta^{n}}{\Delta t},\theta^{n+1})_{\Omega_D}+(\frac{\vec{\theta}_1^{n+1}-\vec{\theta}_1^{n}}{\Delta t},\vec{\theta}_1^{n+1})_{\Omega_S}+C_S(\vec{\theta}_1^{n+1}+\vec{\rho}_1^{n+1},\vec{u}_h^{n+1},\vec{\theta}_1^{n+1})\\
&+C_S(\vec{u}_S(t_{n+1}),\vec{\theta}_1^{n+1}+\vec{\rho}_1^{n+1},\vec{\theta}_1^{n+1})+\eta a_D(\theta^{n+1},\theta^{n+1})+a_S(\vec{\theta}_1^{n+1},\vec{\theta}_1^{n+1})\\
&+\eta\langle\vec{\rho}_1^{n+1}\cdot\vec{n}_S+g(\theta^{n+1}+\rho^{n+1}),\theta^{n+1}\rangle+\beta\langle P_\tau\vec{\theta}_1^{n+1},P_\tau\vec{\theta}_1^{n+1}\rangle\\
&+\langle(\vec{\theta}_1^{n+1}+\vec{\rho}_1^{n+1})\cdot\vec{n}_S-g\rho^{n+1},\vec{\theta}_1^{n+1}\cdot\vec{n}_S\rangle\\
&=-\eta(w_1^{n+1},\theta^{n+1})-(\vec{w}_3^{n+1},\vec{\theta}_1^{n+1})-\langle w_4^{n+1},\vec{\theta}_1^{n+1}\cdot\vec{n}_S\rangle-\eta\langle w_2^{n+1},\theta^{n+1}\rangle\\
&+\eta\langle(\vec{\theta}_1^{n}+\vec{\rho}_1^{n})\cdot\vec{n}_S+g(\theta^n+\rho^n),\theta^{n+1}\rangle+\langle\left(\vec{\theta}_1^{n}+\vec{\rho}_1^{n}\right)\cdot\vec{n}_S-g\left(\theta^n+\rho^n\right),\vec{\theta}_1^{n+1}\cdot\vec{n}_S\rangle\\
&-\langle\beta P_\tau\mathbb{K}\nabla\theta^n,P_\tau\vec{\theta}_1^{n+1}\rangle+\langle\beta P_\tau\mathbb{K}\nabla(\mathbb{P}_1\phi_D(t_{n+1})-\mathbb{P}_1\phi_D(t_n)),P_\tau\vec{\theta}_1^{n+1}\rangle.
\end{aligned}
\tag{89}
$$

In order to make use of (4.2) in [8] to deal with the difficulty from the BJ interface condition, we need to add some terms on both sides of the above inequality to re-write it as follows.

$$
\begin{aligned}
&\eta(\frac{\theta^{n+1}-\theta^{n}}{\Delta t},\theta^{n+1})_{\Omega_D}+(\frac{\vec{\theta}_1^{n+1}-\vec{\theta}_1^{n}}{\Delta t},\vec{\theta}_1^{n+1})_{\Omega_S}+\eta a_D(\theta^{n+1},\theta^{n+1})+a_S(\vec{\theta}_1^{n+1},\vec{\theta}_1^{n+1})\\
&+\langle g\theta^{n+1},\vec{\theta}_1^{n+1}\cdot\vec{n}_S\rangle-\eta\langle\vec{\theta}_1^{n+1}\cdot\vec{n}_S,\theta^{n+1}\rangle+\beta\langle P_\tau(\vec{\theta}_1^{n+1}+\mathbb{K}\nabla\theta^{n+1}),P_\tau\vec{\theta}_1^{n+1}\rangle\\
\le&-\eta\langle\vec{\theta}_1^{n+1}\cdot\vec{n}_S+g\theta^{n+1},\theta^{n+1}\rangle-\eta\langle\vec{\rho}_1^{n+1}\cdot\vec{n}_S+g\rho^{n+1},\theta^{n+1}\rangle\\
&-\langle\vec{\theta}_1^{n+1}\cdot\vec{n}_S-g\theta^{n+1},\vec{\theta}_1^{n+1}\cdot\vec{n}_S\rangle-\langle\vec{\rho}_1^{n+1}\cdot\vec{n}_S-g\rho^{n+1},\vec{\theta}_1^{n+1}\cdot\vec{n}_S\rangle\\
&+|C_S(\vec{\theta}_1^{n+1}+\vec{\rho}_1^{n+1},\vec{u}_h^{n+1},\vec{\theta}_1^{n+1})|+|C_S(\vec{u}_S(t_{n+1}),\vec{\theta}_1^{n+1}+\vec{\rho}_1^{n+1},\vec{\theta}_1^{n+1})|\\
&-\eta(w_1^{n+1},\theta^{n+1})-\eta\langle w_2^{n+1},\theta^{n+1}\rangle-(\vec{w}_3^{n+1},\vec{\theta}_1^{n+1})-\langle w_4^{n+1},\vec{\theta}_1^{n+1}\cdot\vec{n}_S\rangle\\
&+\eta\langle\vec{\theta}_1^{n}\cdot\vec{n}_S+g\theta^n,\theta^{n+1}\rangle+\eta\langle\vec{\rho}_1^{n}\cdot\vec{n}_S+g\rho^n,\theta^{n+1}\rangle\\
&+\langle\vec{\theta}_1^{n}\cdot\vec{n}_S-g\theta^n,\vec{\theta}_1^{n+1}\cdot\vec{n}_S\rangle+\langle\vec{\rho}_1^{n}\cdot\vec{n}_S-g\rho^n,\vec{\theta}_1^{n+1}\cdot\vec{n}_S\rangle
\end{aligned}
$$

$$
\begin{aligned}
&+\beta\langle P_\tau \mathbb{K}\nabla\theta^{n+1}, P_\tau \overrightarrow{\theta}_1^{n+1}\rangle - \langle \beta P_\tau \mathbb{K}\nabla\theta^n, P_\tau \overrightarrow{\theta}_1^{n+1}\rangle \\
&+\langle \beta P_\tau \mathbb{K}\nabla(\mathbb{P}_1\phi_D(t_{n+1}) - \mathbb{P}_1\phi_D(t_n)), P_\tau \overrightarrow{\theta}_1^{n+1}\rangle.
\end{aligned}
$$

By using the Schwarz inequalities and Young inequalities, we can get

$$
\begin{aligned}
&\eta\langle \overrightarrow{\theta}_1^{n+1}\cdot\overrightarrow{n}_S + g\theta^{n+1}, \theta^{n+1}\rangle \le \eta\left[\frac{1}{\epsilon_3}\|\overrightarrow{\theta}_1^{n+1}\|_{0,\Gamma}^2 + 2\epsilon_3\|\theta^{n+1}\|_{0,\Gamma}^2 + \frac{g^2}{\epsilon_3}\|\theta^{n+1}\|_{0,\Gamma}^2\right],\\
&\eta\langle \overrightarrow{\rho}_1^{n+1}\cdot\overrightarrow{n}_S + g\rho^{n+1}, \theta^{n+1}\rangle \le \eta\left[\frac{1}{\epsilon_2}\|\overrightarrow{\rho}_1^{n+1}\|_{0,\Gamma}^2 + 2\epsilon_2\|\theta^{n+1}\|_{0,\Gamma}^2 + \frac{g^2}{\epsilon_2}\|\rho^{n+1}\|_{0,\Gamma}^2\right],\\
&\langle \overrightarrow{\theta}_1^{n+1}\cdot\overrightarrow{n}_S - g\theta^{n+1}, \overrightarrow{\theta}_1^{n+1}\cdot\overrightarrow{n}_S\rangle \le \frac{1}{\epsilon_3}\|\overrightarrow{\theta}_1^{n+1}\|_{0,\Gamma}^2 + 2\epsilon_3\|\overrightarrow{\theta}_1^{n+1}\|_{0,\Gamma}^2 + \frac{g^2}{\epsilon_3}\|\theta^{n+1}\|_{0,\Gamma}^2,\\
&\langle \overrightarrow{\rho}_1^{n+1}\cdot\overrightarrow{n}_S - g\rho^{n+1}, \overrightarrow{\theta}_1^{n+1}\cdot\overrightarrow{n}_S\rangle \le \frac{1}{\epsilon_2}\|\overrightarrow{\rho}_1^{n+1}\|_{0,\Gamma}^2 + 2\epsilon_2\|\overrightarrow{\theta}_1^{n+1}\|_{0,\Gamma}^2 + \frac{g^2}{\epsilon_2}\|\rho^{n+1}\|_{0,\Gamma}^2,\\
&\eta\langle \overrightarrow{\theta}_1^{n}\cdot\overrightarrow{n}_S + g\theta^{n}, \theta^{n+1}\rangle \le \eta\left[\frac{1}{\epsilon_3}\|\overrightarrow{\theta}_1^{n}\|_{0,\Gamma}^2 + 2\epsilon_3\|\theta^{n}\|_{0,\Gamma}^2 + \frac{g^2}{\epsilon_3}\|\theta^{n}\|_{0,\Gamma}^2\right],\\
&\eta\langle \overrightarrow{\rho}_1^{n}\cdot\overrightarrow{n}_S + g\rho^{n}, \theta^{n+1}\rangle \le \eta\left[\frac{1}{\epsilon_2}\|\overrightarrow{\rho}_1^{n}\|_{0,\Gamma}^2 + 2\epsilon_2\|\theta^{n+1}\|_{0,\Gamma}^2 + \frac{g^2}{\epsilon_2}\|\rho^{n}\|_{0,\Gamma}^2\right],\\
&\langle \overrightarrow{\theta}_1^{n}\cdot\overrightarrow{n}_S - g\theta^{n}, \overrightarrow{\theta}_1^{n+1}\cdot\overrightarrow{n}_S\rangle \le \frac{1}{\epsilon_3}\|\overrightarrow{\theta}_1^{n}\|_{0,\Gamma}^2 + 2\epsilon_3\|\overrightarrow{\theta}_1^{n+1}\|_{0,\Gamma}^2 + \frac{g^2}{\epsilon_3}\|\theta^{n}\|_{0,\Gamma}^2,\\
&\langle \overrightarrow{\rho}_1^{n}\cdot\overrightarrow{n}_S - g\rho^{n}, \overrightarrow{\theta}_1^{n+1}\cdot\overrightarrow{n}_S\rangle \le \frac{1}{\epsilon_2}\|\overrightarrow{\rho}_1^{n}\|_{0,\Gamma}^2 + 2\epsilon_2\|\overrightarrow{\theta}_1^{n+1}\|_{0,\Gamma}^2 + \frac{g^2}{\epsilon_2}\|\rho^{n+1}\|_{0,\Gamma}^2.
\end{aligned}
$$

By using the above inequalities, we have

$$
\begin{aligned}
&\eta(\frac{\theta^{n+1}-\theta^n}{\Delta t}, \theta^{n+1})_{\Omega_D} + (\frac{\overrightarrow{\theta}_1^{n+1}-\overrightarrow{\theta}_1^{n}}{\Delta t}, \overrightarrow{\theta}_1^{n+1})_{\Omega_S} + \eta a_D(\theta^{n+1},\theta^{n+1}) + a_S(\overrightarrow{\theta}_1^{n+1}, \overrightarrow{\theta}_1^{n+1})\\
&+\langle g\theta^{n+1}, \overrightarrow{\theta}_1^{n+1}\cdot\overrightarrow{n}_S\rangle - \eta\langle \overrightarrow{\theta}_1^{n+1}\cdot\overrightarrow{n}_S, \theta^{n+1}\rangle + \beta\langle P_\tau(\overrightarrow{\theta}_1^{n+1} + \mathbb{K}\nabla\theta^{n+1}), P_\tau\overrightarrow{\theta}_1^{n+1}\rangle\\
\le\, &|C_S(\overrightarrow{\theta}_1^{n+1} + \overrightarrow{\rho}_1^{n+1}, \overrightarrow{u}_h^{n+1}, \overrightarrow{\theta}_1^{n+1})| + |C_S(\overrightarrow{u}_S(t_{n+1}), \overrightarrow{\theta}_1^{n+1} + \overrightarrow{\rho}_1^{n+1}, \overrightarrow{\theta}_1^{n+1})|\\
&+\eta\left[\frac{1}{\epsilon_3}\|\overrightarrow{\theta}_1^{n+1}\|_{0,\Gamma}^2 + 2\epsilon_3\|\theta^{n+1}\|_{0,\Gamma}^2 + \frac{g^2}{\epsilon_3}\|\theta^{n+1}\|_{0,\Gamma}^2\right]\\
&+\eta\left[\frac{1}{\epsilon_2}\|\overrightarrow{\rho}_1^{n+1}\|_{0,\Gamma}^2 + 2\epsilon_2\|\theta^{n+1}\|_{0,\Gamma}^2 + \frac{g^2}{\epsilon_2}\|\rho^{n+1}\|_{0,\Gamma}^2\right]\\
&+\left[\frac{1}{\epsilon_3}\|\overrightarrow{\theta}_1^{n+1}\|_{0,\Gamma}^2 + 2\epsilon_3\|\overrightarrow{\theta}_1^{n+1}\|_{0,\Gamma}^2 + \frac{g^2}{\epsilon_3}\|\theta^{n+1}\|_{0,\Gamma}^2\right]\\
&+\left[\frac{1}{\epsilon_2}\|\overrightarrow{\rho}_1^{n+1}\|_{0,\Gamma}^2 + 2\epsilon_2\|\overrightarrow{\theta}_1^{n+1}\|_{0,\Gamma}^2 + \frac{g^2}{\epsilon_2}\|\rho^{n+1}\|_{0,\Gamma}^2\right]\\
&+\eta[\frac{1}{\epsilon_1}\|w_1^{n+1}\|_0^2 + \epsilon_1\|\theta^{n+1}\|_0^2] + \frac{1}{\epsilon_1}\|\overrightarrow{w}_3^{n+1}\|_0^2 + \epsilon_1\|\overrightarrow{\theta}_1^{n+1}\|_0^2 + \eta[\frac{1}{\epsilon_2}\|w_2^{n+1}\|_{0,\Gamma}^2 + \epsilon_2\|\theta^{n+1}\|_{0,\Gamma}^2]\\
&+\frac{1}{\epsilon_2}\|w_4^{n+1}\|_{0,\Gamma}^2 + \epsilon_2\|\overrightarrow{\theta}_1^{n+1}\|_{0,\Gamma}^2 + \eta\left[\frac{1}{\epsilon_3}\|\overrightarrow{\theta}_1^{n}\|_{0,\Gamma}^2 + 2\epsilon_3\|\theta^{n+1}\|_{0,\Gamma}^2 + \frac{g^2}{\epsilon_3}\|\theta^{n}\|_{0,\Gamma}^2\right]\\
&+\eta\left[\frac{1}{\epsilon_2}\|\overrightarrow{\rho}_1^{n}\|_{0,\Gamma}^2 + 2\epsilon_2\|\theta^{n+1}\|_{0,\Gamma}^2 + \frac{g^2}{\epsilon_2}\|\rho^{n}\|_{0,\Gamma}^2\right] + \left[\frac{1}{\epsilon_3}\|\overrightarrow{\theta}_1^{n}\|_{0,\Gamma}^2 + 2\epsilon_3\|\overrightarrow{\theta}_1^{n+1}\|_{0,\Gamma}^2 + \frac{g^2}{\epsilon_3}\|\theta^{n}\|_{0,\Gamma}^2\right]\\
&+\left[\frac{1}{\epsilon_2}\|\overrightarrow{\rho}_1^{n}\|_{0,\Gamma}^2 + 2\epsilon_2\|\overrightarrow{\theta}_1^{n+1}\|_{0,\Gamma}^2 + \frac{g^2}{\epsilon_2}\|\rho^{n}\|_{0,\Gamma}^2\right] + \beta\langle P_\tau\mathbb{K}\nabla\theta^{n+1}, P_\tau\overrightarrow{\theta}_1^{n+1}\rangle\\
&-\langle \beta P_\tau\mathbb{K}\nabla\theta^n, P_\tau\overrightarrow{\theta}_1^{n+1}\rangle + \langle \beta P_\tau\mathbb{K}\nabla(\mathbb{P}_1\phi_D(t_{n+1}) - \mathbb{P}_1\phi_D(t_n)), P_\tau\overrightarrow{\theta}_1^{n+1}\rangle.
\end{aligned}
\tag{90}
$$

Now we need to handle the difficulties arising from the nonlinear terms and interface terms. For the nonlinear terms in the above inequality, we follow the idea of (66)-(67) in semi-discretization analysis to obtain the inequalities as follows.

$$
\begin{aligned}
&|C_S(\vec{\rho}_1^{\,n+1}, \vec{u}_h^{\,n+1}, \vec{\theta}_1^{\,n+1}) + C_S(\vec{u}_S(t_{n+1}), \vec{\rho}_1^{\,n+1}, \vec{\theta}_1^{\,n+1})| \\
\le& \frac{\nu}{8}\|\nabla\vec{\rho}_1^{\,n+1}\|_0\|\nabla\vec{\theta}_1^{\,n+1}\|_0 + \frac{\nu}{8}\|\nabla\vec{\rho}_1^{\,n+1}\|_0\|\nabla\vec{\theta}_1^{\,n+1}\|_0 \\
=& \frac{\nu}{4}\|\nabla\vec{\rho}_1^{\,n+1}\|_0\|\nabla\vec{\theta}_1^{\,n+1}\|_0 \\
\le& \frac{\nu}{8}\|\nabla\vec{\theta}_1^{\,n+1}\|_0^2 + \frac{\nu}{8}\|\nabla\vec{\rho}_1^{\,n+1}\|_0^2,
\end{aligned} \tag{91}
$$

$$
\begin{aligned}
&|C_S(\vec{\theta}_1^{\,n+1}, \vec{u}_h^{\,n+1}, \vec{\theta}_1^{\,n+1}) + C_S(\vec{u}_S(t_{n+1}), \vec{\theta}_1^{\,n+1}, \vec{\theta}_1^{\,n+1})| \\
\le& \frac{\nu}{8}\|\nabla\vec{\theta}_1^{\,n+1}\|_0^2 + \frac{\nu}{8}\|\nabla\vec{\theta}_1^{\,n+1}\|_0^2 \\
=& \frac{\nu}{4}\|\nabla\vec{\theta}_1^{\,n+1}\|_0^2.
\end{aligned} \tag{92}
$$

Also, by trace theory and Young's inequality, we have the following estimates for interface terms.

$$
\begin{aligned}
&\|\theta^{n+1}\|_{0,\Gamma}^2 \le C\|\theta^{n+1}\|_0\|\nabla\theta^{n+1}\|_0 \le \frac{C}{2}\left[\frac{1}{\varepsilon_4}\|\theta^{n+1}\|_0^2 + \varepsilon_4\|\nabla\theta^{n+1}\|_0^2\right], \\
&\|\vec{\theta}_1^{\,n+1}\|_{0,\Gamma}^2 \le C\|\vec{\theta}_1^{\,n+1}\|_0\|\nabla\vec{\theta}_1^{\,n+1}\|_0 \le \frac{C}{2}\left[\frac{1}{\varepsilon_4}\|\vec{\theta}_1^{\,n+1}\|_0^2 + \varepsilon_4\|\nabla\vec{\theta}_1^{\,n+1}\|_0^2\right], \\
&\|\theta^{n}\|_{0,\Gamma}^2 \le C\|\theta^{n}\|_0\|\nabla\theta^{n}\|_0 \le \frac{C}{2}\left[\frac{1}{\varepsilon_4}\|\theta^{n}\|_0^2 + \varepsilon_4\|\nabla\theta^{n}\|_0^2\right], \\
&\|\vec{\theta}_1^{\,n}\|_{0,\Gamma}^2 \le C\|\vec{\theta}_1^{\,n}\|_0\|\nabla\vec{\theta}_1^{\,n}\|_0 \le \frac{C}{2}\left[\frac{1}{\varepsilon_4}\|\vec{\theta}_1^{\,n}\|_0^2 + \varepsilon_4\|\nabla\vec{\theta}_1^{\,n}\|_0^2\right].
\end{aligned} \tag{93}
$$

Similar to the proof of (4.2) in [8], we have

$$
\begin{aligned}
&|\beta\langle P_\tau\mathbb{K}\nabla\theta^{n+1}, P_\tau\vec{\theta}_1^{\,n+1}\rangle| + |\beta\langle P_\tau(\mathbb{K}\nabla\theta^{n}), P_\tau\vec{\theta}_1^{\,n+1}\rangle| \\
&+|\langle\beta P_\tau\mathbb{K}\nabla(\mathbb{P}_1\phi_D(t_{n+1}) - \mathbb{P}_1\phi_D(t_n)), P_\tau\vec{\theta}_1^{\,n+1}\rangle| \\
\le& C_k\nu\|\nabla\theta^{n+1}\|_0\|\nabla\vec{\theta}_1^{\,n+1}\|_0 + C_k\nu\|\nabla\theta^{n}\|_0\|\nabla\vec{\theta}_1^{\,n+1}\|_0 \\
&+C_k\nu\|\nabla(\mathbb{P}_1\phi_D(t_{n+1}) - \mathbb{P}_1\phi_D(t_n))\|_0\|\nabla\vec{\theta}_1^{\,n+1}\|_0 \\
\le& \frac{C_k\nu}{2}\|\nabla\theta^{n+1}\|_0^2 + \frac{3C_k\nu}{2}\|\nabla\vec{\theta}_1^{\,n+1}\|_0^2 + \frac{C_k\nu}{2}\|\nabla\theta^{n}\|_0^2 + \frac{\Delta t C_k\nu}{2}\int_{t_n}^{t_{n+1}}\|\nabla\phi_{D,t}\|_0^2\,dt
\end{aligned}
$$

where C_k is proportional to $\sqrt{k}$ and (4.13) in [36] is used in the last step.

Hence, multiplying (90) with $2\Delta t$ and using the above inequalities, we have

$$
\begin{aligned}
&\eta\|\theta^{n+1}\|_0^2 - \eta\|\theta^{n}\|_0^2 + \|\vec{\theta}_1^{\,n+1}\|_0^2 - \|\vec{\theta}_1^{\,n}\|_0^2 + 2\eta\Delta t a_D(\theta^{n+1},\theta^{n+1}) + 2\Delta t a_S(\vec{\theta}_1^{\,n+1}, \vec{\theta}_1^{\,n+1}) \\
&-2\eta\Delta t\langle\vec{\theta}_1^{\,n+1}\cdot\vec{n}_S, \theta^{n+1}\rangle + 2\Delta t\langle g\theta^{n+1}, \vec{\theta}_1^{\,n+1}\cdot\vec{n}_S\rangle + 2\Delta t\beta\langle P_\tau(\vec{\theta}_1^{\,n+1} + \mathbb{K}\nabla\theta^{n+1}), P_\tau\vec{\theta}_1^{\,n+1}\rangle \\
\le& C\Delta t\left[\|w_1^{n+1}\|_0^2 + \|w_3^{n+1}\|_0^2 + \|w_2^{n+1}\|_{0,\Gamma}^2 + \|w_4^{n+1}\|_{0,\Gamma}^2\right] \\
&+\Delta t\left[\frac{3\nu}{4} + C(5\epsilon_2 + 4\epsilon_3)\epsilon_4 + \frac{C(1+\eta)\epsilon_4}{\epsilon_3} + 3C_k\nu\right]\|\nabla\vec{\theta}_1^{\,n+1}\|_0^2 \\
&+\Delta t\left[C\eta(5\epsilon_2 + 4\epsilon_3)\epsilon_4 + \frac{C(1+\eta)g^2\epsilon_4}{\epsilon_3} + C_k\nu\right]\|\nabla\theta^{n+1}\|_0^2 \\
&\Delta t\left[2\epsilon_1 + \frac{C(5\epsilon_2 + 4\epsilon_3)}{\epsilon_4} + \frac{C(1+\eta)}{\epsilon_3\epsilon_4}\right]\|\vec{\theta}_1^{\,n+1}\|_0^2 \\
&+\Delta t\left[2\eta\epsilon_1 + \frac{C\eta(5\epsilon_2 + 4\epsilon_3)}{\epsilon_4} + \frac{C(1+\eta)g^2}{\epsilon_3\epsilon_4}\right]\|\theta^{n+1}\|_0^2 \\
&+\Delta t\frac{C(1+\eta)\epsilon_4}{\epsilon_3}\|\nabla\vec{\theta}_1^{\,n}\|_0^2 + \Delta t\left[\frac{C(1+\eta)g^2\epsilon_4}{\epsilon_3} + C_k\nu\right]\|\nabla\theta^{n}\|_0^2 \\
&+\Delta t\frac{C(1+\eta)}{\epsilon_3\epsilon_4}\|\vec{\theta}_1^{\,n}\|_0^2 + \Delta t\frac{C(1+\eta)g^2}{\epsilon_3\epsilon_4}\|\theta^{n}\|_0^2
\end{aligned}
$$

$$
\begin{aligned}
&+C\Delta t\left[\|\overrightarrow{\rho}_1^{n+1}\|_{0,\Gamma}^2+\|\rho^{n+1}\|_{0,\Gamma}^2+\|\overrightarrow{\rho}_1^{n}\|_{0,\Gamma}^2+\|\rho^{n}\|_{0,\Gamma}^2+\|\nabla\overrightarrow{\rho}_1^{n+1}\|_0^2\right]\\
&+\Delta t^2C_k\nu\int_{t_n}^{t_{n+1}}\|\nabla\phi_{D,t}\|_0^2\,dt\\
\le{}& C\Delta t\left[\|w_1^{n+1}\|_0^2+\|w_3^{n+1}\|_0^2+\|w_2^{n+1}\|_{0,\Gamma}^2+\|w_4^{n+1}\|_{0,\Gamma}^2\right]\\
&+\Delta t\left[\frac{3\nu}{4}+C(5\epsilon_2+4\epsilon_3)\epsilon_4+\frac{C(1+\eta)\epsilon_4}{\epsilon_3}+3C_k\nu\right]\|\nabla\overrightarrow{\theta}_1^{n+1}\|_0^2\\
&+\Delta t\left[C\eta(5\epsilon_2+4\epsilon_3)\epsilon_4+\frac{C(1+\eta)\epsilon_4+\epsilon_3C_k\nu}{\epsilon_3}\right]\|\nabla\theta^{n+1}\|_0^2\\
&\Delta t\left[2\epsilon_1+\frac{C(5\epsilon_2+4\epsilon_3)}{\epsilon_4}+\frac{C(1+\eta)}{\epsilon_3\epsilon_4}\right]\|\overrightarrow{\theta}_1^{n+1}\|_0^2\\
&+\Delta t\left[2\eta\epsilon_1+\frac{C\eta(5\epsilon_2+4\epsilon_3)}{\epsilon_4}+\frac{C(1+\eta)g^2}{\epsilon_3\epsilon_4}\right]\|\theta^{n+1}\|_0^2\\
&+\Delta t\frac{C(1+\eta)\epsilon_4}{\epsilon_3}\|\nabla\overrightarrow{\theta}_1^{n}\|_0^2+\Delta t\left[\frac{C(1+\eta)\epsilon_4+\epsilon_3C_k\nu}{\epsilon_3}\right]\|\nabla\theta^n\|_0^2\\
&+\Delta t\frac{C(1+\eta)}{\epsilon_3\epsilon_4}\|\overrightarrow{\theta}_1^{n}\|_0^2+\Delta t\frac{C(1+\eta)g^2}{\epsilon_3\epsilon_4}\|\theta^n\|_0^2\\
&+C\Delta t\left[\|\overrightarrow{\rho}_1^{n+1}\|_{0,\Gamma}^2+\|\rho^{n+1}\|_{0,\Gamma}^2+\|\overrightarrow{\rho}_1^{n}\|_{0,\Gamma}^2+\|\rho^{n}\|_{0,\Gamma}^2+\|\nabla\overrightarrow{\rho}_1^{n+1}\|_0^2\right]\\
&+\Delta t^2C_k\nu\int_{t_n}^{t_{n+1}}\|\nabla\phi_{D,t}\|_0^2\,dt
\end{aligned}
\tag{94}
$$

Based on the above preparation for the treatment of BJ interface condition, we can utilize (4.2) in [8] to advance the proof as follows. For small enough ν, we can choose ϵ_2,ϵ_3 and ϵ_4 to obtain

$$
\begin{aligned}
&2\eta a_D(\theta^{n+1},\theta^{n+1})+2a_S(\overrightarrow{\theta}_1^{n+1},\overrightarrow{\theta}_1^{n+1})-2\eta\langle\overrightarrow{\theta}_1^{n+1}\cdot\overrightarrow{n}_S,\theta^{n+1}\rangle+2\langle g\theta^{n+1},\overrightarrow{\theta}_1^{n+1}\cdot\overrightarrow{n}_S\rangle\\
&+2\beta\langle P_\tau(\overrightarrow{\theta}_1^{n+1}+\mathbb{K}\nabla\theta^{n+1}),P_\tau\overrightarrow{\theta}_1^{n+1}\rangle\\
\ge{}&\left[C\eta(5\epsilon_2+4\epsilon_3)\epsilon_4+\frac{2C(1+\eta)\epsilon_4+2C_k\nu\epsilon_3}{\epsilon_3}\right]\|\nabla\theta^{n+1}\|_0^2\\
&+\left[\frac{3\nu}{4}+C(5\epsilon_2+4\epsilon_3)\epsilon_4+\frac{2C(1+\eta)\epsilon_4+3C_k\nu\epsilon_3}{\epsilon_3}\right]\|\nabla\overrightarrow{\theta}_1^{n+1}\|_0^2\\
&-C_3\|\theta^{n+1}\|_0^2-C_3\|\overrightarrow{\theta}_1^{n+1}\|_0^2
\end{aligned}
\tag{95}
$$

Then, substituting (95) into (94), we have

$$
\begin{aligned}
&\eta\|\theta^{n+1}\|_0^2+\|\overrightarrow{\theta}_1^{n+1}\|_0^2+\Delta t\left[\frac{C(1+\eta)\epsilon_4+C_k\nu\epsilon_3}{\epsilon_3}\right]\|\nabla\theta^{n+1}\|_0^2\\
&+\Delta t\left[\frac{C(1+\eta)\epsilon_4}{\epsilon_3}\right]\|\nabla\overrightarrow{\theta}_1^{n+1}\|_0^2\\
\le{}&(1+C\Delta t)(\eta\|\theta^n\|_0^2+\|\overrightarrow{\theta}_1^{n}\|_0^2)+\Delta t\left[\frac{C(1+\eta)\epsilon_4+C_k\nu\epsilon_3}{\epsilon_3}\right]\|\nabla\theta^n\|_0^2\\
&+\Delta t\left[\frac{C(1+\eta)\epsilon_4}{\epsilon_3}\right]\|\nabla\overrightarrow{\theta}_1^{n}\|_0^2\\
&+C\Delta t(\|w_1^{n+1}\|_0^2+\|w_3^{n+1}\|_0^2+\|w_2^{n+1}\|_{0,\Gamma}^2+\|w_4^{n+1}\|_{0,\Gamma}^2)\\
&+C\Delta t(\|\overrightarrow{\rho}_1^{n+1}\|_{0,\Gamma}^2+\|\rho^{n+1}\|_{0,\Gamma}^2+\|\overrightarrow{\rho}_1^{n}\|_{0,\Gamma}^2+\|\rho^{n}\|_{0,\Gamma}^2+\|\nabla\overrightarrow{\rho}_1^{n+1}\|_0^2)\\
&+C\Delta t(\|\theta^{n+1}\|_0^2+\|\overrightarrow{\theta}_1^{n+1}\|_0^2)+C\Delta t^2\int_{t_n}^{t_{n+1}}\|\nabla\phi_{D,t}\|_0^2\,dt
\end{aligned}
$$

Setting $\epsilon = \frac{C(1+\eta)\epsilon_4 + C_k \nu \epsilon_3}{\epsilon_3}$, $\tilde{\epsilon} = \frac{C(1+\eta)\epsilon_4}{\epsilon_3}$, then we can consider the special norm $\eta\|\theta^n\|_0^2 + \|\overrightarrow{\theta}_1^n\|_0^2 + \epsilon\Delta t\|\nabla\theta^n\|_0^2 + \tilde{\epsilon}\Delta t\|\nabla\overrightarrow{\theta}_1^n\|_0^2$ $(n = 1, \ldots, N)$ for the discrete Gronwall's inequality as follows. Summing from $n = 0$ to $N-1$, we have

$$
\begin{aligned}
&\eta\|\theta^N\|_0^2 + \|\overrightarrow{\theta}_1^N\|_0^2 + \epsilon\Delta t\|\nabla\theta^N\|_0^2 + \tilde{\epsilon}\Delta t\|\nabla\overrightarrow{\theta}_1^N\|_0^2 \\
\le{}& (1 + C\Delta t)^N[\eta\|\theta^0\|_0^2 + \|\overrightarrow{\theta}_1^0\|_0^2 + \epsilon\Delta t\|\nabla\theta^0\|_0^2 + \tilde{\epsilon}\Delta t\|\nabla\overrightarrow{\theta}_1^0\|_0^2] \\
&+C\Delta t\sum_{j=0}^{N-1}(1 + C\Delta t)^j[\|w_1^{j+1}\|_0^2 + \|w_2^{j+1}\|_{0,\Gamma}^2 + \|w_3^{j+1}\|_0^2 + \|w_4^{j+1}\|_{0,\Gamma}^2 \\
&+\|\overrightarrow{\rho}_1^{j+1}\|_{0,\Gamma}^2 + \|\rho^{j+1}\|_{0,\Gamma}^2 + \|\overrightarrow{\rho}_1^{j}\|_{0,\Gamma}^2 + \|\rho^{j}\|_{0,\Gamma}^2 + \|\nabla\overrightarrow{\rho}_1^{j+1}\|_0^2] \\
&+C\Delta t\sum_{j=0}^{N-1}(1 + C\Delta t)^j[\|\theta^{j+1}\|_0^2 + \|\overrightarrow{\theta}_1^{j+1}\|_0^2] \\
&+C\Delta t^2\sum_{j=0}^{N-1}(1 + C\Delta t)^j\int_{t_j}^{t_{j+1}}\|\nabla\phi_{D,t}\|_0^2\,dt \\
\le{}& Ce^{CT}\Big[\eta\|\theta^0\|_0^2 + \|\overrightarrow{\theta}_1^0\|_0^2 + \tilde{\epsilon}\Delta t\|\overrightarrow{\theta}_1^0\|_1^2 + \epsilon\Delta t\|\theta^0\|_1^2 \\
&+\Delta t\sum_{j=0}^{N-1}\Big(\|w_1^{j+1}\|_0^2 + \|w_2^{j+1}\|_{0,\Gamma}^2 + \|w_3^{j+1}\|_0^2 + \|w_4^{j+1}\|_{0,\Gamma}^2 + \Delta t\int_{t_j}^{t_{j+1}}\|\nabla\phi_{D,t}\|_0^2\,dt \\
&+\|\overrightarrow{\rho}_1^{j+1}\|_{0,\Gamma}^2 + \|\rho^{j+1}\|_{0,\Gamma}^2 + \|\overrightarrow{\rho}_1^{j}\|_{0,\Gamma}^2 + \|\rho^{j}\|_{0,\Gamma}^2 + \|\nabla\overrightarrow{\rho}_1^{j+1}\|_0^2\Big)\Big].
\end{aligned} \tag{96}
$$

Then by Lemma 2, we complete the proof of (69).

5. The Lagrange multiplier method under the framework of domain decomposition for Navier-Stokes-Darcy system with the defective boundary

With the foundation built up in the previous two sections, in this section we propose the Lagrange multiplier method under the framework of the parallel, non-iterative, multi-physics domain decomposition for the Navier-Stokes-Darcy system with the defective boundary condition. Based on the weak formulation with Lagrange multipliers, we utilize the three-step backward differentiation scheme for the temporal discretization and finite elements for the spatial discretization.

First, we present the following coupled weak formulation with Lagrange multipliers for Navier-Stokes-Darcy system with the defective boundary which is defined in section 2.2: find $(\overrightarrow{u}_S, p_S) \in H^1(0,T;X_S,X_S') \times L^2(0,T;Q_S)$, $\phi_D \in H^1(0,T;X_D,X_D')$ and $\lambda = \{\lambda_i(t)\}_{i=0}^m \in L^2(0,T)^{m+1}$ such that

$$
\begin{aligned}
&(\frac{\partial\overrightarrow{u}_S}{\partial t}, \overrightarrow{v})_{\Omega_S} + \eta(\frac{\partial\phi_D}{\partial t}, \psi)_{\Omega_D} + \sum_{i=0}^m\lambda_i(t)\int_{S_i}\overrightarrow{v}\cdot\overrightarrow{n}_S\,ds + C_S(\overrightarrow{u}_S, \overrightarrow{u}_S, \overrightarrow{v}) + a_S(\overrightarrow{u}_S, \overrightarrow{v}) \\
&+b_S(\overrightarrow{v}, p_S) + \eta/Sa_D(\phi_D, \psi) + \langle g\phi_D, \overrightarrow{v}\cdot\overrightarrow{n}_S\rangle - \eta/S\langle\overrightarrow{u}_S\cdot\overrightarrow{n}_S, \psi\rangle \\
&+\frac{\alpha\nu\sqrt{d}}{\sqrt{\mathrm{trace}(\prod)}}\langle P_\tau(\overrightarrow{u}_S + \mathbb{K}\nabla\phi_D), P_\tau\overrightarrow{v}\rangle \\
={}&\eta/S(f_D, \psi)_{\Omega_D} + (\overrightarrow{f}_S, \overrightarrow{v})_{\Omega_S},\ \forall\,\overrightarrow{v}\in X_S,\ \ \psi\in X_D, \\
&\sum_{i=0}^m\mu_i(t)\int_{S_i}\overrightarrow{v}\cdot\overrightarrow{n}_S\,ds + b_S(\overrightarrow{u}_S, q) = \sum_{i=0}^m\mu_iQ_i,\ \forall\,q\in Q_S, \mu = \{\mu_i(t)\}_{i=0}^m\in L^2(0,T)^{m+1}.
\end{aligned} \tag{97}
$$

For the problem (97) to be solvable, the following compatibility conditions must be satisfied [64]:

$$
\int_{S_i}\overrightarrow{u}_0\cdot\overrightarrow{n}_S ds = Q_i(0), \qquad i = 0, 1, \cdots, n. \tag{98}
$$

Second, based on the Robin boundary conditions (16) and (12)-(13), we propose the following decoupled weak formulation with Lagrange multipliers for Navier-Stokes-Darcy system with the defective boundary: for $\forall\ \overrightarrow{v}\in X_S, \forall\ q\in Q_S$,

$$\eta(\frac{\partial \phi_D}{\partial t}, \psi)_{\Omega_D} + \eta/S a_D(\phi_D, \psi) + \eta/S\langle g\phi_D, \psi\rangle = \eta/S(f_D, \psi)_{\Omega_D} + \eta/S\langle \xi_D, \psi\rangle, \ \forall\, \psi \in X_D. \tag{99}$$

$$\begin{aligned}
&(\frac{\partial \vec{u}_S}{\partial t}, \vec{v})_{\Omega_S} + \sum_{i=0}^{m} \lambda_i(t) \int_{S_i} \vec{v}\cdot\vec{n}_S\, ds + C_S(\vec{u}_S, \vec{u}_S, \vec{v}) + a_S(\vec{u}_S, \vec{v}) + b_S(\vec{v}, p_S)\\
&+\langle \vec{u}_S\cdot\vec{n}_S, \vec{v}\cdot\vec{n}_S\rangle + \frac{\alpha\nu\sqrt{d}}{\sqrt{\mathrm{trace}(\prod)}}\langle P_\tau \vec{u}_S, P_\tau \vec{v}\rangle\\
&= (\vec{f}_S, \vec{v})_{\Omega_S} + \langle \xi_S, \vec{v}\cdot\vec{n}_S\rangle - \langle \xi_{S\tau}, P_\tau\vec{v}\rangle,
\end{aligned} \tag{100}$$

$$\sum_{i=0}^{m} \mu_i(t) \int_{S_i} \vec{v}\cdot\vec{n}_S\, ds + b_S(\vec{u}_S, q) = \sum_{i=0}^{m} \mu_i Q_i, \quad \mu = \{\mu_i(t)\}_{i=0}^{m} \in L^2(0,T)^{m+1}. \tag{101}$$

Furthermore, compared with the backward Euler scheme for the temporal discretization in section 3.3, we utilize the three-step backward differentiation in order to improve the accuracy of the temporal discretization from the first order to the third order. The spatial discretization is still effected by finite elements. The full discretization scheme of the Lagrange multiplier method under the framework of the parallel, non-iterative, multi-physics domain decomposition is defined as follows: At the nth$(n = 2, \cdots)$ step, set

$$\xi_D^n = \vec{u}_h^n\cdot\vec{n}_S + g\phi_h^n, \quad \xi_S^n = \vec{u}_h^n\cdot\vec{n}_S - g\phi_h^n, \quad \xi_{S\tau}^n = \frac{\alpha\nu\sqrt{d}}{\sqrt{\mathrm{trace}(\prod)}} P_\tau(\mathbb{K}\nabla\phi_h^n) \tag{102}$$

and then independently solve

$$\begin{aligned}
&\eta(\frac{11\phi_h^{n+1} - 18\phi_h^n + 9\phi_h^{n-1} - 2\phi_h^{n-2}}{6\triangle t}, \psi_h)_{\Omega_D} + \eta/S a_D(\phi_h^{n+1}, \psi_h) + \eta/S\langle g\phi_h^{n+1}, \psi_h\rangle\\
&= \eta/S(f_D^{n+1}, \psi_h)_{\Omega_D} + \eta/S\langle \xi_D^n, \psi_h\rangle, \ \forall\, \psi_h \in X_{Dh},
\end{aligned} \tag{103}$$

$$\begin{aligned}
&(\frac{11\vec{u}_h^{n+1} - 18\vec{u}_h^n + 9\vec{u}_h^{n-1} - 2\vec{u}_h^{n-2}}{6\triangle t}, \vec{v}_h)_{\Omega_S} + \sum_{i=0}^{m} \lambda_i^{n+1} \int_{S_i} \vec{v}_h\cdot\vec{n}_S\, ds\\
&+C_S(\vec{u}_h^{n+1}, \vec{u}_h^{n+1}, \vec{v}_h) + a_S(\vec{u}_h^{n+1}, \vec{v}_h) + b_S(\vec{v}_h, p_h^{n+1}) + \langle \vec{u}_h^{n+1}\cdot\vec{n}_S, \vec{v}_h\cdot\vec{n}_S\rangle\\
&+\frac{\alpha\nu\sqrt{d}}{\sqrt{\mathrm{trace}(\prod)}}\langle P_\tau\vec{u}_h^{n+1}, P_\tau\vec{v}_h\rangle\\
&= (\vec{f}_S^{n+1}, \vec{v}_h)_{\Omega_S} + \langle \xi_S^n, \vec{v}_h\cdot\vec{n}_S\rangle - \langle \xi_{S\tau}^n, P_\tau\vec{v}_h\rangle, \ \forall\, \vec{v}_h \in X_{Sh}
\end{aligned} \tag{104}$$

$$\sum_{i=0}^{m} \mu_i^{n+1} \int_{S_i} \vec{v}_h\cdot\vec{n}_S\, ds + b_S(\vec{u}_h^{n+1}, q_h) = \sum_{i=0}^{m} \mu_i^{n+1} Q_i, \ \forall\, q_h \in Q_{Sh}, \ \mu_i^{n+1} \in R^{n+1} \tag{105}$$

for λ_i^{n+1} $(i = 0, \cdots, m)$, ϕ_h^{n+1}, $\vec{u}_h^{n+1}$, p_h^{n+1} and μ_h^{n+1}.

6. Numerical examples

In this section, we will present two examples to illustrate the features of the two algorithms proposed in section 3 and section 5.

Example 1. Consider the Navier-Stokes-Darcy model in section 2.1 on the domain $\Omega = [0,1]\times[-0.25, 0.75]$, where $\Omega_D = [0,1]\times[0, 0.75]$, and $\Omega_S = [0,1]\times[-0.25, 0]$. Choose $\alpha = 1$, $\nu = 1$, $g = 1$, $z = 0$, and $\mathbb{K} = kI$ where I the identity matrix and $k = 1$. The boundary condition functions and the source terms are chosen such that the exact solutions are

$$\phi_D = [2 - \pi\, sin(\pi x)][-y + cos(\pi(1-y))]cos(2\pi t), \tag{106}$$

$$\vec{u}_S = [x^2y^2 + e^{-y}, -\frac{2}{3}xy^3 + 2 - \pi\, sin(\pi x)]^T cos(2\pi t), \tag{107}$$

$$p_S = -[2 - \pi\, sin(\pi x)]cos(2\pi y)cos(2\pi t), \tag{108}$$

which satisfy the interface conditions (6)-(7), including the Beavers-Joseph interface condition. The Taylor-Hood elements are used for the Navier-Stokes equations and the quadratic finite elements are used for the second order formulations of

Table 1
Errors of the first non-iterative DDM with backward Euler for $\triangle t = 8h^3$.

h	$\|u_h - u\|_0$	order	$\|u_h - u\|_1$	order	$\|p_h - p\|_0$	order	$\|\phi_h - \phi\|_0$	order	$\|\phi_h - \phi\|_1$	order
1/4	3.0184×10^{-2}	–	6.2839×10^{-1}	–	2.6888×10^{-1}	–	2.6574×10^{-1}	–	6.2396×10^{-1}	–
1/8	3.7366×10^{-3}	3.01	1.6705×10^{-1}	1.91	2.8863×10^{-2}	3.21	3.7408×10^{-2}	2.83	1.0425×10^{-1}	2.51
1/16	4.6077×10^{-4}	3.02	4.2861×10^{-2}	1.96	3.2376×10^{-3}	3.15	4.7426×10^{-3}	2.98	1.9657×10^{-2}	2.40
1/32	5.6977×10^{-5}	3.02	1.0812×10^{-2}	1.99	3.7882×10^{-4}	3.09	5.9374×10^{-4}	3.00	4.4097×10^{-3}	2.15

Table 2
Errors of the first non-iterative DDM with backward Euler for $\triangle t = h$.

h	$\|u_h - u\|_0$	order	$\|u_h - u\|_1$	order	$\|p_h - p\|_0$	order	$\|\phi_h - \phi\|_0$	order	$\|\phi_h - \phi\|_1$	order
1/8	2.8012×10^{-2}	–	4.0061×10^{-1}	–	5.0047×10^{-1}	–	2.5251×10^{-2}	–	1.4670×10^{-1}	–
1/16	1.0208×10^{-2}	1.45	1.2970×10^{-1}	1.62	1.5062×10^{-1}	1.73	1.5081×10^{-2}	0.75	7.5080×10^{-2}	0.96
1/32	4.1315×10^{-3}	1.30	4.6703×10^{-2}	1.47	5.6933×10^{-2}	1.40	8.4446×10^{-3}	0.84	3.9880×10^{-2}	0.92
1/64	1.8419×10^{-3}	1.16	1.9169×10^{-2}	1.28	2.5420×10^{-2}	1.16	4.4881×10^{-3}	0.91	2.0805×10^{-2}	0.94

Table 3
Errors of the non-iterative DDM with three-step BDF for $\triangle t = h$.

h	$\|u_h - u\|_0$	order	$\|u_h - u\|_1$	order	$\|p_h - p\|_0$	order	$\|\phi_h - \phi\|_0$	order	$\|\phi_h - \phi\|_1$	order
1/8	4.8057×10^{-2}	–	7.4713×10^{-1}	–	8.0323×10^{-1}	–	6.2662×10^{-3}	–	7.9365×10^{-2}	–
1/16	2.9286×10^{-3}	4.03	4.5974×10^{-2}	4.0	5.1411×10^{-2}	3.96	7.2423×10^{-4}	3.11	1.8559×10^{-2}	2.09
1/32	2.2501×10^{-4}	3.70	3.5787×10^{-3}	3.68	3.6113×10^{-3}	3.83	1.3492×10^{-4}	2.64	4.5594×10^{-3}	2.02
1/64	1.9783×10^{-5}	3.50	3.9761×10^{-4}	3.17	3.7601×10^{-4}	3.26	1.9332×10^{-5}	2.80	1.1335×10^{-3}	2.00

the Darcy equation. Newton iteration is used to deal with the nonlinear advection. In the following, we will provide the numerical results at $T = 1$ for the algorithm in section 3.

Table 1 provides the numerical solution errors with $\triangle t = 8h^3$. We can find the non-iterative domain decomposition algorithm is stable, and the accuracy order is about $O(h^3 + \triangle t) = O(h^3)$. Table 2 provides the numerical solution errors with $\triangle t = h$. From this table, we can see that the non-iterative domain decomposition algorithm is still stable, but the accuracy order is about first order. These results numerically verify the expected optimal accuracy orders arising from backward Euler scheme, Taylor-Hood elements and quadratic elements.

In order to improve the accuracy order of the temporal discretization, we use the three-step backward differentiation scheme to replace the backward Euler scheme in the algorithm of section 3. Then we list the corresponding numerical solution errors for $\triangle t = h$ in Table 3. These results in the table are consistent with the expected optimal accuracy orders arising from the three-step backward differentiation scheme, Taylor-Hood elements and quadratic elements. In particular, we see the optimal accuracy order of $O(h^3 + \triangle t^3) = O(h^3)$ with respect to L^2 norms for $\overrightarrow{u}$ and ϕ since we use quadratic finite elements for them.

Example 2. Consider the time-dependent Navier-Stokes-Darcy model with defective boundary conditions and BJ interface condition on the following domain. Let $\Omega = [0, 1] \times [0, 1]$. We choose Ω_S to be the polygon $\overline{ABCDEFGHIJ}$ where $A = (0, 1), B = (0, 3/4), C = (1/2, 1/4), D = (1/2, 0), E = (3/4, 0), F = (3/4, 1/4), G = (1, 1/4), H = (1, 1/2), I = (3/4, 1/2)$ and $J = (1/4, 1)$. Let $\Omega_D = \Omega/\Omega_S$, $S_0 = \overline{AB} \cup \overline{JA}$, $S_1 = \overline{DE}$, and $S_2 = \overline{GH}$.

Set $T = 1$, $\alpha = 1$, $\nu = 1$, $g = 1$, $z = 0$, and $\mathbb{K} = K\mathbb{I}$, where $\mathbb{I}$ denotes the identity matrix and $K = 1$. The boundary condition data and source terms are chosen to be 0 except Q_i on S_i $(i = 0, 1, 2)$. We subdivide Ω into rectangle of height and width $h = 1/M$, where M denotes a positive integer, and then subdivide each rectangle into two triangles by drawing a diagonal. For this numerical experiment, we choose $M = 32$ and $\triangle t = h$. The Taylor-Hood elements are used for the Navier-Stokes equations and the quadratic finite elements are used for the second order formulations of the Darcy equation. In the following, we will provide the numerical results at $T = 1$ for the algorithm in section 5.

In the first test, we set $Q_1 = Q_2 = -1$ and $Q_0 = 2$ so that the total inflow rate is equal to the total outflow rate. In the second test, we keep the same Q_1 and Q_2 but set $Q_0 = 1$ so that the total inflow rate is larger than the total outflow rate. In the third test, we keep the same Q_1 and Q_2 but set $Q_0 = 3$ so that the total inflow rate is smaller than the total outflow rate. Figs. 3-5 illustrate the numerical solutions at the end time $T = 1$ for these three tests. These physically valid velocity fields verify the effectiveness of the proposed Lagrange multiplier method under the framework of the parallel, non-iterative, multi-physics domain decomposition. Particularly, compare with Fig. 3, we observe more flow from the conduit to the porous media in Fig. 4 and more flow from the porous media to the conduit in Fig. 5, especially in the area around the outflow boundary S_0. These phenomena are expected due to the chosen unbalanced inflow and outflow rates for the conduit. Furthermore, we consider the more realistic parameters for hydraulic conductivity tensor. Fig. 6 illustrate that the proposed method works well for some smaller parameters.

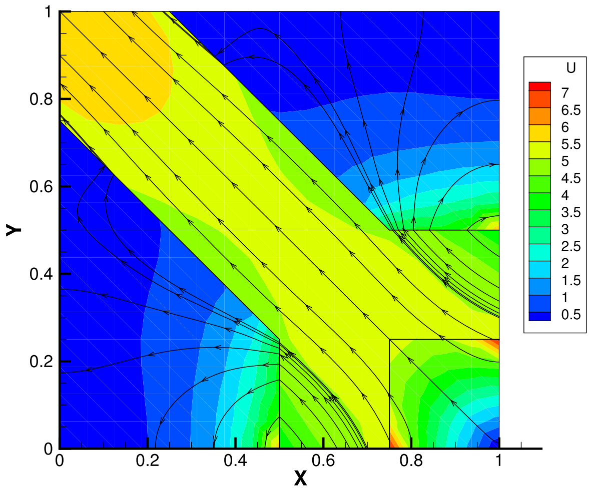

Fig. 3. Plot of the speed and the velocity streamlines for $Q_0 = 2$, $Q_1 = -1$, and $Q_2 = -1$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

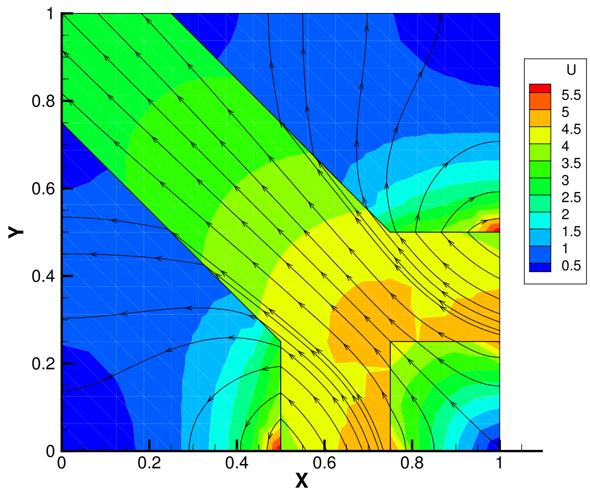

Fig. 4. Plot of the speed and the velocity streamlines for $Q_0 = 1$, $Q_1 = -1$, and $Q_2 = -1$.

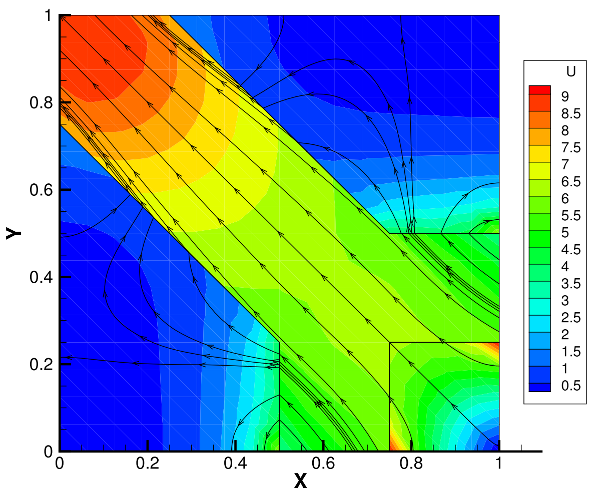

Fig. 5. Plot of the speed and the velocity streamlines for $Q_0 = 3$, $Q_1 = -1$, and $Q_2 = -1$.

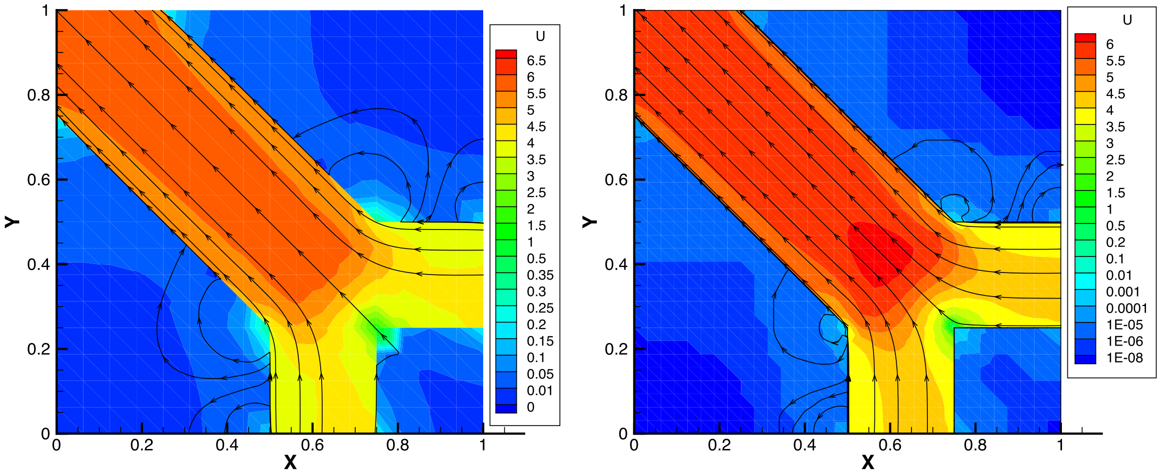

Fig. 6. Plot of the speed and the velocity streamlines with $k = 10^{-4}$ (left) and $k = 10^{-8}$ (right) for $Q_0 = 2$, $Q_1 = -1$, and $Q_2 = -1$.

7. Conclusion

In this article we first develop and analyze a parallel, non-iterative, multi-physics domain decomposition method for the time-dependent Navier-Stokes-Darcy system with BJ interface condition. Then we develop a Lagrange multiplier method under the framework of the proposed domain decomposition method to solve the time-dependent Navier-Stokes-Darcy system with defective boundary condition and BJ interface condition. Finite elements are used for the spatial discretization. Backward Euler and three-step backward differentiation schemes are used for the temporal discretization. Numerical examples are provided to illustrate the features of the proposed method.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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