Locally p-admissible measures on $\mathbf{R}$

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Abstract. In this note we show that locally p-admissible measures on $\mathbf{R}$ necessarily come from local Muckenhoupt A_p weights. In the proof we employ the corresponding characterization of global p-admissible measures on $\mathbf{R}$ in terms of global A_p weights due to Björn, Buckley and Keith, together with tools from analysis in metric spaces, more specifically preservation of the doubling condition and Poincaré inequalities under flattening, due to Durand-Cartagena and Li.

As a consequence, the class of locally p-admissible weights on $\mathbf{R}$ is invariant under addition and satisfies the lattice property. We also show that measures that are p-admissible on an interval can be partially extended by periodical reflections to global p-admissible measures. Surprisingly, the p-admissibility has to hold on a larger interval than the reflected one, and an example shows that this is necessary.

Key words and phrases: local Muckenhoupt A_p weight, locally doubling measure, locally p-admissible measure, local Poincaré inequality.

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1. Introduction

Globally p-admissible weights for Sobolev spaces and differential equations on $\mathbf{R}^n$ were introduced in Heinonen–Kilpeläinen–Martio [16]. Four conditions were imposed on such weights, which were later reduced to the following two conditions (the remaining two being redundant), see [16, 2nd ed., Section 20]. Even earlier, such weights were used to study regularity of linear degenerate elliptic equations (with $p = 2$) in Fabes–Jerison–Kenig [12] and Fabes–Kenig–Serapioni [13].

Definition 1.1. A measure μ on $\mathbf{R}^n$ is *globally p-admissible*, $1 \le p < \infty$, if it is globally doubling and supports a global p-Poincaré inequality. If $d\mu = w\,dx$ we also say that the weight w is *globally p-admissible*.

Muckenhoupt A_p weights are globally p-admissible (see [16, Theorem 15.21] and [6, Theorem 4]), but the converse is not true in $\mathbf{R}^n$, $n \ge 2$. On the other hand, on $\mathbf{R}$ even globally p-admissible *measures* are given by global A_p weights, as was shown in Björn–Buckley–Keith [7, Theorem 2].

In many situations it is local, rather than global, properties that are relevant, especially when dealing with local properties such as regularity of solutions to differential equations. In [8], Chua and Wheeden extensively studied which weights support various types of Poincaré inequalities on intervals, and also obtained optimal constants using variants of the Muckenhaupt condition. However, the conditions given in [8] allow the weights to vanish to high orders at the end points of the interval. Hence, such p-admissible weights on a bounded interval are *not* necessarily A_p weights on that interval, only on its subintervals that stay away from its boundary, see [8, point 5 of p. 152], Example 4.7 and also Theorem 4.6.

The principal aim of this paper is to obtain the following characterization of locally p-admissible measures on $\mathbf{R}$, see Section 3 for the relevant definitions. There are several different possibilities for formulating local doubling conditions and local Poincaré inequalities. The conditions we impose on the measure do not require any uniformity in the constants nor in the radii involved, and are thus truly local. However, uniformity is natural in many situations, and then we are able to conclude slightly more, see Section 7.

Theorem 1.2. *Let $p \ge 1$ and let μ be a measure on $\mathbf{R}$. Then the following are equivalent*:

(i) *μ is locally p-admissible*;
(ii) *$d\mu = w\,dx$, where w is a local A_p weight*;
(iii) *$d\mu = w\,dx$, and for each bounded interval $I \subset \mathbf{R}$ there is a global A_p weight $\widetilde{w}$ on $\mathbf{R}$ such that $\widetilde{w} = w$ on I.*

As a consequence of these characterizations we obtain the lattice property for locally p-admissible weights on $\mathbf{R}$, as well as the preservation of local p-admissibility when taking finite sums of measures, see Section 5. This complements some results in Kilpeläinen–Koskela–Masaoka [21], where such questions were studied for global A_p weights and globally p-admissible measures on $\mathbf{R}^n$. As a byproduct, we provide an elementary proof of [21, Proposition 4.3], see Lemma 5.3.

This note is a continuation of the systematic development of local and semilocal doubling measures and Poincaré inequalities on metric spaces from Björn–Björn [3] and [4]. Local assumptions are also natural for studying p-harmonic and quasiharmonic functions, as in [3], [4], Danielli–Garofalo–Marola [9], Garofalo–Marola [14] and Holopainen–Shanmugalingam [19]. Theorem 1.2 also plays a role in Liouville type theorems on the real line, see Björn–Björn–Shanmugalingam [5].

The proof of Theorem 1.2 turned out to be more complicated than we had expected, and considerably more involved than the proof of the corresponding global result in Björn–Buckley–Keith [7, Theorem 2]. In addition to careful estimates, we also use the metric space theory. More specifically, to show that a locally p-admissible measure μ is absolutely continuous, we create a suitable p-admissible measure on the circle $\mathbf{S}^1$, and then use a flattening argument due to Durand-Cartagena–Li [11] to obtain a globally q-admissible measure $\hat{\mu}$ on $\mathbf{R}$ for some q. We then have at our disposal the global result in [7, Theorem 2] which yields that $\hat{\mu}$ is absolutely continuous and that the corresponding weight $\widehat{w}$ (given by $d\hat{\mu} = \widehat{w}\,dx$) is an A_q weight. The number q obtained from [11] can be larger than p (and depends on μ as well as on the interval used in constructing $\hat{\mu}$). However, the only consequence we need from this step is that $\hat{\mu}$ is absolutely continuous and hence so is μ. Once the absolute continuity of μ is in place we instead use a direct argument to deduce the local A_p condition. To complete the proof we also need the fact from Björn–Björn [3] that locally p-admissible measures are semilocally p-admissible.

Having characterized the locally p-admissible measures μ on $\mathbf{R}$, it is also interesting to know how the minimal p-weak upper gradients behave for functions in the Newtonian Sobolev space $N^{1,p}(\mathbf{R},\mu)$. If u is locally absolutely continuous on some interval, then the fundamental theorem of calculus shows that $|u'|$ is an

upper gradient for u, and thus $g_u \le |u'|$ a.e. For Lipschitz functions u and arbitrary measures on $\mathbf{R}$, the minimal p-weak upper gradient g_u has been fully described in Di Marino–Speight [10, Theorem 2]. The following result addresses this question for general Newtonian functions and weights on $\mathbf{R}$.

Proposition 1.3. *Let μ be a measure on $\mathbf{R}$ and $1 < p < \infty$. Assume that $d\mu = w\,dx$ and $w, w^{1/(1-p)} \in L^1_{\mathrm{loc}}(I)$ for some (not necessarily open) interval $I \subset \mathbf{R}$. Then every $u \in N^{1,p}_{\mathrm{loc}}(I,\mu)$ is locally absolutely continuous on I and $g_u = |u'|$ a.e.*

In particular, the proposition applies if w is locally p-admissible on $\mathbf{R}$ (and thus a local A_p weight, by Theorem 1.2) with $p > 1$. Note that $N^{1,p}_{\mathrm{loc}}(I,\mu)$ is a refinement of the standard Sobolev space $W^{1,p}_{\mathrm{loc}}(I,w)$ considered in Heinonen–Kilpeläinen–Martio [16], see the discussion at the end of Section 2.

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2. Metric spaces

We are primarily interested in measures and weights on $\mathbf{R}$, but we will also need to use tools from first-order analysis on metric spaces. In this section we discuss the definitions used in metric spaces. For proofs of the facts stated in this section we refer the reader to Björn–Björn [2] and Heinonen–Koskela–Shanmugalingam–Tyson [18].

We assume throughout the paper that $1 \le p < \infty$ and that $X = (X, d, \mu)$ is a metric space equipped with a metric d and a positive complete Borel measure μ such that $0 < \mu(B) < \infty$ for all balls $B \subset X$. We assume throughout the paper that balls are open. We let $B = B(x,r) = \{y \in X : d(x,y) < r\}$ denote the ball with centre x and radius $r > 0$, and let $\lambda B = B(x, \lambda r)$. In metric spaces it can happen that balls with different centres and/or radii denote the same set; we will however adopt the convention that a ball B comes with a predetermined centre x_B and radius r_B.

We primarily deal with X being the real line $\mathbf{R}$, and in this case balls and bounded open intervals are the same objects. We will use both nomenclatures and notations.

We follow Heinonen and Koskela [17] in introducing upper gradients as follows (in [17] they are referred to as very weak gradients). A *curve* is a continuous mapping from an interval. We will only consider curves which are nonconstant, compact and rectifiable, i.e. of finite length. A curve can thus be parameterized by its arc length ds.

Definition 2.1. A nonnegative Borel function g on X is an *upper gradient* of an extended real-valued function u on X if for all curves $\gamma : [0, l_\gamma] \to X$,

$$|u(\gamma(0)) - u(\gamma(l_\gamma))| \le \int_\gamma g\,ds, \tag{2.1}$$

where we follow the convention that the left-hand side is considered to be ∞ whenever at least one of the terms therein is $\pm\infty$. If g is a nonnegative measurable function on X and if (2.1) holds for p-almost every curve (see below), then g is a *p-weak upper gradient* of u.

We say that a property holds for *p-almost every curve* if it fails only for a curve family Γ with *zero p-modulus*, i.e. there is a Borel function $0 \le \rho \in L^p(X)$ such that $\int_\gamma \rho\,ds = \infty$ for every curve $\gamma \in \Gamma$. The p-weak upper gradients were introduced in Koskela–MacManus [23]. It was also shown therein that if $g \in L^p_{\rm loc}(X)$ is a p-weak upper gradient of u, then one can find a sequence $\{g_j\}_{j=1}^\infty$ of upper gradients of u such that $\|g_j - g\|_{L^p(X)} \to 0$.

If u has an upper gradient in $L^p_{\rm loc}(X)$, then it has a *minimal p-weak upper gradient* $g_u \in L^p_{\rm loc}(X)$ in the sense that for every p-weak upper gradient $g \in L^p_{\rm loc}(X)$ of u we have $g_u \le g$ a.e., see Shanmugalingam [26]. The minimal p-weak upper gradient is well defined up to a set of measure zero.

Following Shanmugalingam [25], the *Newtonian space* $N^{1,p}(X) = N^{1,p}(X,\mu)$ is the collection of all measurable functions $u : X \to [-\infty,\infty]$, having an upper gradient in $L^p(X)$, such that

$$\|u\|_{N^{1,p}(X)} := \left(\int_X |u|^p \,d\mu + \int_X g_u^p \,d\mu \right)^{1/p} < \infty.$$

The space $N^{1,p}(X)/{\sim}$, where $u \sim v$ if and only if $\|u-v\|_{N^{1,p}(X)} = 0$, is a Banach space and a lattice, see [25]. Contrary to the usual a.e.-defined Sobolev functions, the functions in $N^{1,p}(X)$ are defined everywhere (with values in $[-\infty,\infty]$), and $u \sim v$ if and only if $u = v$ outside a set of p-capacity zero, which is important in Proposition 1.3.

In this paper, the letter C will denote various positive constants whose values may vary even within a line.

3. Doubling and Poincaré inequalities

We will discuss several notions of locally p-admissible measures on the real line, relations between them, and connections to local and global Muckenhoupt A_p weights. We give the following definitions. Let $B_0 = B(x_0, r_0)$.

The measure μ *is doubling within* B_0 if there is $C > 0$ (depending on x_0 and r_0) such that

$$\mu(2B) \le C\mu(B) \tag{3.1}$$

for all balls $B \subset B_0$.

The *p-Poincaré inequality holds within* B_0 if there are constants $C > 0$ and $\lambda \ge 1$ (depending on x_0 and r_0) such that for all balls $B \subset B_0$, all integrable functions u on λB, and all upper gradients g of u (or equivalently all p-weak upper gradients g of u),

$$\fint_B |u - u_B| \,d\mu \le C r_B \left(\fint_{\lambda B} g^p \,d\mu \right)^{1/p}, \tag{3.2}$$

where $u_B = \fint_B u\,d\mu = \int_B u\,d\mu/\mu(B)$. We also say that the *Lipschitz p-Poincaré inequality holds within* B_0 if (3.2) holds for all Lipschitz functions u on λB with

$$g(x) = \operatorname{Lip} u(x) = \limsup_{y \to x} \frac{|u(y) - u(x)|}{d(y,x)}.$$

The measure μ is (*Lipschitz*) *p-admissible within* B_0 if it is doubling and supports a (Lipschitz) p-Poincaré inequality within B_0. Moreover, w is an *A_p weight within* B_0 if

$$\fint_B w\,dx < C \begin{cases} \left(\fint_B w^{1/(1-p)}\,dx \right)^{1-p}, & 1 < p < \infty, \\ \operatorname*{ess\,inf}_B w, & p = 1, \end{cases} \tag{3.3}$$

for all balls $B \subset B_0$.

Note that whether a property holds within a ball or not depends also on the ambient space X, since $2B \setminus B_0$ or $\lambda B \setminus B_0$ may be nonempty. Unless said otherwise, the ambient space will be assumed to be $\mathbf{R}$ in this paper.

Furthermore, a property such as these above is *local* if for every $x \in X$ there are $R_x, C_x > 0$ and $\lambda_x \geq 1$ such that it holds within the ball $B(x, R_x)$ with constants C_x and λ_x. If it holds within every ball $B_0 \subset X$, with C and λ depending on B_0, it is *semilocal*. If, moreover, the constants C and λ are independent of B_0 then the property is *global*. If $d\mu = w\,dx$, we will sometimes say that w has a property if μ has it.

As $\operatorname{Lip} u$ is an upper gradient of u, the Lipschitz p-Poincaré inequality trivially follows from the standard p-Poincaré inequality (3.2) within any ball. If μ is globally doubling on a complete metric space, then μ supports a global Lipschitz p-Poincaré inequality if and only if it supports the global standard p-Poincaré inequality (3.2), by Keith [20, Theorem 2] (or [18, Theorem 8.4.2]). As we shall see, the corresponding equivalence is true also in the local (and semilocal) case on $\mathbf{R}$.

The p-Poincaré inequality (20.3) in Heinonen–Kilpeläinen–Martio [16, 2nd ed.] is weaker than the standard p-Poincaré inequality (3.2) but stronger than the Lipschitz p-Poincaré inequality defined above. On $\mathbf{R}$, in view of [20, Theorem 2] (or [18, Theorem 8.4.2]) and Theorem 4.1 below, the p-Poincaré inequality (20.3) in [16, 2nd ed.] is equivalent to both p-Poincaré inequalities considered in this paper, provided that μ is locally doubling.

It was shown in Hajłasz–Koskela [15] that in geodesic spaces with a doubling measure supporting a p-Poincaré inequality, the dilation constant λ in (3.2) can be taken to be 1. This is true also under the local assumptions on the doubling and Poincaré inequality considered here, and without assuming completeness, both for the standard and the Lipschitz p-Poincaré inequality, cf. Björn–Björn [3, Theorem 5.1]. In the rest of this paper, we will mainly be concerned with the real line $\mathbf{R}$, and in this case this can be deduced much more simply and generally. In particular, the doubling assumption is not required.

Proposition 3.1. *Let μ be a measure on $\mathbf{R}$. Assume that $I \subset \mathbf{R}$ is a bounded open interval such that the p-Poincaré inequality (3.2) holds for $B = I$ with dilation constant $\lambda \geq 1$ and constant $C_{\rm PI}$. Then (3.2) also holds for $B = I$ with dilation constant 1 and the same constant $C_{\rm PI}$.*

The proof can be easily modified for Lipschitz Poincaré inequalities, and also for so-called (q,p)-Poincaré inequalities. We assume implicitly, as always in this paper, that balls have finite and positive measure.

Proof. We may assume that $I = (-1, 1)$. Let u be a bounded integrable function on I and g be an upper gradient of u on I. For $0 < \varepsilon < 1$, we let

$$u_\varepsilon(x) = \begin{cases} u(x), & \text{if } |x| \leq 1-\varepsilon, \\ u(\pm(1-\varepsilon)), & \text{if } \pm x \geq 1-\varepsilon, \end{cases} \quad \text{and} \quad g_\varepsilon(x) = \begin{cases} g(x), & \text{if } |x| \leq 1-\varepsilon, \\ 0, & \text{if } |x| > 1-\varepsilon. \end{cases}$$

It is easy to see that g_ε is an upper gradient of u_ε. By dominated convergence, as u is bounded,

$$\begin{aligned} \fint_I |u - u_I|\,d\mu &= \lim_{\varepsilon\to 0} \fint_I |u_\varepsilon - (u_\varepsilon)_I|\,d\mu \leq \lim_{\varepsilon\to 0} C_{\rm PI} \biggl(\fint_{\lambda I} g_\varepsilon^p\,d\mu \biggr)^{1/p} \\ &\leq \lim_{\varepsilon\to 0} C_{\rm PI} \biggl(\fint_I g_\varepsilon^p\,d\mu \biggr)^{1/p} \leq C_{\rm PI} \biggl(\fint_I g^p\,d\mu \biggr)^{1/p}. \end{aligned}$$

By [2, Proposition 4.13], the p-Poincaré inequality holds also for unbounded integrable u. □

4. Proof of Theorem 1.2

The aim of this section is to show the following characterization of (semi)locally p-admissible measures by means of A_p weights. The corresponding global characterization was given in Björn–Buckley–Keith [7, Theorem 2].

Theorem 4.1. *Let μ be a measure on $\mathbf{R}$. Then the following are equivalent*:
(a) *μ is locally Lipschitz p-admissible*;
(b) *μ is locally p-admissible*;
(c) *μ is semilocally p-admissible*;
(d) *$d\mu = w\,dx$, where w is a local A_p weight*;
(e) *$d\mu = w\,dx$, where w is a semilocal A_p weight*;
(f) *$d\mu = w\,dx$, and for each bounded interval $I \subset \mathbf{R}$ there is a global A_p weight $\widetilde{w}$ on $\mathbf{R}$ such that $\widetilde{w} = w$ on I.*

Our first goal is to justify the implications (e) $\Rightarrow$ (f) and (d) $\Rightarrow$ (b). This will be implied by the following lemma and its corollary.

Lemma 4.2. *Assume that w satisfies the A_p condition (3.3) within the interval $I_0 = (0, M)$. Then the periodically reflected weight*

$$\widehat{w}(x) = \begin{cases} w(2kM - x), & x \in [(2k-1)M, 2kM], \\ w(x - 2kM), & x \in [2kM, (2k+1)M], \end{cases} \qquad k \in \mathbf{Z},$$

is a global A_p weight on $\mathbf{R}$.

For weights on $\mathbf{R}^n$, similar reflection results were obtained in Björn [1, Proposition 10.5] and Rychkov [24, Lemma 1.1], but since the proof on $\mathbf{R}$ becomes especially simple, we provide it here for the reader's convenience.

Proof. Let $I \subset \mathbf{R}$ be a bounded open interval. If $|I| \le M$ then I intersects at most two copies $I_k := (kM, (k+1)M)$ and I_{k+1} of I_0. We can assume that $k = 0$ and that $|I \cap I_0| \ge |I \cap I_1|$. Using the A_p condition (3.3) for w on $B = I \cap I_0$, we see that for $p > 1$,

$$\begin{aligned} \int_I \widehat{w}\,dx \biggl(\int_I \widehat{w}^{1/(1-p)}\,dx \biggr)^{p-1} &\le 2 \int_{I\cap I_0} w\,dx \biggl(2 \int_{I\cap I_0} w^{1/(1-p)}\,dx \biggr)^{p-1} \\ &\le 2^p C |I \cap I_0|^p \le 2^p C |I|^p. \end{aligned}$$

On the other hand, if $|I| > M$, then by translating I, we can assume that $I \subset (0, nM)$ with $nM < 3|I|$. Then

$$\begin{aligned} \int_I \widehat{w}\,dx \biggl(\int_I \widehat{w}^{1/(1-p)}\,dx \biggr)^{p-1} &\le n \int_{I_0} w\,dx \biggl(n \int_{I_0} w^{1/(1-p)}\,dx \biggr)^{p-1} \\ &\le n^p C |I_0|^p \le C |I|^p. \end{aligned}$$

After division by $|I|^p$, we obtain (3.3) for $\widehat{w}$ on both types of I.

For $p = 1$, the proof is similar using that $\operatorname{ess\,inf}_I w = \operatorname{ess\,inf}_{I\cap I_0} w$. □

Corollary 4.3. *If $d\mu = w\,dx$ satisfies the A_p condition (3.3) within a bounded open interval I_0 then μ is p-admissible within $\frac{1}{2}I_0$.*

For the converse implication see Theorem 4.6. In view of Proposition 3.1, we assume that the dilation constant $\lambda = 1$ from now on.

Proof. The extension $\widehat{w}$ of w, provided by Lemma 4.2, is a global A_p weight and thus globally p-admissible on $\mathbf{R}$, by Theorem 15.21 in Heinonen–Kilpeläinen–Martio [16] (for $p > 1$) and Theorem 4 in Björn [6] (for $p \ge 1$). It then follows that for every interval $I \subset \frac{1}{2}I_0$, we have $2I \subset I_0$ and thus

$$\mu(2I) = \hat{\mu}(2I) \le C\hat{\mu}(I) = C\mu(I),$$

by the doubling condition for $d\hat{\mu} = \widehat{w}\,dx$ within I_0. Moreover, the p-Poincaré inequality (3.2) holds for $\widehat{w}$ (and thus for w) on I. □

Most of the rest of this section is devoted to showing that (a) ⇒ (d), and simultaneously that (c) ⇒ (e), in Theorem 4.1.

Globally p-admissible measures on $\mathbf{R}$ are known to be global A_p weights and, in particular, absolutely continuous with respect to the Lebesgue measure, cf. Björn–Buckley–Keith [7]. Next, we obtain a similar characterization for locally Lipschitz p-admissible measures. This will be done using reflections and a flattening argument from Durand-Cartagena–Li [11]. Verifying p-admissibility for reflected measures turns out to be more involved than for the A_p condition in Lemma 4.2.

Lemma 4.4. *Assume that μ is Lipschitz p-admissible within the interval $(-M, 2M) \subset \mathbf{R}$ and define the measure $\hat{\mu}$ on $[-M, M]$ by*

$$\hat{\mu}(A) = \mu(A \cap [0, M]) + \mu(-A \cap [0, M]), \quad \text{where} \quad -A = \{x \in \mathbf{R} : -x \in A\}.$$

Let the metric space $(X, d, \hat{\mu})$ be obtained by identifying the endpoints $\pm M$ of the interval $[-M, M]$ with each other and inheriting the length metric from the circle of radius M/π. Then $\hat{\mu}$ is doubling and supports a p-Poincaré inequality on X.

Because of the local doubling property, μ and thus $\hat{\mu}$ is nonatomic. Note that the doubling and Poincaré constants for $\hat{\mu}$ depend on those for μ within $(-M, 2M)$. Example 4.7 below shows that it is not enough to assume that μ is p-admissible within the interval $(0, M)$, even though $\hat{\mu}$ only depends on $\mu|_{[0,M]}$.

Equivalently, $(X, d, \hat{\mu})$ can be obtained by letting

$$\hat{\mu}(A) = \mu(A \cap [0, M]) + \mu((M - A) \cap [0, M]),$$

where $A \subset [0, 2M]$ and $M - A = \{x \in \mathbf{R} : M - x \in A\}$, and identifying the points 0 and $2M$. Thus, the reflection points 0 and M play symmetric roles.

Proof. We begin by proving the doubling condition (3.1) and the p-Poincaré inequality (3.2) for $\hat{\mu}$ and $I = (x - r, x + r)$, where $x \in [-\frac{1}{2}M, \frac{1}{2}M]$ and $0 < r \le \frac{1}{4}M$. Intervals centred at $x \in X \setminus [-\frac{1}{2}M, \frac{1}{2}M]$, and of length at most $\frac{1}{2}M$, can be treated similarly by reflecting at M. Since X is compact, the global doubling and p-Poincaré inequality then follow for all $I \subset X$, by Proposition 1.2 and Theorem 1.3 in Björn–Björn [3].

By symmetry, we can assume that $0 \le x \le 2r$. (If $2r < x \le \frac{1}{2}M$ then $2I \subset [0, M]$ and the doubling property for $\hat{\mu}$ and I is immediate.) From the doubling property of μ within $(-M, 2M)$ it follows that the measures $\mu((0, a))$ and $\mu((-a, 0))$ are comparable for every $0 < a < M$. Namely, with C_d being the doubling constant within $(-M, 2M)$,

$$\mu((0, a)) \le \mu((-a, a)) \le C_d\mu\big(\big(-\tfrac{1}{2}a, \tfrac{1}{2}a\big)\big) \le C_d\mu\big(\big(-\tfrac{3}{2}a, \tfrac{1}{2}a\big)\big) \le C_d^2\mu((-a, 0)),$$

and similarly $\mu((-a, 0)) \le C_d^2\mu((0, a))$. Hence,

$$\hat{\mu}(2I) \le 2\mu((0, x + 2r)) \le 2\mu(2I) \le 2C_d\mu(I) \le 4C_d^3\mu(I \cap [0, M)) \le 4C_d^3\hat{\mu}(I)$$

and thus $\hat\mu$ is doubling on X.

We shall now prove the p-Poincaré inequality for $\hat\mu$. As above, and by symmetry, we let $I = (x-r, x+r)$ with $0 \le x \le \frac12 M$ and $r \le \frac14 M$. Since X is complete, it suffices to verify the Lipschitz p-Poincaré inequality on I, cf. Keith [20, Theorem 2] (or [18, Theorem 8.4.2]).

If $r \le x$, then $I \subset [0,M]$ and the Lipschitz p-Poincaré inequality for $\hat\mu$ follows directly from the one for μ. Assume therefore that $0 \le x < r \le \frac14 M$. Let u be a Lipschitz function on I. We can also assume that $u(0) = 0$ and thus $\hat u := u\chi_{(0,x+r)}$ is also Lipschitz on I. Let $I' = (-r, x+r)$. Since $\mu((-r,0))$, $\mu(I')$ and $\mu(I)$ are all comparable, because of the doubling property of μ, the proof of Lemma 2.1 in Kinnunen–Shanmugalingam [22] shows that

$$\fint_I |\hat u|\,d\hat\mu \le C \fint_{I'} |\hat u|\,d\mu \le C|I'| \biggl(\fint_{I'} (\operatorname{Lip}\hat u)^p\,d\mu \biggr)^{1/p} \le C|I| \biggl(\fint_I (\operatorname{Lip} u)^p d\hat\mu \biggr)^{1/p}.$$

The integral of $\tilde u := u\chi_{(x-r,0)}$ over I is estimated similarly using reflection, and hence

$$\fint_I |u-u(0)|\,d\hat\mu \le C \biggl(\fint_I |\hat u|\,d\hat\mu + \fint_I |\tilde u|\,d\hat\mu \biggr) \le C|I| \biggl(\fint_I (\operatorname{Lip} u)^p\,d\hat\mu \biggr)^{1/p}.$$

Finally, we note that $\fint_I |u-u_{I,\hat\mu}|\,d\hat\mu \le 2 \fint_I |u-u(0)|\,d\hat\mu$, see [2, Lemma 4.17]. □

Corollary 4.5. *If μ is Lipschitz p-admissible within an open interval $I_0 \subset \mathbf{R}$, then it is absolutely continuous with respect to the Lebesgue measure on I_0.*

Proof. Let $x \in I_0$ be arbitrary. By translation, we can assume that $x = 0$ and that $(-M, 2M) \subset I_0$ for some $M > 0$. Lemma 4.4 then shows that the metric space $(X, d, \hat\mu)$, obtained by reflecting μ at 0 and identifying the points $\pm M$ with each other, supports a p-Poincaré inequality with $\hat\mu$ doubling. Now, flattening $(X, d, \hat\mu)$ as in Durand-Cartagena–Li [11], we obtain $\mathbf{R}$ with the measure

$$d\tilde\mu(y) = \frac{d\hat\mu(y)}{\hat\mu((-|y|,|y|))} = \frac{d\hat\mu(y)}{2\mu((0,|y|))} \quad \text{for } y \in X,$$

which, by [11, Theorem 4.1], is q-admissible for some sufficiently large q. Theorem 2 in Björn–Buckley–Keith [7] then implies that $\tilde\mu$ must be an A_q weight on $\mathbf{R}$ and, in particular, it is absolutely continuous with respect to the Lebesgue measure. It follows that also $\hat\mu$ is absolutely continuous with respect to the Lebesgue measure on $[0,M]$. Applying this to all $x \in I_0$, with corresponding M, shows that μ is absolutely continuous with respect to the Lebesgue measure on I_0. □

Theorem 4.6. *Assume that μ is Lipschitz p-admissible within a bounded open interval I_0 and let $\theta > 1$. Then $d\mu = w\,dt$ for some nonnegative weight w and w is an A_p weight within $\theta^{-1}I_0$, with an A_p constant depending on θ and μ.*

Proof. Corollary 4.5 shows that $d\mu = w\,dt$ for some weight function w on I_0. Let $I = (x-r, x+r) \subset \theta^{-1}I_0$. Then $\theta I \subset I_0$.

First, we consider the case $p > 1$ and test the Lipschitz p-Poincaré inequality on θI with the Lipschitz function

$$u(y) := \int_{-\infty}^{y} \frac{w(t)\chi_I(t)}{(w(t)+\varepsilon)^{p/(p-1)}}\,dt,$$

where $\varepsilon > 0$ is fixed but arbitrary. Note that $u \equiv 0$ on the left component $I_L = (x-\theta r, x-r]$ of $\theta I \setminus I$ and that $u \equiv u(x+r)$ on the right component $I_R = [x+r, x+\theta r)$ of $\theta I \setminus I$. Hence, at least one of the following holds

$$|u-u_{\theta I}| \ge \tfrac12 u(x+r) \text{ on } I_L \quad \text{or} \quad |u-u_{\theta I}| \ge \tfrac12 u(x+r) \text{ on } I_R.$$

Since $\mu(I_L)$ and $\mu(I_R)$ are comparable to $\mu(\theta I)$, with comparison constant depending on θ, this implies that the left-hand side of the Lipschitz p-Poincaré inequality on θI is

$$\fint_{\theta I} |u - u_{\theta I}|\,d\mu \geq Cu(x+r) = C\int_I \frac{d\mu}{(w+\varepsilon)^{p/(p-1)}}.$$

At the same time, $\operatorname{Lip} u = |u'| \leq (w+\varepsilon)^{-1/(p-1)}\chi_I$ a.e. (by the fundamental theorem of calculus) and thus the right-hand side of the Lipschitz p-Poincaré inequality is

$$Cr\biggl(\fint_{\theta I} (\operatorname{Lip} u)^p\,d\mu\biggr)^{1/p} \leq Cr\biggl(\frac{1}{\mu(\theta I)}\int_I \frac{d\mu}{(w+\varepsilon)^{p/(p-1)}}\biggr)^{1/p}.$$

Combining the last two estimates with the Lipschitz p-Poincaré inequality yields

$$\mu(\theta I)^{1/p} \leq Cr\biggl(\int_I \frac{d\mu}{(w+\varepsilon)^{p/(p-1)}}\biggr)^{1/p-1} = Cr\biggl(\int_I \frac{w(t)\,dt}{(w(t)+\varepsilon)^{p/(p-1)}}\biggr)^{1/p-1}.$$

Raising the last estimate to the pth power, writing $\int_I w(t)\,dt = \mu(I) \leq \mu(\theta I)$, dividing by $|I| = 2r$ and letting $\varepsilon \to 0$ we obtain (3.3) for I, by monotone convergence.

Now, we consider $p = 1$ and let $m = \operatorname{ess\,inf}_I w$. Test the Lipschitz 1-Poincaré inequality on θI with the Lipschitz function

$$u(y) := \int_{-\infty}^{y} \chi_{E_\varepsilon}(t)\,dt,$$

where $E_\varepsilon = \{t \in I : w(t) < m+\varepsilon\}$ and $\varepsilon > 0$ is fixed but arbitrary. Then, as above, $u(x+r) = |E_\varepsilon|$ is majorized by the right-hand side in the Lipschitz 1-Poincaré inequality, i.e.

$$0 < |E_\varepsilon| \leq \frac{Cr}{\mu(\theta I)}\int_{\theta I} \chi_{E_\varepsilon}(t) w(t)\,dt \leq Cr\frac{(m+\varepsilon)|E_\varepsilon|}{\mu(I)}.$$

Dividing by $|E_\varepsilon| > 0$ and letting $\varepsilon \to 0$, yields (3.3) for I, i.e. within $\theta^{-1}I_0$. □

Proof of Theorem 4.1. (e) ⇒ (f) This follows from Lemma 4.2.

(f) ⇒ (e) ⇒ (d) and (b) ⇒ (a) These implications are trivial.

(d) ⇒ (b) This follows from Corollary 4.3.

(b) ⇒ (c) This follows from Proposition 1.2 and Theorem 1.3 in Björn–Björn [3].

(a) ⇒ (d) and (c) ⇒ (e) These implications follow from Theorem 4.6. □

Lemma 4.2 and Theorem 4.6 imply that if μ is p-admissible within $(-\theta M, \theta M)$ for some $\theta > 1$, then the periodically repeated reflections of $\mu|_{[-M,M]}$ provide a p-admissible measure on $\mathbf{R}$. For the above arguments to hold it is important that p-admissibility is assumed within a larger interval. Next, we give an example showing that it is not enough to assume p-admissibility within $(-M, M)$.

Example 4.7. Let $X = [0,1]$, $w(x) = x^\alpha$, $\alpha \geq 0$, and $d\mu = w\,dx$. Then μ is doubling on X. By Chua–Wheeden [8, Theorem 1.4], μ supports a 1-Poincaré inequality for the interval $(a,b) \subset [0,1]$ with $\lambda = 1$ and the optimal constant

$$\begin{aligned} C &= \frac{2}{(b-a)\mu(a,b)}\biggl\|\frac{\mu((a,x))\mu((x,b))}{w(x)}\biggr\|_{L^\infty(a,b)} \leq \frac{2}{(b-a)}\biggl\|\frac{\mu((a,x))}{w(x)}\biggr\|_{L^\infty(a,b)} \\ &= \frac{2}{(b-a)}\biggl\|\frac{x^{1+\alpha}-a^{1+\alpha}}{(1+\alpha)w(x)}\biggr\|_{L^\infty(a,b)} = \frac{2}{(b-a)}\biggl\|\frac{(x-a)\xi_x^\alpha}{x^\alpha}\biggr\|_{L^\infty(a,b)} \leq 2, \end{aligned}$$

where $\xi_x \in (a,x)$ comes from the mean-value theorem. As this holds for all intervals $(a,b) \subset [0,1]$, μ supports a 1-Poincaré inequality with respect to the metric space

$[0,1]$, with constant 2. It follows that μ, extended by 0 outside $[0,1]$, is p-admissible within $(0,1)$.

If $p < 1+\alpha$, then $w^{1/(1-p)}$ is not integrable at 0, and hence the A_p condition (3.3) does not hold for w within $(0,1)$. It is also easily verified that the set $\{0\}$ has zero capacity with respect to the metric space $[0,1]$. This implies that the collection of all nonconstant compact rectifiable curves in $[0,1]$ starting at 0 has p-modulus zero (see [2, Proposition 1.48]). Hence, $\chi_{(0,\infty)}$ has 0 as a p-weak upper gradient on $(\mathbf{R},\hat{\mu})$ for any extension $\hat{\mu}$ of μ to $\mathbf{R}$, which violates the p-Poincaré inequality on the interval $(-1,1)$. Thus, μ is *not* a restriction of any measure on $\mathbf{R}$ supporting a p-Poincaré inequality.

5. Consequences of Theorem 1.2

Corollary 5.1. *Let μ_j, $j = 1,2$, be locally p-admissible measures on $\mathbf{R}$. Then $\mu = \mu_1 + \mu_2$ is locally p-admissible on $\mathbf{R}$.*

Corollary 5.2. *Let w_j, $j = 1,2$, be locally p-admissible weights on $\mathbf{R}$. Then $\max\{w_1,w_2\}$ and $\min\{w_1,w_2\}$ are locally p-admissible on $\mathbf{R}$. Moreover, for $p > 1$, the weight $w_1^{1/(1-p)}$ is locally $p/(p-1)$-admissible on $\mathbf{R}$.*

These statements follow directly from the characterizations in Theorem 1.2 together with similar statements for global A_p weights. The lattice property of global A_p weights on $\mathbf{R}^n$ was proved in Kilpeläinen–Koskela–Masaoka [21, Proposition 4.3] using nontrivial characterizations of A_p and A_∞ weights. Here we seize the opportunity to provide an elementary proof, including $p = 1$ and also covering the local case.

It is straightforward that the A_p condition

$$\fint_B w\,dx < C \begin{cases} \left(\fint_B w^{1/(1-p)}\,dx\right)^{1-p}, & 1 < p < \infty, \\ \operatorname*{ess\,inf}_B w, & p = 1, \end{cases} \tag{5.1}$$

for w is precisely the $A_{p/(p-1)}$ condition for the conjugate weight $w^{1/(1-p)}$ with the $A_{p/(p-1)}$ constant $C^{1/(p-1)}$ when $p > 1$.

Lemma 5.3. *Assume that the A_p condition holds for w_1 and w_2 with a constant C in some ball $B \subset \mathbf{R}^n$. Then it holds also for w_1+w_2, $\max\{w_1,w_2\}$ and $\min\{w_1,w_2\}$ with constants $2C$, $2C$ and $2^{p-1}C$, respectively.*

Proof. We have

$$\fint_B \max\{w_1,w_2\}\,dx \le \fint_B (w_1+w_2)\,dx = \fint_B w_1\,dx + \fint_B w_2\,dx,$$
$$\fint_B \min\{w_1,w_2\}\,dx \le \min\left\{\fint_B w_1\,dx, \fint_B w_2\,dx\right\}.$$

For $p = 1$, (5.1) then follows directly from the facts that

$$\operatorname*{ess\,inf}_B w_1 + \operatorname*{ess\,inf}_B w_2 \le 2\operatorname*{ess\,inf}_B \max\{w_1,w_2\} \le 2\operatorname*{ess\,inf}_B (w_1+w_2),$$
$$\min\left\{\operatorname*{ess\,inf}_B w_1, \operatorname*{ess\,inf}_B w_1\right\} = \operatorname*{ess\,inf}_B \min\{w_1,w_2\}.$$

For $p > 1$, we have

$$\fint_B w_1\,dx + \fint_B w_2\,dx < C\left(\fint_B w_1^{1/(1-p)}\,dx\right)^{1-p} + C\left(\fint_B w_2^{1/(1-p)}\,dx\right)^{1-p}.$$

Since $1-p<0$ and

$$\fint_B w_j^{1/(1-p)}\,dx \ge \fint_B \max\{w_1,w_2\}^{1/(1-p)}\,dx \ge \fint_B (w_1+w_2)^{1/(1-p)}\,dx$$

for $j=1,2$, this proves (5.1) for w_1+w_2 and $\max\{w_1,w_2\}$.

To prove (5.1) for $\min\{w_1,w_2\}$, we note that

$$\min\{w_1,w_2\}^{1/(1-p)} = \max\bigl\{w_1^{1/(1-p)}, w_2^{1/(1-p)}\bigr\},$$

which by the above argument satisfies the $A_{p/(p-1)}$ condition with $2C^{1/(p-1)}$. The duality between (5.1) and the $A_{p/(p-1)}$ condition concludes the proof. □

6. Proof of Proposition 1.3

Proof of Proposition 1.3. By considering a smaller interval if necessary, we can assume that I is closed, $u\in N^{1,p}(I,\mu)$ and $w, w^{1/(1-p)}\in L^1(I)$. Let $g\in L^p(I,\mu)$ be an upper gradient of u. Let $\varepsilon>0$ be arbitrary and find $\delta>0$ so that

$$\int_E w^{1/(1-p)}\,dx<\varepsilon \quad \text{whenever } E\subset I \text{ and } |E|<\delta. \tag{6.1}$$

Consider finitely many pairwise disjoint intervals $(a_j,b_j)\subset I$ with

$$\sum_j |b_j-a_j|<\delta.$$

The Hölder inequality then yields for each j,

$$|u(b_j)-u(a_j)| \le \int_{a_j}^{b_j} g\,dx \le \biggl(\int_{a_j}^{b_j} g^p w\,dx\biggr)^{1/p}\biggl(\int_{a_j}^{b_j} w^{1/(1-p)}\,dx\biggr)^{1-1/p}. \tag{6.2}$$

Summing over all j and using (6.1) and the Hölder inequality for sums, we conclude that

$$\begin{aligned}\sum_j |u(b_j)-u(a_j)| &\le \biggl(\sum_j \int_{a_j}^{b_j} g^p\,d\mu\biggr)^{1/p}\biggl(\sum_j \int_{a_j}^{b_j} w^{1/(1-p)}\,dx\biggr)^{1-1/p}\\ &\le \biggl(\int_I g^p\,d\mu\biggr)^{1/p}\varepsilon^{1-1/p}.\end{aligned}$$

Since $\varepsilon>0$ was arbitrary and $g\in L^p(I,\mu)$, we conclude that u is locally absolutely continuous (and thus a.e. differentiable) on I.

It remains to show that $|u'|\le g_u$ a.e., since the converse inequality is trivial. Let $x\in\operatorname{int} I$. Replacing (a_j,b_j) in (6.2) by $(x-h,x+h)\subset I$, we see that for all upper gradients $g\in L^p(I,\mu)$,

$$\frac{|u(x+h)-u(x-h)|}{2h} \le \biggl(\frac{1}{2h}\int_{x-h}^{x+h} g^p w\,dx\biggr)^{1/p}\biggl(\frac{1}{2h}\int_{x-h}^{x+h} w^{1/(1-p)}\,dx\biggr)^{1-1/p}.$$

Letting $h\to 0$, together with the observation that a.e. $x\in I$ is a point of differentiability of u as well as a Lebesgue point both of $g^p w$ and of $w^{1/(1-p)}$, shows that $|u'|\le g$ a.e. As this holds for all upper gradients g of u, and there is a sequence $\{g_j\}_{j=1}^\infty$ of upper gradients tending to g_u pointwise a.e., we conclude that $|u'|\le g_u$ a.e., and thus $|u'|\in L^p(I,\mu)$. □

7. Uniform assumptions

Sometimes it can be of interest to consider (semi)locally admissible measures with uniform constants. We therefore introduce the following notions.

Definition 7.1. Any of the properties considered in Section 3 is *uniformly local* if there are $R, C > 0$ and $\lambda \ge 1$ such that the property holds within every ball $B_0 \subset X$ of radius R, with the same constants C and λ.

The property is *semiuniformly local* if for every x it holds within some ball $B(x, R_x)$ with constants C and λ independent of x and R_x.

If it holds within every ball B_0 with C and λ depending on the radius (but not the centre) of B_0, then it is *uniformly semilocal*.

Uniformly local A_p weights were studied by Rychkov [24] under the name "local A_p weights" (primarily for the specific radius $R = 1$).

A careful analysis of the proofs in this paper shows that the involved constants depend on each other in a controllable way. This, in particular, means that the implications (f) $\Leftrightarrow$ (e) $\Rightarrow$ (d) $\Leftrightarrow$ (b) $\Leftrightarrow$ (a) and (c) $\Rightarrow$ (e) in Theorem 4.1 hold also if the (semi)local notions are replaced by uniformly (semi)local ones, and (f) is replaced by its uniform version, where the global A_p constant of the extension $\widetilde{w}$ may depend on r_B, but not on the centre of B.

Moreover, the covers used in the proofs of [3, Proposition 1.2 and Theorem 1.3] (leading to the implication (b) $\Rightarrow$ (c) in Theorem 4.1) can be controlled by constants which only depend on C, λ and the involved radii, but not on x. Since the other estimates therein are quantitative as well, also the implication (b) $\Rightarrow$ (c) in Theorem 4.1 holds for uniformly (semi)locally p-admissible measures on $\mathbf{R}$.

Note that the "uniform" properties require uniformity both in C and R, while the semiuniformity allows R_x to depend on x. In Björn–Björn [3, Section 6], this property was shown to be sufficient for many qualitative, as well as some quantitative, properties of p-harmonic functions, but it is not strong enough for the uniform conclusions above. In fact, any positive continuous weight on $\mathbf{R}$ is semiuniformly locally p-admissible, but the weight $e^{e^{|x|}}$ is not even uniformly locally doubling.

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