This article has been accepted for inclusion in a future issue of this journal. Content is final as presented, with the exception of pagination.

MSE Analysis of a Multi-Loop LMS Pseudo-Random Noise Canceler for Mixed-Signal Circuit Calibration

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***Abstract*—This paper applies new analytical techniques to evaluate the stability and mean-square error (MSE) convergence of a multi-loop LMS pseudo-random noise canceller which applies to a variety of known mixed-signal circuit calibration techniques. To the authors' knowledge, it is the first published MSE analysis of any multi-loop LMS system, and, unlike most published MSE analyses of single-loop LMS systems, it does not make any simplifying assumptions. The analysis proves that the noise canceler can be made unconditionally stable by design, and provides guidance on how to choose design parameters to achieve a desired level of noise cancellation.**

***Index Terms*—LMS, adaptive filters.**

I. Introduction

Statistical calibration techniques are widely used in mixed-signal circuit blocks, including ADCs, DACs, and PLLs, to suppress error that would otherwise be caused by non-ideal analog circuit behavior such as component mismatches, gain errors, and nonlinearity. Typically, they use pseudo-random calibration sequences or pseudo-random component scrambling to cause the targeted types of non-ideal analog circuit behavior to contribute noise that they correlate against and cancel in the digital domain.

The signal processing performed by several known statistical calibration techniques can be modeled as special cases of the multi-loop least mean square (LMS) noise canceler shown in Fig. 1. The system's objective is to adaptively cancel a noise sequence, $r_e[n]$, from its input, $r_{ideal}[n] + r_e[n]$, such that its output, $r[n]$, well approximates $r_{ideal}[n]$. The noise sequence has the form

$$r_e(t) = \sum_{k=1}^{L} b_k[n] * S_k[n-P], \tag{1}$$

where P is a non-negative design-dependent integer, $b_k[n]$ for each k is the impulse response of a stable linear time-invariant (LTI) system, and $S_k[n-P]$ for each k is a zero-mean random sequence. The signs of $S_k[n-P]$ for all n and k are zero-mean independent random variables. The magnitudes of $S_k[n-P]$ for all n and k are 0 or 1 and independent of

Manuscript received October 18, 2019; revised January 21, 2020 and March 27, 2020; accepted March 30, 2020. This work was supported by the National Science Foundation under Grant 1909678. This article was recommended by Associate Editor P. Rombouts. *(Corresponding author: Ian Galton.)*

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Digital Object Identifier 10.1109/TCSI.2020.2985277

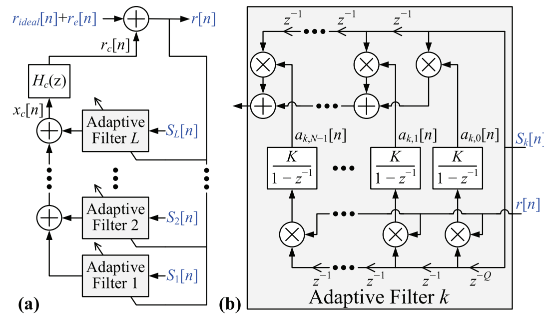

Fig. 1. a) Multi-loop LMS noise canceler, b) details of the kth adaptive filter.

$r_{ideal}[n] - \mathrm{E}\{r_{ideal}[n]\}$ although they may be a function of $\mathrm{E}\{r_{ideal}[n]\}$, where $\mathrm{E}\{r_{ideal}[n]\}$ denotes the expected value of $r_{ideal}[n]$. The $S_k[n]$ sequences are known to the system a priori, but the $b_k[n]$ impulse responses are not known to the system a priori, and in some cases neither is the transfer function, $H_c(z)$, of the shared portion of the LMS feedback loops. The purpose of the kth adaptive filter's feedback loop is to cancel the kth term in (1).

The system of Fig. 1 reduces to the classical single-loop LMS noise canceler when $L = 1$ and $H_c(z) = z^{-1}$ [1]–[3]. Unfortunately, the system is much harder to analyze when $L > 1$ or $H_c(z)$ is other than a unit delay. The interactions and potential correlations among the multiple feedback loops greatly complicate the analysis when $L > 1$, and the stability of the system is much harder to analyze when $H_c(z)$ introduces multiple delays.

To the knowledge of the authors, [4] presents the only previously published mathematical analysis of any multi-loop LMS noise canceler. It proves that that the expected value of each $a_{k,m}[n]$ coefficient in Fig. 1 converges to its ideal value for sufficiently small positive values of K, but neither it nor any other prior publication has analyzed the mean square error (MSE) of these coefficients. This is a significant limitation because the accuracy of the noise cancellation is directly related to these MSEs. Moreover, the analysis in [4] does not exclude the possibility that the MSEs are unbounded, so it leaves open the possibility that the system is not even stable in some cases.

Without analytical results, computer simulations are the only way to evaluate the stability and MSE of mixed-signal circuit calibration techniques that implement the noise canceler. Unfortunately, complicated mixed-signal circuits such as calibrated ADCs and DACs are notoriously time-consuming to simulate, so simulation only allows spot-check predictions of noise canceler stability and MSE performance for specific use cases. This is undesirable in practice, because it adds uncertainty to design-phase predictions of performance and yield.

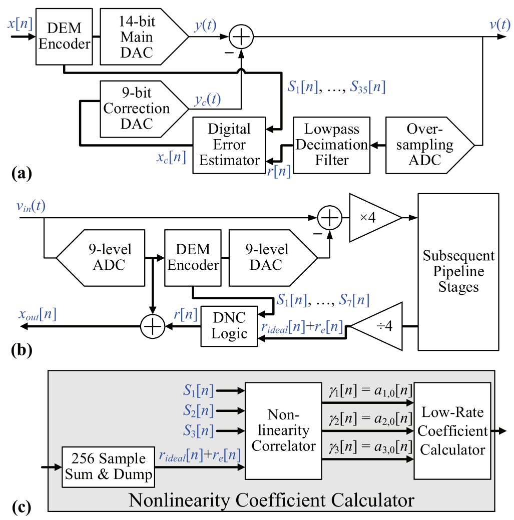

Fig. 2. Example applications a) continuous-time DAC, b) pipelined ADC, and c) nonlinearity coefficient calculator for a VCO-based ADC.

This paper addresses this issue. It applies a new analysis methodology to derive tight upper bounds on the $a_{k,m}[n]$ coefficient MSEs, and estimates the corresponding data converter signal-to-noise ratio (SNR) limit from residual post cancellation noise for an example application. The results prove that the noise canceler can be made unconditionally stable by design, and provide guidance on how to choose design parameters to achieve a desired level of noise cancellation. Unlike most prior adaptive filter analyses, the analysis in the paper is completely rigorous and does not rely on simplifying assumptions.

II. Example Applications

Fig. 2 shows high-level details of three example systems enabled by calibration techniques that implement the noise canceler shown in Fig. 1. They are the subjects of several prior publications, so only the details necessary to demonstrate their relevance to the results of this paper are repeated below.

Fig. 2a shows the high-level structure of a 14-bit main DAC with a mismatch noise cancellation (MNC) feedback path that measures and cancels the main DAC's static and dynamic error from component mismatches over its first Nyquist band [4], [5]. Unlike other published DAC calibration techniques that address both static and dynamic error from component mismatches such as [6]–[9], it can be used for background calibration in addition to foreground calibration.

The MNC feedback path implements the L feedback paths in the noise canceler of Fig. 1a: the digital error estimator implements the L adaptive filters, and the transfer function between the correction DAC input and decimation filter output is $-H_c(z)$. The decimation filter output can be written as $r[n] = r_{ideal}[n] + r_e[n] + r_c[n]$, where $r_{ideal}[n]$ is the decimation filter output sequence that would have occurred in the absence of both the MNC feedback loop and the main DAC's component mismatches, $r_e[n]$ is pseudo-random noise caused by the main DAC's component mismatches, and $r_c[n]$ is the MNC feedback path's noise cancellation sequence.

The main DAC's dynamic element matching (DEM) encoder causes $r_e[n]$ to have the form of (1). The $S_k[n]$ sequences, which are known because they are explicitly generated within the DEM encoder, have the properties described in the Introduction and, additionally, are restricted to values of -1, 0 and 1. The $b_k[n]$ impulse responses are not known a priori because they depend on the component mismatches. The $H_c(z)$ transfer function is not known precisely because it depends on analog circuitry, but it can be determined approximately from transistor-level simulations performed during design. The integer Q in the z^{-Q} delay in Fig. 1b is the value of n that maximizes $|h_c[n]|$, where $h_c[n]$ is the inverse z-transform of $H_c(z)$.

In the IC presented in [5], $K = 6 \cdot 10^{-7}$, $L = 35$, $N = 9$, $P = 3$, and $Q = 21$. As shown in [5], the system is insensitive to error from both the 9-bit correction DAC and the oversampling ADC, so its behavior well-approximates that of the noise canceler of Fig. 1 with these values of K, L, N, P, and Q.

Fig. 2b shows a simplified diagram of a pipelined ADC with a calibration technique called DAC noise cancellation (DNC) applied to the first stage [10]–[12]. The 9-level first-stage is shown explicitly, and the subsequent stages, which together act as a moderate-resolution ADC in their own right, are lumped together in the figure. In such pipelined ADCs, the 9-level DAC's component mismatches typically limit overall ADC accuracy to less than about 10 bits unless the error is canceled.

The DNC logic performs such cancellation. It can be implemented in the form of the noise canceler of Fig. 1. The mismatch noise introduced by the 9-level DAC as seen by the DNC logic, i.e., $r_e[n]$, has the form of (1) with $L = 7$ and $S_k[n]$ sequences like those in the example system of Fig 2a. Most pipelined ADCs are based on switched-capacitor circuitry in which case $b_k[n]$ is non-zero for only one value of n, so each of the adaptive filters in Fig. 1 only needs one tap, i.e., $N = 1$. As the DNC logic is entirely digital, it is possible to implement it with a single loop delay, i.e., with $H_c(z) = z^{-1}$. However, in practice it is often convenient to use $H_c(z) = z^{-Q}$ with $Q > 1$ to relax digital circuit timing constraints.

Fig. 2c shows a portion of the nonlinearity calibration technique used in the VCO-based ADC presented in [13]. The calibration technique measures the ADC's gain error and its second-order and third-order nonlinearity coefficients by applying a calibration sequence, $c[n] = t_1[n] + t_2[n] + t_3[n]$, to the ADC's input, and correlating successive sums of 256 ADC output values against $S_1[n] = t_1[n], S_2[n] = t_1[n]t_2[n]$, and $S_3[n] = t_1[n]t_2[n]t_3[n]$, where $t_1[n]$, $t_2[n]$, and $t_3[n]$ are two-level pseudo-random sequences that are well-modeled as sequences of independent, zero-mean random variables.

The nonlinearity correlator within Fig. 2c can be implemented as the noise canceler of Fig. 1 with $L = 3$. The ADC presented in [13] is such that $r_e[n]$ has the form of (1) with $L = 3$ and each $b_k[n]$ is non-zero for only one value of n, so $N = 1$. As with the DNC technique, the noise canceler is entirely digital, and $H_c(z) = z^{-Q}$ with $Q > 1$ can be used to relax digital timing constraints. Instead of using $r[n]$ directly, the calibration technique uses the accumulator outputs, $a_{1,0}[n]$, $a_{2,0}[n]$, and $a_{3,0}[n]$, to calculate coefficients with which it subsequently inverts the ADC's nonlinearity.

In addition to the three example systems shown in Fig. 2, the results of this paper are applicable to the widely used gain error correction (GEC) calibration technique for pipelined ADCs [11]. With minor modifications they can also be applied to other applications such as those described in [14] and [15].

III. Mean-Square Convergence Analysis

This section and the appendices apply a new analysis methodology to evaluate the simultaneous MSE convergence of all $L \times N$ adaptive filter coefficients in the noise canceler of Fig. 1. The analysis bounds the maximum combined MSE among all L adaptive filters at each time n. The underlying difference equations are too complicated to write explicitly, particularly when $H_c(z)$ is not a simple delay, so to make the analysis tractable only those of their properties that are necessary to prove the final results are derived. The difference equations are time-varying and contain a huge number of correlated terms, many of which oppose convergence. The analysis decomposes them into infinite sums of homogeneous difference equations, each with one of the correlated terms as an initial condition, and applies several induction-based techniques to bound their contribution to the overall system's MSE evolution considering their correlations.

The *convergence error* of each accumulator in Fig. 1b is

$$z_{k,m}[n] = a_{k,m}[n] - a'_{k,m}, \tag{2}$$

where $a'_{k,m}$ is the expected value of $a_{k,m}[n]$ in the limit as $n \to \infty$ [4]. It is convenient for the following analysis to group the $z_{k,m}[n]$ sequences into L N-dimensional vectors given by

$$\mathbf{z}_k[n] = \begin{bmatrix} z_{k,0}[n] & z_{k,1}[n+1] & \cdots & z_{k,N-1}[n+N-1] \end{bmatrix}^T, \tag{3}$$

for $k = 1, 2, \ldots, L$, and for any real random vector $\mathbf{x} = [x_1 x_2 \cdots x_N]^T$ to define the corresponding *RMS vector* as

$$\boldsymbol{\sigma}(\mathbf{x}) = \left[\sqrt{\mathrm{E}\left\{x_1^2\right\}} \quad \sqrt{\mathrm{E}\left\{x_2^2\right\}} \quad \cdots \quad \sqrt{\mathrm{E}\left\{x_N^2\right\}} \right]^T, \tag{4}$$

where $\mathrm{E}\{x_i^2\}$ is the expected value of x_i^2. The objective of the analysis is to evaluate the steady-state behavior of $||\boldsymbol{\sigma}(\mathbf{z}_k[n])||^2$, i.e., the behavior of $||\boldsymbol{\sigma}(\mathbf{z}_k[n])||^2$ for large enough n that $\mathrm{E}\{\mathbf{z}_k[n]\}$ has converged to zero, where

$$\|\boldsymbol{\sigma}(\mathbf{z}_k[n])\| = \sqrt{\sum_{m=0}^{N-1} \mathrm{E}\left\{z_{k,m}^2[n+m]\right\}} \tag{5}$$

is the *L2 norm* of $\boldsymbol{\sigma}(\mathbf{z}_k[n])$. The different time shifts of the elements of $\mathbf{z}_k[n]$ are present only because they simplify the subsequent analysis. They do not affect the results of the analysis because the objective is to analyze the steady-state behavior of the system.

The analysis uses the matrix spectral norm, which for any real $N \times N$ matrix $\mathbf{A}$, is denoted as $||\mathbf{A}||_2$ and is defined as the square root of the maximum Eigenvalue of $\mathbf{A}\mathbf{A}^{\mathrm{H}}$, where $\mathbf{A}^{\mathrm{H}}$ is the conjugate transpose of $\mathbf{A}$ [16]. The definition implies that

$$\|\mathbf{A}\|_2 = \max_{\|\mathbf{v}\| \neq 0} \left\{ \|\mathbf{A}\mathbf{v}\| / \|\mathbf{v}\| \right\}, \tag{6}$$

where the maximum is taken over all deterministic non-zero real $N \times 1$ vectors, $\mathbf{v}$ [16].

Without loss of generality, the $S_k[n]$ sequences in (1) are taken to satisfy

$$S_k[n] = 0 \quad \text{for } n < -N. \tag{7}$$

As can be verified from Fig. 1, (7) implies that $r_c[n] = 0$ and the accumulators in Fig. 1 remain unchanged for all $n < -N$. Hence, the condition can be viewed as "turning on" the noise canceler at time $n = -N$. Otherwise, no assumptions are made regarding the initial conditions of the system.

It follows from Fig. 1 and (1) that

$$a_{k,m}[n] = a_{k,m}[n-1] + K u_{k,m}[n] \tag{8}$$

for each $k = 1, 2, \ldots, L$, each $m = 0, 1, \ldots, N-1$, and every $n \geq 0$, where

$$\begin{aligned} u_{k,m}[n] &= S_k[n-m-Q]\Bigg(r_{ideal}[n] + \sum_{l=1}^{L}\sum_{i=-\infty}^{\infty} b_l[i] S_l[n-P-i] \\ &\quad - \sum_{l=1}^{L}\sum_{i=-\infty}^{\infty}\sum_{j=0}^{N-1} h_c[i] a_{l,j}[n-i] S_l[n-i-j] \Bigg), \end{aligned} \tag{9}$$

and $h_c[n]$ is the inverse z-transform of $H_c(z)$. The system is causal and the feedback loops must be delay-free, so $h_c[n] = 0$ and $b_k[n] = 0$, for all $n \leq 0$.

As described above, the noise canceler is "turned on" at time $n = -N$ with otherwise arbitrary initial conditions. Therefore, the $N \cdot L$ difference equations given by (8) together with initial conditions

$$\{a_{l,m}[p];\ l = 1, 2, \ldots, L,\ m = 0, 1, \ldots, N-1,\ -N \leq p \leq -1\} \tag{10}$$

completely describe the $N \cdot L$ $a_{k,m}[n]$ sequences for $n \geq 0$.

Combining (2), (3), (8), and (9) gives

$$\begin{aligned} \mathbf{z}_k[n] &= \mathbf{z}_k[n-1] + K\mathbf{e}_{k,n} \\ &\quad - K \sum_{i=-\infty}^{\infty}\sum_{l=1}^{L} S_k[n-Q]\, S_l[n-i]\, \mathbf{H}_{\mathbf{c}\_i} \mathbf{z}_l[n-i], \end{aligned} \tag{11}$$

for each $k = 1, 2, \ldots, L$ and all $n \geq 0$, where

$$\begin{aligned} \mathbf{e}_{k,n} &= S_k[n-Q]\, \mathbf{r}[n] \\ &\quad + \sum_{i=-\infty}^{\infty}\sum_{l=1}^{L} S_k[n-Q]\, S_l[n-i]\, (\mathbf{b}_{l\_i} - \mathbf{H}_{\mathbf{c}\_i}\mathbf{a}'_l), \end{aligned} \tag{12}$$

$$\mathbf{b}_{l\_i} = \begin{bmatrix} b_l[i-P] & b_l[i-P+1] & \cdots & b_l[i-P+N-1] \end{bmatrix}^T, \tag{13}$$

$$\mathbf{r}[n] = \begin{bmatrix} r_{ideal}[n] & r_{ideal}[n+1] & \cdots & r_{ideal}[n+N-1] \end{bmatrix}^T, \tag{14}$$

$$\mathbf{a}'_l = \begin{bmatrix} a'_{l,0} & a'_{l,1} & \cdots & a'_{l,N-1} \end{bmatrix}^T, \tag{15}$$

and

$$\mathbf{H}_{\mathbf{c}\_i} = \begin{bmatrix} h_c[i] & h_c[i-1] & \cdots & h_c[i-N+1] \\ h_c[i+1] & h_c[i] & \cdots & h_c[i-N+2] \\ \vdots & \vdots & \ddots & \vdots \\ h_c[i+N-1] & h_c[i+N-2] & \cdots & h_c[i] \end{bmatrix}. \tag{16}$$

Equation (2) implies that the elements of $\mathbf{a}'_l$ are the lth adaptive filter's optimal coefficients in the sense that they are the values of $a_{l,m}[n]$ for which $\mathbf{z}_l[n] = \mathbf{0}$. As shown in [4], $\mathbf{a}'_l$ satisfies

$$\mathbf{b}_{l\_Q} - \mathbf{H}_{\mathbf{c}\_Q}\mathbf{a}'_l = \mathbf{0} \tag{17}$$

because, as implied by (12), this causes $\mathrm{E}\{\mathbf{e}_{l,n}\} = \mathbf{0}$.

Each term in (11) that contains $\mathbf{z}_l[n-i]$ has a factor of $S_l[n-i]$, and (7) implies that $S_l[n-i] = 0$ when $n-i < -N$, so (11) has no dependence on $\mathbf{z}_l[p]$ for any $p < -N$. Therefore, without loss of generality, $\mathbf{z}_l[p]$ can be *defined* as $\mathbf{0}$ for all $p < -N$. It follows that $\mathbf{z}_k[n]$ for each $k = 1, 2, \ldots, L$

is specified for all $n \geq 0$ by the vector difference equation (11) with initial conditions

$$\mathbf{z}_l[p] = \begin{cases} \mathbf{z}_{l,p}, & \text{for} -N \leq p \leq -1 \\ \mathbf{0}, & \text{for } p < -N \end{cases} \quad l = 1, 2, \ldots, L. \tag{18}$$

Definitions (2) and (3) imply that each $\mathbf{z}_{l,p}$ equals $-\mathbf{a}'_l$ plus a vector, each element of which is either one of the initial conditions given by (10) or one of the values given by

$$\{a_{l,m}[p]; \ l = 1, 2, \ldots, L, \ m = 0, 1, \ldots, N-1, \ 0 \leq p \leq m-1\}. \tag{19}$$

Each element of (19) can be obtained by starting from the initial conditions in (10) and recursively applying (8) to itself a finite number of times, so all the $\mathbf{z}_{l,p}$ vectors are finite.

The difference equation (11) provides an expression for $\mathbf{z}_k[n]$ that contains terms proportional to $\mathbf{z}_l[p]$ where $p = n - i$, but $p \geq n$ for some of these terms. This complicates the solution of (11). In principle, (11) can be substituted recursively into itself to obtain a modified difference equation that only contains terms proportional to $\mathbf{z}_l[p]$ with $p < n$, but the modified difference equation is intractably complicated. Instead, the following lemma, which is proven in Appendix A, provides the pertinent features of the modified difference equation without requiring its full derivation.

Lemma 1: If $|b_k[n]|$ for $k = 1, 2, \ldots, L$ and $|h_c[n]|$ are each bounded by a sequence that decays exponentially to zero as $n \to \infty$, then for each $k = 1, 2, \ldots, L$, any given set of initial conditions (18), and all $n \geq 0$, (11) can be written as

$$\mathbf{z}_k[n] = \mathbf{z}_k[n-1] - \sum_{r=1}^{N} K^r \mathbf{H}_{r,k}\left(\mathbf{z}_l[p]|_{p \leq n-1, 1 \leq l \leq L}\right) + K\mathbf{e}_{k,n} + K^2\mathbf{v}_{k,n} \tag{20}$$

and (12) can be written as

$$\mathbf{e}_{k,n} = S_k[n-Q]\left(\mathbf{r}[n] + \sum_{l=1}^{L}\sum_{i=-n-N}^{N-2} S_l[n+i]\left(\mathbf{b}_{l\_-i} - \mathbf{H}_{\mathbf{c}\_-i}\mathbf{a}'_l\right)\right), \tag{21}$$

where

(i) $\mathbf{H}_{r,k}(\mathbf{z}_l[p]|_{p \leq n-1, 1 \leq l \leq L})$ for $n \geq 0$ can be written as

$$\sum_{u=1}^{M_r}\sum_{l=1}^{L}\sum_{i=1}^{n+N} s_{r,k,u,l,i}[n]\mathbf{D}_{r,u,i}\mathbf{z}_l[n-i], \tag{22}$$

where M_r is an integer, $s_{r,k,u,l,i}[n]$ is a product of $S_{l'}[n-m]$ sequences, each with $l' \in \{1, 2, \ldots, L\}$ and $m \in \{-(N-2), -(N-3), \ldots, i + Q - 1\}$ including $S_k[n-Q]$, and each $\mathbf{D}_{r,u,i}$ is an $N \times N$ deterministic matrix with a spectral norm which is bounded by a sequence that decays to zero exponentially as $i \to \infty$,

(ii) for the special case of $r = 1$

$$\mathbf{H}_{1,k}\left(\mathbf{z}_l[p]|_{p \leq n-1, 1 \leq l \leq L}\right) = S_k[n-Q]\sum_{l=1}^{L}\left(\sum_{j=1}^{n+N} S_l[n-j]\mathbf{H}_{\mathbf{c}\_j}\mathbf{z}_l[n-j] + \sum_{j=0}^{N-2} S_l[n+j]\mathbf{H}_{\mathbf{c}\_-j}\mathbf{z}_l[n-1]\right) \tag{23}$$

for $n \geq 0$, and

(iii) $\mathbf{v}_{k,n}$ for each k and n is an $N \times 1$ real vector that is independent of $S_l[n+J]$ for all l and all $J \geq N-1$, and there exists a constant B such that $||\boldsymbol{\sigma}(\mathbf{v}_{k,n})|| < B$ for all k and n.

Lemma 1 implies that for each $k = 1, 2, \ldots, L$, and all $n \geq 0$, $\mathbf{z}_k[n]$ is uniquely determined by the difference equation (20) and initial conditions (18). The lemma only specifies $\mathbf{H}_{r,k}(\mathbf{z}_l[p]|_{p \leq n-1, 1 \leq l \leq L})$ for $n \geq 0$, but for notational convenience in the remainder of the paper and without loss of generality, the definition is made that

$$\mathbf{H}_{r,k}\left(\mathbf{z}_l[p]|_{p \leq n-1, 1 \leq l \leq L}\right) = \mathbf{0} \quad \text{for all } n < 0. \tag{24}$$

The following two theorems provide the key results of the paper. Both refer to h_K, which is defined as

$$h_K = \min_{0 < \alpha \leq K}\left\{\left(1 - \left\|\mathbf{I} - \alpha\mathbf{H}_{\mathbf{c}\_Q}\right\|_2\right)/\alpha\right\}, \tag{25}$$

where $\mathbf{H}_{\mathbf{c}\_Q}$ is given by (16) with $i = Q$.

Theorem 1: If $|b_k[n]|$ for $k = 1, 2, \ldots, L$ and $|h_c[n]|$ are each bounded by a sequence that decays exponentially to zero as $n \to \infty$, if there exists a positive value, K', such that $h_K > 0$ when $K = K'$, and if there exists a non-zero positive constant c and integer M such that $\mathrm{E}\{S_k^2[n]\} > c$ for every k occurs at least once every consecutive M samples, then, for any positive ε, there exists a positive δ such that

$$\limsup_{n \to \infty}\|\boldsymbol{\sigma}(\mathbf{z}_k[n])\| < \varepsilon \tag{26}$$

for all $0 < K < \delta$ and each $k = 1, 2, \ldots, L$.

Proof: Given that $\mathbf{H}_{r,k}$, as defined in Lemma 1, is a linear function, (18), (20), and (24) imply that

$$\mathbf{z}_k[n] = \sum_{q=-N}^{n}\mathbf{y}_{k,q}[n], \tag{27}$$

where $\mathbf{y}_{k,q}[n]$ for $k = 1, 2, \ldots, L$ are the solution of the L difference equations given by

$$\mathbf{x}_k[n] = \mathbf{x}_k[n-1] - \sum_{r=1}^{N} K^r\mathbf{H}_{r,k}\left(\mathbf{x}_l[p]|_{p \leq n-1, 1 \leq l \leq L}\right) \tag{28}$$

when $n > q$ and

$$\mathbf{x}_k[n] = \begin{cases} \mathbf{x}_{k,q}, & \text{if } n = q, \\ \mathbf{0}, & \text{if } n < q, \end{cases} \tag{29}$$

when $n \leq q$, with $\mathbf{x}_k[n]$ replaced by $\mathbf{y}_{k,q}[n]$ and

$$\mathbf{x}_{k,q} = \begin{cases} \mathbf{z}_k[q] - \mathbf{z}_k[q-1], & \text{if } q < 0, \\ K\mathbf{e}_{k,q} + K^2\mathbf{v}_{k,q}, & \text{otherwise.} \end{cases} \tag{30}$$

Equations (27) and the linearity of difference equations (28) with (29) imply that

$$\mathbf{z}_k[n] = \sum_{q=-N}^{-1}\mathbf{y}_{k,q}[n] + \sum_{q=0}^{n}\left(K^2\mathbf{a}_{k,q}[n] + K\mathbf{b}_{k,q}[n]\right), \tag{31}$$

where $\mathbf{a}_{k,q}[n]$ and $\mathbf{b}_{k,q}[n]$ for $k = 1, 2, \ldots, L$ are the solution of (28) with $\mathbf{x}_k[n]$ replaced by $\mathbf{a}_{k,q}[n]$ and $\mathbf{b}_{k,q}[n]$, respectively, and (29) for $\mathbf{x}_{k,q} = \mathbf{v}_{k,q}$ and $\mathbf{x}_{k,q} = \mathbf{e}_{k,q}$, respectively.

Given that $\mathbf{a}'_k$ satisfies (17) for each k, and $\mathbf{b}_{k,q}[q] = \mathbf{e}_{k,q}$ as implied by the definition of $\mathbf{b}_{k,q}[n]$, it follows from (21) that

$$\mathbf{b}_{k,q}[q] = S_k[q-Q]\mathbf{v}_q^{+} + S_k[q-Q]\mathbf{v}_q^{-}, \tag{32}$$

where

$$\mathbf{v}_q{}^{+} = \mathbf{r}[q] + \sum_{l=1}^{L}\sum_{i=-Q+1}^{N-2} S_l[q+i]\mathbf{c}_{q,l,i}, \quad (33)$$

$$\mathbf{v}_q{}^{-} = \sum_{l=1}^{L}\sum_{i=-q-N}^{-Q-1} S_l[q+i]\mathbf{c}_{q,l,i}, \quad (34)$$

and

$$\mathbf{c}_{q,l,i} = \begin{cases} \mathbf{b}_{l\_-i} - \mathbf{H}_{\mathbf{c}\_-i}\mathbf{a}'_l, & \text{if } i \geq -q-N, \\ \mathbf{0}, & \text{otherwise.} \end{cases} \quad (35)$$

Therefore, the linearity of difference equations (28) with (29) implies that $\mathbf{b}_{k,q}[n]$ can be further decomposed as $\mathbf{b}_{k,q}[n] = \mathbf{b}_{k,q}{}^{+}[n] + \mathbf{b}_{k,q}{}^{-}[n]$, where

$$\mathbf{b}_{k,q}{}^{+}[n] = \sum_{r=1}^{L} S_r[q-Q]\mathbf{b}_{k,q,r}{}^{+}[n], \quad (36)$$

$\mathbf{b}_{k,q}{}^{-}[n]$ is given by (36) with each $+$ replaced by $-$, $\mathbf{b}_{k,q,r}{}^{+}[n]$ and $\mathbf{b}_{k,q,r}{}^{-}[n]$ for $k = 1, 2, \ldots, L$ are the solutions of (28) with $\mathbf{x}_k[n]$ replaced by $\mathbf{b}_{k,q,r}{}^{+}[n]$ and $\mathbf{b}_{k,q,r}{}^{-}[n]$, respectively, and (29) for $\mathbf{x}_{k,q} = \delta_{k,r}\mathbf{v}_q{}^{+}$ and $\mathbf{x}_{k,q} = \delta_{k,r}\mathbf{v}_q{}^{-}$, respectively, in which $\delta_{k,r}$ is the Kronecker delta function ($\delta_{k,r}$ is one for $k = p$ and zero otherwise).

Equations (28) with $\mathbf{x}_k[n]$ replaced by $\mathbf{b}_{k,q,r}{}^{+}[n]$ contains an $\mathbf{H}_{r,k}(\mathbf{b}_{k,q,r}{}^{+}[p]|_{p\leq n-1, 1\leq l\leq L})$ term, which has the form of (22) with $\mathbf{z}_{k,q}[n-i]$ replaced by $\mathbf{b}_{k,q,r}{}^{+}[n-i]$. Lemma 1 implies that each $s_{r,k,u,l,i}[n]$ factor in (22) is independent of $S_{l'}[n-J]$ for $J \geq i+Q$ and all l', and (29) with $\mathbf{x}_k[n]$ replaced by $\mathbf{b}_{k,q,r}{}^{+}[n]$ implies $\mathbf{b}_{k,q,r}{}^{+}[n-i] = \mathbf{0}$ for all $i > n-q$. This shows that the $s_{r,k,u,l,i}[n]$ factor in each nonzero term of $\mathbf{H}_{r,k}(\mathbf{b}_{k,q,r}{}^{+}[p]|_{p\leq n-1, 1\leq l\leq L})$ is independent of $S_{l'}[n-J]$ for all $J \geq n-q+Q$ and all l', or, equivalently, independent of $S_{l'}[q-J]$ for all $J \geq Q$ and all l'. Given that $\mathbf{b}_{k,q,r}{}^{+}[q] = \delta_{k,r}\mathbf{v}_q{}^{+}$ and both $\mathbf{r}[q]$ and $\mathbf{c}_{q,l,i}$ are independent of $S_{l'}[p]$ for all l' and p, (33) implies that $\mathbf{b}_{k,q,r}{}^{+}[q]$ also is independent of $S_l[q-J]$ for all $J \geq Q$ and all l. Hence, (28) with (29) imply that $\mathbf{b}_{k,q,r}{}^{+}[n]$ is independent of $S_l[q-J]$ for all $J \geq Q$ and all $n \geq q$. It follows that, for any $n \geq q_2 > q_1$, both $\mathbf{b}_{k,q1,r}{}^{+}[n]$ and $\mathbf{b}_{k,q2,r}{}^{+}[n]$ are independent of $S_l[q_1-Q]$. Definition (36) implies that $(\mathbf{b}_{k,q1}{}^{+}[n])^{\mathrm{T}}\mathbf{b}_{k,q2}{}^{+}[n]$ is a sum of terms, each of the form $S_r[q_1-Q]S_{r'}[q_2-Q](\mathbf{b}_{k,q1,r}{}^{+}[n])^{\mathrm{T}}\mathbf{b}_{k,q2,r'}{}^{+}[n]$ where $1 \leq r, r' \leq L$. Given that $S_r[q_1-Q]$ is zero-mean and independent of both $S_{r'}[q_2-Q]$ and $(\mathbf{b}_{k,q1,r}{}^{+}[n])^{\mathrm{T}}\mathbf{b}_{k,q2,r'}{}^{+}[n]$, it follows that $\mathrm{E}\{(\mathbf{b}_{k,q1}{}^{+}[n])^{\mathrm{T}}\mathbf{b}_{k,q2}{}^{+}[n]\} = 0$.

To prove the corresponding result for $\mathbf{b}_{k,q}{}^{-}[n]$, it is necessary to invoke the linearity of (28) with (29), (34), and the definition of $\mathbf{b}_{k,q,r}{}^{-}[n]$ to further decompose $\mathbf{b}_{k,q,r}{}^{-}[n]$ as

$$\mathbf{b}_{k,q,r}{}^{-}[n] = \sum_{l=1}^{L}\sum_{i=-q-N}^{-Q-1} S_l[q+i]\mathbf{b}_{k,q,r,l,i}[n], \quad (37)$$

where $\mathbf{b}_{k,q,r,l,i}[n]$ for $k = 1, 2, \ldots, L$ are the solution of (28) with $\mathbf{x}_k[n]$ replaced by $\mathbf{b}_{k,q,r,l,i}[n]$ and (29) for $\mathbf{x}_{k,q} = \delta_{k,r}\mathbf{c}_{q,l,i}$. Given that $\mathbf{x}_{k,q}$ and the $s_{r,k,u,l,i}[n]$ factor in each nonzero term of $\mathbf{H}_{r,k}(\mathbf{b}_{k,q,r,l,i}[p]|_{p\leq n-1, 1\leq l\leq L})$ are independent of $S_{l'}[q-J]$ for all $J \geq Q$ and all l', (28) implies that $\mathbf{b}_{k,q,r,l,i}[n]$ is also independent of $S_{l'}[q-J]$ for all $J \geq Q$ and all l'. Substituting (37) into (36) with each $+$ replaced by $-$ shows that $(\mathbf{b}_{k,q1}{}^{-}[n])^{T}\mathbf{b}_{k,q2}{}^{-}[n]$ for any q_1 and q_2 is a sum of terms, each of the form $S_r[q_1-Q]S_{r'}[q_2-Q]S_l[q_1+i]S_{l'}[q_2+i'](\mathbf{b}_{k,q1,r,l,i}[n])^{T}\mathbf{b}_{k,q2,r',l',i'}[n]$, with $1 \leq r, r', l, l' \leq L$, and $i, i' \leq -Q-1$. Suppose $q_2 > q_1$. Then $(\mathbf{b}_{k,q1,r,l,i}[n])^{\mathrm{T}}\mathbf{b}_{k,q2,r',l',i'}[n]$ is independent of $S_{l''}[q_1-J]$ for all $J \geq Q$ and all l'', and the range of i and i' further implies that $q_1+i < q_1-Q < q_2-Q$. Therefore, if $q_1+i = q_2+i'$, then $S_r[q_1-Q]$ is independent of each of $S_l[q_1+i]$, $S_{l'}[q_2+i']$, $S_{r'}[q_2-Q]$ and $(\mathbf{b}_{k,q1,r,l,i}[n])^{T}\mathbf{b}_{k,q2,r',l',i'}[n]$. If $q_1+i < q_2+i'$, then $S_l[q_1+i]$ is independent of each of $S_{l'}[q_2+i']$, $S_r[q_1-Q]$, $S_{r'}[q_2-Q]$, $(\mathbf{b}_{k,q1,r,l,i}[n])^{\mathrm{T}}\mathbf{b}_{k,q2,r',l',i'}[n]$. If $q_2+i' < q_1+i$, then $S_{l'}[q_2+i']$ is independent of each of $S_l[q_1+i]$, $S_r[q_1-Q]$, $S_{r'}[q_2-Q]$, and $(\mathbf{b}_{k,q1,r,l,i}[n])^{\mathrm{T}}\mathbf{b}_{k,q2,r',l',i'}[n]$. Thus, in each of the above cases, the product of the terms is zero mean, so $\mathrm{E}\{(\mathbf{b}_{k,q1}{}^{-}[n])^{\mathrm{T}}\mathbf{b}_{k,q2}{}^{-}[n]\} = 0$.

Consequently, applying Lemma C1 in Appendix C to (31) with $\mathbf{b}_{k,q}[n] = \mathbf{b}_{k,q}{}^{+}[n] + \mathbf{b}_{k,q}{}^{-}[n]$ yields

$$\begin{aligned}\|\boldsymbol{\sigma}(\mathbf{z}_k[n])\| \leq{}& K^2 \sum_{q=0}^{n}\left(\left\|\boldsymbol{\sigma}\left(\mathbf{a}_{k,q}[n]\right)\right\|\right) + \sum_{q=-N}^{-1}\left\|\boldsymbol{\sigma}\left(\mathbf{y}_{k,q}[n]\right)\right\| \\ &+ K\sqrt{\sum_{q=0}^{n}\left\|\boldsymbol{\sigma}\left(\mathbf{b}_{k,q}{}^{+}[n]\right)\right\|^2} \\ &+ K\sqrt{\sum_{q=0}^{n}\left\|\boldsymbol{\sigma}\left(\mathbf{b}_{k,q}{}^{-}[n]\right)\right\|^2}. \end{aligned} \quad (38)$$

Given that $\mathbf{a}_{k,q}[q] = \mathbf{v}_{k,q}$, it follows from Lemma 1, Part (iii) that Lemma B1 in Appendix B is applicable to $||\boldsymbol{\sigma}(\mathbf{a}_{k,q}[n])||$. Definition (36) and $\mathbf{b}_{k,q,r}{}^{+}[q] = \delta_{k,r}\mathbf{v}_q{}^{+}$ imply

$$\mathbf{b}_{k,q}{}^{+}[q] = S_k[q-Q]\,\mathbf{v}_q{}^{+}. \quad (39)$$

Similarly, the definition of $\mathbf{b}_{k,q}{}^{-}[n]$ and $\mathbf{b}_{k,q,r}{}^{-}[q] = \delta_{k,r}\mathbf{v}_q{}^{-}$ imply

$$\mathbf{b}_{k,q}{}^{-}[q] = S_k[q-Q]\,\mathbf{v}_q{}^{-}. \quad (40)$$

Equations (33)-(35), (39), and (40) imply that $\mathbf{b}_{k,q}{}^{+}[q]$ and $\mathbf{b}_{k,q}{}^{-}[q]$ are independent of $S_l[q+J]$ for all l and all $J \geq N-1$, so Lemma B1 is also applicable to $||\boldsymbol{\sigma}(\mathbf{b}_{k,q}{}^{+}[n])||$ and $||\boldsymbol{\sigma}(\mathbf{b}_{k,q}{}^{-}[n])||$. Therefore, to apply Lemma B1 to (38) it remains to show that it is applicable to $||\boldsymbol{\sigma}(\mathbf{y}_{k,q}[n])||$ for $q = -N, -N+1, \ldots, -1$.

Definitions (2), (3), and (15) imply that $\mathbf{z}_k[q]$ can be written as $\mathbf{z}_k[q] = a_{k,0}[q]\mathbf{i}^{(0)} + a_{k,1}[q+1]\mathbf{i}^{(1)} + \ldots + a_{k,N-1}[q+N-1]\mathbf{i}^{(N)} - \mathbf{a}'_k$, where $\mathbf{i}^{(m)}$ for each $m = 0, 1, \ldots, N-1$ is an $N \times 1$ vector whose elements are zero except for the $(m+1)$th element which is 1. Initial conditions (10) are deterministic, so $a_{k,m}[q+m]\mathbf{i}^{(m)}$ for each $-N \leq q+m \leq -1$ is deterministic as is $\mathbf{a}'_k$. For $0 \leq q+m \leq N-2$, $a_{k,m}[q+m]$ can be calculated by repeatedly applying (8) with (9)-(10) and n replaced by $n+m$ for $n = -m, -m+1, \ldots, q$. The lower limit of the summations over i in (9) can be changed to $i = 1$, because $h_c[i]$ and $b_k[i]$ are zero for all $n \leq 0$. Therefore, (9) implies that $u_{k,m}[n+m]$ is independent of $S_l[n+J]$ for all l and all $J \geq m$ for each $n = -m, -m+1, \ldots, q$. Given that $0 \leq m \leq N-1$, this with (8) implies that $a_{k,m}[q+m]\mathbf{i}^{(m)}$ is independent of $S_l[q+J]$ for all l and all $J \geq N-1$.

It follows that $a_{k,m}[q+m]\mathbf{i}^{(m)}$ for both $-N \leq q+m \leq -1$ and $0 \leq q+m \leq N-2$ are independent of $S_l[q+J]$ for all l and all $J \geq N-1$. Hence, $\mathbf{z}_k[q]$ for $q = -N, -N+1, \ldots, -1$ are independent of $S_l[q+J]$ for all l and all $J \geq N-1$. The same is true of $\mathbf{z}_k[q-1]$ because (18) implies $\mathbf{z}_k[-N-1]=\mathbf{0}$. Given that $\mathbf{y}_{k,q}[q]=\mathbf{z}_k[q]-\mathbf{z}_k[q-1]$, it follows that Lemma B1 is applicable to each $||\boldsymbol{\sigma}(\mathbf{y}_{k,q}[n])||$ term in (38).

When applicable, Lemma B1 implies that for all sufficiently small $K>0$ there exist positive constants A and b such that

$$\left\|\boldsymbol{\sigma}\left(\mathbf{w}_{k,q}[n]\right)\right\| \le A\left(1-bh_K K\right)^{\lfloor (n-q)/M\rfloor} \max_{1\le l\le L}\left\|\boldsymbol{\sigma}\left(\mathbf{w}_{l,q}[q]\right)\right\|, \tag{41}$$

where $\mathbf{w}_{k,q}[n]$ for $k=1,2,\ldots,L$ are the solution of (28) and (29) with $\mathbf{x}_k[n]$ replaced by $\mathbf{w}_{k,q}[n]$ for any particular value of $q\ge -N$. The definition of h_K, i.e., (25), implies that h_K does not decrease as K decreases, so (41) and the geometric series formula imply that there must exist positive constants A' and A'' such that for all sufficiently small $K>0$ and all integers $n\ge 0$,

$$\sum_{q=0}^{n}\left\|\boldsymbol{\sigma}\left(\mathbf{w}_{k,q}[n]\right)\right\| \le \frac{A'}{K}\max_{0\le q\le n}\left\{\max_{1\le l\le L}\left\|\boldsymbol{\sigma}\left(\mathbf{w}_{l,q}[q]\right)\right\|\right\}, \tag{42}$$

and

$$\sqrt{\sum_{q=0}^{n}\left\|\boldsymbol{\sigma}\left(\mathbf{w}_{k,q}[n]\right)\right\|^2} \le \frac{A''}{\sqrt{K}}\max_{0\le q\le n}\left\{\max_{1\le l\le L}\left\|\boldsymbol{\sigma}\left(\mathbf{w}_{l,q}[q]\right)\right\|\right\}. \tag{43}$$

As shown above, Lemma B1 is applicable to all the terms in (38), so it follows from (38), (41) with $\mathbf{w}_{k,q}[n]$ replaced by $\mathbf{y}_{k,q}[n]$, (42) with $\mathbf{w}_{k,q}[n]$ replaced by $\mathbf{a}_{k,q}[n]$, and (43) with $\mathbf{w}_{k,q}[n]$ replaced by $\mathbf{b}_{k,q}{}^{+}[n]$ and $\mathbf{b}_{k,q}{}^{-}[n]$ that the left side of (26) is upper bounded by a constant times K plus another constant times $\sqrt{K}$. Hence, δ can be made sufficiently small that (26) holds for all $0<K<\delta$ and each $k=1,2,\ldots,L$. □

Theorem 2: If $|b_k[n]|$ for $k=1,2,\ldots,L$ and $|h_c[n]|$ are each bounded by a sequence that decays exponentially to zero as $n\to\infty$, if there exists a positive value, K', such that $h_K>0$ when $K=K'$, and if there exists a non-zero positive constant c such that $\mathrm{E}\{S_k^2[n]\}>c$ for all n and k, then for any positive ε there exists a positive δ such that

$$\begin{aligned}\limsup_{n\to\infty}\left\|\boldsymbol{\sigma}\left(\mathbf{z}_k[n]\right)\right\| \le \limsup_{n\to\infty}\Bigg(&\sqrt{K\cdot f_k\left(b^{+}[n]\right)+\varepsilon K}\\ &+\sqrt{K\cdot f_k\left(b^{-}[n]\right)+\varepsilon K}+\sqrt{\varepsilon K}\Bigg)\end{aligned} \tag{44}$$

for all $0<K<\delta$ and each $k=1,2,\ldots,L$, where

$$b^{+}[n]=\left\|\boldsymbol{\sigma}\left(\mathbf{r}[n]\right)\right\|^2+\sum_{i=-Q+1}^{N-2}\sum_{l=1}^{L}\left\|\mathbf{b}_{l\_-i}-\mathbf{H}_{\mathbf{c}\_-i}\mathbf{a}'_l\right\|^2, \tag{45}$$

$$b^{-}[n]=\sum_{i=-\infty}^{-Q-1}\sum_{l=1}^{L}\left\|\mathbf{b}_{l\_-i}-\mathbf{H}_{\mathbf{c}\_-i}\mathbf{a}'_l\right\|^2, \tag{46}$$

$$f_k\left(b^{\pm}[n]\right)=K\sum_{q=0}^{n}\mathrm{E}\{S_k^2[q-Q]\}b^{\pm}[q]\prod_{m=q'}^{n}R_k[m], \tag{47}$$

$q'=q+Q+N-1$, $R_k[n]=1-2Kh_K\mathrm{E}\{S_k^2[n-Q]\}$, and the $\pm$ superscript notation is used to indicate that (47) holds when both $\pm$ symbols are replaced by either $+$ or $-$. In the special case where the first and second-order statistics of $\mathbf{r}[n]$ and $S_k[n]$ for each k do not change over time and $\mathrm{E}\{S_k^2[n]\}\ne 0$ for all k, then $b^{\pm}[n]$ is independent of n and (47) reduces to

$$f_k\left(b^{\pm}[n]\right)\le b^{\pm}[n]/(2h_K)+\varepsilon' K \tag{48}$$

for a constant ε'.

Proof: The hypothesis of Theorem 2 is a special case of that of Theorem 1, so results from the proof of Theorem 1 hold under the hypothesis of Theorem 2. In particular, the limit superior of $||\boldsymbol{\sigma}(\mathbf{z}_k[n])||$ as $n\to\infty$ is upper bounded by the sum of those of the terms on the right side of (38). This with the same reasoning that concluded the proof of Theorem 1 implies there exists a positive constant B' such that

$$\begin{aligned}\limsup_{n\to\infty}\left\|\boldsymbol{\sigma}\left(\mathbf{z}_k[n]\right)\right\| &\le KB'+K\limsup_{n\to\infty}\sqrt{\sum_{q=0}^{n}\left\|\boldsymbol{\sigma}\left(\mathbf{b}_{k,q}{}^{+}[n]\right)\right\|^2}\\ &\quad+K\limsup_{n\to\infty}\sqrt{\sum_{q=0}^{n}\left\|\boldsymbol{\sigma}\left(\mathbf{b}_{k,q}{}^{-}[n]\right)\right\|^2}.\end{aligned} \tag{49}$$

Given that $\mathrm{E}\{S_k^2[n]\}>0$ for all k, (33), (34), (39), and (40) imply that $||\boldsymbol{\sigma}(\mathbf{b}_{1,q}{}^{+}[q])||, ||\boldsymbol{\sigma}(\mathbf{b}_{2,q}{}^{+}[q])||, \ldots, ||\boldsymbol{\sigma}(\mathbf{b}_{L,q}{}^{+}[q])||$ are either all non-zero or all zero, and the same is true of $||\boldsymbol{\sigma}(\mathbf{b}_{1,q}{}^{-}[q])||, ||\boldsymbol{\sigma}(\mathbf{b}_{2,q}{}^{-}[q])||, \ldots, ||\boldsymbol{\sigma}(\mathbf{b}_{L,q}{}^{-}[q])||$. Therefore, Lemma B2 in Appendix B is applicable to the last two terms on the right side of (49). The lemma implies that for all sufficiently small $K>0$,

$$\begin{aligned}&\left\|\boldsymbol{\sigma}\left(\mathbf{b}_{k,q}^{\pm}[n]\right)\right\|^2\\ &\le (1+AK)\left(\prod_{m=q'}^{n}R_k[m]+\beta_{k,n-q'}\right)\left\|\boldsymbol{\sigma}\left(\mathbf{b}_{k,q}^{\pm}[q]\right)\right\|^2,\end{aligned} \tag{50}$$

where A is a positive constant, and $\beta_{k,m}$ is as defined in the lemma statement. Applying Lemma C1 to (33)-(35), (39), and (40) yields $||\boldsymbol{\sigma}(\mathbf{b}_{k,q}^{\pm}[q])||^2\le \mathrm{E}\{S_k^2[q-Q]\}b^{\pm}[q]$, where $b^{+}[q]$ and $b^{-}[q]$ are given by (45) and (46), respectively, with n replaced by q. Substituting this, (45)-(46) into (50), summing both sides of the result for $q=0,1,\ldots,n$, and applying the properties of $\beta_{k,m}$ described in the statement of Lemma B2 yields

$$\begin{aligned}&\sum_{q=0}^{n}\left\|\boldsymbol{\sigma}\left(\mathbf{b}_{k,q}^{\pm}[n]\right)\right\|^2\\ &\le\frac{1+AK}{K}\left\{f_k\left(b^{\pm}[n]\right)+\frac{\beta A'\max_{n\ge 0}\left(b^{\pm}[n]\right)}{2(1-\beta)h_K c_{\min}}\right\},\end{aligned} \tag{51}$$

where $f_k(b^{\pm}[n])$ is given by (47) and $c_{\min}>0$ as defined in the statement of Lemma B2. Since $0<R_k[n]<1$ and $0<\mathrm{E}\{S_k^2[n]\}\le 1$ for all k and n, and $b^{\pm}[n]$ is bounded for all n, this and the properties of geometric series imply that $f_k(b^{\pm}[n])$ is less than some constant for sufficiently small positive values of K. Hence, (51) implies that there exists a constant ε that

$$\sum_{q=0}^{n}\left\|\boldsymbol{\sigma}\left(\mathbf{b}_{k,q}^{\pm}[n]\right)\right\|^2\le\left(f_k\left(b^{\pm}[n]\right)+\varepsilon\right)/K \tag{52}$$

where ε is a function of K and β and can be made arbitrarily close to 0 by reducing β and K. Lemma B2 holds for any $0<\beta<1$ if K is sufficiently small, so ε can be made arbitrarily close to 0 by reducing K. Therefore, (49) and (52) imply that the left side of (49) can be upper bounded by the limit superior of

$$\sqrt{Kf_k\left(b^+[n]\right)+K\varepsilon}+\sqrt{Kf_k\left(b^-[n]\right)+\varepsilon K}+KB' \quad (53)$$

for all k and all $0 < K < \delta$. If $KB' > \sqrt{K\varepsilon}$, then δ can be further reduced but kept positive such that $KB' \le \sqrt{K\varepsilon}$ for all $0 < K < \delta$. Substituting (45)-(46) into (53) yields the right side of (44). The derivation of (48) in the special case of Theorem 2 directly follows from (47) and the properties of geometric series. □

IV. Implications

The limit superior of $||\boldsymbol{\sigma}(\mathbf{z}_k[n])||$ bounds the steady-state MSE of the kth adaptive filter's coefficients. Theorem 1 implies that it can be made arbitrarily small by reducing K toward 0. This proves that the system in Fig. 1 can be made unconditionally stable for sufficiently small K. Like all stable LMS systems, reducing K reduces the convergence rate of the adaptive filter coefficients in addition to improving MSE performance, so the choice of K represents a tradeoff between convergence rate and accuracy. This tradeoff is quantified for the system of Fig. 1 by the results of this paper together with those of [4].

Theorem 2 provides further insight into the relationship between $||\boldsymbol{\sigma}(\mathbf{z}_k[n])||$ and various system parameters. It follows from (44)-(48) that $||\boldsymbol{\sigma}(\mathbf{z}_k[n])||$ can be decreased by increasing h_K. For example, in the case of the MNC technique of Fig. 2a, this provides guidance regarding the design of the ADC and decimation filter. If $h_c[n]$ were such that $h_c[n] = 0$ for all $n \neq Q$ and $h_c[Q] > 0$, then (16) and (25) would imply that $h_K = h_c[Q]$. Equations (16) and (25) further imply that for a given value of $h_c[Q]$, h_K tends to be less than $h_c[Q]$ if $h_c[n] \neq 0$ for any $n \neq Q$. Therefore, to minimize $||\boldsymbol{\sigma}(\mathbf{z}_k[n])||$, Q should be chosen such that $h_c[Q]$ is the peak value of $h_c[n]$, and the ADC and decimation filter should be designed such that their combined passband is as flat as possible over the DAC's first Nyquist band to ensure that $h_c[n] \cong 0$ for all $n \neq Q$.

Theorem 2 also provides a means with which to evaluate the effect of MNC convergence error on output SNR as demonstrated in the remainder of this section for an example in which MNC is run in foreground calibration mode [4]. If foreground calibration is run for a large enough number of clock periods under conditions that satisfy the hypothesis of Theorem 2, the theorem implies that $||\boldsymbol{\sigma}(\mathbf{z}_k[n])||$ is approximately upper bounded by the right side of (44) (wherein $||\boldsymbol{\sigma}(\mathbf{r}[n])||$ is that which occurred during foreground calibration). In foreground mode the main DAC's input sequence is approximately constant, so $||\boldsymbol{\sigma}(\mathbf{r}[n])||$ is approximately constant, and each $\mathrm{E}\{S_k^2[n]\}$ is constant and greater than 0. Hence, (44)-(46) and (48) can be used to derive the steady-state mean-square error. It follows from (14) and the RMS vector definition that $||\boldsymbol{\sigma}(\mathbf{r}[n])|| = \sqrt{N}\boldsymbol{\sigma}(r_{ideal}[n])$, where $\boldsymbol{\sigma}(r_{ideal}[n])$ is the RMS value of $r_{ideal}[n]$. Furthermore, the terms in (44) that depend on $\mathbf{b}_{l\_-i}-\mathbf{H}_{\mathbf{c}\_-i}\mathbf{a}'_l$ are typically much smaller than the term that depends on $\mathbf{r}[n]$, because $\mathbf{b}_{l\_-i}-\mathbf{H}_{\mathbf{c}\_-i}\mathbf{a}'_l$ is a function of the mismatch noise whereas $\mathbf{r}[n]$ is a function of the main DAC's input sequence during foreground calibration and noise introduced by the ADC. For sufficiently small K, ε in (44) can be neglected, so (44)-(46) and (48) imply

$$\limsup_{n\to\infty} \|\boldsymbol{\sigma}\left(\mathbf{z}_k[n]\right)\| \le \boldsymbol{\sigma}\left(r_{ideal}[n]\right)\sqrt{NK/(2h_K)}. \quad (54)$$

Once foreground calibration mode completes, the system enters normal DAC operation wherein the adaptive filter coefficients measured during foreground calibration are frozen so that each adaptive filter becomes a fixed FIR filter. It follows from Figures 1 and 2a that the error at the input of the correction DAC during normal DAC operation resulting from imperfect foreground calibration convergence is

$$e[n] = \sum_{k=1}^{L}\sum_{m=0}^{N-1} S_k[n-m]z_{k,m}, \quad (55)$$

where $z_{k,m}$ is the deviation of the final foreground calibration mode value of the mth accumulator in kth adaptive filter from its ideal value of $a'_{k,m}$.

The correction DAC converts $e[n]$ to an output error waveform given by $e(t) = \alpha_c(t)e[n_t]$, where $\alpha_c(t)$ is the correction DAC's T_s-periodic pulse shaping waveform, and, n_t is the largest integer less than or equal to t/T_s [4]. Laboratory spectrum analyzers estimate time-average power spectra, the two-sided version of which for $e(t)$ can be written as

$$S_{ee}(\omega) = \lim_{M\to\infty}\frac{1}{MT_s}\left|E_M(j\omega)\right|^2, \quad (56)$$

where $E_M(j\omega)$ is the continuous-time Fourier transform of $e(t)$ restricted to the time interval $0 \le t \le MT_s$ [17]. Its definition implies that $E_M(j\omega)$ can be expressed as the continuous-time Fourier transform of $e_M(t) = \alpha_c(t)e_M[n_t]$, where

$$e_M[n] = \begin{cases} e[n], & \text{if } 0 \le n \le M, \\ 0, & \text{otherwise.}\end{cases} \quad (57)$$

Hence, $E_M(j\omega) = A_c(j\omega)E_M(e^{j\omega T_s})$, where $A_c(j\omega)$ is the continuous-time Fourier transform of $a_c(t)$ restricted to the interval $0 \le t \le T_s$, and $E_M(e^{j\omega T_s})$ is the discrete-time Fourier transform of $e_M[n]$ [4]. This with (56) implies the time-average MSE from $e(t)$ over the first Nyquist band is

$$p_e = \frac{1}{2\pi}\int_{-\pi f_s}^{\pi f_s}\lim_{M\to\infty}\frac{1}{MT_s}\left|A_c(j\omega)E_M(e^{j\omega T_s})\right|^2 d\omega. \quad (58)$$

The pulse-shaping waveform of a Nyquist-rate DAC generally is such that $|A_c(j\omega)| \le |A_c(0)|$ for all ω (e.g., $A_c(j\omega)$ typically approximates a sinc function). Given that $A_c(0)$ can be written as $T_s\,\bar{a}_c$, where,

$$\bar{a}_c = \frac{1}{T_s}\int_0^{T_s} a_c(t)dt, \quad (59)$$

it follows from (58) that

$$p_e \le \frac{T_s\bar{a}_c^2}{2\pi}\lim_{M\to\infty}\frac{1}{M}\int_{-\pi f_s}^{\pi f_s}\left|E_M(e^{j\omega T_s})\right|^2 d\omega. \quad (60)$$

This with Parseval's theorem implies

$$p_e \le \bar{a}_c^2\lim_{M\to\infty}\frac{1}{M}\sum_{n=-\infty}^{\infty} e_M^2[n]. \quad (61)$$

Given that $S_j[q]$ and $S_k[r]$ are uncorrelated if $j \neq k$ or $q \neq r$, (55) and (57) imply that

$$\lim_{M\to\infty}\frac{1}{M}\sum_{n=-\infty}^{\infty} e_M^2[n] = \sum_{k=1}^{L}\left\langle S_k^2\right\rangle\sum_{m=0}^{N-1} z_{k,m}^2, \quad (62)$$

where $\langle S_k^2 \rangle$ is the time average of $S_k^2[n]$ during normal DAC operation. Given that each $\langle S_k^2 \rangle$ is independent of each $z_{k,m}$, the RMS vector and L2 norm definitions with (61) and (62) imply that the expected value of p_e satisfies

$$\mathrm{E}\left\{p_e\right\} \leq \bar{a}_c^2 \sum_{k=1}^{L} \mathrm{E}\left\{\left\langle S_k^2 \right\rangle\right\} \left\|\boldsymbol{\sigma}\left(\mathbf{z}_k\right)\right\|^2. \tag{63}$$

This and (54) imply that the ratio of the output signal power to the expected power of the Nyquist-band noise resulting from MNC convergence error is approximately lower bounded by

$$SNR_c > 10 \log \left(\frac{2 h_K \, p_{signal}}{K \bar{a}_c^2 N \boldsymbol{\sigma}^2\left(r_{ideal}\right) \sum_{k=1}^{L} \mathrm{E}\left\{\left\langle S_k^2 \right\rangle\right\}} \right), \tag{64}$$

where $\boldsymbol{\sigma}(r_{ideal})$ is that which occurred during foreground calibration, and p_{signal} is the mean square value of the DAC output signal during normal DAC operation in the absence of non-ideal DAC behavior.

V. Simulation Results

This section describes behavioral computer simulations performed to compare against the MSE values predicted by (44) for the MNC and DNC techniques, and the SNR_c values predicted by (64) for the MNC technique. The simulated $||\boldsymbol{\sigma}(\mathbf{z}_k[n])||$ values were obtained via (5) except with each expected value replaced by the average of several hundred values of the sequence, $z_{k,m}^2[iJ+n_0]$, for $i = 0, 1, 2, \ldots$, obtained via simulation, with J equal to $1/K$ rounded to the nearest integer and n_0 large enough that the means of $z_{k,m}[n]$ for $n \geq n_0$ were negligible.

The MNC simulations model the 14-bit DAC IC with foreground mode MNC presented in [5], the high-level view of which is shown in Fig. 2a. In that IC, and, hence, in the simulations described below, foreground mode MNC was run for enough clock periods that the means of the adaptive filter coefficient errors, $z_{k,m}[n]$, were negligible, and then the coefficients were frozen and subsequently used for normal DAC operation. The ADC and decimation filter were designed in line with the guidelines described in Section IV. The resulting impulse response, $h_c[n]$, was extracted from transistor-level simulations and used to compute h_K in (25), which was found to be 0.6 and nearly independent of K over the range of K values that were simulated. The simulations model non-ideal circuit behavior, including component mismatches, that are in line with transistor-level simulations of and measured results from the IC presented in [5].

Fig. 3a shows the simulated $||\boldsymbol{\sigma}(\mathbf{z}_k[n])||$ values as well as the corresponding calculated upper bounds predicted by (54) with $K = 2 \times 10^{-5}$ and three different values of $\boldsymbol{\sigma}(r_{ideal}[n])$ for each of the $L = 35$ adaptive filters, all with $N = 9$ coefficients. The three sets of curves from top to bottom correspond to $\boldsymbol{\sigma}(r_{ideal}[n]) = 47$, 26, and 17, respectively, in units equal to the minimum step size of the main DAC. Fig. 3b shows corresponding results with $\boldsymbol{\sigma}(r_{ideal}[n]) = 17$ and 3 different values of K. The three sets of curves from top to bottom correspond to $K = 2 \times 10^{-5}$, 0.5×10^{-5}, and 1.25×10^{-6}, respectively. The simulated and calculated results are in close agreement and suggest that the analytically predicted bound is quite tight. Although not indicated in the figure, the simulation results are consistent with the expected tradeoff between convergence rate and accuracy as a function of K. For example, the simulated convergence rate for $K = 2 \times 10^{-5}$is 16 times that for $K = 1.25 \times 10^{-6}$.

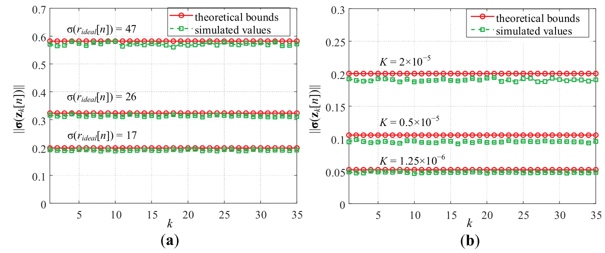

Fig. 3. Simulated MNC adaptive filter MSEs and their theoretical bounds for different values of a) $\sigma(r_{ideal}[n])$, and b) K.

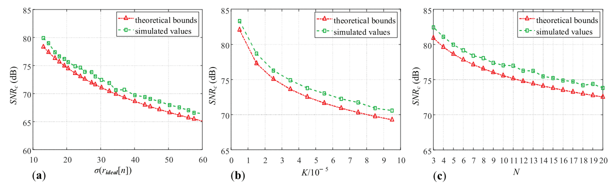

Fig. 4. Simulated MNC DAC SNRc values and their theoretical bounds for different values of a) $\sigma(r_{ideal}[n])$, b) K, and c) N.

Fig. 4 shows simulated values of SNR_c along with the corresponding analytical bound values predicted by (64) for the MNC example during normal DAC operation with a full-scale, 179.4 MHz sinusoidal main DAC input sequence. The value of $\bar{a}_c$ in (64) was found via circuit simulation to be 0.8. The values of the adaptive filter coefficients obtained during foreground mode vary slightly across simulation runs as expected, because $||\boldsymbol{\sigma}(\mathbf{z}_k[n])|| \neq 0$. The variations caused SNR_c to vary within approximately a ± 1 dB range about its average. Each SNR_c value shown in Fig. 4 is the average SNR_c over 10 simulation runs. Fig. 4a) shows simulated and calculated SNR_c results for $K = 2 \times 10^{-5}$ and $N = 9$ versus $\boldsymbol{\sigma}(r_{ideal}[n])$, where $\boldsymbol{\sigma}(r_{ideal}[n])$ was varied by changing the quantization step size of the ADC and is expressed in units of the main DAC's minimum step-size. Fig. 4b) shows corresponding results for $\boldsymbol{\sigma}(r_{ideal}[n]) = 17$ and $N = 9$ as a function of K. Fig. 4c) shows corresponding results for $K = 2 \times 10^{-5}$ and $\boldsymbol{\sigma}(r_{ideal}[n]) = 17$ as a function of N. The simulation results are in line with the theoretical bound values given by (64) as expected. The bound values are not as tight as those shown in Fig. 3 because the inequality $|A_c(j\omega)| \leq |A_c(0)|$ used to derive (60) from (58) is pessimistic.

Fig. 5 shows simulated and calculated MSE results for a 14-bit pipelined ADC of the form shown in Fig. 2b with DNC. The unit element DAC mismatches in the 9-level DEM DAC were chosen as samples of a Gaussian random variable with a standard deviation of 1%. The noise canceler was implemented with $L = 7$, $N = 1$, $P = 0$, $b_k[n] = \Delta_k \delta[n-1]$, and $h_c[n] = \delta[n-Q]$, where $Q = 4$. Hence, (16) and (25) imply that $h_K = 1$, and (17) implies that the terms of (44) which depend on $\mathbf{b}_{l\_-i} - \mathbf{H}_{\mathbf{c}\_-i} \mathbf{a}_l'$ are 0.

The simulated $||\boldsymbol{\sigma}(\mathbf{z}_k[n])||$ values for each of the $L = 7$ LMS feedback loops are shown in Fig. 5 along with the corresponding values predicted by (44) with $\varepsilon = 0$. Fig. 5a shows results for $K = 5 \times 10^{-3}$ and three different values of $\boldsymbol{\sigma}(r_{ideal}[n])$. The three sets of curves from top to bottom correspond to $\boldsymbol{\sigma}(r_{ideal}[n]) = 1$, 2, and 4, respectively, where $r_{ideal}[n]$ is normalized to the minimum step-size of the overall 14-b pipelined ADC. Fig. 5b shows results for $\boldsymbol{\sigma}(r_{ideal}[n]) = 2$ and three different values of K. The three sets of curves from top to bottom correspond to $K = 10^{-2}$, 5×10^{-3}, and 2×10^{-3},

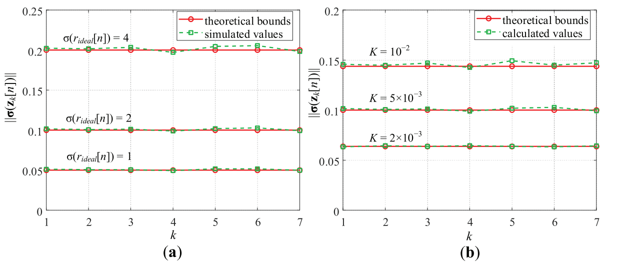

Fig. 5. Simulated DNC adaptive filter MSEs and their theoretical bound for different values of a) $\sigma(r_{ideal}[n])$, and b) K.

respectively. As indicated in the figure, a few of the simulated values of $||\boldsymbol{\sigma}(\mathbf{z}_k[n])||$ slightly exceed the upper bound of (44). This happened because the bounds were calculated with $\varepsilon = 0$. As implied by Theorem 2, the actual value of ε decreases with K but is non-zero in practice. Nevertheless, the theoretical results closely match the corresponding simulated results as expected.

Appendix A

Definition A1: In this paper, a deterministic $N \times N$ real matrix $\mathbf{A}$ is called a *sub-j matrix* if its elements satisfy

$$(\mathbf{A})_{m,n} = 0 \quad \text{if } m - n \le j, \tag{65}$$

for $1 \le n, m \le N$ and $-N+1 \le j \le N-2$, i.e., all elements that lie on and above the jth subdiagonal, the main diagonal, or the $|j|$th superdiagonal of $\mathbf{A}$ are zero if $j > 0$, $j = 0$, or $j < 0$, respectively.

Lemma A1: If $\mathbf{A}$ is an $N \times N$ sub-j matrix and $\mathbf{B}$ is an $N \times N$ sub-i matrix, then $\mathbf{AB}$ is a sub-$(i + j + 1)$ matrix if $-N \le i + j \le N-3$ and $\mathbf{AB} = \mathbf{0}$ if $i + j \ge N-2$.

Proof: By the definition of matrix multiplication, the element on the mth row and nth column of $\mathbf{AB}$ is

$$(\mathbf{AB})_{m,n} = \sum_{r=1}^{N} (\mathbf{A})_{m,r} (\mathbf{B})_{r,n}. \tag{66}$$

Given that $\mathbf{A}$ and $\mathbf{B}$ are sub-j and sub-i matrices, respectively, $(\mathbf{A})_{m,r} = 0$ if $m - r \le j$ and $(\mathbf{B})_{r,n} = 0$ if $r - n \le i$, for each $r = 1, 2, \ldots, N$. Hence, $(\mathbf{AB})_{m,n} = 0$ if either inequality holds, or, equivalently, if either $r \ge m - j$ or $r \le n + i$. Given that r is integer-valued, either $r \le n + i$ or $r \ge n + i + 1$ must hold. Suppose $m - n \le i + j + 1$. Then, $r \ge n + i + 1$ implies $r \ge m - j$, so $(\mathbf{AB})_{m,n} = 0$ if either $r \ge n + i + 1$ or $r \le n + i$, one of which must hold. It follows that

$$(\mathbf{AB})_{m,n} = 0 \text{ if } m - n \le i + j + 1, \tag{67}$$

so $\mathbf{AB}$ fits the definition of a sub-$(i+j+1)$ matrix if $-N+1 \le i + j + 1 \le N - 2$, or, equivalently, if $-N \le i + j \le N-3$. If $i + j \ge N-2$, then (67) implies that $(\mathbf{AB})_{m,n} = 0$ for $m - n \le N - 1$, but given that $1 \le m, n \le N$, this implies that $\mathbf{AB} = \mathbf{0}$.

□

Proof of Lemma 1: Given that $h_c[n] = 0$ for $n \le 0$, $b_l[n] = 0$ for $n \le 0$ and $P \ge 0$, (16) and (13) imply that $\mathbf{H}_{\mathbf{c}\_-j} = \mathbf{0}$ when $j \ge N - 1$ and $\mathbf{b}_{l\_-j} = \mathbf{0}$ when $j \ge N - 1$. This and the definition of $S_l[n] = 0$ when $n < -N$ imply that (11) can be written as

$$\mathbf{z}_k[n] = \mathbf{z}_k[n - 1] + K\mathbf{e}_{k,n} - K\mathbf{f}_k[n], \tag{68}$$

where $\mathbf{e}_{k,n}$ is given by (21) and

$$\mathbf{f}_k[n] = \sum_{l=1}^{L} \sum_{j=-n-N}^{N-2} S_{k,l,j}[n]\mathbf{H}_{\mathbf{c}\_-j}\mathbf{z}_l[n + j], \tag{69}$$

with

$$S_{k,l,j}[n] = S_k[n - Q]\, S_l[n + j]. \tag{70}$$

In the case of $N = 1$, (69) is equivalent to $\mathbf{H}_{1,k}(\mathbf{z}_l[p]|_{p\le n-1, 1\le l\le L})$ of (23) which also has the form of (22), thus the result of Lemma 1 holds for $N = 1$ with $\mathbf{v}_{k,n} = \mathbf{0}$. The remainder of the analysis proves the case for $N \ge 2$.

It is shown below by mathematical induction that

$$\mathbf{z}_k[n] = \mathbf{z}_k[n - 1] + K\mathbf{e}_{k,n} + \sum_{q=1}^{r} K^q \left[K\mathbf{u}_{q,k} - \mathbf{H}_{q,k}\left(\mathbf{z}_l[p]|_{p\le n-1, 1\le l\le L}\right)\right] + K^{r+1}\mathbf{f}_{r,k} \tag{71}$$

for all $n \ge 0$ and each $r = 1, 2, \ldots, N$, where $\mathbf{u}_{q,k}$ has the same form and properties as $\mathbf{v}_{k,n}$ in the lemma statement, $\mathbf{H}_{q,k}(\mathbf{z}_l[p]|_{p\le n-1, 1\le l\le L})$ is as described in the lemma statement, $\mathbf{f}_{r,k}$ for $1 \le r \le N - 1$ is either $\mathbf{0}$ or a finite sum of terms each of which is equal to $\mathbf{f}_l[n + m]$ scaled by a sub-$(m + r - 1)$ matrix and a variable S_Q for some

$$0 \le m \le N - 1 - r, \tag{72}$$

and $1 \le l \le L$ and $\mathbf{f}_{N,k} = \mathbf{0}$, where S_Q represents any product of $S_p[n - J]$ sequences for $p \in \{1, 2, \ldots, L\}$ and $J \in \{-(N-2), -(N - 3), \ldots, Q\}$ as well as $S_k[n - Q]$.

Recursively substituting (68) into itself results in

$$\mathbf{z}_k[n+m+j] = \mathbf{z}_k[n-1] + K\sum_{i=0}^{m+j}\left(\mathbf{e}_{k,n+i} - \mathbf{f}_k[n+i]\right) \tag{73}$$

for $m + j \ge 0$. Equation (69) with n replaced by $n + m$, for any non-negative integer m can be written as

$$\begin{aligned}\mathbf{f}_k[n + m] &= \sum_{l=1}^{L} \sum_{j=-m}^{N-2} S_{k,l,j}[n + m]\mathbf{H}_{\mathbf{c}\_-j}\mathbf{z}_l[n + m + j] \\ &+ \sum_{l=1}^{L} \sum_{j=-n-m-N}^{-m-1} S_{k,l,j}[n + m]\mathbf{H}_{\mathbf{c}\_-j}\mathbf{z}_l[n + m + j].\end{aligned} \tag{74}$$

Substituting (73) with $k = l$ into the first double sum of (74) gives

$$\begin{aligned}\mathbf{f}_k[n + m] &= \sum_{l=1}^{L} \sum_{j=-n-m-N}^{-m-1} S_{k,l,j}[n + m]\mathbf{H}_{\mathbf{c}\_-j}\mathbf{z}_l[n + m + j] \\ &+ \sum_{l=1}^{L} \sum_{j=-m}^{N-2} S_{k,l,j}[n + m]\mathbf{H}_{\mathbf{c}\_-j}\mathbf{z}_l[n - 1] \\ &+ K\sum_{l=1}^{L} \sum_{j=-m}^{N-2} S_{k,l,j}[n+m]\mathbf{H}_{\mathbf{c}\_-j}\sum_{i=0}^{m+j}\left(\mathbf{e}_{l,n+i} - \mathbf{f}_l[n+i]\right).\end{aligned} \tag{75}$$

Substituting (75) with $m = 0$ into (68) gives

$$\begin{aligned}\mathbf{z}_k[n] &= \mathbf{z}_k[n - 1] + K\mathbf{e}_{k,n} \\ &+ K^2\mathbf{u}_{1,k} - K\mathbf{H}_{1,k}\left(\mathbf{z}_l[p]|_{p\le n-1, 1\le l\le L}\right) + K^2\mathbf{f}_{1,k},\end{aligned} \tag{76}$$

where $\mathbf{H}_{1,k}(\mathbf{z}_l[p]|_{p\le n-1,1\le l\le L})$ is given by (23),

$$\mathbf{u}_{1,k} = -\sum_{l=1}^{L}\sum_{j=0}^{N-2} S_{k,l,j}[n]\mathbf{H}_{\mathbf{c}\_-j}\sum_{i=0}^{j}\mathbf{e}_{l,n+i}, \quad (77)$$

$$\mathbf{f}_{1,k} = \sum_{l=1}^{L}\sum_{j=0}^{N-2} S_{k,l,j}[n]\mathbf{H}_{\mathbf{c}\_-j}\sum_{i=0}^{j}\mathbf{f}_l[n+i]. \quad (78)$$

Given that $h_c[n] = 0$ for $n \le 0$, (16) implies that $\mathbf{H}_{\mathbf{c}\_-j}$ is a sub-j matrix when $-N+1 \le j \le N-2$. By definition, a sub-j matrix is also a sub-i matrix if $i \le j$, so each term in the finite sum given by (78) is proportional to a sub-i matrix times $\mathbf{f}_l[n+i]$, where $0 \le i \le N-2$. Furthermore, (23) has the form of (22) given that each of its terms is proportional to $\mathbf{H}_{\mathbf{c}\_j}\mathbf{z}_l[n-i]$ with $i \ge 1$, and each element in $\mathbf{H}_{\mathbf{c}\_j}$ is bounded by a curve that decays exponentially to zero as $j \to \infty$, so the spectral norm of $\mathbf{H}_{\mathbf{c}\_j}$ is bounded by a curve that decays exponentially to zero as $j \to \infty$.

Substituting (21) into (77) and (70) into the result, and applying Lemma A1 and $b_l[n] = 0$ for $n \le 0$, reveals that $\mathbf{u}_{1,k}$ is a finite sum of $N \times 1$ real vectors each of which is independent of $S_l[n+J]$ for all l and all $J \ge N-1$. Furthermore, as $||\mathbf{b}_{l\_j}||$ and $||\mathbf{H}_{\mathbf{c}\_j}||_2$ both are bounded by curves that decay exponentially to zero as $j \to \infty$, so $||\boldsymbol{\sigma}(\mathbf{u}_{1,k})||$ is less than some constant for all n. Hence, $\mathbf{u}_{1,k}$ has the same form and properties as $\mathbf{v}_{k,n}$ in the lemma statement.

Therefore, (76) is equivalent to (71) for the special case of $r = 1$, which proves the induction base case.

To prove the induction step case, suppose (71) holds for one of $r = 1, 2, \ldots, N-1$. Substituting (75) with k replaced by l in place of each $\mathbf{f}_l[n+m]$ in $\mathbf{f}_{r,k}$ results in an expression for $\mathbf{f}_{r,k}$ of the form

$$\mathbf{f}_{r,k} = \mathbf{a}_{r,k} + K\mathbf{b}_{r,k} + K\mathbf{c}_{r,k}, \quad (79)$$

where $\mathbf{a}_{r,k}$ is the first two double sums in (75) scaled by a sub-$(m+r-1)$ matrix and S_Q, $\mathbf{b}_{r,k}$ is the sum of all terms proportional to $\mathbf{H}_{\mathbf{c}\_-j}\mathbf{e}_{l,n+i}$ in the third double sum in (75) scaled by a sub-$(m+r-1)$ matrix and s_Q, and $\mathbf{c}_{r,k}$ is the sum of all terms proportional to $\mathbf{H}_{\mathbf{c}\_-j}\mathbf{f}_l[n+i]$ in the third double sum in (75) scaled by a sub-$(m+r-1)$ matrix and S_Q. Therefore, each term of $\mathbf{a}_{r,k}$, $\mathbf{b}_{r,k}$, $\mathbf{c}_{r,k}$ is proportional to a sub-$(m+r-1)$ matrix times $\mathbf{H}_{\mathbf{c}\_-j}$, where the range of j is given by (75). If $N-2-m < j \le N-2$, it follows that $\mathbf{H}_{\mathbf{c}\_-j}$ is a sub-j matrix and $(m+r-1)+j \ge N-1+r-1 \ge N-1$, thus Lemma A1 implies that the product of any sub-$(m+r-1)$ matrix and $\mathbf{H}_{\mathbf{c}\_-j}$ is $\mathbf{0}$ for $N-2-m < j \le N-2$. Therefore, each $\mathbf{H}_{\mathbf{c}\_-j}$ in the nonzero terms of $\mathbf{a}_{r,k}$, $\mathbf{b}_{r,k}$, $\mathbf{c}_{r,k}$ has the upper index range of j further limited to $N-3-m$.

By definition, each nonzero term of $\mathbf{a}_{r,k}$ is equal to a factor that satisfies the definition of $s_{r,k,u,l,i}[n]$ in (22) times a bounded sub-$(m+r-1)$ matrix times $\mathbf{H}_{\mathbf{c}\_-j}\mathbf{z}_l[n-i]$ with $i \ge 1$ and $(-j) \ge -(N-2)$. Given that the spectral norm $\mathbf{H}_{\mathbf{c}\_j}$ is bounded by a sequence that decays exponentially to zero as $j \to \infty$, the product of bounded matrices times $\mathbf{H}_{\mathbf{c}\_j}$ is also bounded by a sequence that decays exponentially to zero as $j \to \infty$. Therefore, $\mathbf{a}_{r,k}$ satisfies the lemma statement's requirements for $\mathbf{H}_{r+1,k}(\mathbf{z}_l[p]|_{p\le n-1,1\le l\le L})$. Essentially the same argument presented above to prove that $\mathbf{u}_{1,k}$ has the same form and properties as $\mathbf{v}_{k,n}$ also shows $\mathbf{b}_{r,k}$ has the same form and properties as $\mathbf{v}_{k,n}$. Hence, $\mathbf{b}_{r,k}$ satisfies the definition of $\mathbf{u}_{r+1,k}$. The above analysis and (75) imply that each $\mathbf{H}_{\mathbf{c}\_-j}\mathbf{f}_l[n+i]$ in the nonzero terms of $\mathbf{c}_{r,k}$ is restricted to the index range of

$$-m \le j \le N-3-m, \quad \text{and } 0 \le i \le m+j. \quad (80)$$

Within the above index range, the product of S_Q and the scaling factor $S_{k,l,j}[n+m]$ in the triple sum of (75) has the form of S_Q. Therefore, by definition, each nonzero term of $\mathbf{c}_{r,k}$ is equal to $\mathbf{H}_{\mathbf{c}\_-j}\mathbf{f}_l[n+i]$ scaled by a sub-$(m+r-1)$ matrix and S_Q. Given that $\mathbf{H}_{\mathbf{c}\_-j}$ is a sub-j matrix, this with Lemma A1 implies that each term in $\mathbf{c}_{r,k}$ is equal to $\mathbf{f}_l[n+i]$ scaled by a sub-$(m+r+j)$ matrix and S_Q for

$$-N \le (m+r-1)+j \le N-3, \quad (81)$$

and $\mathbf{c}_{r,k} = \mathbf{0}$, otherwise. Specifically, $\mathbf{c}_{r,k} = \mathbf{0}$ if $r = N-1$. Hence, $\mathbf{c}_{r,k}$ satisfies the definition of $\mathbf{f}_{r+1,k}$ for $r = N-1$. For $r \le N-2$, it follows from (80) and (81) that $0 \le i \le N-1-(r+1)$, which is the range given by (72) with m replaced by i and r replaced by $r+1$. Given that $i \le m+j$, the sub-$(m+r+j)$ matrix is, by definition, also a sub-$(i+r)$ matrix. Hence, each term in $\mathbf{c}_{r,k}$ is equal to $\mathbf{f}_l[n+i]$ scaled by a sub-$(i+(r+1)-1)$ matrix and S_Q. Hence, $\mathbf{c}_{r,k}$ satisfies the definition of $\mathbf{f}_{r+1,k}$ for $r \le N-2$.

The above analysis implies that (71) holds with r replaced by $r+1$. It follows from mathematical induction that (71) holds for $r = 1, 2, \ldots, N$. Equation (71) with $r = N$ and $\mathbf{f}_{N,k} = \mathbf{0}$ directly yields (20) with

$$\mathbf{v}_{k,n} = \sum_{q=1}^{N} K^{q-1}\mathbf{u}_{q,k}, \quad (82)$$

which has the form and properties of $\mathbf{v}_{k,n}$ described in the lemma statement because each $\mathbf{u}_{q,k}$ term has that form. □

Appendix B

All Appendix B results relate to the L difference equations given by (28) for $n > q$ and $q \ge -N$, and (29) for $n \le q$. The results are restricted to cases where $\mathbf{x}_k[q]$, or, equivalently, $\mathbf{x}_{k,q}$ is independent of $S_l[q+J]$ for all l and all $J \ge N-1$. The lemmas make use of the following definitions:

$$x_{\max}[n] = \max_{1\le k\le L} \|\boldsymbol{\sigma}(\mathbf{x}_k[n])\|, \quad (83)$$

$$c_k[n] = \mathrm{E}\left\{S_k^2[n]\right\}, \quad (84)$$

$$K_k[n] = 2h_K K c_k[n-Q], \quad (85)$$

$$K_{\min}[n] = \min_{1\le k\le L}(K_k[n]). \quad (86)$$

Lemma B1: The hypothesis of Theorem 1 implies that there exists a positive δ less than or equal to both K' and $1/(4||\mathbf{H}_{\mathbf{c}\_Q}||_2)$, and positive constants b and B such that

$$\|\boldsymbol{\sigma}(\mathbf{x}_k[n])\| \le \gamma[n]x_{\max}[q] \quad (87)$$

for all $k = 1, 2, \ldots, L$, $0 < K < \delta$ and $n > q$, where

$$\gamma[n] = (1+BK)(1-bh_K K)^{\lfloor (n-q')/M \rfloor}, \quad (88)$$

and $q' = q+Q+N-1$.

Proof: Suppose β satisfies $0 < \beta < 1$, and δ is no larger than that described in Lemma B3. Recursively substituting (104) of Lemma B3 into itself for $n \ge q'$ yields

$$x_{\max}^2[n] \le x_{\max}^2[q'-1]\prod_{j=q'}^{n}(1-(1-\beta)K_{\min}[n]) \quad (89)$$

for all $0 < K < \delta$. Definitions (84), (85), and (86) imply

$$K_{\min}[n] = \min_{1\le k\le L}\left(2h_K K E\left\{S_k^2[n-Q]\right\}\right), \quad (90)$$

so the hypothesis of Theorem 1 ensures that there exists a positive constant b such that $(1-\beta)K_{\min}[n] > bh_K K$ at least once every M consecutive values of n. Hence, (89) implies

$$x_{\max}^2[n] \le x_{\max}^2[q'-1]\,(1-bh_K K)^{\lfloor (n-q')/M\rfloor}. \quad (91)$$

By definition, $||\boldsymbol{\sigma}(\mathbf{x}_k[n])|| \le x_{\max}[n]$, so substituting (104) of Lemma B3 with $n = q'-1$ into (91) gives (87). □

Lemma B2: If the hypothesis of Theorem 2 holds, and if $||\boldsymbol{\sigma}(\mathbf{x}_{1,q})||, ||\boldsymbol{\sigma}(\mathbf{x}_{2,q})||, \ldots, ||\boldsymbol{\sigma}(\mathbf{x}_{L,q})||$ are either all non-zero or all zero, then for each β in the range $0 < \beta < 1$, there exists a positive δ less than or equal to both K' and $1/(4||\mathbf{H}_{\mathbf{c}\_Q}||_2)$ such that

$$\|\boldsymbol{\sigma}(\mathbf{x}_k[n])\|^2 \le (1+AK)\left(\prod_{m=q'}^{n} R_k[m] + \beta_{k,n-q'}\right)\|\boldsymbol{\sigma}(\mathbf{x}_k[q])\|^2 \quad (92)$$

for all $0 < K < \delta$ and $n > q$, where $R_k[n] = 1 - 2Kh_K\mathrm{E}\{S_k^2[n-Q]\}$, $q' = q+Q+N-1$, $\beta_{k,m} = 0$ for $m < 0$, $\beta_{k,m} \ge 0$ for $m \ge 0$,

$$\sum_{m=0}^{\infty}\beta_{k,m} \le \beta A'/\left(2(1-\beta)Kh_K c_{\min}\right), \quad (93)$$

and A and A' are positive constants, where $c_{\min}$ is the minimum value of $\mathrm{E}\{S_k^2[n]\}$ over $k = 1, 2, \ldots, L$ and all n.

Proof: The hypothesis of Theorem 2 implies that $\mathrm{E}\{S_k^2[n]\}$ is non-zero for all n for all k. Therefore, (90) implies $K_{\min}[n] \ge 2h_K K c_{\min}$. Choose δ to be no larger than those described in Lemmas B1 and B3 and small enough that $0 < K_k[n] < 1$ for all n and all $0 < K < \delta$. Suppose K is within this range, so $0 < R_k[n] < 1$, $0 < R < 1$ and $0 < U < 1$, where $R = 1 - 2Kh_K c_{\min}$ and $U = 1 - 2(1-\beta)Kh_K c_{\min}$.

Given that $||\boldsymbol{\sigma}(\mathbf{x}_{1,q})||, ||\boldsymbol{\sigma}(\mathbf{x}_{2,q})||, \ldots, ||\boldsymbol{\sigma}(\mathbf{x}_{L,q})||$ are either all non-zero or all zero, it follows that there exists a positive constant $C \ge 1$ such that $C||\boldsymbol{\sigma}(\mathbf{x}_k[q])||^2 \ge x_{\max}^2[q]$ for all k, thus (105) of Lemma B3 with $n = q'-1$ implies

$$\|\boldsymbol{\sigma}(\mathbf{x}_k[n])\|^2 \le (1+AK)\,\|\boldsymbol{\sigma}(\mathbf{x}_k[q])\|^2 \text{ for } q \le n \le q'-1, \quad (94)$$

where $A = BC \ge B$.

Given that $K_k[n] = 1 - R_k[n]$, inequality (105) of Lemma B3 for $n \ge q'$ implies

$$\|\boldsymbol{\sigma}(\mathbf{x}_k[n])\|^2 \le R_k[n]\,\|\boldsymbol{\sigma}(\mathbf{x}_k[n-1])\|^2 + \beta(1-R_k[n])\,x_{\max}^2[n-1] \quad (95)$$

for all $n \ge q'$. Recursively substituting (95) into itself and applying the triangle inequality and $0 < R_k[n] \le R < 1$ gives

$$\|\boldsymbol{\sigma}(\mathbf{x}_k[n])\|^2 \le \prod_{m=q'}^{n} R_k[m]\left\|\boldsymbol{\sigma}\left(\mathbf{x}_k[q'-1]\right)\right\|^2 + \upsilon_k[n], \quad (96)$$

for $n \ge q'$, where

$$\upsilon_k[n] = \beta\sum_{i=q'-1}^{n-1} R^{n-1-i}x_{\max}[i]. \quad (97)$$

Substituting (94) with $n = q'-1$ into (96) implies

$$\|\boldsymbol{\sigma}(\mathbf{x}_k[n])\|^2 \le (1+AK)\prod_{m=q'}^{n} R_k[m]\,\|\boldsymbol{\sigma}(\mathbf{x}_k[q])\|^2 + \upsilon_k[n] \quad (98)$$

for all $n \ge q'$. Substituting (104) of Lemma B3 with $n = q'-1$ into (89) with $K_{\min}[n] \ge 2h_K K c_{\min}$ yields

$$x_{\max}^2[n] \le (1+AK)U^{n-q'+1}x_{\max}^2[q] \quad (99)$$

for $n \ge q'-1$, because $A \ge B$. Given that $C||\boldsymbol{\sigma}(\mathbf{x}_k[q])||^2 \ge x_{\max}^2[q]$, substituting (99) into (97) yields

$$\upsilon_k[n] \le \beta(1+AK)\,C\,\|\boldsymbol{\sigma}(\mathbf{x}_k[q])\|^2 R^{n-q'}\sum_{i=0}^{n-q'}\left(\frac{U}{R}\right)^i \quad (100)$$

for $n \ge q'$. The summation on the right side of (100) is upper bounded by $(1-(U/R)^{n-q'+1})/(1-U/R)$, so (98) and (100) imply

$$\|\boldsymbol{\sigma}(\mathbf{x}_k[n])\|^2 \le (1+AK)\left(\prod_{m=q'}^{n} R_k[m] + \beta_{k,n-q'}\right)\|\boldsymbol{\sigma}(\mathbf{x}_k[q])\|^2 \quad (101)$$

for $n \ge q'$, where

$$\beta_{k,m} = \frac{\beta CU\left(R^{m+1} - U^{m+1}\right)}{R-U}, \quad (102)$$

when $m \ge 0$. It follows from (94) and $0 < R_k[m] < 1$ that (101) also holds for $q \le n \le q'-1$ with $\beta_{k,m} = 0$ for $m < 0$, and it follows from $\mathbf{x}_k[n] = \mathbf{0}$ for $n < q$ that (101) also holds for $n < q$.

Inequality (101) is equivalent to (92), so it remains to show that $\beta_{k,m}$ satisfies (93). The sum of a^{m+1} for $m = 0, 1, 2, \ldots$ is upper bounded by $a/(1-a) < 1/(1-a)$ if $0 < a < 1$, and, by definition, $0 < R < U < 1$, so $\beta_{k,m} \ge 0$ for $m \ge 0$ and

$$\sum_{m=0}^{\infty}\beta_{k,m} \le \frac{\beta CU}{(1-U)(1-R)}, \quad (103)$$

which implies (93) with $A' \ge CU/(1-R)$. □

Lemma B3: The hypothesis of Theorem 1 implies that for each β in the range $0 < \beta < 1$, there exists a positive constant B, and a positive constant δ less than K' and $1/(4||\mathbf{H}_{\mathbf{c}\_Q}||_2)$ such that

$$x_{\max}^2[n] \le \begin{cases}(1-(1-\beta)K_{\min}[n])\,x_{\max}^2[n-1], & \text{if } n \ge q',\\ (1+BK)x_{\max}^2[q], & \text{if } q \le n \le q'-1,\end{cases} \quad (104)$$

and

$$\|\boldsymbol{\sigma}(\mathbf{x}_k[n])\|^2 \le \begin{cases}(1-K_k[n])\,\|\boldsymbol{\sigma}(\mathbf{x}_k[n-1])\|^2 \\ \quad +\beta K_k[n]x_{\max}^2[n-1], & \text{if } n \ge q',\\ \|\boldsymbol{\sigma}(\mathbf{x}_k[q])\|^2 + BKx_{\max}^2[q], & \text{if } q \le n \le q'-1,\end{cases} \quad (105)$$

for all $k = 1, 2, \ldots, L$ and all $0 < K < \delta$, where $q' = q+Q+N-1$.

Proof: Lemma 1 implies that $\mathbf{H}_{r,k}(\mathbf{x}_l[p]|_{p\le n-1, 1\le l\le L})$ for each n is a finite sum of terms, and each term has the form $s_{r,k,u,l,i}[n]\mathbf{D}_{r,u,i}\mathbf{x}_l[n-i]$, where $|s_{r,k,u,l,i}[n]| \le 1$ and $\mathbf{D}_{r,u,i}$ is a bounded matrix. If $x_{\max}[q] = 0$, then the probability of $\mathbf{x}_k[q]$ being non-zero for any k is zero, so (28) and (29) imply that the probability of $\mathbf{x}_k[q+1]$ being non-zero for any k is zero and, hence, $x_{\max}[q+1] = 0$. Continuing this argument inductively for all $n = q, q+1, q+2, \ldots, q'-1$ implies that if $x_{\max}[q] = 0$, then $x_{\max}[n] = 0$. If $x_{\max}[q] \neq 0$, (28), (29), and (83) imply that there exist constants δ and B' such that for all $0 < K < \delta$, $|x_{\max}[n] - x_{\max}[n-1]| < KB'x_{\max}[q]$ for $n = q+1, q+2, \ldots, q'-1$. Since $|x_{\max}[n] - x_{\max}[q]|$ is less than or equal to the sum of $|x_{\max}[p] - x_{\max}[p-1]|$ for $p = q+1, q+2, \ldots, n$, it follows that $|x_{\max}[n] - x_{\max}[q]| < K(Q+N-2)B'x_{\max}[q]$ for all $n = q+1, q+2, \ldots, q'-1$, which also means that $|x_{\max}[n] - x_{\max}[q]| < B''Kx_{\max}[q]$ for $B'' = (Q+N-2)B'$. It follows that for any $\mathbf{x}_k[q]$, there exist positive constants B and $\delta < 1/B$ such that

$$(1 - BK)\, x_{\max}^2[q] \le x_{\max}^2[n] \le (1 + BK)\, x_{\max}^2[q] \tag{106}$$

for all $q \le n \le q'-1$ and $0 < K < \delta$. The right inequality of (106) implies (104) for $q \le n \le q'-1$.

Given that $x_{\max}[n] = 0$ for $q \le n \le q'-1$ if $x_{\max}[q] = 0$, as explained above, and $||\boldsymbol{\sigma}(\mathbf{x}_k[n])|| \le x_{\max}[n]$, it can be verified by inspection that (105) holds for $q \le n \le q'-1$ when $x_{\max}[q] = 0$. The properties of $\mathbf{H}_{r,k}(\mathbf{x}_l[p]|_{p\le n-1,\, 1\le l\le L})$ mentioned above imply that $||\boldsymbol{\sigma}(\mathbf{H}_{r,k}(\mathbf{x}_l[p]|_{p\le n-1,\, 1\le l\le L}))||$ is finite for all $q \le n \le q'-1$, all k, and all r. Therefore, if $x_{\max}[q] \neq 0$, applying Lemma C1 to (28) implies that $||\boldsymbol{\sigma}(\mathbf{x}_k[n])|| \le ||\boldsymbol{\sigma}(\mathbf{x}_k[n-1])|| + KCx_{\max}[q]$ for all $q < n \le q'-1$, where C is a positive constant. Recursively substituting this inequality into itself a finite number of times results in $||\boldsymbol{\sigma}(\mathbf{x}_k[n])|| \le ||\boldsymbol{\sigma}(\mathbf{x}_k[q])|| + KC'x_{\max}[q]$ for all $q \le n \le q'-1$, where C' is a positive constant. Squaring this inequality yields $||\boldsymbol{\sigma}(\mathbf{x}_k[n])||^2 \le ||\boldsymbol{\sigma}(\mathbf{x}_k[q])||^2 + KC''x_{\max}^2[q]$ for all $q \le n \le q'-1$, where C'' is a positive constant. If B in (106) is less than C'', it can be enlarged to equal C'' (in which case (106) will still hold for all $q \le n \le q'-1$, all $0 < K < \delta$, and some $\delta < 1/B$). This proves that (105) holds for $q \le n \le q'-1$ when $x_{\max}[q] \neq 0$.

Equation (29) implies that $x_{\max}[p] = 0$ for all $p < q$. This and (106) imply that (107) of Lemma B4 holds for all $p < q'$ for some constant B' when K is sufficiently small. Hence, Lemma B4 implies that (104), (105), and (107) hold for $p = q'$. Mathematical induction with this as the base case and Lemma B4 as step case implies that (104) and (105) hold for all $n \ge q'$. □

Lemma B4: The hypothesis of Theorem 1 implies that for any positive constant $B' \ge 4||\mathbf{H}_{\mathbf{c}\_Q}||_2$ and each β in the range $0 < \beta < 1$, there exists a positive δ less than $1/B'$ and K' such that if

$$x_{\max}^2[p] \ge \left(1 - B'K\right) x_{\max}^2[p-1] \tag{107}$$

holds for some K in the range $0 < K < \delta$ and all $p < n$, where $n \ge q'$ and $q' = q + Q + N - 1$, then (107) also holds for $p = n$ and (104)-(105) hold for the chosen value of n.

Proof: Choose any $n \ge q'$. Writing $\mathbf{x}_k[n-1] - \mathbf{x}_k[n-Q]$ as a telescopic sum gives

$$\mathbf{x}_k[n-1] - \mathbf{x}_k[n-Q] = \sum_{i=1}^{Q-1} \mathbf{x}_k[n-i] - \mathbf{x}_k[n-i-1]. \tag{108}$$

Given that $n \ge q'$ and $q' = q + Q + N - 1$, it follows that $n - i > q$ for all $1 \le i \le Q-1$. Substituting (28) with n replaced by $n-i$ into the right side of (108) yields

$$\mathbf{x}_k[n-1] = \mathbf{x}_k[n-Q] - \sum_{i=1}^{Q-1}\sum_{r=1}^{N} K^r \mathbf{H}_{r,k}\left(\mathbf{x}_l\,[p]|_{p\le n-i-1, 1\le l\le L}\right). \tag{109}$$

Respectively subtracting and adding $Kc_k[n-Q]\mathbf{H}_{\mathbf{c}\_Q}$ times the left and right sides of (109) to and from the right side of (28) yields

$$\mathbf{x}_k[n] = \left(\mathbf{I} - Kc_k[n-Q]\mathbf{H}_{\mathbf{c}\_Q}\right)\mathbf{x}_k[n-1] - K\boldsymbol{\alpha}_k[n] - K^2\boldsymbol{\beta}_k[n], \tag{110}$$

where

$$\boldsymbol{\alpha}_k[n] = \mathbf{H}_{1,k}\left(\mathbf{x}_l\,[p]|_{p\le n-1, 1\le l\le L}\right) - c_k[n-Q]\mathbf{H}_{\mathbf{c}\_Q}\mathbf{x}_k[n-Q], \tag{111}$$

and

$$\boldsymbol{\beta}_k[n] = \sum_{r=2}^{N} K^{r-2}\mathbf{H}_{r,k}\left(\mathbf{x}_l\,[p]|_{p\le n-1, 1\le l\le L}\right) + c_k[n-Q]\mathbf{H}_{\mathbf{c}\_Q}\sum_{i=1}^{Q-1}\sum_{r=1}^{N} K^{r-1}\mathbf{H}_{r,k}\left(\mathbf{x}_l\,[p]|_{p\le n-i-1, 1\le l\le L}\right). \tag{112}$$

The RMS vector definition, i.e., (4), and the L2 norm definition imply that $||\boldsymbol{\sigma}(\mathbf{v})||^2 = \mathrm{E}\{\mathbf{v}^{\mathrm{T}}\mathbf{v}\}$ for any real vector $\mathbf{v}$. Therefore, substituting (110) into $\mathrm{E}\{\mathbf{x}_k^{\mathrm{T}}[n]\mathbf{x}_k[n]\}$ and applying the triangle inequality gives

$$\|\boldsymbol{\sigma}\left(\mathbf{x}_k[n]\right)\|^2 \le a_0 + 2\left(a_1K + a_2K^2 + a_4K^3\right) + a_3K^2 + a_5K^4 \tag{113}$$

and

$$\|\boldsymbol{\sigma}\left(\mathbf{x}_k[n]\right)\|^2 \ge a_0 - 2\left(a_1K + a_2K^2 + a_4K^3\right) - a_3K^2 - a_5K^4, \tag{114}$$

where

$$a_0 = \left\|\boldsymbol{\sigma}\left(\left(\mathbf{I} - Kc_k[n-Q]\mathbf{H}_{\mathbf{c}\_Q}\right)\mathbf{x}_k[n-1]\right)\right\|^2, \tag{115}$$

$$a_1 = \left|\mathrm{E}\left\{\boldsymbol{\alpha}_k^T[n]\left(\mathbf{I} - Kc_k[n-Q]\mathbf{H}_{\mathbf{c}\_Q}\right)\mathbf{x}_k[n-1]\right\}\right|, \tag{116}$$

$$a_2 = \left|\mathrm{E}\left\{\boldsymbol{\beta}_k^T[n]\left(\mathbf{I} - Kc_k[n-Q]\mathbf{H}_{\mathbf{c}\_Q}\right)\mathbf{x}_k[n-1]\right\}\right|, \tag{117}$$

$$a_3 = \mathrm{E}\left\{\boldsymbol{\alpha}_k^T[n]\boldsymbol{\alpha}_k[n]\right\}, \tag{118}$$

$$a_4 = \left|\mathrm{E}\left\{\boldsymbol{\beta}_k^T[n]\boldsymbol{\alpha}_k[n]\right\}\right|, \quad \text{and } a_5 = \mathrm{E}\left\{\boldsymbol{\beta}_k^T[n]\boldsymbol{\beta}_k[n]\right\}. \tag{119}$$

Applying Lemma C1 to (115) gives

$$\sqrt{a_0} \ge \|\boldsymbol{\sigma}\left(\mathbf{x}_k[n-1]\right)\| - \left\|\boldsymbol{\sigma}\left(Kc_k[n-Q]\mathbf{H}_{\mathbf{c}\_Q}\mathbf{x}_k[n-1]\right)\right\|. \tag{120}$$

Lemma C2 and the homogeneity property of matrix norms imply that the second term on the right side of (120) can be replaced by $Kc_k[n-Q]||\mathbf{H}_{\mathbf{c}\_Q}||_2||\boldsymbol{\sigma}(\mathbf{x}_k[n-1])||$ without violating the inequality, so

$$a_0 \ge \left(1 - Kc_k[n-Q]\left\|\mathbf{H}_{\mathbf{c}\_Q}\right\|_2\right)^2 \|\boldsymbol{\sigma}\left(\mathbf{x}_k[n-1]\right)\|^2. \tag{121}$$

By the hypothesis of Theorem 1, there exists a positive K' such that $h_K > 0$ when $K = K'$. The definition of h_K, i.e., (25), implies that h_K does not decrease as K decreases, so $h_K > 0$ for all $0 < K < K'$. The definition of h_K further implies that

$$\alpha h_K \le 1 - \left\|\mathbf{I} - \alpha\mathbf{H}_{\mathbf{c}\_Q}\right\|_2 \tag{122}$$

for any α in the range $0 < \alpha \le K$, and it can be verified by inspection that (122) also holds when $\alpha = 0$. Applying Lemma C2 to (115) gives

$$a_0 \le \left\|\mathbf{I} - Kc_k[n-Q]\mathbf{H}_{\mathbf{c}\_Q}\right\|_2^2 \|\boldsymbol{\sigma}(\mathbf{x}_k[n-1])\|^2, \quad (123)$$

so (122) with $\alpha = Kc_k[n-Q]$ implies

$$a_0 \le (1 - Kc_k[n-Q]h_K)^2 \|\boldsymbol{\sigma}(\mathbf{x}_k[n-1])\|^2 \quad (124)$$

for all $0 < K < \delta$ and any δ that satisfies $\delta < K'$.

It follows from (111), (112), and the properties of $\mathbf{H}_{r,k}(\mathbf{x}_l[p]|_{p\le n-1, 1\le l\le L})$ stated in Lemma 1 that $\boldsymbol{\alpha}_k[n]$ and $\boldsymbol{\beta}_k[n]$ each have the form

$$\sum_{u=1}^{M}\sum_{l=1}^{L}\sum_{i\ge 1} a_{k,u,l,i}[n]\mathbf{F}_{u,i,n}\mathbf{x}_l[n-i], \quad (125)$$

where M is an integer, each $\mathbf{F}_{u,i,n}$ is an $N \times N$ deterministic matrix with a spectral norm that is bounded by a sequence that decays to zero exponentially as $i \to \infty$, and each $a_{k,u,l,i}[n]$ is a product of one or more of $S_r[n-J]$, $(S_r^2[n-J]-c_r[n-J])/2$, and $c_r[n-J]$ for any J and r, and one factor of the product is $S_k[n-Q]$, $(S_k^2[n-Q]-c_k[n-Q])/2$, or $c_k[n-Q]$. Hence, (118), (119), and the triangle inequality imply that a_3, a_4, and a_5 are bounded by sums of terms wherein each term has the form $|\mathrm{E}\{\mathbf{v}_{k,u,l,i}^{\mathrm{T}}[n]\mathbf{w}_{k,u',l',i'}[n]\}|$, where $\mathbf{v}_{k,u,l,i}[n]$ and $\mathbf{w}_{k,u',l',i'}[n]$ each have the form of one of the terms in (125), i.e., $\mathbf{v}_{k,u,l,i}[n] = a_{k,u,l,i}[n]\mathbf{F}_{u,i,n}\mathbf{x}_l[n-i]$ and $\mathbf{w}_{k,u',l',i'}[n] = a'_{k,u',l',i'}[n]\mathbf{F}'_{u',i',n}\mathbf{x}_{l'}[n-i']$, where $a'_{k,u',l',i'}[n]$ and $\mathbf{F}'_{u',i',n}$ have the respective properties of $a_{k,u,l,i}[n]$ and $\mathbf{F}_{u,i,n}$ described above. It follows that $a_{k,u,l,i}[n]a'_{k,u',l',i'}[n]$ has the properties of $a_k[n]$ stated in the hypothesis of Lemma B5, so $|\mathrm{E}\{\mathbf{v}_{k,u,l,i}^{\mathrm{T}}[n]\mathbf{w}_{k,u',l',i'}[n]\}|$ has the properties of $|\mathrm{E}\{a_k[n](\mathbf{A}\mathbf{x}_l[n-i])^{\mathrm{T}}\mathbf{B}\mathbf{x}_{l'}[n-j]\}|$ stated in the hypothesis of Lemma B5. Thus, Lemma B5 implies that each term in the sums that bound a_3, a_4, and a_5 is less than or equal to

$$\frac{c_k[n-Q]\left(1+B''K\right)\left\|\mathbf{F}_{u,i,n}\right\|_2\left\|\mathbf{F}'_{u',i',n}\right\|_2 x_{\max}^2[n-1]}{(1-B'K)^{\frac{i+i'-2}{2}}} \quad (126)$$

for $0 < K < \delta$, where δ, B' and B'' are positive constants. Given that $||\mathbf{F}_{u,i,n}||_2$ and $||\mathbf{F}'_{u',i',n}||_2$ are bounded by sequences that decay exponentially to zero as $i, i' \to \infty$ and the rate of decay of the denominator of (126) decreases with K, there must exist a positive δ that is less than $1/B'$ such that if $0 < K < \delta$, the sums of these terms are finite and satisfy

$$a_i \le A_i c_k[n-Q]x_{\max}^2[n-1] \quad (127)$$

for $i = 3, 4$, and 5, where A_i is a positive constant.

Similarly, (117) implies that a_2 is bounded by a sum whose terms each have the form $|\mathrm{E}\{(a_{k,u,l,i}[n]\mathbf{F}_{u,i,n}\mathbf{x}_l[n-i])^{\mathrm{T}}\mathbf{B}\mathbf{x}_k[n-1]\}|$, where $\mathbf{B} = \mathbf{I} - Kc_k[n-Q]u_D[n]\mathbf{H}_{\mathbf{c}\_Q}$. Hence, (127) holds for $i = 2$ when $0 < K < \delta$ for some positive $\delta < 1/B'$ by the same argument presented above to prove that it holds for $i = 3$, 4, and 5.

Equations (23), and (111) imply that $\boldsymbol{\alpha}_k[n]$ is a sum of terms each of which has a factor of $a_{k,u,l,j}[n] = S_k[n-Q]S_l[n-j]$ when $l \ne k$ or $j \ne Q$, or $a_{k,u,l,j}[n] = (S_k^2[n-Q]-c_k[n-Q])/2$ otherwise. Therefore, (116) implies that a_1 is bounded by a sum whose terms each have the form $|\mathrm{E}\{(a_{k,u,l,j}[n]\mathbf{F}_{u,i,n}\mathbf{x}_l[n-i])^{\mathrm{T}}\mathbf{B}\mathbf{x}_k[n-1]\}|$, where $\mathbf{B} = \mathbf{I} - Kc_k[n-Q]\mathbf{H}_{\mathbf{c}\_Q}$. Applying Lemma B5 and using the same argument presented above for a_2, a_3, a_4, and a_5, but this time with $\rho = 0$, leads to the conclusion that there exists a positive δ that is less than $1/B'$ such that

$$a_1 \le KA_1 c_k[n-Q]x_{\max}^2[n-1] \quad (128)$$

if $0 < K < \delta$, where A_1 is a positive constant.

Substituting (121), (124), (127) for $i = 2, 3, 4$, and 5, and (128) into (113) and (114), combining all terms proportional to K^j for $j \ge 2$, and applying (85), (83), and $0 < h_K < 1$ gives

$$\|\boldsymbol{\sigma}(\mathbf{x}_k[n])\|^2 \le (1 - K_k[n])\|\boldsymbol{\sigma}(\mathbf{x}_k[n-1])\|^2 + \frac{CK}{h_K}K_k[n]x_{\max}^2[n-1] \quad (129)$$

and

$$\|\boldsymbol{\sigma}(\mathbf{x}_k[n])\|^2 \ge \left(1 - 2Kc_k[n-Q]\left\|\mathbf{H}_{\mathbf{c}\_Q}\right\|_2\right)\|\boldsymbol{\sigma}(\mathbf{x}_k[n-1])\|^2 - 2CK^2c_k[n-Q]x_{\max}^2[n-1] \quad (130)$$

for $0 < K < \delta$ and some positive δ, where C is a positive constant. As (129) and (130) hold for all k and (83) implies $||\boldsymbol{\sigma}(\mathbf{x}_k[n])|| \le x_{\max}[n]$ for each k, $||\boldsymbol{\sigma}(\mathbf{x}_l[n])|| = x_{\max}[n]$ for some l, and $||\boldsymbol{\sigma}(\mathbf{x}_{l'}[n-1])|| = x_{\max}[n-1]$ for some l', (129) with k replaced by l implies

$$x_{\max}^2[n] \le \left(1 - \left(1 - \frac{CK}{h_K}\right)K_l[n]\right)x_{\max}^2[n-1], \quad (131)$$

and (130) with k replaced by l' implies

$$x_{\max}^2[n] \ge \left(1 - 2Kc_{l'}[n-Q]\left(\left\|\mathbf{H}_{\mathbf{c}\_Q}\right\|_2 + CK\right)\right)x_{\max}^2[n-1]. \quad (132)$$

If $0 < K < ||\mathbf{H}_{\mathbf{c}\_Q}||_2/C$, (132) implies (107) with $B' \ge 4||\mathbf{H}_{\mathbf{c}\_Q}||_2$. Hence, there exists a positive δ such that (107) with $p = n$ holds for all $0 < K < \delta$. Replacing C in (129) and (131) with a larger constant does not violate either inequality, and given that h_K does not decrease as K decreases, CK/h_K can be made arbitrarily small by reducing K toward 0. Consequently, for any β in the range $0 < \beta < 1$, there exists a positive δ such that both (129) and (131) with CK/h_K replaced by β hold for all $0 < K < \delta$. Moreover, the definitions of $K_{\min}[n]$ imply that (131) holds with $K_l[n]$ replaced by $K_{\min}[n]$. Given that $n \ge q'$, thus, (104) is implied by (131) and (105) is implied by (129) for the chosen value of n. □

Lemma B5: Suppose $n \ge q'$, where $q' = q + Q + N - 1$, and there exist positive constants $B' \ge 4||\mathbf{H}_{\mathbf{c}\_Q}||_2$ and $\delta < 1/B'$ such that $\mathbf{x}_k[n]$ satisfies (107) for all $0 < K < \delta$, all k, and all $p < n$, $\mathbf{A}$ and $\mathbf{B}$ are any $N \times N$ deterministic matrixes, and $a_k[n]$ is a product of one or more of $S_r[n-J]$, $(S_r^2[n-J]-c_r[n-J])/2$, and $c_r[n-J]$ for any J and r, where one factor of the product is $S_k[n-Q]$, $(S_k^2[n-Q]-c_k[n-Q])/2$, or $c_k[n-Q]$. Then there exist a positive constant $\delta' < 1/B'$ and a positive constant B'' such that for any K in the range $0 < K < \delta'$, any l, any l', and any i, $j \ge 1$,

$$\left|\mathrm{E}\left\{a_k[n](\mathbf{A}\mathbf{x}_l[n-i])^{\mathrm{T}}\mathbf{B}\mathbf{x}_{l'}[n-j]\right\}\right| \le \frac{c_k[n-Q]\left(\rho + B''K\right)\|\mathbf{A}\|_2\|\mathbf{B}\|_2 x_{\max}^2[n-1]}{(1-B'K)^{\frac{i+j-2}{2}}} \quad (133)$$

holds with $\rho = 1$. It also holds with $\rho = 0$ if $a_k[n] = (S_k^2[n-Q] - c_k[n-Q])/2$ or $a_k[n] = S_k[n-Q]S_l[n-j]$ with $l \ne k$ or $j \ne Q$.

Proof: Lemma 1 implies that $\mathbf{H}_{r,k}(\mathbf{x}_l[p]|_{p\le n-1, 1\le l\le L})$ is a sum of terms each of which has the form

$s_{r,k,u,l,i}[n]\mathbf{D}_{r,u,i}\mathbf{x}_l[n-i]$, where $s_{r,k,u,l,i}[n]\mathbf{D}_{r,u,i}$ is independent of $S_l[n+J]$ for $J \geq N-1$ and all l. Therefore, recursively applying (28) with (29) and the stipulation that $\mathbf{x}_{k,q}$ is independent of $S_l[q+J]$ for all $J \geq N-1$, all l and all k shows that $\mathbf{x}_k[n]$ is independent of $S_l[n+J]$ for all $J \geq N-1$, all l and all k, or, equivalently, that $\mathbf{x}_k[n-i]$ for any i is independent of $S_l[m]$ for all $m \geq n-i+N-1$, all l, and all k. In particular, this implies that if $i \geq Q+N-1$, then $\mathbf{x}_k[n-i]$ is independent of $S_l[n-Q]$ for all l and k. By definition, $a_k[n]$ can be written as $a_k[n] = a'_k[n]a''_k[n]$, where $a'_k[n]$ is the product of all $S_l[n-Q]$, $0.5(S_l^2[n-Q]-c_l[n-Q])$, and $c_l[n-Q]$ factors for any l in $a_k[n]$, and $a''_k[n]$ comprises the remaining factors in $a_k[n]$. It follows that $a'_k[n]$ is statistically independent of $a''_k[n]$, and it is statistically independent of $\mathbf{x}_k[n-i]$ for all $i \geq Q+N-1$, all l, and all k.

Choose any $n \geq q'$ and any $i,\ j \geq Q+N-1$, any l, any l' and any k. The above definitions and their independence properties imply

$$\begin{aligned}&\left|\mathrm{E}\left\{a_k[n]\left(\mathbf{A}\mathbf{x}_l[n-i]\right)^{\mathrm{T}}\mathbf{B}\mathbf{x}_{l'}[n-j]\right\}\right| \\ &\quad = \left|\mathrm{E}\left\{a'_k[n]\right\}\right|\left|\mathrm{E}\left\{a''_k[n]\left(\mathbf{A}\mathbf{x}_l[n-i]\right)^{\mathrm{T}}\mathbf{B}\mathbf{x}_{l'}[n-j]\right\}\right|. \quad (134)\end{aligned}$$

By definition, $|a''[n]| \leq 1$, so applying Lemma C3 to (134) yields

$$\begin{aligned}&\left|\mathrm{E}\left\{a_k[n](\mathbf{A}\mathbf{x}_l[n-i])^{\mathrm{T}}\mathbf{B}\mathbf{x}_{l'}[n-j]\right\}\right| \\ &\quad \leq \left|\mathrm{E}\left\{a'_k[n]\right\}\right| \|\boldsymbol{\sigma}\left(\mathbf{A}\mathbf{x}_l[n-i]\right)\| \, \|\boldsymbol{\sigma}\left(\mathbf{B}\mathbf{x}_{l'}[n-j]\right)\|. \quad (135)\end{aligned}$$

If $a_k[n] = S_k[n-Q]S_l[n-j]$ with $l \neq k$ or $j \neq Q$, then $\mathrm{E}\{a'_k[n]\} = 0$ because $S_k[n-Q]$ and $S_l[n-j]$ are independent and $\mathrm{E}\{S_k[n-Q]\} = 0$. If $a_k[n] = (S_k^2[n-Q]-c_k[n-Q])/2$, then $\mathrm{E}\{a'_k[n]\} = 0$, because $c_k[n-Q] = \mathrm{E}\{S_k^2[n-Q]\}$. For every other case, $a'_k[n]$ can be written as $a'''_k[n]S_k[n-Q]$, $a'''_k[n](S_k^2[n-Q]-c_k[n-Q])/2$, or $a'''_k[n]c_k[n-Q]$, where $|a'''_k[n]| \leq 1$, so $0 \leq |\mathrm{E}\{a'_k[n]\}| \leq c_k[n-Q]$ because $c_k[n-Q] = \mathrm{E}\{S_k^2[n-Q]\}$. Therefore, $|\mathrm{E}\{a'_k[n]\}| \leq \rho c_k[n-Q]$ where ρ is as defined in the lemma statement. Substituting this into (135) and applying Lemma C2 yields

$$\begin{aligned}&\left|\mathrm{E}\left\{a_k[n](\mathbf{A}\mathbf{x}_l[n-i])^{\mathrm{T}}\mathbf{B}\mathbf{x}_{l'}[n-j]\right\}\right| \\ &\quad \leq \rho c_k[n-Q]\,\|\mathbf{A}\|_2\,\|\boldsymbol{\sigma}\left(\mathbf{x}_l[n-i]\right)\|\,\|\mathbf{B}\|_2\,\|\boldsymbol{\sigma}\left(\mathbf{x}_{l'}[n-j]\right)\|. \quad (136)\end{aligned}$$

For any $m \geq 2$, recursively substituting (107), which holds by the lemma's hypothesis, into itself for $p = n-1$, $n-2$, …, $n-m+1$ yields

$$x_{\max}[n-m] \leq x_{\max}[n-1]\left(1-B'K\right)^{-(m-1)/2}. \quad (137)$$

By inspection, (137) also holds for $m = 1$. By definition, $||\boldsymbol{\sigma}(\mathbf{x}_k[r])|| \leq x_{\max}[r]$ for all k and r, so $||\boldsymbol{\sigma}(\mathbf{x}_l[n-i])||$ and $||\boldsymbol{\sigma}(\mathbf{x}_{l'}[n-j])||$ in (136) can be replaced by $x_{\max}[n-i]$ and $x_{\max}[n-j]$, respectively. Making these substitutions and substituting (137) into the result gives (133) for the special case of $B'' = 0$. This proves that the lemma is true for all i, $j \geq Q+N-1$, all l, and all k.

It is next shown by mathematical induction that the lemma also is true for all $1 \leq i,\ j < Q+N-1$. The induction base case is that the lemma is true for $i,\ j = Q+N-1$, all l, and all k, as proven above.

The induction step hypothesis is that the lemma is true for $i \geq i'$ and $j \geq j'$ for some i' and j' in the range $2 \leq i'$, $j' \leq Q+N-1$. Given that $n \geq q'$ and $q' = q+Q+N-1$, it follows that $n-(j'-1) > q$. Therefore, (28) with n replaced by $n-(j'-1)$ holds, i.e.,

$$\begin{aligned}&\mathbf{x}_{l'}\left[n-\left(j'-1\right)\right] \\ &\quad = \mathbf{x}_{l'}\left[n-j'\right] - K\sum_{r=1}^{N} K^{r-1}\mathbf{H}_{r,l'}\left(\mathbf{x}_l\left[p\right]|_{p\leq n-j',1\leq l\leq L}\right) \\ &\qquad\qquad (138)\end{aligned}$$

for $l' = 1, 2, \ldots, L$. As $\mathbf{H}_{r,k}(\mathbf{z}_l[p]|_{p\leq n-1,\,1\leq l\leq L})$ has the form of (22), the summation term in (138) has the form

$$\begin{aligned}\mathbf{a}[n] = \sum_{r=1}^{N}\sum_{u=1}^{M_r}\sum_{l=1}^{L}\sum_{i=1}^{n-j'+1+N} & K^{r-1}s_{r,l',u,l,i}[n-j'+1] \\ &\times \mathbf{D}_{r,u,i}\mathbf{x}_l\left[n-j'+1-i\right]. \quad (139)\end{aligned}$$

Therefore, (138) and the triangle inequality imply

$$\begin{aligned}&\left|\mathrm{E}\left\{a_k[n]\left(\mathbf{A}\mathbf{x}_l\left[n-i'\right]\right)^{\mathrm{T}}\mathbf{B}\mathbf{x}_{l'}\left[n-\left(j'-1\right)\right]\right\}\right| \\ &\quad \leq \left|\mathrm{E}\left\{a_k[n]\left(\mathbf{A}\mathbf{x}_l\left[n-i'\right]\right)^{\mathrm{T}}\mathbf{B}\mathbf{x}_{l'}\left[n-j'\right]\right\}\right| \\ &\quad\quad + \left|\mathrm{E}\left\{Ka_k[n]\left(\mathbf{A}\mathbf{x}_l\left[n-i'\right]\right)^{\mathrm{T}}\mathbf{B}\mathbf{a}[n]\right\}\right|. \quad (140)\end{aligned}$$

It follows from (139) and the triangle inequality that the right-most expectation on the right side of (140) is upper bounded in magnitude by a sum of terms each of which is given by K^r for $r \geq 1$ times the left side of (133) with $a_k[n]s_{r,l',u,l,i}[n-j'+1]$ in place of $a_k[n]$, $\mathbf{BD}_{r,u,i}$ in place of $\mathbf{B}$, i' in place of i, and $j'-1+i$ in place of j. By definition $a_k[n]s_{r,l',u,l,i}[n-j'+1]$ has the general form of $a_k[n]$ given in the lemma statement, and $j'-1+i \geq j'$, so the induction step hypothesis implies that each term is upper bounded by a power of K times the right side of (133) with the substitutions described above. Given that $||\mathbf{BD}_{r,u,i}||_2 \leq ||\mathbf{B}||_2||\mathbf{D}_{r,u,i}||_2$ and $||\mathbf{D}_{r,u,i}||_2$ is upper bounded by a sequence that decays to zero exponentially as $i \to \infty$, this and (107) imply that provided K is a sufficiently small positive value there must exist a positive constant, A, such that the right-most expectation in (140) is upper bounded in magnitude by $A{\cdot}K$ times the right side of (133) with i and j replaced by i' and $j'-1$, respectively. The induction step hypothesis also implies that the first expectation on the right side of (140) is upper bounded in magnitude by the right side of (133) with i and j replaced by i' and j', respectively. Hence, (140) implies that B'' can be enlarged by a finite amount such that (133) holds with i and j replaced by i' and $j'-1$, respectively.

Thus, the lemma is true for $i = i'$ and $j = j'-1$, and nearly the same argument shows that it is also true for $i = i'-1$ and $j = j'$. This proves the induction step case, so mathematical induction implies that the lemma is true for all $1 \leq i,\ j < Q+N-1$. □

Appendix C

Lemma C1: If $\mathbf{v}$ and $\mathbf{w}$ are any N-dimensional real random vectors, then

$$\|\boldsymbol{\sigma}\left(\mathbf{v}\right)\| - \|\boldsymbol{\sigma}\left(\mathbf{w}\right)\| \leq \|\boldsymbol{\sigma}\left(\mathbf{v}+\mathbf{w}\right)\| \leq \|\boldsymbol{\sigma}\left(\mathbf{v}\right)\| + \|\boldsymbol{\sigma}\left(\mathbf{w}\right)\|, \quad (141)$$

and if $\mathrm{E}\{\mathbf{v}^T\mathbf{w}\} = 0$, then

$$\|\boldsymbol{\sigma}\left(\mathbf{v}+\mathbf{w}\right)\|^2 = \|\boldsymbol{\sigma}\left(\mathbf{v}\right)\|^2 + \|\boldsymbol{\sigma}\left(\mathbf{w}\right)\|^2. \quad (142)$$

*Proof:*It follows from the RMS vector definition, (4), and the L2 norm definition that

$$\|\boldsymbol{\sigma}\,(\mathbf{v}+\mathbf{w})\|^2=\sum_{j=1}^{N}\mathrm{E}\left\{\upsilon_j^2\right\}+\mathrm{E}\left\{w_j^2\right\}+2\mathrm{E}\left\{\upsilon_j w_j\right\}, \qquad (143)$$

where υ_j and w_j are the jth elements of $\mathbf{v}$ and $\mathbf{w}$, respectively. The Cauchy-Schwarz inequality implies that the $|\mathrm{E}\{\upsilon_j w_j\}|^2 \leq \mathrm{E}\{\upsilon_j^2\}\mathrm{E}\{w_j^2\}$, so (143) implies

$$\begin{aligned}\|\boldsymbol{\sigma}\,(\mathbf{v}+\mathbf{w})\|^2 &\leq \sum_{j=1}^{N}\mathrm{E}\left\{\upsilon_j^2\right\}+\mathrm{E}\left\{w_j^2\right\}+2\sqrt{\mathrm{E}\left\{\upsilon_j^2\right\}\mathrm{E}\left\{w_j^2\right\}}\\ &= \sum_{j=1}^{N}\left(\sqrt{\mathrm{E}\left\{\upsilon_j^2\right\}}+\sqrt{\mathrm{E}\left\{w_j^2\right\}}\right)^2 = \|\boldsymbol{\sigma}\,(\mathbf{v})+\boldsymbol{\sigma}\,(\mathbf{w})\|^2 .\end{aligned} \qquad (144)$$

Taking the square root of (144) and applying the triangle inequality of norms gives the right-most inequality of (141). Replacing $\mathbf{w}$ with $-\mathbf{w}$ and then $\mathbf{v}$ with $\mathbf{v}+\mathbf{w}$ in this inequality yields the left-most inequality of (141). Equation (142) follows directly from (143) when $\mathrm{E}\{\mathbf{v}^T\mathbf{w}\}=0$, because $\mathrm{E}\{\mathbf{v}^T\mathbf{w}\}$ is the sum of $\mathrm{E}\{\upsilon_j w_j\}$ over $j=1,2,\ldots,N$. □

Lemma C2: If $\mathbf{v}$ is any $N\times 1$ real random vector and $\mathbf{D}$ is any $N\times N$ deterministic real matrix, then

$$\|\boldsymbol{\sigma}\,(\mathbf{D}\mathbf{v})\| \leq \|\mathbf{D}\|_2\,\|\boldsymbol{\sigma}\,(\mathbf{v})\| . \qquad (145)$$

Proof: For any possible value that the random vector $\mathbf{v}$ can take on, it follows from the properties of L2 norm that $||\mathbf{D}\mathbf{v}|| \leq ||\mathbf{D}||_2||\mathbf{v}||$, where $||\mathbf{v}||$ is L2 norm of $\mathbf{v}$. Since $\mathbf{D}$ is deterministic, it follows that

$$\mathrm{E}\left\{\|\mathbf{D}\mathbf{v}\|^2\right\} \leq \mathrm{E}\left\{\|\mathbf{D}\|_2^2\,\|\mathbf{v}\|^2\right\} = \|\mathbf{D}\|_2^2\,\mathrm{E}\left\{\|\mathbf{v}\|^2\right\}. \qquad (146)$$

Taking square root on both sides of (146) yields (145). □

Lemma C3: For any real scalar a that satisfies $|a|\leq 1$ and any $N\times 1$ real random vectors $\mathbf{v}$ and $\mathbf{w}$,

$$\left|\mathrm{E}\left\{a\mathbf{v}^{\mathrm{T}}\mathbf{w}\right\}\right| \leq \|\boldsymbol{\sigma}(\mathbf{v})\|\cdot\|\boldsymbol{\sigma}(\mathbf{w})\| . \qquad (147)$$

Proof: The triangle and Cauchy–Schwarz inequalities imply

$$\left|\mathrm{E}\{a\mathbf{v}^{\mathrm{T}}\mathbf{w}\}\right| \leq \sum_{j=1}^{N}\left|\mathrm{E}\left\{a\upsilon_j w_j\right\}\right| \leq \sum_{j=1}^{N}\sqrt{\mathrm{E}\left\{a^2\upsilon_j^2\right\}}\sqrt{\mathrm{E}\left\{w_j^2\right\}}. \qquad (148)$$

Given that $|a|\leq 1$, this implies

$$\left|\mathrm{E}\{a\mathbf{v}^{\mathrm{T}}\mathbf{w}\}\right| \leq \sum_{j=1}^{N}\sqrt{\mathrm{E}\left\{\upsilon_j^2\right\}}\sqrt{\mathrm{E}\left\{w_j^2\right\}}. \qquad (149)$$

The right side of (149) is the dot product of $\boldsymbol{\sigma}(\mathbf{v})$ and $\boldsymbol{\sigma}(\mathbf{w})$, which the Cauchy–Schwarz inequality implies is less than or equal to $||\boldsymbol{\sigma}(\mathbf{v})||\cdot||\boldsymbol{\sigma}(\mathbf{w})||$. □

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