Predictions of quantum gravity in inflationary cosmology: effects of the Weyl-squared term

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ABSTRACT: We derive the predictions of quantum gravity with fakeons on the amplitudes and spectral indices of the scalar and tensor fluctuations in inflationary cosmology. The action is $R + R^2$ plus the Weyl-squared term. The ghost is eliminated by turning it into a fakeon, that is to say a purely virtual particle. We work to the next-to-leading order of the expansion around the de Sitter background. The consistency of the approach puts a lower bound ($m_\chi > m_\phi/4$) on the mass m_χ of the fakeon with respect to the mass m_ϕ of the inflaton. The tensor-to-scalar ratio r is predicted within less than an order of magnitude ($4/3 < N^2 r < 12$ to the leading order in the number of e-foldings N). Moreover, the relation $r \simeq -8n_T$ is not affected by the Weyl-squared term. No vector and no other scalar/tensor degree of freedom is present.

Contents

1 Introduction

Inflation is a theory of accelerated expansion of the early universe [1–8], which accounts for the origin of the present large-scale structure. It explains the approximate isotropy of the cosmic microwave background radiation and allows us to study the quantum fluctuations as sources of the cosmological perturbations that seed the formation of the structures of the cosmos [9–15]. It also provides a rich environment where we can develop knowledge that might allow us to establish a nontrivial connection between high-energy physics and the physics of large scales.

Inflationary cosmology is often studied with the help of a matter field that drives the expansion by rolling down a potential $V(\phi)$ (for reviews, see [16–18]). Alternatively, gravity itself can drive the expansion, as in the Starobinsky $R+R^2$ model [2] and the $f(R)$

theories [19, 20]. The predictions end up depending strongly on the model, specifically the choices of $V(\phi)$ and $f(R)$. In single-field slow-roll inflation, potentials with a plateau lead to a scalar power spectrum that is compatible with current observations [21–23].

In particular, the Starobinsky $R + R^2$ model works well at the phenomenological level. However, once R^2 is introduced, it is hard to justify why the square $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ of the Weyl tensor $C_{\mu\nu\rho\sigma}$ is not included as well, since it has the same dimension in units of mass. We can spare the other quadratic combinations, such as $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, since they are related to R^2 and $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ by algebraic identities and the Gauss-Bonnet theorem. Thus, we are lead to consider the action

$$S = -\frac{M_{\rm Pl}^2}{16\pi} \int {\rm d}^4x \sqrt{-g} \left(R + \alpha R^2 + \beta C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} \right), \qquad (1.1)$$

which we briefly refer to as "$R + R^2 + C^2$ theory". The trouble with (1.1) is that the C^2 term is normally responsible for the presence of ghosts. Immediate ways out are to expand the physical quantities in powers of β [24], which is equivalent to assume that the ghosts are very heavy, and/or restrict to situations where the ghosts are short-lived. This approach amounts to "living with ghosts" [25], but does not eliminate the problem.

If we want to work with the $R + R^2 + C^2$ theory, we must explain how to treat C^2 in order to remove the ghosts, at least perturbatively and at the level of the cosmological perturbations. Here we use the procedure of eliminating them in favor of purely virtual particles [26, 27]. This procedure originates in high-energy physics, where the requirements of locality, renormalizability and unitarity result in consistency contraints on perturbative quantum field theory.

The simplest way to think of the idea is as follows. A normal particle can be real or virtual, depending on whether it is observed or not. As far as we know, a particle that is always real does not exist. What about a particle that is always virtual and can never become real? We can think of it as a purely virtual quantum [28] or a fake particle, i.e., a particle that mediates interactions among other particles, but is invisible to our detectors. And by that we mean invisible in principle, not just in practice.

Perturbative quantum gravity can be formulated as a unitary theory of scattering if the action (1.1) is quantized in a new way [26], by eliminating the would-be ghost in favor of a purely virtual particle, called *fakeon* [27]. In the expansion around flat space, the fakeon is introduced by replacing the Feynman $i\epsilon$ prescription (for a pole of the free propagator) with an alternative prescription that allows us to project the corresponding degree of freedom away consistently with the optical theorem. This means that the loop corrections are unable to resuscitate the degree of freedom back. Moreover, the prescription is compatible with renormalizability [26, 27]. A fakeon mediates interactions, but does not belong to the spectrum of asymptotic states. In this sense it is a "fake degree of freedom". Note that it removes a ghost at the fundamental level, without advocating its irrelevance for practical purposes. Incidentally, the calculations of Feynman diagrams with the fakeon prescription in quantum gravity [29, 30] are not harder than analogous calculations for the standard model.

Nevertheless, quantum field theory is formulated perturbatively, commonly around flat space. To study inflation and cosmology it is necessary to work on nontrivial back-

grounds. This raises the issue of understanding purely virtual quanta in curved space. A simplification comes from the fact that in cosmology we do not need to go as far as computing loop corrections, as argued in ref. [31], although we have to study the quantum fluctuations. In this paper, we show that we can work with the classical limit of the fakeon prescription/projection, which amounts to taking the average of the retarded and advanced potentials as Green function G_{f} for the fake particles [32],

$$G_{\mathrm{f}} = \frac{1}{2}\left(G_{\mathrm{Ret}} + G_{\mathrm{Adv}}\right), \tag{1.2}$$

combined with a certain wealth of knowledge on how to use this formula and interpret its consequences. Note that the quantum fakeon prescription cannot be inferred from (1.2), because (1.2) is not a good propagator in Feynman diagrams [28].

As said, the predictions of the popular models of inflation are model dependent. On the other hand, in high-energy physics the constraints of locality, unitarity and renormalizability leave room for a limited class of interactions, scalar potentials, and so on, to the extent that the theory of quantum gravity emerging from the idea of fake particle is essentially unique (when matter is switched off) and contains just two independent parameters more than Einstein gravity. They can be identified as the masses m_ϕ and m_χ of a scalar field ϕ (the inflaton) and a spin-2 fakeon $\chi_{\mu\nu}$. The triplet graviton-scalar-fakeon exhausts the set of degrees of freedom of the theory. From the cosmological point of view the physical modes are the usual curvature perturbation $\mathcal{R}$ and the tensor fluctuations. The extra degrees of freedom are turned into fake ones and projected away. In particular, no vector fluctuations, or additional scalar and tensor fluctuations survive.

We show that the consistency of the picture in curved space leads to a lower bound $m_\chi > m_\phi/4$ on the mass m_χ of the fakeon with respect to the mass m_ϕ of the inflaton. To the next-to-leading order, the amplitude $A_\mathcal{R}$ and the spectral index $n_\mathcal{R} - 1$ of the scalar fluctuations depend only on m_ϕ (and the number N of e-foldings). Instead, the amplitude A_T and the spectral index n_T of the tensor fluctuations do depend on m_χ. The bound $m_\chi > m_\phi/4$ narrows the window of allowed values of n_T and the tensor-to-scalar ratio $r = A_T/A_\mathcal{R}$ to less than one order of magnitude and makes the predictions quite precise, even before knowing the actual values of m_ϕ and m_χ.

Inflationary cosmology in higher-derivative gravity with ghosts have been studied in refs. [33–39]. Typically, the ghost sector is quantized by means of negative norms. Extra spectra are predicted, which may or may not be suppressed on superhorizon scales. Inflation has been considered in nonlocal theories of gravity as well [40], where the classical action contains infinitely many free parameters. The cosmological perturbations in those scenarios have been studied in [41, 42].

The gain achieved by means of fakeons is that no ghosts are present and the number of independent parameters is kept to a minimum. Whenever there is an overlap, we find agreement with the results derived in the other approaches. This occurs, for example, when H/m_χ or m_ϕ/m_χ are sufficiently small to suppress the effects of the fakeons in our theory and the effects of the ghosts in the theories of refs. [38, 39], where H is the value of the Hubble parameter during inflation. Even when H or m_χ are not large, we can still relate

some results, due to the universality of the low-energy expansion. For example, we can do so for any quantity that has a convergent, resummable expansion for small H/m_χ or m_ϕ/m_χ. In the limit $m_\chi/m_\phi \to \infty$, the results we find agree with those of the theory $R + R^2$ [19, 43].

We make the calculations in two frameworks and show that the final results match. In the first approach, which we call *inflaton framework*, the scalar field ϕ is introduced explicitly to eliminate the R^2 term, while the C^2 term is unmodified. The scalar potential coincides with the Starobinsky one. In the second approach, which we call *geometric framework*, both R^2 and C^2 are present. The C^2 term does not affect the FLRW metric, so in both approaches the background metric coincides with the one of the Starobinsky theory. The differences arise in the action of the fluctuations over the background. The map relating the two frameworks is a field-dependent conformal transformation, combined with a time reparametrization. A third formulation, where the scalar ϕ and a spin-2 fakeon $\chi_{\mu\nu}$ are introduced explicitly in order to eliminate both higher-derivative terms R^2 and C^2 is also available [30], but will not be studied here.

The paper is organized as follows. In section 2, we briefly review the formulation of quantum gravity with fakeons and present the two frameworks just mentioned. In section 3, we study the tensor and scalar fluctuations in the inflaton framework. The fakeon projection, which allows us to make sense of the term C^2, is briefly introduced in section 2 and discussed in detail in section 4. In section 5, we make the calculations in the geometric framework. In section 6, we study the vector fluctuations and show that they are projected away altogether at the quadratic level. Section 7 contains the summary of our predictions and section 8 contains the conclusions. In appendix A, we derive the map relating the inflaton framework to the geometric framework and show that the results agree. In appendix B we show that the curvature perturbation $\mathcal{R}$ can be considered constant on superhorizon scales for adiabatic fluctuations of the energy-momentum tensor.

2 Quantum gravity with fakeons

In this section we introduce the theory and the two frameworks we are going to work with. We begin by recalling a few basic features of the fakeons. Being purely virtual quanta, they are particles that mediate interactions, but do not belong to the physical spectrum of asymptotic states. Expanding around flat space, they are introduced by means of a new quantization prescription for the poles of the free propagators [26], alternative to the Feynman $i\epsilon$ prescription. The physical subspace V is obtained by projecting the fake degrees of freedom away. The theory is unitary in V, where the optical theorem holds. What makes the projection consistent to all orders [27] is that the fakeon prescription does not allow the loop corrections to resuscitate back the states that have been projected away.

The prescription makes sense irrespective of the sign of the residue at the pole of the propagator. Yet, it requires that the real part of the squared mass be positive. Indeed, fakeons cannot cure tachyons, but only ghosts. The no-tachyon condition is the main requirement we have to fulfill and its analogue on nontrivial backgrounds is going to play an important role.

The projection must also be performed at the classical level. An action like (1.1) is physically unacceptable as the classical limit of quantum gravity, because it has undesirable solutions. Yet, (1.1) is the starting point to formulate quantum gravity as a quantum field theory. It is local and provides the Feynman rules that allow us (together with the Feynman prescription for physical particles and the fakeon prescription for fake particles), to calculate the loop diagrams and the S matrix. An action of this type is called "interim" classical action [32].

The true classical action S_{class} is obtained from the interim classical action S_{inter} by projecting the fake degrees of freedom away. At the classical level, the projection is achieved by means of the classical limit of the fakeon prescription. Precisely, S_{class} is obtained by: (i) solving the field equations of the fakeons (derived from S_{inter}) by means of the fakeon Green function; and (ii) inserting the solutions back into S_{inter}. In the perturbative expansion around flat space, the fakeon Green function is the arithmetic average of the retarded and advanced potentials [32]. We will see that this piece of information is enough to derive the fakeon Green function on nontrivial backgrounds.

The plan of the paper is to calculate the effects of inflationary cosmology on the fluctuations of the cosmic microwave background radiation at the quadratic level. Since we do not need to work out loop corrections, we can quantize the projected action S_{class}, rather than projecting the quantum version of S_{inter}. This simplification saves us a lot of effort.

The good feature of S_{class} is that it no longer contains the fake degrees of freedom, by construction, so in principle it can be quantized with the usual methods. The nontrivial counterpart is that S_{class} is not fully local, due to the nonlocal remnants left by the fakeon projection. Because of this, the quantization of S_{class} is not as simple as usual, also taking into account that we must perform it on a nontrivial background. However, in a variety of lucky cases, which include those studied in this paper, it is possible to treat the nonlocal sector of S_{class} in a relatively simple way and extract physical predictions with the procedure described above, either because the nonlocal sector of S_{class} does not affect the quantities we are interested in, or because it affects them only at higher orders.

Summarizing, the simplest way to proceed, which we adopt in the paper, is as follows. First, we work out the classical action S_{class} of quantum gravity, by projecting the interim action S_{inter}. Second, we quantize S_{class} with the usual methods, paying special attention to the nonlocal sector, anticipating that in the end it does not create too serious difficulties.

Now we give the interim classical actions S_{inter} of quantum gravity in the two approaches we study in the paper. The projection $S_{\text{inter}} \to S_{\text{class}}$ and the quantization of S_{class} will be performed in the next sections, after expanding around the de Sitter background.

The higher-derivative form of the interim classical action is

$$S_{\text{geom}}(g, \Phi) = -\frac{M_{\text{Pl}}^2}{16\pi} \int \mathrm{d}^4x \sqrt{-g} \left[R + \frac{1}{2m_\chi^2} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} - \frac{R^2}{6m_\phi^2} \right] + S_{\mathfrak{m}}(g, \Phi), \qquad (2.1)$$

where $C_{\mu\nu\rho\sigma}$ denotes the Weyl tensor, $M_{\text{Pl}} = 1/\sqrt{G}$ is the Planck mass, Φ are the matter fields and $S_{\mathfrak{m}}$ is the action of the matter sector. The no-tachyon condition (i.e., the requirement that the free propagator around flat space does not have tachyonic poles) determines the signs in front of $C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$ and R^2.

The degrees of freedom of the gravitational sector are the graviton, a scalar field ϕ of mass m_ϕ and a spin-2 fakeon $\chi_{\mu\nu}$ of mass m_χ. The reason why $\chi_{\mu\nu}$ must be quantized as a fakeon is that the residue of the free propagator has the wrong sign at the $\chi_{\mu\nu}$ pole, so the Feynman prescription would turn it into a ghost, causing the violation of unitarity. On the other hand, ϕ can be quantized either as a fakeon or a physical particle, because the residue at the ϕ pole has the correct sign. In this paper, we assume that ϕ is a physical particle (the inflaton).

For simplicity, we have omitted the cosmological term in (2.1). We will do the same throughout the paper. Once it is included, the theory is manifestly renormalizable, like Stelle's theory [44], because the fakeon prescription does not modify the ultraviolet divergences [26, 27].

With the help of an auxiliary field φ, we can write S_{QG} in the equivalent form

$$
\begin{aligned}
S_{\mathrm{geom}} = &-\frac{M_{\mathrm{Pl}}^2}{16\pi} \int \mathrm{d}^4 x \sqrt{-g} \left(R + \frac{1}{2m_\chi^2} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \right) \\
&+ \frac{M_{\mathrm{Pl}}^2}{96\pi m_\phi^2} \int \mathrm{d}^4 x \sqrt{-g} (2R - \varphi)\varphi + S_{\mathfrak{m}}(g, \Phi).
\end{aligned}
\tag{2.2}
$$

Making the Weyl transformation

$$
g_{\mu\nu} \to g_{\mu\nu} \mathrm{e}^{\hat{\kappa}\phi}, \qquad \phi = -\frac{1}{\hat{\kappa}} \ln \left(1 - \frac{\varphi}{3m_\phi^2} \right),
\tag{2.3}
$$

where $\hat{\kappa} = M_{\mathrm{Pl}}^{-1} \sqrt{16\pi/3}$, we can diagonalize the quadratic part and obtain the new action

$$
S_{\mathrm{infl}} = -\frac{M_{\mathrm{Pl}}^2}{16\pi} \int \mathrm{d}^4 x \sqrt{-g} \left(R + \frac{1}{2m_\chi^2} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \right) + S_\phi(g, \phi) + S_{\mathfrak{m}}(g\mathrm{e}^{\hat{\kappa}\phi}, \Phi),
\tag{2.4}
$$

where

$$
S_\phi(g, \phi) = \frac{1}{2} \int \mathrm{d}^4 x \sqrt{-g} \left(D_\mu \phi D^\mu \phi - 2V(\phi) \right)
\tag{2.5}
$$

and

$$
V(\phi) = \frac{m_\phi^2}{2\hat{\kappa}^2} \left(1 - \mathrm{e}^{\hat{\kappa}\phi} \right)^2
\tag{2.6}
$$

is the Starobinsky potential. The action (2.4) is not manifestly renormalizable. In fact, it is as renormalizable as (2.1) — once the cosmological term is reinstated –, because it is related to (2.1) by a (perturbative and nonderivative) field redefinition.

The geometric framework is defined by the interim actions (2.1) or (2.2), while the inflaton framework is defined by (2.4). In the rest of the paper, we switch the matter sector $S_{\mathfrak{m}}$ off. If needed, its effects can be studied along the guidelines outlined in the next sections. We do not review the details on the parametrizations of the fluctuations and their transformations under diffeomorphisms, which are easy to find in the literature (see for example [17–19]).

3 Inflaton framework (R+scalar+C^2)

In this section, we study the tensor and scalar fluctuations in the inflaton framework. The action is (2.4), with the potential (2.6). The Friedmann equations are

$$\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} = -4\pi G \dot{\phi}^2, \qquad \frac{\dot{a}^2}{a^2} = \frac{4\pi G}{3}\left(\dot{\phi}^2 + 2V(\phi)\right), \qquad \ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} = -V'(\phi), \qquad (3.1)$$

where $V'(\phi) = \mathrm{d}V(\phi)/\mathrm{d}\phi$. We define the usual quantities

$$\varepsilon = -\frac{\dot{H}}{H^2}, \qquad \eta = 2\varepsilon - \frac{\dot{\varepsilon}}{2H\varepsilon}, \qquad (3.2)$$

where $H = \dot{a}/a$ is the Hubble parameter.

The de Sitter limit is the one where H is approximately constant. It is easy to show that the constant value it tends to is $m_\phi/2$. Indeed, $\dot{H} \simeq 0$ in the first equation (3.1) gives $\dot{\phi} \simeq 0$. On the other hand, if we insert $\dot{\phi} \simeq 0$ (and so $\ddot{\phi} \simeq 0$) in the third equation (3.1) we obtain $V'(\phi) \simeq 0$, which has two solutions: $\phi \simeq 0$ and $\phi \to -\infty$. The first possibility gives the trivial case, since $\phi \simeq 0$, $\dot{\phi} \simeq 0$ in the second equation (3.1) give $H \simeq 0$. The second possibility is the right one, since $\phi \to -\infty$, $\dot{\phi} \simeq 0$ in the second equation (3.1) give $H \simeq m_\phi/2$.

The expansion around the de Sitter background is an expansion in powers of $\sqrt{\varepsilon}$. This can be proved by studying the solution of the equations (3.1) around the de Sitter metric. Leaving the details to appendix A, here we just mention the properties that we need to proceed. It is possible to show that $\eta = \mathcal{O}(\sqrt{\varepsilon})$ and

$$\frac{\mathrm{d}^n \varepsilon}{\mathrm{d}t^n} = H^n \mathcal{O}(\varepsilon^{\frac{n+2}{2}}). \qquad (3.3)$$

In other words, each time derivative raises the order by $\sqrt{\varepsilon}$, so the expansion in powers of $\sqrt{\varepsilon}$ is also an expansion of slow time dependence. Moreover, we have

$$H = \frac{m_\phi}{2}\left(1 - \frac{\sqrt{3\varepsilon}}{2} + \frac{7\varepsilon}{12} + \mathcal{O}(\varepsilon^{3/2})\right),$$

$$\eta = -2\sqrt{\frac{\varepsilon}{3}} + \frac{13}{9}\varepsilon + \mathcal{O}(\varepsilon^{3/2}), \qquad (3.4)$$

$$-aH\tau = 1 + \varepsilon + \mathcal{O}(\varepsilon^{3/2}).$$

(see formulas (A.7), suppressing bars). The last line is the expansion of $-aH\tau$, where τ is the conformal time, defined by

$$\tau = -\int_t^{+\infty} \frac{\mathrm{d}t'}{a(t')}, \qquad (3.5)$$

with the initial condition chosen to have $\tau = -1/(aH)$ in the de Sitter limit $\varepsilon \to 0$.

3.1 Tensor fluctuations

To study the tensor fluctuations, it is convenient to parametrize the metric as

$$g_{\mu\nu} = \mathrm{diag}(1, -a^2, -a^2, -a^2) - 2a^2\left(u\delta_\mu^1\delta_\nu^1 - u\delta_\mu^2\delta_\nu^2 + v\delta_\mu^1\delta_\nu^2 + v\delta_\mu^2\delta_\nu^1\right), \qquad (3.6)$$

where $u = u(t,z)$ and $v = v(t,z)$ are the graviton modes.

Let $u_{\mathbf{k}}(t)$ denote the Fourier transform of $u(t, z)$ with respect to the coordinate z, where $\mathbf{k}$ is the space momentum. The quadratic Lagrangian obtained from (2.4) is

$$(8\pi G)\frac{\mathcal{L}_t}{a^3} = \dot{u}^2 - \frac{k^2}{a^2}u^2 - \frac{1}{m_\chi^2}\left[\ddot{u}^2 - 2\left(H^2 - 2\pi G\dot{\phi}^2 + \frac{k^2}{a^2}\right)\dot{u}^2 + \frac{k^4}{a^4}u^2\right], \qquad (3.7)$$

plus an identical contribution for v, where $k = |\mathbf{k}|$. To simplify the notation, we understand that u^2 stands for $u_{-\mathbf{k}}u_{\mathbf{k}}$, $\dot{u}^2$ for $\dot{u}_{-\mathbf{k}}\dot{u}_{\mathbf{k}}$, etc. We extend this convention to mixed products such as $u\dot{u}$, which can be interpreted either as $u_{-\mathbf{k}}\dot{u}_{\mathbf{k}}$ or $\dot{u}_{-\mathbf{k}}u_{\mathbf{k}}$.

It is possible to eliminate the higher derivatives by considering the extended Lagrangian

$$\mathcal{L}_t' = \mathcal{L}_t + \Delta\mathcal{L}_t, \qquad (3.8)$$

where

$$(8\pi Gm_\chi^2)\frac{\Delta\mathcal{L}_t}{a^3} = (S - \ddot{u} - f\dot{u} - hu)^2. \qquad (3.9)$$

Here $f(t)$, $h(t)$ are functions to be determined, and S, which may stand for $S_{-\mathbf{k}}(t)$ or $S_{\mathbf{k}}(t)$, denotes an auxiliary field. The equivalence of $\mathcal{L}_t'$ and $\mathcal{L}_t$ is due to the fact that $\mathcal{L}_t' = \mathcal{L}_t$ when S is replaced by the solution of its own field equation. The higher derivatives disappear in the sum $\mathcal{L}_t + \Delta\mathcal{L}_t$, because the term proportional to $\ddot{u}^2$ cancels out.

Next, we perform the field redefinitions

$$u = U + \alpha V, \qquad\qquad S = V + \beta U, \qquad (3.10)$$

where $\alpha(t)$ and $\beta(t)$ are other functions to be determined. We use the freedom to choose f, h, α and β to write $\mathcal{L}_t'$ in a convenient form, such that it contains a unique, nonderivative term mixing U and V. Specifically, we reduce the Lagrangian $\mathcal{L}_t'$ to the form

$$\mathcal{L}_t' = \mathcal{L}_t^{(U)} + \mathcal{L}_t^{(V)} + \mathcal{L}_t^{(UV)}, \qquad (3.11)$$

where

$$(8\pi G)\frac{\mathcal{L}_t^{(U)}}{a^3\gamma} = \dot{U}^2 - \omega^2 U^2, \qquad (8\pi Gm_\chi^2 M^2)\frac{\mathcal{L}_t^{(UV)}}{a^3\gamma} = 2\sigma UV,$$

$$(8\pi GM^4)\frac{\mathcal{L}_t^{(V)}}{a^3\gamma} = -\dot{V}^2 + \Omega^2 V^2, \qquad (3.12)$$

and γ, ω^2, Ω^2 and σ are other functions of time, while M is constant and has the dimension of a mass. Since γ is going to be positive, V is the fakeon and U is the physical excitation, up to the mixing due to $\mathcal{L}_t^{(UV)}$.

The fakeon projection amounts to solving the V field equations by means of the fakeon prescription and inserting the solution back into $\mathcal{L}_t'$. In all the cases considered here, this is achieved by determining the solution $G_f(t, t')$ of $\Sigma G_f(t, t') = \delta(t - t')$ as the arithmetic average of the retarded and advanced potentials, where Σ is an operator of the form

$$\Sigma \equiv F_2(t)\frac{\mathrm{d}^2}{\mathrm{d}t^2} + F_1(t)\frac{\mathrm{d}}{\mathrm{d}t} + F_0(t), \qquad (3.13)$$

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$F_i(t)$ being functions of time. A certain detour allows us to get to the results we need here without even knowing the explicit expression of $G_f(t, t')$, which is derived in section 4, where the projection is discussed in detail.

If we take

$$\alpha = \frac{1}{M^2}, \qquad \beta = M^2 + m_\chi^2 + 2H_0^2, \qquad f = 3H,$$
$$h = M^2 + m_\chi^2 + H_0^2 + H^2(1+\varepsilon) + \frac{k^2}{a^2}, \tag{3.14}$$

where H_0 is a constant, we obtain the decomposition (3.11) with

$$\gamma = 1 + 2\frac{H_0^2}{m_\chi^2}, \qquad \omega^2 = h - M^2 - \frac{\sigma}{m_\chi^2} - m_\chi^2\gamma, \qquad \Omega^2 = h - M^2 + \frac{\sigma}{m_\chi^2},$$
$$\sigma\gamma = (1+\varepsilon)(1-2\varepsilon)(1-3\varepsilon)H^4 + \dot{\varepsilon}(1-6\varepsilon)H^3 \tag{3.15}$$
$$+ \left[m_\chi^2(1+\varepsilon) + \ddot{\varepsilon} + 4\varepsilon\frac{k^2}{a^2}\right]H^2 - H_0^2(H_0^2 + m_\chi^2).$$

The constant H_0 is in principle arbitrary, but a remarkable choice, $H_0 = m_\phi/2$, makes $\mathcal{L}_t^{(UV)}$ vanish in the de Sitter limit. There, U and V decouple and

$$(8\pi G)\frac{\mathcal{L}_t^{(U)}}{a^3\gamma} = \dot{U}^2 - \frac{k^2}{a^2}U^2, \qquad (8\pi GM^4)\frac{\mathcal{L}_t^{(V)}}{a^3\gamma} = -\dot{V}^2 + \left(\gamma m_\chi^2 + \frac{k^2}{a^2}\right)V^2. \tag{3.16}$$

It is relatively straightforward to derive the power spectrum of the fluctuations in this limit. The V equation of motion is

$$\ddot{V} + \frac{3}{2}m_\phi\dot{V} + \left(m_\chi^2 + \frac{m_\phi^2}{2} + \frac{k^2}{a^2}\right)V = 0. \tag{3.17}$$

As said, we need to solve it by means of the fakeon prescription and insert the solution back into the action. Since (3.17) is homogeneous and U-independent, the solution is just $V = 0$.

Using $V = 0$, formula (3.10) gives $u = U$, so we obtain a Mukhanov action

$$\mathcal{L}_t^{(U)} = \left(\frac{m_\phi^2 + 2m_\chi^2}{2m_\chi^2}\right)\mathcal{L}_{t\,E}^{(U)}$$

that coincides with the one of Einstein gravity with a scalar field, apart from the overall factor. The u two-point function in the de Sitter limit is

$$\langle u_{\mathbf{k}}(\tau)u_{\mathbf{k'}}(\tau)\rangle = \frac{2m_\chi^2}{m_\phi^2 + 2m_\chi^2}\langle u_{\mathbf{k}}(\tau)u_{\mathbf{k'}}(\tau)\rangle_E, \tag{3.18}$$

where

$$\langle u_{\mathbf{k}}(\tau)u_{\mathbf{k'}}(\tau)\rangle_E = (2\pi)^3\delta^{(3)}(\mathbf{k}+\mathbf{k'})\frac{\pi Gm_\phi^2}{2k^3}(1 + k^2\tau^2).$$

Details on the derivation of (3.18) are given below. Formula (3.18) makes us already appreciate that the result depends on the mass m_χ of the fakeon in a nontrivial way.

Quasi de Sitter expansion. Formulas (3.14) and (3.15) are exact, i.e., they do not assume ε small. From now on, we work to the first order in ε, where we can use approximate formulas. Observe that (3.14) and (3.15) depend on m_χ, H, ε and m_ϕ (through $H_0 = m_\phi/2$). However, the last three quantities are related by (3.4), so we can eliminate one of them. The price of this is that we introduce terms proportional to $\sqrt{\varepsilon}$, which are unnecessary at this level. It is possible to avoid it by switching to a slightly different parametrization. Specifically, if we choose

$$\alpha = \frac{1}{M^2}, \qquad \beta = M^2 + m_\chi^2 \gamma, \qquad f = 3H - \frac{4\varepsilon H^3}{m_\chi^2 \gamma},$$
$$h = M^2 + m_\chi^2 \gamma + \frac{k^2}{a^2} + \varepsilon \frac{H^2(m_\chi^2 - 4H^2)}{m_\chi^2 \gamma}, \qquad \gamma = 1 + 2\frac{H^2}{m_\chi^2}, \tag{3.19}$$

and

$$\omega^2 = h - M^2 - \frac{\sigma}{m_\chi^2} - m_\chi^2 \gamma, \qquad \Omega^2 = h - M^2 + \frac{\sigma}{m_\chi^2},$$
$$\sigma\gamma = \varepsilon H^2 \left(m_\chi^2 - 4H^2 + \frac{4k^2}{\gamma a^2} \right), \tag{3.20}$$

we find the Lagrangian

$$(8\pi G)\frac{\mathcal{L}_\mathrm{t}'}{a^3 \gamma} = \dot{U}^2 - \omega^2 U^2 + \frac{1}{M^4}\left(-\dot{V}^2 + \Omega^2 V^2\right) + \frac{2\sigma}{m_\chi^2 M^2}UV. \tag{3.21}$$

The V equation of motion is now

$$\Sigma V = -\frac{\sigma M^2}{m_\chi^2}U, \tag{3.22}$$

where

$$\Sigma \equiv \Sigma_0 + \gamma m_\chi^2, \qquad \Sigma_0 \equiv \frac{\mathrm{d}^2}{\mathrm{d}t^2} + 3H\frac{\mathrm{d}}{\mathrm{d}t} + \frac{k^2}{a^2}. \tag{3.23}$$

Anticipating that the solution for V is of order ε, we have dropped higher-order terms proportional to εV, $\varepsilon\dot{V}$ from (3.22). Let $\Sigma^{-1}|_\mathrm{f}$ denote the fakeon Green function $G_\mathrm{f}(t,t')$, i.e., the solution of $\Sigma G_\mathrm{f}(t,t') = \delta(t - t')$ defined by the fakeon prescription (see section 4). Then the solution of (3.22) can be written as

$$\frac{V(t)}{M^2} = -\frac{1}{m_\chi^2}\int_{-\infty}^{+\infty} \mathrm{d}t' G_\mathrm{f}(t,t')\sigma(t')U(t') = -\frac{1}{m_\chi^2} \Sigma^{-1}\big|_\mathrm{f}\sigma U. \tag{3.24}$$

Inserting this expression into the Lagrangian (3.21), we can see that the nonlocal contribution due to $\mathcal{L}_\mathrm{t}^{(UV)}$ is of order ε^2, so we can drop it. The projected Lagrangian is

$$(8\pi G)\frac{\mathcal{L}_\mathrm{t}^\mathrm{prj}}{a^3 \gamma} = \dot{U}^2 - \frac{\bar{k}^2}{a^2}U^2, \qquad \bar{k} = k\left(1 - \frac{2\varepsilon H^2}{m_\chi^2 \gamma^2}\right). \tag{3.25}$$

At this point, it is straightforward to work out the Mukhanov action. Defining

$$w = \frac{a\sqrt{\gamma}}{\sqrt{4\pi G}}U, \qquad \nu_\mathrm{t} = \frac{3}{2} + \frac{\varepsilon}{\gamma}, \tag{3.26}$$

and switching to the conformal time (3.5), the w action to order ε derived from (3.25) reads

$$S_{\mathrm{t}}^{\mathrm{prj}} = \frac{1}{2} \int \mathrm{d}\tau \left[w'^2 - \bar{k}^2 w^2 + \frac{w^2}{\tau^2} \left(\nu_{\mathrm{t}}^2 - \frac{1}{4} \right) \right], \tag{3.27}$$

where the prime denotes the derivative with respect to τ.

Power spectrum and spectral index. Formula (3.27) tells us that the conjugate momentum of w is $p = w'$, so after turning w, p into operators $\hat{w}$, $\hat{p}$, we impose the equal time quantization condition

$$[\hat{p}_{\mathbf{k}}(\tau), \hat{w}_{\mathbf{k}'}(\tau)] = -i\delta^{(3)}\left(\mathbf{k} - \mathbf{k}'\right), \tag{3.28}$$

where we have reinstated the subscripts $\mathbf{k}$. As usual, we write the Fourier decomposition

$$w_{\mathbf{k}}(\tau) = v_{\mathbf{k}}(\tau)\hat{a}_{\mathbf{k}} + v_{-\mathbf{k}}^*(\tau)\hat{a}_{-\mathbf{k}}^\dagger, \qquad [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}'), \tag{3.29}$$

where $\hat{a}_{\mathbf{k}}^\dagger$ and $\hat{a}_{\mathbf{k}}$ are creation and annihilation operators.

The limit $k/(aH) \to \infty$ of (3.27) allows us to define the Bunch-Davies vacuum state $|0\rangle$. From formula (3.27), we see that the only difference with respect to the result obtained in the de Sitter limit is a rescaling of k. Thus, we require

$$v_{\mathbf{k}} \to \frac{\mathrm{e}^{-i\bar{k}\tau}}{\sqrt{2\bar{k}}} \qquad \text{for } \frac{k}{aH} \to \infty. \tag{3.30}$$

Using the condition (3.30), we can work out the modes $v_{\mathbf{k}}$ and obtain

$$U_{\mathbf{k}} = \pi H(1-\varepsilon)|\tau|^{3/2}\sqrt{\frac{G}{\gamma}} \left[\mathrm{e}^{i\pi(2\nu_{\mathrm{t}}+1)/4} H_{\nu_{\mathrm{t}}}^{(1)}(|\bar{k}\tau|)\hat{a}_{\mathbf{k}} + \mathrm{e}^{-i\pi(2\nu_{\mathrm{t}}+1)/4} H_{\nu_{\mathrm{t}}}^{(2)}(|\bar{k}\tau|)\hat{a}_{-\mathbf{k}}^\dagger \right], \tag{3.31}$$

having used the third formula of (3.4), where $H_{\nu_{\mathrm{t}}}^{(1,2)}$ are the Hankel functions. For the purpose of computing the power spectrum, we need to work out the leading behavior in the superhorizon limit $|k\tau| \to 0$. There we have

$$U_{\mathbf{k}} = (1-\varepsilon) \left(\frac{|\bar{k}\tau|}{2} \right)^{(3-2\nu_{\mathrm{t}})/2} \frac{H\Gamma(\nu_{\mathrm{t}})}{\bar{k}^{3/2}} \sqrt{\frac{8G}{\gamma}} \left[\mathrm{e}^{i\pi(2\nu_{\mathrm{t}}-1)/4}\hat{a}_{\mathbf{k}} + \mathrm{e}^{-i\pi(2\nu_{\mathrm{t}}-1)/4}\hat{a}_{-\mathbf{k}}^\dagger \right]. \tag{3.32}$$

The redefinitions (3.10) tell us that to compute the u two-point function we also need the fakeon $V_{\mathbf{k}}$, which is given by formula (3.24). While the general discussion of the fakeon Green function is left to section 4, here we can quickly get to the result we need as follows. In the superhorizon limit $|k\tau| \to 0$ we can ignore the term proportional to k^2/a^2 in the expression (3.20) of σ. Once we do this, we can commute σ and $\Sigma^{-1}|_{\mathrm{f}}$ in (3.24), because the commutator gives corrections of higher orders in ε. Moreover, recalling that $\Sigma_0 U_{\mathbf{k}} = \mathcal{O}(\varepsilon)$, because $U_{\mathbf{k}}$ solves the Mukhanov equation of the projected Lagrangian $\mathcal{L}_{\mathrm{t}}^{\mathrm{prj}}$ of formula (3.25), $\Sigma^{-1}|_{\mathrm{f}}$ just multiplies $U_{\mathbf{k}}$ by $1/(\gamma m_\chi^2)$. Collecting these facts, we have, in the superhorizon limit and discarding higher orders,

$$\frac{V_{\mathbf{k}}}{M^2} = -\frac{1}{m_\chi^2} \left. \Sigma^{-1} \right|_{\mathrm{f}} \sigma U_{\mathbf{k}} = -\frac{\sigma}{m_\chi^2} \left. \Sigma^{-1} \right|_{\mathrm{f}} U_{\mathbf{k}} = -\frac{\sigma}{m_\chi^2} \left. \frac{1}{\Sigma_0 + \gamma m_\chi^2} \right|_{\mathrm{f}} U_{\mathbf{k}} = -\frac{\sigma}{m_\chi^2} \frac{1}{\gamma m_\chi^2} U_{\mathbf{k}},$$

that is to say,

$$\frac{V_{\mathbf{k}}}{M^2} = -\frac{\varepsilon H^2}{m_\chi^4 \gamma^2}(m_\chi^2 - 4H^2)U_{\mathbf{k}}. \tag{3.33}$$

The power spectrum $\mathcal{P}_u$ of each graviton polarization is defined by

$$\langle u_{\mathbf{k}}(\tau)u_{\mathbf{k}'}(\tau)\rangle = (2\pi)^3 \delta^{(3)}(\mathbf{k} + \mathbf{k}')\frac{2\pi^2}{k^3}\mathcal{P}_u. \tag{3.34}$$

The two-point function can be evaluated in the superhorizon limit from (3.10), (3.31) and (3.33). We find

$$\mathcal{P}_u = \frac{GH^2}{\pi\gamma}\left(1 - \frac{2\varepsilon}{\gamma} + \frac{2\varepsilon}{\gamma}\psi_0(3/2)\right)\left(\frac{k|\tau|}{2}\right)^{-2\varepsilon/\gamma},$$

where ψ_0 is the digamma function.

The power spectrum of the tensor fluctuations, matched with the usual conventions, is $\mathcal{P}_T = 16\mathcal{P}_u$. Replacing $|\tau|$ by $1/k_*$, where k_* is a reference scale, it is common to write

$$\ln \mathcal{P}_T(k) = \ln A_T + n_T \ln \frac{k}{k_*}, \tag{3.35}$$

where A_T and n_T are called amplitude and spectral index (or tilt), respectively. We find

$$A_T = \frac{8Gm_\chi^2 m_\phi^2}{\pi(m_\phi^2 + 2m_\chi^2)}\left(1 - \frac{2\sqrt{3\varepsilon}m_\chi^2}{m_\phi^2 + 2m_\chi^2} - \frac{\varepsilon m_\chi^2(2m_\chi^2 + 37m_\phi^2)}{6(m_\phi^2 + 2m_\chi^2)^2} - n_T(2 - \gamma_E - \ln 2)\right),$$

$$n_T = 3 - 2\nu_{\mathrm{t}} = -\frac{4\varepsilon m_\chi^2}{m_\phi^2 + 2m_\chi^2}, \tag{3.36}$$

where γ_E is the Euler-Mascheroni constant and we have used the first formula of (3.4) to eliminate H.

3.2 Scalar fluctuations

Now we study the scalar fluctuations in the inflaton framework. We work in the comoving gauge, where the ϕ fluctuation $\delta\phi$ is set to zero and the metric reads

$$g_{\mu\nu} = \mathrm{diag}(1, -a^2, -a^2, -a^2) + 2\mathrm{diag}(\Phi, a^2\Psi, a^2\Psi, a^2\Psi) - \delta_\mu^0\delta_\nu^i\partial_i B - \delta_\mu^i\delta_\nu^0\partial_i B. \tag{3.37}$$

After Fourier transforming the space coordinates, (2.4) gives the quadratic Lagrangian

$$(8\pi G)\frac{\mathcal{L}_{\mathrm{s}}}{a^3} = -3(\dot{\Psi} + H\Phi)^2 + 4\pi G\dot{\phi}^2\Phi^2 + \frac{k^2}{a^2}\left[2B(\dot{\Psi} + H\Phi) + \Psi(\Psi - 2\Phi)\right]$$

$$-\frac{k^4}{3a^4 m_\chi^2}\left[(\dot{B} + \Phi + \Psi)^2 - 2BH(\Phi + \Psi) - 4\pi G\dot{\phi}^2 B^2\right]. \tag{3.38}$$

As before, Ψ^2 stands for $\Psi_{-\mathbf{k}}\Psi_{\mathbf{k}}$, $\dot{\Psi}^2$ for $\dot{\Psi}_{-\mathbf{k}}\dot{\Psi}_{\mathbf{k}}$, and so on.

Since Φ appears algebraically, we eliminate it by means of its own field equation. We remain with a Lagrangian that depends only on B and Ψ. The field redefinitions

$$\Psi = \frac{U}{\sqrt{\varepsilon}}, \qquad B = \frac{a^2}{k^2}V + \frac{3U}{\sqrt{\varepsilon}H(3 - \varepsilon)}, \tag{3.39}$$

allow us to decompose $\mathcal{L}_\mathrm{s}$ as

$$\mathcal{L}_\mathrm{s} = \mathcal{L}_\mathrm{s}^{(U)} + \mathcal{L}_\mathrm{s}^{(V)} + \mathcal{L}_\mathrm{s}^{(UV)}, \tag{3.40}$$

where $\mathcal{L}_\mathrm{s}^{(UV)}$ is the sum of a term proportional to UV plus one proportional to $\dot{U}V$. In addition, $\mathcal{L}_\mathrm{s}^{(UV)}$ vanishes in the de Sitter limit.

We do not give the full expression of $\mathcal{L}_\mathrm{s}$ here, but stress that after the redefinition (3.39) it admits a series expansion in powers of k and $\sqrt{\varepsilon}$. In particular,

$$\lim_{\varepsilon \to 0, k \to 0} (8\pi G)\frac{\mathcal{L}_\mathrm{s}}{a^3} = \dot{U}^2 - \frac{1}{3m_\chi^2}\left(\dot{V}^2 - m_\chi^2 \gamma V^2\right),$$

where γ is defined in (3.19). As before, V is the fakeon and U is the physical excitation.

Quasi de Sitter expansion. In the de Sitter limit $\varepsilon \to 0$, we find

$$(8\pi G)\frac{\mathcal{L}_\mathrm{s}^{(U)}}{a^3} = \dot{U}^2 - \frac{k^2}{a^2}U^2, \qquad \mathcal{L}_\mathrm{s}^{(UV)} = 0, \tag{3.41}$$

$$(24\pi G m_\chi^2 \upsilon)\frac{\mathcal{L}_\mathrm{s}^{(V)}}{a^3} = -\dot{V}^2 + \left[18H^4 + 3H^2 m_\chi^2(3 - \hat{k}^2 + 2\hat{k}^4) + \hat{k}^4 m_\chi^4(1 + \hat{k}^2)\right]\frac{V^2}{9\upsilon H^2},$$

where

$$\upsilon = 1 + \frac{\hat{k}^4 m_\chi^2}{9H^2}, \qquad \hat{k} = \frac{k}{m_\chi a}.$$

Note that υ and the coefficient of V^2 in (3.41) are positive definite.

Again, we see that the fakeon V decouples. Its own equation of motion sets it to zero, so the Lagrangian of U coincides with the usual Mukhanov expression, normalization included. This means that the power spectrum of the scalar fluctuations coincides with the one of Einstein gravity in this limit.

To order $\eta \sim \sqrt{\varepsilon}$, we find

$$(8\pi G)\frac{\mathcal{L}_\mathrm{s}^{(U)}}{a^3} = \dot{U}^2 - \frac{k^2}{a^2}U^2 + 2\sqrt{3\varepsilon}H^2 U^2, \tag{3.42}$$

$$(8\pi G)\frac{\mathcal{L}_\mathrm{s}^{(UV)}}{a^3} = \frac{2\sqrt{\varepsilon}V}{9\upsilon}\left[(2\hat{k}^2 - 3)\dot{U} + \frac{\hat{k}^2 m_\chi^2}{9\upsilon H^3}\left(9H^2 - 12\hat{k}^2 H^2 + 8\hat{k}^4 H^2 + \hat{k}^4 m_\chi^2\right)U\right].$$

Since the V equation of motion implies $V = \mathcal{O}(\sqrt{\varepsilon})$, $\mathcal{L}_\mathrm{s}^{(V)}$ remains the one of formula (3.41) to the order we are considering. Moreover, after integrating V out, the projected U Lagrangian is just $\mathcal{L}_\mathrm{s}^{(U)}$, since the V-dependent corrections are $\mathcal{O}(\varepsilon)$.

From $\mathcal{L}_\mathrm{s}^{(U)}$, we can derive the Mukhanov action by following the steps from (3.26) to (3.32), with the replacements $\bar{k} \to k$, $\gamma \to 1$ and $\nu_\mathrm{t} \to \nu_\mathrm{s}$, where

$$\nu_\mathrm{s} = \frac{3}{2} + 2\sqrt{\frac{\varepsilon}{3}}. \tag{3.43}$$

Recalling that in the comoving gauge the curvature perturbation $\mathcal{R}$ coincides with Ψ, we can derive the power spectrum $\mathcal{P}_\mathcal{R}$, defined by

$$\langle \mathcal{R}_\mathbf{k}(\tau)\mathcal{R}_{\mathbf{k}'}(\tau)\rangle = (2\pi)^3 \delta^{(3)}(\mathbf{k} + \mathbf{k}')\frac{2\pi^2}{k^3}\mathcal{P}_\mathcal{R}. \tag{3.44}$$

Inserting the solution for U into the left formula of (3.39), we find

$$\ln \mathcal{P}_{\mathcal{R}}(k) = \ln A_{\mathcal{R}} + (n_{\mathcal{R}} - 1) \ln \frac{k}{k_*}, \tag{3.45}$$

where the amplitude $A_{\mathcal{R}}$ and the spectral index $n_{\mathcal{R}} - 1$ are

$$A_{\mathcal{R}} = \frac{G m_\phi^2}{4\pi\varepsilon} \left(1 - \sqrt{3\varepsilon} - (n_{\mathcal{R}} - 1)(2 - \gamma_E - \ln 2) \right), \tag{3.46}$$

$$n_{\mathcal{R}} - 1 = 3 - 2\nu_{\rm s} = -4\sqrt{\frac{\varepsilon}{3}}, \tag{3.47}$$

respectively. We see that the mass m_χ of the fakeon does not affect the result to the order we are considering.

Finally, from (3.36) we derive the tensor-to-scalar ratio

$$r = \frac{A_T}{A_{\mathcal{R}}} = \frac{32\varepsilon m_\chi^2}{m_\phi^2 + 2m_\chi^2} \left(1 + \frac{\sqrt{3\varepsilon} m_\phi^2}{m_\phi^2 + 2m_\chi^2} + (n_{\mathcal{R}} - 1 - n_T)(2 - \gamma_E - \ln 2) \right). \tag{3.48}$$

4 The fakeon projection

In this section we discuss the fakeon projection, starting from the tensor fluctuations. The Lagrangian $\mathcal{L}_{\rm t}^{(V)}$ of formula (3.21) leads to the V equation of motion (3.22). The fakeon Green function $G_{\rm f}(t, t')$ is the solution of $\Sigma G_{\rm f}(t, t') = \delta(t - t')$, defined by the fakeon prescription, where Σ is given in formula (3.23). For the purposes of this paper, it is sufficient to invert Σ in the de Sitter limit $a(t) = e^{Ht}$, where H is treated as a constant. We keep H generic to make the discussion easily adaptable to the geometric framework. We will use the information that H is $m_\phi/2$ in the de Sitter limit (in the inflaton framework) only later.

It is convenient to switch to a symmetric operator by noting that

$$\Sigma a^{-3} = a^{-3/2} \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + m_\chi^2 - \frac{H^2}{4} + \frac{k^2}{a^2} \right) a^{-3/2}. \tag{4.1}$$

We want to prove that the fakeon solution $\hat{G}_{\rm f}(t, t')$ of

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + m_\chi^2 - \frac{H^2}{4} + \frac{k^2}{a^2} \right) \hat{G}_{\rm f}(t, t') = \delta(t - t') \tag{4.2}$$

is

$$\hat{G}_{\rm f}(t, t') = \frac{i\pi \, \mathrm{sgn}(t - t')}{4H \sinh(n_\chi \pi)} \left[J_{in_\chi}(\check{k}) J_{-in_\chi}(\check{k}') - J_{in_\chi}(\check{k}') J_{-in_\chi}(\check{k}) \right], \tag{4.3}$$

where $\mathrm{sgn}(t)$ is the sign function, J_n denotes the Bessel function of the first kind and

$$n_\chi = \sqrt{\frac{m_\chi^2}{H^2} - \frac{1}{4}}, \qquad \check{k} = \frac{k}{a(t)H}, \qquad \check{k}' = \frac{k}{a(t')H}. \tag{4.4}$$

In principle, we could add solutions of the homogeneous equation, which are the functions $J_{\pm in_\chi}(\check{k})$, multiplied by constants. The job of the projection is to determine those

constants uniquely. Because it comes from quantum field theory, the fakeon projection is known perturbatively around flat space, in four-momentum space. However, a notion of four-momentum is not immediately available in curved space.

Fortunately, there are three limits where $\hat{G}_f$ is known, which are $k/(aH) \to \infty$, $k/(aH) = 0$ and $a = $ constant. The limit $k/(aH) \to \infty$ gives the flat-space case once we switch to conformal time. The limit $k/(aH) \to 0$ gives the flat-space case if we keep the cosmological time. The case $a = $ constant is precisely flat space, but is not relevant here, since we are interested in the de Sitter background. Hence, necessary conditions are that the solution (4.3) reduces to the known expressions [32, 45] in both cases $k/(aH) \to \infty$ and $k/(aH) = 0$. Any of these two conditions is also sufficient. The other condition can be seen as a consistency check.

Switching to conformal time $\tau = -1/(aH)$, equation (4.2) can be written as

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} + k^2 + \frac{m_\chi^2}{\tau^2 H^2} \right) \left(H\sqrt{\tau\tau'}\hat{G}_f \right) = \delta(\tau - \tau').$$

For $k|\tau|$ large we obtain

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} + k^2 \right) \left(H\sqrt{\tau\tau'}\hat{G}_f \right) \simeq \delta(\tau - \tau'). \tag{4.5}$$

Solving it by means of the arithmetic average of the retarded and advanced potentials, we find [32, 45]

$$\hat{G}_f \simeq \frac{1}{2Hk\sqrt{\tau\tau'}} \sin\left(k|\tau - \tau'| \right). \tag{4.6}$$

It is easy to check that (4.3) does satisfy (4.6) when $k|\tau|, k|\tau'| \gg 1$.

As said, the most general solution of (4.2) is equal to (4.3) plus solutions of the homogeneous equation, multiplied by constant coefficients c_1 and c_2. Now we know that those coefficients must vanish, to match (4.6) for $k|\tau|, k|\tau'|$ large. This proves that (4.3) is the correct fakeon Green function.

A consistency check is given by the limit $k \to 0$. There, (4.2) turns into an equation similar to (4.5), provided we keep the cosmological time t instead of switching to τ. Consequently, the solution (4.3) must tend to [32, 45]

$$\frac{1}{2Hn_\chi} \sin\left(Hn_\chi |t - t'| \right). \tag{4.7}$$

It is easy to check that this is indeed the $k \to 0$ limit of (4.3).

From (4.1) we derive the fakeon Green function

$$G_f(t, t') = \frac{i\pi\mathrm{sgn}(t - t')\mathrm{e}^{-3H(t-t')/2}}{4H\sinh\left(n_\chi \pi \right)} \left[J_{in_\chi}(\check{k})J_{-in_\chi}(\check{k}') - J_{in_\chi}(\check{k}')J_{-in_\chi}(\check{k}) \right]. \tag{4.8}$$

4.1 Consistency condition

We have determined the fakeon Green function in curved space by referring to two situations where the problem becomes a flat-space one, which are $k/(aH) \to \infty$ and $k/(aH) \to 0$.

As mentioned in the introduction, purely virtual particles are subject to a consistency (no-tachyon) condition in flat-space, i.e., their squared mass should be positive. Formula (4.6) shows that this requirement is always satisfied for $k/(aH) \to \infty$, while formula (4.7) shows that it is satisfied for $k/(aH) \to 0$ if n_χ is real. Recalling that H is $m_\phi/2$ in the inflaton framework, the condition reads

$$m_\chi > \frac{m_\phi}{4}, \tag{4.9}$$

which is a lower bound on the mass of the fakeon with respect to the mass of the inflaton. When (4.9) holds, the oscillating behavior of (4.7) suppresses the contributions with

$$|t - t'| \gg \frac{1}{Hn_\chi} = \frac{4}{\sqrt{16m_\chi^2 - m_\phi^2}}.$$

One may wonder if it is meaningful to impose a condition stronger than (4.9), for example require that the time-dependent squared mass be positive for all values of $k/(aH)$. To discuss this issue, let us consider the Lagrangian that gives the fakeon Green function of formula (4.2), which is

$$\hat{L} = -\frac{1}{2}\left(\frac{\mathrm{d}\hat{V}}{\mathrm{d}t}\right)^2 + \frac{m(t)^2}{2}\hat{V}^2, \qquad m(t)^2 = m_\chi^2 - \frac{H^2}{4} + \frac{k^2}{a^2}. \tag{4.10}$$

A redefinition $t = h(t')$, $\hat{V}(t) = f(t')\tilde{V}(t')$, with $\mathrm{d}h/\mathrm{d}t' = f^2$, leaves the kinetic term invariant, but changes the squared mass. Specifically, the transformed Lagrangian reads

$$\tilde{L} = -\frac{1}{2}\left(\frac{\mathrm{d}\tilde{V}}{\mathrm{d}t'}\right)^2 + \frac{M(t')^2}{2}\tilde{V}^2,$$

where

$$M^2 = f^4 m^2 - f\frac{\mathrm{d}^2}{\mathrm{d}t'^2}\left(\frac{1}{f}\right) = f^4 m^2 + f^3\frac{\mathrm{d}^2 f}{\mathrm{d}t^2}. \tag{4.11}$$

This transformation law shows that the signs of $m(t)^2$ and $M(t')^2$ do not have a reparametrization-independent meaning, in general, so a squared mass that becomes negative in some time interval is not necessarily a sign of a lack of consistency.

In passing, it is easy to verify that if the masses are independent of time, then the condition of positive square mass is independent of the parametrization. Indeed, if $m(t)^2$ is t-independent and positive, the most general reparametrization $f(t')$ that leaves $M(t')^2$ t'-independent has

$$f(t'(t))^2 = \sqrt{\rho^2 + \frac{M^2}{m^2}} + \rho\cos(2mt + \theta),$$

where ρ and θ are arbitrary real constants of integration. Since f^2 must be real and identically positive, M^2 must also be positive.

Summarizing, a necessary condition for the fakeon projection in the inflationary scenario is that the fakeon squared mass be positive in the superhorizon limit:

$$m(t)^2\big|_{k/(aH)\to 0} > 0. \tag{4.12}$$

This condition also leads to (4.9) in the case of the scalar fluctuations. Indeed, consider the Lagrangian $\mathcal{L}_\mathrm{s}^{(V)}$ given in formula (3.41). Making the change of variables

$$V(t) = \frac{\sqrt{\upsilon}}{a^{3/2}} W(t), \tag{4.13}$$

the W equation of motion takes the form

$$\ddot{W} + m(t)^2 W = \mathcal{O}(\sqrt{\varepsilon}), \tag{4.14}$$

for some involved rational function $m(t)^2$ of H^2/m_χ^2 and $k^2/(aH)^2$, equal to $(4m_\chi^2 - H^2)/4$ in the superhorizon limit. Thus, (4.12) gives again the bound (4.9).

As we show in section 6, the vector fluctuations give the same bound. The same bound is also found in the geometric framework. It is conceivable that, if (4.9) were violated, the theory would predict a rather different large-scale structure of the universe, or a different scenario would have to be envisaged to produce the present situation.

The stronger requirement that $m(t)^2$ be positive for every k makes sense if we believe that the cosmological time plays a special role. Then we still find the bound (4.9) for the tensor fluctuations, while a stronger bound is obtained in the case of the scalar fluctuations. Studying the coefficient $m(t)^2$ of W in (4.14) numerically, we find that it is positive for all values of $k^2/(aH)^2$ if

$$m_\chi \geqslant 0.312 m_\phi. \tag{4.15}$$

As soon as $m_\chi \lesssim 0.312 m_\phi$, there exists a finite k domain where $m(t)^2$ has negative values. When m_χ satisfies (4.9) but not (4.15), there is a time interval $\Delta t \sim \ln(k/m_\phi)/m_\phi$, comparable with the duration of inflation, where the fakeon Green function is "tachyonic" and its nonlocal contribution is no longer negligible.

In the rest of the paper, we take (4.9) as the consistency condition for the fakeon projection in inflationary cosmology, because it is universal and reparametrization independent. Yet, the issues just mentioned suggest that there is a chance that it might be conservative. The formulas of the power spectra do not depend on it, but (4.15) narrows the window of allowed values of the tensor-to-scalar ratio r a little bit more than (4.9) (see section 7).

5 Geometric framework $(R + R^2 + C^2)$

In this section we study the geometric framework, which is sometimes known in the literature as Jordan frame. The higher-derivative equations of the background metric, derived from (2.1) with the FLRW ansatz, can be written in the simple form

$$\frac{\dot{\varepsilon}}{H} = -3\varepsilon\left(1 - \frac{\varepsilon}{2}\right) + \frac{m_\phi^2}{2H^2}, \tag{5.1}$$

where ε is again $-\dot{H}/H^2$. It is worth to stress that ε, H, a and the cosmological time t are different from those of the inflaton framework, although we denote them by means of the same symbols. The match between the two frameworks is worked out in detail in appendix A.

The quasi de Sitter approximation of (5.1) requires $\varepsilon \sim m_\phi^2/(6H^2) \ll 1$, so H is no longer related to m_ϕ in the de Sitter limit, where actually $m_\phi \ll H$. As far as the mass m_χ is concerned, it can be either of order H or of order m_ϕ. This means that we have two types of quasi de Sitter expansions, depending on whether $m_\chi \sim H$ or $m_\chi \sim m_\phi$. We study the scalar and tensor fluctuations in both.

The two possibilities can also be understood as follows. The de Sitter metric is not an exact solution of the field equations of the theory $R + R^2 + C^2$. It is an exact solution in two cases: (i) when we ignore both R and C^2; and (ii) when we ignore just R. In other words, the term R^2 is leading with respect to the term R, while the term C^2 can either be of order R or of order R^2 (as far as the fluctuations are concerned). The first case is studied by expanding in powers of ε with $\xi = H^2/m_\chi^2$ fixed. The second case is studied by expanding in powers of ε with $\zeta \equiv m_\chi^2/m_\phi^2$ fixed.

The relation between m_ϕ, H and ε is

$$\frac{m_\phi^2}{H^2} = \varepsilon \left(6 + \varepsilon - \frac{2}{3}\varepsilon^2\right) + \mathcal{O}(\varepsilon^4). \tag{5.2}$$

It can be found by writing down the most general expansion for m_ϕ^2/H^2 in powers of ε, differentiating it and applying (5.1) to determine the coefficients. If needed, (5.2) can be extended to arbitrarily high orders (an asymptotic series being obtained).

5.1 $m_\chi \sim H$: tensor fluctuations

We start from the tensor fluctuations. Parametrizing the metric as in (3.6), the quadratic Lagrangian obtained from (2.1) is

$$(8\pi G)\frac{\mathcal{L}_{\mathrm{t}}}{a^3} = \dot{u}^2 \left(1 + \frac{2\Upsilon}{m_\phi^2} + \frac{\Upsilon}{m_\chi^2} + \frac{2k^2}{a^2 m_\chi^2}\right) - \frac{k^2}{a^2}\left(1 + \frac{2\Upsilon}{m_\phi^2} + \frac{k^2}{a^2 m_\chi^2}\right)u^2 - \frac{\ddot{u}^2}{m_\chi^2}, \tag{5.3}$$

plus an identical contribution for v, where

$$\Upsilon \equiv 2H^2 + \dot{H}. \tag{5.4}$$

Expanding around the de Sitter background with $\xi = H^2/m_\chi^2$ fixed, the first nonvanishing contribution to the spectral index n_T turns out to be $\mathcal{O}(\varepsilon^2)$. For this reason, we work out the predictions to the second order in ε included. Expanding the Lagrangian (5.3), we find

$$(8\pi G)\frac{\mathcal{L}_{\mathrm{t}}}{a^3} = \dot{U}^2 \left(1 + \frac{5\varepsilon}{6} + \frac{2\varepsilon^2}{9} + 3\varepsilon\xi + \frac{15}{2}\varepsilon^2\xi + \frac{3k^2\varepsilon}{a^2 m_\chi^2}\right) - (3 + 2\varepsilon + 9\varepsilon\xi)\,\varepsilon H^2 U^2$$
$$- \frac{k^2 U^2}{a^2}\left(1 + \frac{5\varepsilon}{6} + \frac{2\varepsilon^2}{9} + 3\varepsilon^2\xi + \frac{3k^2\varepsilon}{2a^2 m_\chi^2}\right) - \frac{3\varepsilon\ddot{U}^2}{2m_\chi^2}, \tag{5.5}$$

where

$$U = \sqrt{\frac{2}{3\varepsilon}}\,u.$$

The important point of (5.5) is that the unique higher-derivative term $\ddot{U}^2$ is multiplied by ε, so the fakeon projection can be handled iteratively. The change of variables

$$U = E\left(1 - \frac{5\varepsilon}{12} + \frac{43\varepsilon^2}{288}\right) - \frac{3}{2}\varepsilon\xi E\left(1 - \frac{7\varepsilon}{4}\right)$$
$$+ \frac{27}{8}\varepsilon^2\xi^2 E - \frac{9\varepsilon\xi}{4H}\dot{E} + \varepsilon\mathcal{O}(k^2|\tau|^2, \varepsilon\dot{E}, \ddot{E}, \cdots) \tag{5.6}$$

allows us to cast the Lagrangian in the form

$$(8\pi G)\frac{\mathcal{L}_t}{a^3} = \dot{E}^2 - \frac{k^2}{a^2}E^2 - 3\varepsilon(1-\varepsilon)H^2E^2.$$

Since $\dot{E} \simeq \varepsilon E$ for $|k\tau|$ small [as in (3.32)], the corrections $\varepsilon\mathcal{O}(k^2|\tau|^2, \varepsilon\dot{E}, \ddot{E}, \cdots)$ of (5.6) are either $\mathcal{O}(\varepsilon^{5/2})$ or give subleading contributions in the superhorizon limit $k|\tau| \ll 1$. This means that we do not need to specify them for our purposes.

At this point, it is sufficient to upgrade the steps from formula (3.26) to formula (3.32) to the appropriate order, with the substitutions $U \to E$, $\bar{k} \to k$, $\gamma \to 1$. We find

$$\nu_t = \frac{3}{2} + 3\varepsilon^2. \tag{5.7}$$

The power spectrum of the tensor fluctuations is $\mathcal{P}_T = 16\mathcal{P}_u$, with $\mathcal{P}_u$ defined by (3.34). Using the definition (3.35), the amplitude and the spectral index are

$$A_T = \frac{24GH^2}{\pi}\varepsilon\left[1 - \frac{17\varepsilon}{6} - 3\varepsilon\xi - \frac{31}{36}\varepsilon^2 + 17\varepsilon^2\xi + 9\varepsilon^2\xi^2 - n_T(2 - \gamma_E - \ln 2)\right], \tag{5.8}$$

$$n_T = \frac{\mathrm{d}\ln\mathcal{P}_T(k)}{\mathrm{d}\ln k} = 3 - 2\nu_t = -6\varepsilon^2. \tag{5.9}$$

5.2 $m_\chi \sim H$: scalar fluctuations

Now we discuss the scalar fluctuations in the geometric framework by expanding in powers of ε to the next-to-leading order with $\xi = H^2/m_\chi^2$ fixed. We switch directly from (2.1) to the action (2.2) (with $S_\mathfrak{m} \to 0$), to remove the higher derivatives without changing the metric that couples to matter. We isolate the background value of φ from its fluctuation Ω by writing

$$\varphi = -6\Upsilon + \Omega, \tag{5.10}$$

where Υ is defined in (5.4). The gauge invariant curvature perturbation $\mathcal{R}$ is

$$\mathcal{R} = \Psi - \frac{H}{6\dot{\Upsilon}}\Omega. \tag{5.11}$$

We work in the spatially-flat gauge, where Ω is an independent field and Ψ is set to zero. This means that the metric is

$$g_{\mu\nu} = \mathrm{diag}(1 + 2\Phi, -a^2, -a^2, -a^2) - \delta_\mu^0\delta_\nu^i\partial_i B - \delta_\mu^i\delta_\nu^0\partial_i B. \tag{5.12}$$

After Fourier transforming the space coordinates, the quadratic Lagrangian reads

$$
(8\pi G)\frac{\mathcal{L}_\mathrm{s}}{a^3} = \frac{1}{m_\phi^2}\left[(\Upsilon - H^2)\Phi\Omega - H\Phi\dot{\Omega} - 3\Upsilon^2\Phi^2 - \frac{\Omega^2}{12}\right] + \frac{k^2}{a^2}HB\Phi
$$
$$
- \frac{k^2}{3Hm_\phi^2 a^2}\left[H\Omega(\Phi + \dot{B}) + 2H^2B\Omega - 3\Upsilon B\Phi(\Upsilon + 2H^2)\right]
$$
$$
- \frac{k^4}{3a^4 m_\chi^2}(\Phi + \dot{B} - BH)^2. \tag{5.13}
$$

The field Φ appears in (5.13) as a Lagrange multiplier, so we integrate it out by solving its own field equation and inserting the solution back into the action. So doing, we obtain a two-derivative quadratic Lagrangian for B and Ω, which we then expand around the de Sitter background by means of (5.2). Making the field redefinitions

$$
\Omega = 12\sqrt{2\varepsilon}H^2 U, \qquad B = \frac{3a^2}{k^2}\sqrt{\frac{\varepsilon}{2}}V, \tag{5.14}
$$

we obtain an action that is regular for $\varepsilon, k \to 0$. Its $\varepsilon = 0$ limit is

$$
(8\pi G)\lim_{\varepsilon\to 0}\frac{\mathcal{L}_\mathrm{s}}{a^3} = \dot{U}^2 + V^2 - \frac{k^2}{a^2}\left(U^2 + \frac{2UV}{3H} - \frac{k^2U^2}{9a^2H^2}\right).
$$

We note that at this level V appears algebraically and can be integrated out. This means that the fakeon projection can be handled iteratively in ε.

After integrating V out, every m_χ dependence disappears to the first order in ε. In particular, if we define

$$
U = \left(1 - \frac{5}{12}\varepsilon\right)E, \tag{5.15}
$$

the action becomes

$$
(8\pi G)\frac{\mathcal{L}_\mathrm{s}}{a^3} = \dot{E}^2 - \frac{k^2}{a^2}E^2 + 3\varepsilon H^2 E^2. \tag{5.16}
$$

The redefinition (3.26) with $U \to E$, $\gamma \to 1$ and $\nu_\mathrm{t} \to \nu_\mathrm{s}$, where

$$
\nu_\mathrm{s} = \frac{3}{2} + 2\varepsilon, \tag{5.17}
$$

gives the action (3.27) with $\bar{k} \to k$. Inserting the solution for E into (5.15), (5.14) and then (5.11), and using the definition (3.45), we find in the superhorizon limit,

$$
A_\mathcal{R} = \frac{GH^2}{2\pi\varepsilon}\left(1 - \frac{17}{6}\varepsilon - (n_\mathcal{R} - 1)(2 - \gamma_E - \ln 2)\right), \tag{5.18}
$$

$$
n_\mathcal{R} - 1 = \frac{\mathrm{d}\ln\mathcal{P}_\mathcal{R}}{\mathrm{d}\ln k} = 3 - 2\nu_\mathrm{s} = -4\varepsilon. \tag{5.19}
$$

Together with (5.8), formula (5.18) gives the tensor-to-scalar ratio

$$
r = \frac{A_T}{A_\mathcal{R}} = 48\varepsilon^2\left(1 - 3\varepsilon\xi - 4\varepsilon(2 - \gamma_E - \ln 2)\right), \tag{5.20}
$$

to the next-to-leading order in ε. More explicitly, we get, after inverting (5.2),

$$r = \frac{96 m_\chi^2 \varepsilon^2}{m_\phi^2 + 2m_\chi^2} \left(1 - 4\varepsilon(2 - \gamma_E - \ln 2)\right). \tag{5.21}$$

So far, we have assumed ε small and ξ arbitrary. However, we see from (5.8) and (5.20) that higher orders of ε carry higher powers of ξ. To write (5.21) we have used

$$\frac{1}{1 + 3\varepsilon\xi} = 1 - 3\varepsilon\xi + 9\varepsilon^2\xi^2 + \mathcal{O}(\varepsilon^3\xi^3).$$

Conservatively, formula (5.21) is reliable as long as $3\varepsilon\xi \simeq m_\phi^2/(2m_\chi^2)$ is reasonably smaller than one. However, we may argue that the overall factor in front of (5.21) is exact. In the next two subsections we show that it is indeed so.

5.3 $m_\chi \sim m_\phi$: tensor fluctuations

Now we study the tensor fluctuations in the geometric framework with $\zeta = m_\chi^2/m_\phi^2$ fixed. The metric is still parametrized as (3.6) and the quadratic Lagrangian obtained from (2.1) is (5.3), plus an identical contribution for v. After replacing m_χ^2 with $m_\phi^2\zeta$, we use (5.2) to eliminate m_ϕ^2 and then expand in ε. We work out the leading and next-to-leading orders in ε.

As in subsection 3.1, we eliminate the higher derivatives of (5.3) by considering the extended Lagrangian $\mathcal{L}_t' = \mathcal{L}_t + \Delta\mathcal{L}_t$, where $\Delta\mathcal{L}_t$ is defined in (3.9). If we perform the redefinitions

$$u = \sqrt{\frac{3\varepsilon\zeta}{1 + 2\zeta}} \left(1 - \frac{5\varepsilon}{12}\right)(U + V), \qquad S = 2\sqrt{3\varepsilon\zeta(1 + 2\zeta)} H^2 \left(1 - \frac{5\varepsilon}{12}\right)(U - \varepsilon V),$$

and choose

$$f = 3H, \qquad h = 2(1 + 2\zeta)H^2 + \frac{k^2}{a^2}, \tag{5.22}$$

we obtain

$$(8\pi G)\frac{\mathcal{L}_t^{(U)}}{a^3} = \dot{U}^2 - \frac{k^2}{a^2}U^2\left(1 - \frac{2\varepsilon}{1 + 2\zeta}\right) - 3\varepsilon H^2 U^2,$$

$$(8\pi G)\frac{\mathcal{L}_t^{(V)}}{a^3} = -\dot{V}^2 + hV^2, \qquad (8\pi G)\frac{\mathcal{L}_t^{(UV)}}{a^3} = 4\varepsilon V\left(H\dot{U} + \frac{k^2}{a^2}\frac{U}{1 + 2\zeta}\right).$$

As usual, we have just written the $\varepsilon \to 0$ limit of $\mathcal{L}_t^{(V)}$, since the fakeon projection implies $V = \mathcal{O}(\varepsilon)$. This means that, to the order of approximation we are considering, we can drop both $\mathcal{L}_t^{(V)}$ and $\mathcal{L}_t^{(UV)}$, so the projected Lagrangian is just $\mathcal{L}_t^{(U)}$.

Switching to conformal time and defining

$$w = \frac{aU}{\sqrt{4\pi G}}, \qquad \nu_t = \frac{3}{2}, \qquad \bar{k} = k\left(1 - \frac{\varepsilon}{1 + 2\zeta}\right), \tag{5.23}$$

the Mukhanov action is (3.27). The fakeon Green function $G_{\mathrm{f}}(t,t')$ can be discussed as in subsection 4, with the replacements

$$m_\chi^2 \to 4\zeta H^2, \qquad n_\chi \to \sqrt{\frac{4m_\chi^2}{m_\phi^2} - \frac{1}{4}}. \tag{5.24}$$

and the solution is still (4.8). The consistency condition (4.12) gives again (4.9). Aside from the changes (5.24), everything works as before and we find, from the V field equation of $\mathcal{L}_{\mathrm{t}}'$,

$$V(t) = -2\varepsilon \int_{-\infty}^{+\infty} \mathrm{d}t'\, G_{\mathrm{f}}(t,t') \left[H\dot{U}(t') + \frac{k^2}{a(t')^2} \frac{U(t')}{1+2\zeta} \right]. \tag{5.25}$$

The fakeon average can be worked out with the procedure of subsection 3.1. Recalling that the terms in the square bracket of (5.25) are subleading or of higher orders in ε, we obtain that V does not contribute in the superhorizon limit $|k\tau| \ll 1$.

Inverting (5.2) to restore the m_ϕ^2 dependence of the overall factor, the power spectrum $\mathcal{P}_T = 16\mathcal{P}_u$ of the tensor fluctuations gives the amplitude

$$A_T = \frac{8G}{\pi} \frac{m_\chi^2 m_\phi^2}{m_\phi^2 + 2m_\chi^2} \left(1 - \frac{6\varepsilon m_\chi^2}{m_\phi^2 + 2m_\chi^2} \right), \tag{5.26}$$

while the spectral index n_T is $\mathcal{O}(\varepsilon^2)$.

5.4 $m_\chi \sim m_\phi$: scalar fluctuations

Now we study the scalar fluctuations in the geometric framework with $\zeta = m_\chi^2/m_\phi^2$ fixed. We replace m_χ^2 with $m_\phi^2 \zeta$, use (5.2) to eliminate m_ϕ^2 and then expand in powers of ε. We work to the next-to-leading order in ε.

We eliminate the higher derivatives of (2.1) by means of (2.2). The metric is still parametrized as (5.12) in the spatially-flat gauge $\Psi = 0$. The curvature perturbation is (5.11) and the φ fluctuation Ω is defined by (5.10). The quadratic Lagrangian obtained from (2.2) is (5.13). Defining

$$\Omega = 12\sqrt{2\varepsilon} H^2 U \left(1 - \frac{5\varepsilon}{12} \right), \qquad B = \frac{V}{\sqrt{2}k^2 H} \sqrt{\varepsilon\left(k^4 + 36\zeta a^4 H^4\right)} + \sqrt{\frac{\varepsilon}{2}} \frac{U}{H} \left(1 + \frac{7\varepsilon}{12} \right),$$

and expanding to the next-to-leading order in ε, we obtain the decomposition (3.40) with

$$
\begin{aligned}
(8\pi G)\frac{\mathcal{L}_{\mathrm{s}}^{(U)}}{a^3} &= \dot{U}^2 - \frac{k^2}{a^2}U^2 + 3\varepsilon H^2 U^2, \\
(8\pi G)\frac{\mathcal{L}_{\mathrm{s}}^{(V)}}{a^3} &= -\dot{V}^2 + \left[2(1+2\zeta) + \frac{k^2}{a^2 H^2} + g_1 \right] H^2 V^2, \\
(8\pi G)\frac{\mathcal{L}_{\mathrm{s}}^{(UV)}}{a^3} &= \varepsilon V \left(g_2 H\dot{U} + \frac{k^2}{a^2} g_3 U \right),
\end{aligned}
\tag{5.27}
$$

where g_i, $i = 1, 2, 3$, are regular functions of $k/(aH)$ and ζ, which tend to finite values in both limits $k/(aH) \to 0, \infty$. Moreover, g_1 tends to zero for $k/(aH) \to 0$ and g_3 tends to zero for $k/(aH) \to \infty$.

The expression of $\mathcal{L}_s^{(V)}$ of (5.27) is to the leading order, which is sufficient for our present purposes. The discussion about the fakeon Green function proceeds as before. It is easy to check that the consistency condition (4.12) coincides with (4.9). Clearly, the fakeon projection implies $V = \mathcal{O}(\varepsilon)$. This means that the projected Lagrangian is just $\mathcal{L}_s^{(U)}$ to the order of approximation we are considering, i.e., we can drop both $\mathcal{L}_s^{(V)}$ and $\mathcal{L}_s^{(UV)}$.

The Mukhanov action is (3.27) with $\gamma \to 1$, $\bar{k} \to k$ and $\nu_t \to \nu_s = (3/2) + 2\varepsilon$. The power spectrum $\mathcal{P}_\mathcal{R}$ gives the amplitude

$$A_\mathcal{R} = \frac{m_\phi^2 G}{12\pi\varepsilon^2}\left(1 - 3\varepsilon - (n_\mathcal{R} - 1)(2 - \gamma_E - \ln 2)\right) \tag{5.28}$$

and the spectral index $n_\mathcal{R} - 1 = -4\varepsilon$. Again, the dependence on m_χ drops out. Combining this result with (5.26), the tensor-to-scalar ratio is

$$r = \frac{96 m_\chi^2 \varepsilon^2}{m_\phi^2 + 2m_\chi^2}\left(1 + \frac{3\varepsilon m_\phi^2}{m_\phi^2 + 2m_\chi^2} - 4\varepsilon(2 - \gamma_E - \ln 2)\right), \tag{5.29}$$

which agrees with (5.21) for ζ large.

6 Vector fluctuations

In this section we study the vector fluctuations and show that they are set to zero by the fakeon projection at the quadratic level. For definiteness, we work in the geometric framework, but equivalent results are obtained in the inflaton framework.

We parametrize the metric as

$$g_{\mu\nu} = \text{diag}(1, -a^2, -a^2, -a^2) - \delta_\mu^0 \delta_\nu^i B_i - \delta_\mu^i \delta_\nu^0 B_i - \delta_\mu^i \delta_\nu^j (\partial_i E_j + \partial_j E_i),$$

where $\partial^i B_i = 0$ and $\partial^i E_i = 0$. A gauge invariant quantity is

$$B_i - \dot{E}_i \tag{6.1}$$

We choose a gauge where $E_i = 0$ and rewrite the metric as

$$g_{\mu\nu} = \text{diag}(1, -a^2, -a^2, -a^2) - \delta_\mu^0 \delta_\nu^1 C - \delta_\mu^0 \delta_\nu^2 D - \delta_\mu^1 \delta_\nu^0 C - \delta_\mu^2 \delta_\nu^0 D, \tag{6.2}$$

where $C = C(t, z)$ and $D = D(t, z)$ are the independent vector modes. After Fourier transforming the space coordinates, the quadratic Lagrangian $\mathcal{L}_v$ obtained from (2.1) is given by

$$(32\pi G m_\chi^2 a)\frac{\mathcal{L}_v}{k^2} = -\dot{C}^2 + \left[m_\chi^2 + (4\zeta + \varepsilon - 2\varepsilon\zeta)H^2 + \frac{k^2}{a^2}\right]C^2 \tag{6.3}$$

plus an identical contribution for D, where $\zeta = m_\chi^2/m_\phi^2$. As before, C^2 stands for $C_{-\mathbf{k}}C_\mathbf{k}$, $\dot{C}^2$ for $\dot{C}_{-\mathbf{k}}\dot{C}_\mathbf{k}$, and so on. After the redefinition

$$\mathcal{V} = \frac{kC}{2m_\chi a^{1/2}},$$

the Lagrangian turns into

$$(8\pi G)\mathcal{L}_v = -\dot{\mathcal{V}}^2 + \left[m_\chi^2 + (16\zeta - 1 + 2\varepsilon - 8\varepsilon\zeta)\frac{H^2}{4} + \frac{k^2}{a^2} \right]\mathcal{V}^2. \qquad (6.4)$$

The kinetic term has the wrong sign, so $\mathcal{V}$ needs to be quantized as a fakeon. Since $\mathcal{V}$ does not couple to any other field at this level, the fakeon projection sets it to zero. Therefore, the vector modes do not contribute to the two-point functions. Note that these conclusions hold without expanding around the de Sitter background.

The consistency condition (4.12) is studied by requiring that the coefficient of $\mathcal{V}^2$ in (6.4) be positive in the superhorizon/de Sitter limit, which gives again (4.9).

7 Summary of predictions and connection with observations

In this section we summarize the predictions and make contact with observations. We express the results in terms of the number of e-foldings, which is defined by

$$N = \int_{t_i}^{t_f} H(t')\mathrm{d}t', \qquad (7.1)$$

where t_i is the time when $\varepsilon(t_i) = \varepsilon$ and t_f is when inflation ends, $\varepsilon(t_f) = 1$. It is convenient to work in the geometric framework, where we can use (5.1) and (5.2). Then we translate the formulas to the inflaton framework by means of the map of appendix A. Expressing every quantity as a function of ε, (7.1) gives

$$N = \int_\varepsilon^1 \frac{H(t'(\varepsilon'))}{\dot{\varepsilon}(t'(\varepsilon'))}\mathrm{d}\varepsilon' = \int_\varepsilon^1 \frac{\mathrm{d}\varepsilon'}{2\varepsilon'^2}\left[1 + \frac{\varepsilon'}{6} + \mathcal{O}(\varepsilon'^2)\right] = \frac{1}{2\varepsilon} - \frac{1}{12}\ln\varepsilon + \mathcal{O}(\varepsilon^0). \qquad (7.2)$$

The $\mathcal{O}(\varepsilon^0)$ corrections are not very meaningful, because they depend on the upper bound of integration and $\varepsilon(t_f) = 1$ is just a conventional choice. To the leading order, we can take

$$N \simeq \frac{1}{2\varepsilon},$$

in the geometric framework. Note that in the inflaton framework we have instead $N \simeq \sqrt{3}/(2\sqrt{\varepsilon})$, as can be shown using (A.6). Once expressed in terms of N, the predictions obtained in the two frameworks agree (see appendix A). Collecting the results of formulas (3.36), (3.46)–(3.47), (5.26) and (5.28), we obtain, to the leading order,

$A_{\mathcal{R}}$	A_T	r	$n_{\mathcal{R}} - 1$	n_T
$\dfrac{m_\phi^2 N^2}{3\pi M_{\mathrm{Pl}}^2}$	$\dfrac{8m_\chi^2 m_\phi^2}{\pi(m_\phi^2 + 2m_\chi^2)M_{\mathrm{Pl}}^2}$	$\dfrac{24m_\chi^2}{N^2(m_\phi^2 + 2m_\chi^2)}$	$-\dfrac{2}{N}$	$-\dfrac{3m_\chi^2}{N^2(m_\phi^2 + 2m_\chi^2)}$

$$(7.3)$$

The formula of n_T comes from (3.36), since in this particular case the inflaton framework is more powerful than the geometric framework.
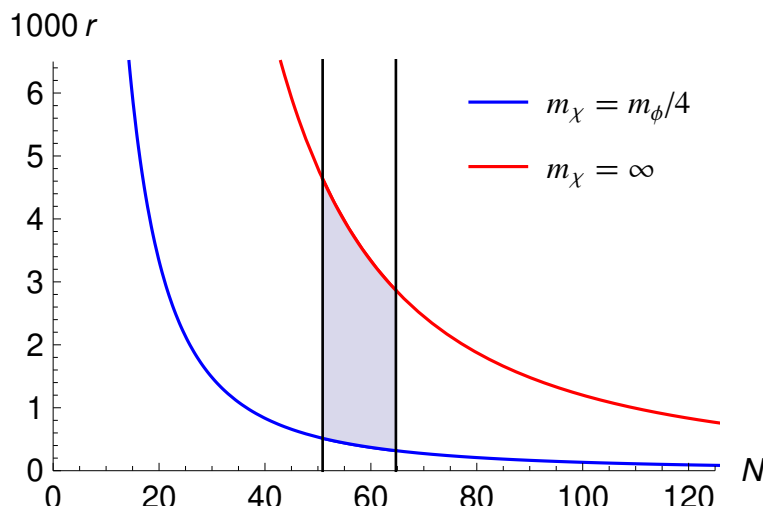
Figure 1. Allowed values of the tensor-to-scalar ratio r.

We see that the predictions for $A_{\mathcal{R}}$ and $n_{\mathcal{R}} - 1$ coincide with the ones of the $R + R^2$ model. Instead, the predictions for A_T, r and n_T are smaller by a factor $2m_\chi^2/(m_\phi^2 + 2m_\chi^2)$. Note that (7.3) implies the relation

$$r \simeq -8n_T, \tag{7.4}$$

which is known to hold in single-field slow-roll models independently of the scalar potential $V(\phi)$.[1] It is a nontrivial fact that it does not depend on m_χ, besides N and m_ϕ.

The bound (4.9) on m_χ is also a prediction of the theory, required by the consistency of the fakeon projection with inflationary cosmology. Because of it, the tensor-to-scalar ratio r and the spectral index n_T are predicted within less than one order of magnitude. Precisely,

$$\frac{1}{9} \lesssim \frac{N^2}{12}r \simeq -\frac{2N^2}{3}n_T \lesssim 1. \tag{7.5}$$

For example, for $N = 60$ we have

$$0.4 \lesssim 1000r \lesssim 3, \qquad -0.4 \lesssim 1000n_T \lesssim -0.05.$$

The allowed values of r are shown in figure 1, where the vertical lines denote the minimum and maximum values of N in the range $n_{\mathcal{R}} = 0.9649 \pm 0.0042$ at 68% CL [23]. The windows (7.5) are compatible with the data available at present, which give $r < 0.1$ [23].

The results of this paper also provide corrections to the amplitudes, which can be used to estimate the theoretical errors. From (7.2) we find

$$\varepsilon = \frac{1}{2N}\left(1 + \frac{\ln N}{12N}\right) + \mathcal{O}\left(\frac{1}{N^2}\right),$$

---

[1]See for example ref. [46].

so we obtain

$$A_{\mathcal{R}} = \frac{GN^2 m_\phi^2}{3\pi} \left(1 - \frac{\ln N}{6N} + \mathcal{O}\left(\frac{1}{N}\right) \right).$$

$$A_T = \frac{8G}{\pi} \frac{m_\chi^2 m_\phi^2}{m_\phi^2 + 2m_\chi^2} \left(1 - \frac{3m_\chi^2}{N(m_\phi^2 + 2m_\chi^2)} \left(1 + \frac{\ln N}{12N} \right) + \mathcal{O}\left(\frac{1}{N^2}\right) \right).$$

With $N = 60$, the first correction to A_T is between 0.3% ($m_\chi = m_\phi/4$) and 2.5% ($m_\chi \to \infty$). Although $A_{\mathcal{R}}$ and $n_{\mathcal{R}} - 1$ do not depend on m_χ in our approximation, they will at higher orders.

8 Conclusions

We have worked out the predictions of quantum gravity with fakeons on inflationary cosmology. By expanding around the de Sitter background the amplitudes and spectral indices of the scalar and tensor fluctuations have been calculated to the next-to-leading orders, comparing different frameworks, which lead to matching results. The physical content of the theory is exhausted by the two power spectra. The vector degrees of freedom, as well as the other scalar and tensor ones, are handled by means of the fakeon prescription and projected away. The methodologies we have developed to deal with this operation appear to be generalizable to higher orders.

The local, renormalizable, unitary, perturbative quantum field theory of gravity considered in this paper depends only on four parameters: the cosmological constant, Newton's constant, m_ϕ and m_χ. The values of the cosmological constant and Newton's constant are known. It will be possible to derive the values of m_ϕ and m_χ from $n_{\mathcal{R}}$ and r once new cosmological data will be available [47]. At that point, the theory will be uniquely determined and all other predictions (tensor tilt, running of the spectral indices, and so on) will be stringent tests of its validity.

The consistency of the approach puts a lower bound on the mass m_χ of the fakeon with respect to the mass m_ϕ of the scalar field. The tensor-to-scalar ratio r is determined within less than an order of magnitude. Moreover, the relation $r = -8n_T$ is not affected by m_χ within our approximation. A separate analysis is required to study the case where the consistency bound on m_χ is violated and work out the consequences of the violation on the physics of the primordial universe. Finally, the investigation of this paper and the results we have obtained shed light on the problem of understanding purely virtual particles in curved space.

Acknowledgments

A   Map relating the inflaton framework to the geometric framework

In this appendix we derive some expansions used in the paper and show that the results of the inflaton and geometric frameworks agree with each other. For definiteness, the quantities with bars ($\bar{a}$, $\bar{t}$, $\bar{H}$, $\bar{\varepsilon}$, $\bar{\eta}$, etc.) refer to the inflaton framework, while the quantities without bars (a, t, H, ε, η, etc.) refer to the geometric framework.

We start by determining $-aH\tau$ in the geometric formalism. Writing the most general expansion

$$- aH\tau = 1 + a_1\varepsilon + a_2\varepsilon^2 + a_3\varepsilon^3 + \cdots , \tag{A.1}$$

the numerical coefficients a_i are calculated by differentiating (A.1) and then using the definition (3.5), the equation (5.1) and (A.1) again. This procedure gives an equality of two power series. Matching the coefficients recursively, we obtain a_i for every i. To the lowest orders, the result is

$$- aH\tau = 1 + \varepsilon + 3\varepsilon^2 + \frac{44}{3}\varepsilon^3 + \mathcal{O}(\varepsilon^4). \tag{A.2}$$

If we continue to arbitrary orders, we find an asymptotic series.

The action (2.4) is obtained from (2.1) by means of the conformal transformation (2.3). If we want to map the parametrizations (3.6) and (3.37) of the background metrics into each other, we need to combine that transformation with a time redefinition $\bar{t}(t)$, so that

$$\frac{\mathrm{d}\bar{t}}{\mathrm{d}t} = \frac{\bar{a}}{a}, \qquad \mathrm{d}\bar{s}^2 = \bar{g}_{\mu\nu}\mathrm{d}\bar{x}^\mu\mathrm{d}\bar{x}^\nu = \mathcal{W}\mathrm{d}s^2 = \mathcal{W}g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu, \qquad \mathcal{W} \equiv 1 - \frac{\varphi}{3m_\phi^2}. \tag{A.3}$$

We split the conformal factor $\mathcal{W}$ into the sum of its background part $\mathcal{W}_0$ and the fluctuation $\delta\mathcal{W}$. Using (5.10) and the second equation of (2.3), it is easy to find

$$\mathcal{W}_0 = 1 + \frac{2\Upsilon}{m_\phi^2}, \qquad \delta\mathcal{W} = -\frac{\Omega}{3m_\phi^2} = -\hat{\kappa}\mathcal{W}_0\delta\phi. \tag{A.4}$$

The transformations of the background quantities are

$$\frac{\mathrm{d}\bar{t}}{\mathrm{d}t} = \frac{\bar{a}}{a} = \sqrt{\mathcal{W}_0}, \qquad \bar{H} = \frac{1}{\sqrt{\mathcal{W}_0}}\left(H + \frac{\dot{\mathcal{W}}_0}{2\mathcal{W}_0} \right), \tag{A.5}$$

plus those of ε and η, which follow directly from their definitions. Using (5.1), (5.4) and (5.2), we find, to the lowest orders,

$$\bar{H} = \frac{m_\phi}{2}\left(1 - \frac{3\varepsilon}{2} + \frac{7}{4}\varepsilon^2 + \mathcal{O}(\varepsilon^3) \right), \qquad \bar{\varepsilon} = -\frac{1}{\bar{H}^2}\frac{\mathrm{d}\bar{H}}{\mathrm{d}\bar{t}} = 3\varepsilon^2 - 2\varepsilon^4 + \mathcal{O}(\varepsilon^5),$$

$$\bar{\eta} = -2\varepsilon + \frac{13}{3}\varepsilon^2 + \mathcal{O}(\varepsilon^3), \qquad H = \frac{m_\phi}{\sqrt{6\varepsilon}}\left(1 - \frac{\varepsilon}{12} + \frac{19\varepsilon^2}{288} + \mathcal{O}(\varepsilon^3) \right). \tag{A.6}$$

The relations can be extended to arbitrary orders, if needed. We find $\bar{\eta} = \mathcal{O}(\bar{\varepsilon}^{1/2})$ and $\mathrm{d}^n\bar{\varepsilon}/\mathrm{d}\bar{t}^n = \bar{H}^n \mathcal{O}(\bar{\varepsilon}^{(n+2)/2})$, which justifies the organization (3.3) of the expansion around the de Sitter background in the inflaton framework. In particular, inverting $\bar{\varepsilon}(\varepsilon)$ we get

$$\bar{H} = \frac{m_\phi}{2}\left(1 - \frac{\sqrt{3\bar{\varepsilon}}}{2} + \frac{7\bar{\varepsilon}}{12} - \frac{47\bar{\varepsilon}^{3/2}}{72\sqrt{3}} + \mathcal{O}(\bar{\varepsilon}^2)\right),$$

$$\bar{\eta} = -2\sqrt{\frac{\bar{\varepsilon}}{3}}\left(1 - \frac{13}{6}\sqrt{\frac{\bar{\varepsilon}}{3}} - \frac{2\bar{\varepsilon}}{27} + \mathcal{O}(\bar{\varepsilon}^{3/2})\right), \tag{A.7}$$

$$-\bar{a}\bar{H}\tau = 1 + \bar{\varepsilon} + \frac{4\bar{\varepsilon}^{3/2}}{\sqrt{3}} + \frac{91}{9}\bar{\varepsilon}^2 + \mathcal{O}(\bar{\varepsilon}^{5/2}).$$

The last formula is derived from (A.2). Without passing through the geometric framework, the relations (A.7) can be worked out directly in the inflaton framework by expanding the equations (3.1) around the de Sitter background.

The map relating the fluctuations can be worked out from (A.3). The tensor modes u and v are clearly invariant,

$$\bar{u} = \mathcal{W}_0\frac{a^2}{\bar{a}^2}u = u, \qquad \bar{v} = \mathcal{W}_0\frac{a^2}{\bar{a}^2}v = v, \tag{A.8}$$

while the scalar fluctuations Ψ and Φ transform as

$$\bar{\Psi} = \Psi + \frac{\Omega}{6(m_\phi^2 + 2\Upsilon)}, \qquad \bar{\Phi} = \Phi - \frac{\Omega}{6(m_\phi^2 + 2\Upsilon)}. \tag{A.9}$$

These formulas are written up to corrections of orders $\mathcal{O}(u\Omega)$, $\mathcal{O}(v\Omega)$, $\mathcal{O}(\Psi\Omega)$ and $\mathcal{O}(\Phi\Omega)$, respectively. We can omit them for our purposes, since they do not affect the quadratic action and the two-point functions. We recall that the action is expanded around a solution of the equations of motion (which is then expanded around the de Sitter metric — which is not an exact solution), so the linear terms in the fluctuations are absent. Switching from one framework to the other, the corrections just mentioned affect the cubic terms, but not the quadratic ones.

From (A.9) we derive the transformation of the curvature perturbation $\mathcal{R}$. Observe that, given a scalar $Y = Y_0 + \delta Y$, where δY denotes the fluctuation around its background value Y_0, the combination

$$\mathcal{R}_Y = \Psi + H\frac{\delta Y}{\dot{Y}_0} \tag{A.10}$$

is invariant under infinitesimal time reparametrizations. If we choose $Y = \mathcal{W}$ and use the relations (A.4), we find, in the geometric framework,

$$\mathcal{R}_\mathcal{W} = \Psi - H\frac{m_\phi^2}{2\dot{\Upsilon}}\frac{\Omega}{3m_\phi^2} = \Psi - H\frac{\Omega}{6\dot{\Upsilon}} = \mathcal{R}, \tag{A.11}$$

the last equality following from (5.11). Using (A.4), (A.5) and (A.9) to rewrite this expression in the inflaton framework, we obtain

$$\mathcal{R}_\mathcal{W} = \bar{\Psi} + \bar{H}\left(\frac{\mathrm{d}\phi_0}{\mathrm{d}\bar{t}}\right)^{-1}\delta\phi = \bar{\mathcal{R}}, \tag{A.12}$$

where ϕ_0 is the background value of $\phi = \phi_0 + \delta\phi$, such that $\mathcal{W}_0 = e^{-\hat{\kappa}\phi_0}$. We recall that in section (3.2) the comoving gauge $\delta\phi = 0$ was used, so we just had $\bar{\mathcal{R}} = \bar{\Psi}$ there. Equations (A.11) and (A.12) prove that $\mathcal{R} = \bar{\mathcal{R}}$, so $\mathcal{R}$ is also invariant when we switch frameworks.

This fact, together with (A.8), ensures that the power spectra calculated in the paper coincide in the two frameworks. We can use the formulas (A.6) to check it explicitly to the orders we have been working with. Comparing (3.36) with (5.8), (5.26) and (5.9), we find

$$\bar{A}_T(\bar{H}, \bar{\varepsilon}) = A_T(H, \varepsilon), \qquad \bar{n}_T(\bar{\varepsilon}) = n_T(\varepsilon).$$

Finally, comparing (3.46) and (3.47) with (5.18), (5.19) and (5.28), it is easy to verify that

$$\bar{A}_{\mathcal{R}}(\bar{H}, \bar{\varepsilon}, \bar{\eta}) = A_{\mathcal{R}}(H, \varepsilon, \eta), \qquad \bar{n}_{\mathcal{R}}(\bar{\varepsilon}, \bar{\eta}) = n_{\mathcal{R}}(\varepsilon, \eta).$$

B Superhorizon evolution

In this appendix we show that the curvature perturbation $\mathcal{R}$ can be considered constant on superhorizon scales for adiabatic fluctuations of the energy-momentum tensor, in particular after the metric fluctuations exit the horizon and before they re-enter it. We start by showing this result in the inflaton framework.

Consider the energy momentum tensor $T_{\mu\nu}$ with components

$$T_{00} = \rho(1+2\Phi)+\delta\rho, \qquad T_{0i} = -\partial_i\delta q, \qquad T_{ij} = a^2\delta_{ij}[p(1-2\Psi)+\delta p] + \left(\partial_i\partial_j - \frac{\triangle}{3}\delta_{ij}\right)\delta\Pi,$$

where $\delta\rho$, δq, δp and $\delta\Pi$ are its scalar fluctuations around the background. The gauge invariant curvature perturbation is

$$\mathcal{R} = \Psi - \frac{H}{\rho + p}(\delta q + pB). \tag{B.1}$$

The unprojected equations derived from the action (2.4) for the metric (3.37) in the Newton gauge ($B = 0$) read

$$2\dot{\Psi} + 2H\Phi - \frac{2\triangle\dot{W}}{3m_\chi^2 a^2} + 8\pi G\delta q = 0,$$

$$\Phi - \Psi + \frac{1}{m_\chi^2}\left(\ddot{W} + H\dot{W} - \frac{\triangle W}{3a^2}\right) + 8\pi G\delta\Pi = 0,$$

$$6H\dot{\Psi} - \frac{2\triangle\Psi}{a^2} + \frac{2\triangle^2 W}{3m_\chi^2 a^4} + 8\pi G(\delta\rho + 2\rho\Phi) = 0,$$

$$\ddot{\Psi} + H(3\dot{\Psi} + \dot{\Phi}) + 2\Phi\dot{H} + 3H^2\Phi + \frac{\triangle(\Phi - \Psi)}{3a^2} - \frac{\triangle^2 W}{9m_\chi^2 a^4} - 4\pi G\delta p = 0, \tag{B.2}$$

where $W = \Psi + \Phi$ and the contributions of the scalar field ϕ are moved into $T_{\mu\nu}$. It is possible to show that formulas (B.2), together with the Friedmann equations

$$H^2 = \frac{8\pi G}{3}\rho, \qquad \dot{H} + \frac{3}{2}H^2 = -4\pi G p,$$

imply the equation

$$\frac{\dot{H}}{H}\dot{\mathcal{R}} = -4\pi G\left(\delta p - \frac{\dot{p}}{\dot{\rho}}\delta\rho\right) - \frac{\triangle}{a^2}\left[\frac{\dot{p}}{\dot{\rho}}\left(\Psi - \frac{\triangle W}{3m_\chi^2 a^2} - \frac{H\dot{W}}{m_\chi^2}\right) - \frac{\dot{H}\dot{W}}{3m_\chi^2 H} + \frac{8\pi G}{3}\delta\Pi\right]. \quad \text{(B.3)}$$

Thus, for adiabatic fluctuations

$$\delta p = \frac{\dot{p}}{\dot{\rho}}\delta\rho$$

and on superhorizon scales $k/(aH) \ll 1$, the scalar $\mathcal{R}$ is practically constant. Since the property holds for the whole set of solutions of the unprojected equations, it also holds for the projected ones. Note that after the end of inflation ε is no longer small, so the $\dot{H}$ factor in front $\dot{\mathcal{R}}$ in (B.3) is not a source of trouble.

In the geometric framework we reach the same conclusions. It is sufficient to work with the action (2.2) and note that the only difference with respect to the formulas just written is a redefinition of $T_{\mu\nu}$, brought by the variation of the terms containing φ with respect to the metric. Since we are considering only scalar quantities here, this is just a redefinition of ρ, p and the fluctuations $\delta\rho$, δq, δp and $\delta\Pi$. Observe that we may need a nontrivial $\delta\Pi$ for this redefinition, which is the reason why we kept it nonzero in the derivation above.

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