An infinite family of two-distance tight frames

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ABSTRACT

The relationship between equiangular tight frames and strongly regular graphs has been known for several years. This relationship has been exploited to construct many of the latest examples of new strongly regular graphs. Recently it was shown that there is a similar relationship between two-distance tight frames and strongly regular graphs. In this paper we present a new tensor like construction of two-distance tight frames, and hence a family of strongly regular graphs. While graphs with these parameters were known to exist, this new construction is very simple, requiring only the existence of an affine plane, whereas the original constructions often require more complicated objects such as generalized quadrangles.

Keywords: tight frames, two-distance, strongly regular graph

1. INTRODUCTION

One area of frame theory that has seen a recent flurry of activity is so-called *optimal projective packings*, that is, collections of unit vectors $\Phi = \{\varphi_i\}_{i=1}^N$ in d-dimensional space such that the *coherence*,

$$\mu(\Phi) := \max_{i\neq j} |\langle \varphi_i, \varphi_j\rangle|,$$

is as small as possible. In particular, several new constructions of frames have recently been found[?,3,4,6,9,11] which saturate the lower bound on the coherence known as the Welch bound. Such a collection is known as an equiangular tight frame (ETF). Of particular interest is the case of ETFs for real vector spaces. These frames are known to be equivalent to a certain well-studied class of graphs. Indeed, constructions of ETFs have provided several new contributions to the graph theory literature.

Recently another connection between frame theory and graph theory was pointed out in.[1] In particular, it was shown that certain frames known as two-distance tight frames are essentially equivalent to strongly regular graphs. Unfortunately, in contrast to equiangular tight frames, we have very few constructions of two-distance tight frames. The goal of this paper is to present one such construction. While the associated strongly regular graphs have the same parameters as known graphs, we believe that the construction here is simpler. Hence this construction is a proof of concept that, like ETFs, two-distance tight frames can be a fruitful way to view strongly regular graphs and hopefully lead to more new constructions.

2. TWO-DISTANCE TIGHT FRAMES AND STRONGLY REGULAR GRAPHS

Let $\Phi = \{\varphi_i\}_{i=1}^N$ be a sequence of vectors in $\mathbb{R}^d$. As is common in frame theory we will also denote the $d\times N$ matrix $[\varphi_1 \cdots \varphi_N]$ with the symbol Φ. If $\|\varphi_i\|^2 = s$ for each $i\in[N]$ and there are two real numbers a and b such that $\langle\varphi_i,\varphi_j\rangle \in \{a,b\}$ for all $i\neq j$, then for any $i \neq j$ we have

$$\|\varphi_i - \varphi_j\|^2 = 2s - 2\langle\varphi_i,\varphi_j\rangle \in \{2s-2\alpha, 2s-2\beta\}.$$

Hence, such a collection Φ is called *two-distance*. If there is some $\alpha > 0$ such that $\Phi\Phi^\top = \alpha I$, then Φ is called a *tight frame*. The main object of interest in this article will be two-distance tight frames. One particularly important example is two-distance tight frames with $a^2 = b^2$. In this case, the acute angle between the lines spanned by two distinct vectors φ_i and φ_j is given by $\frac{1}{s}\arccos|\langle\varphi_i,\varphi_j\rangle| = \frac{1}{s}\arccos|a|$, and hence such we call such a sequence an *equiangular tight frame* (ETF).

In 2009 Waldron[10] showed that an ETF with N vectors in $\mathbb{R}^d$ exists if and only if a certain graph exists. A similar relationship with graphs exists for two-distance tight frames. This relationship was first pointed out by Barg, Glazyrin, Okoudjou, and Yu.[1] In particular, two-distance tight frames are almost equivalent to strongly regular graphs.

Definition 2.1. Let $G = (V, E)$ be a regular graph on v vertices with degree k. We say that G is *strongly regular* if there are numbers λ and μ such that:

(i) Every pair of adjacent vertices has λ common neighbors.

(ii) Every pair of non-adjacent vertices has μ common neighbors.

Such a graph is called an $\mathrm{SRG}(v, k, \lambda, \mu)$.

Proposition 2.2. *Let A be a symmetric $\{0,1\}$-matrix with zero diagonal. The matrix A is the adjacency matrix of an SRG(v, k, λ, μ) if and only if the following two conditions hold:*

(i) Every row and column of A contains k ones, that is, $AJ = kJ = JA$ where J is the $v \times v$ all-ones matrix.

(ii) $A^2 = (\lambda - \mu)A + (k - \mu)I + \mu J = kI + \lambda A + \mu(J - I - A)$

The content of the following theorem is contained in,[1] but we prove it here in order to readily be able to translate back and forth between SRG parameters and two-distance tight frame parameters.

Theorem 2.3. *Let A be the adjacency matrix of an SRG(v, k, λ, μ) and set*

$$\alpha_\pm = \frac{1}{2}\Big(\lambda - \mu \pm \sqrt{(\lambda-\mu)^2 + 4(k-\mu)}\Big).$$

The matrix

$$G = sI + aA + b(J - I - A)$$

where

$$a = 1 - \frac{k + |\alpha_-|}{v}, \qquad b = -\frac{k + |\alpha_-|}{v}, \qquad \text{and} \qquad s = |\alpha_-| - \frac{k + |\alpha_-|}{v},$$

is the Gram matrix of a two-distance tight frame for $\mathbb{R}^d$ where $d = N - 1 - \frac{1}{\lambda-\mu}(k + (N-1)\alpha_+)$.

If $\Phi = \{\varphi_i\}_{i=1}^N$ is a two-distance tight frame for $\mathbb{R}^d$ with inner products a and b and squared norms s, and $a^2 \neq b^2$, then there exists an SRG(v, k, λ, μ) with adjacency matrix A where

$$v = N, \quad k = \frac{1}{a^2 - b^2}\left(\frac{Ns^2}{d} - s^2 - b^2(N-1)\right), \quad \lambda = \frac{1}{(a-b)^2}\left(\frac{Nsa}{d} - b^2N - 2a(s-b) - 2kb(a-b)\right),$$

$$\text{and} \quad \mu = \frac{b}{(a-b)^2}\left(\frac{Ns}{d} - bN - 2(s-b) - 2k(a-b)\right)$$

such that

$$\Phi^\top \Phi = sI + aA + b(J - I - A).$$

Proof. Let $\{x_i\}_{i=1}^N$ be an orthogonal basis of eigenvectors of A with associated eigenvalues $\{\alpha_i\}_{i=1}^N$. We may assume without loss of generality that x_1 is the all-ones vector $\mathbb{1}$, and hence $\alpha_1 = k$. This implies that $\langle x_i, \mathbb{1}\rangle = 0$ and hence $Jx_i = 0$ for $i \geq 2$. From Proposition 2.2 we see that for $i \geq 2$ each number α_i is a root of $x^2 - (\lambda-\mu)x - (k-\mu)$. Note that α_- is the minimum root of this polynomial. Since the trace of

A is zero we must have $k + M\alpha_- + (N-M-1)\alpha_+ = 0$ for some $M \in \mathbb{N}$. This implies $\alpha_- < 0$ and the eigenspace of A associated with α_- has dimension

$$M = \frac{1}{\beta - \alpha}(k + (N-1)\beta) = \frac{1}{\lambda - \mu}(k + (N-1)\alpha_+).$$

Since $\{x_i\}_{i=1}^N$ is an orthogonal basis of eigenvectors for A and J, from the observation that

$$G = (s-b)I + (a-b)A + bJ = |\alpha_-|I + A + bJ$$

we see that the eigenvalues of G are $|\alpha_-| + k + bv = 0$ with multiplicity 1, $|\alpha_-| + \alpha_- = 0$ with multiplicity M, and $|\alpha_-| + \alpha_+ = \alpha_+ - \alpha_-$ with multiplicity $v - M - 1$. This shows that G is a rank $v - M - 1$ symmetric matrix with $G^2 = (\alpha_+ - \alpha_-)G$, and hence the Gram matrix of a tight frame for $\mathbb{R}^{v-1-M}$. Since G clearly has two off-diagonal entries and diagonal s, this completes the proof of this direction.

Let $\Phi = \{\varphi_i\}_{i=1}^N$ be a two-distance tight frame in $\mathbb{R}^d$ with inner products a and b such that $a \neq -b$ and squared norms s. Let $G = \Phi^\top\Phi = (\langle\varphi_j, \varphi_i\rangle)_{i,j=1}^N$ be the Gram matrix of Φ. There is some $\alpha > 0$ such that $\Phi\Phi^\top = \alpha I$. However, since $Ns = \operatorname{tr}(\Phi^\top\Phi) = \operatorname{tr}(\Phi\Phi^\top) = \alpha d$, we see that $\Phi\Phi^\top = \frac{Ns}{d}I$, and $G^2 = \Phi^\top\Phi\Phi^\top\Phi = \frac{Ns}{d}G$.

Let $k(i)$ be the number of entries in row i of G which equal a, hence $N - k(i) - 1$ is the number of entries in the same row which equal b. Computing the ith diagonal entry of G^2 in two different ways we have

$$\frac{Ns^2}{d} = s^2 + k(i)a^2 + (N - k(i) - 1)b^2.$$

Solving for $k(i)$ we obtain

$$k(i) = \frac{1}{a^2 - b^2}\left(\frac{Ns^2}{d} - s^2 - b^2(N-1)\right).$$

This shows that $k(i)$ does not depend on i, so we set $k = k(i)$ for any i. Let A be a $\{0,1\}$-matrix such that $G = sI + aA + b(J - I - A)$. It follows that A is the adjacency matrix of a regular graph with degree k. Finally, we compute

$$\frac{Ns}{d}((s-b)I + (a-b)A + bJ) = \frac{Ns}{d}G = G^2 = ((s-b)I + (a-b)A + bJ)^2.$$

Expanding the right hand side (using the fact that $JA = AJ = kJ$) and solving for A^2 we see that A is the adjacency matrix of a strongly regular graph, and we can then use Proposition 2.2 to solve for λ and μ. $\square$

Unfortunately, the assumption that $a^2 \neq b^2$ in the second half of Theorem 2.3 cannot be removed. In the proof of the second half of Theorem 2.3 we used the fact that $a^2 - b^2 \neq 0$ in order to show that the graph was regular. The next example shows that in the case that $a^2 = b^2$ the graph need not be regular.

Example 2.4. Let

$$\Phi = \begin{bmatrix} +1 & +1 & +1 & -1 \\ +1 & -1 & +1 & +1 \\ +1 & +1 & -1 & +1 \end{bmatrix}.$$

It can easily be verified that Φ is a tight frame, and

$$\Phi^\top\Phi = \begin{bmatrix} 3 & +1 & +1 & +1 \\ +1 & 3 & -1 & -1 \\ +1 & -1 & 3 & -1 \\ +1 & -1 & -1 & 3 \end{bmatrix} = 3\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + (+1)\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} + (-1)\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

However, neither of these last two matrices is the adjacency of a regular graph, let alone a strongly regular graph.

More can be said about the relationship between two-distance tight frames and ETFs. Waldron[10] showed that ETFs are in one-to-one correspondence with a certain class of strongly regular graphs on $N-1$ vertices, as opposed to the N vertices as in Theorem 2.3. Later, Fickus, the second author, Mixon, Peterson, and Watson[5] showed that ETFs with extra symmetry do give rise to strongly regular graphs on N vertices. In this article we will not be considering ETFs, and thus these extra subtleties can be safely ignored.

3. BLOCK DESIGNS

The goal of this paper is to present the construction of an infinite family of two-distance tight frames. The construction is reminiscent of the constructions of SRGs by Goethals and Seidel,[8] as well as the several recent constructions of ETFs.[3,4,7] In each of those constructions an ETF with $d+1$ vectors in d dimensions is embedded into several subspaces of a higher dimensional space in order to build an ETF, though this was not the terminology used by Goethals and Seidel. In this case we will be embedding a two-distance tight frame. Both the two-distance tight frame we embed and the embeddings are constructed using balanced incomplete block designs.

Definition 3.1. Given $v, k, \lambda \in \mathbb{N}$ a *balanced incomplete block design* with parameters (v, k, λ), or BIBD(v, k, λ), is a v element set $\mathcal{V}$ together with a collection $\mathcal{B}$ of k-element subsets of $\mathcal{V}$ with the following properties:

(i) There is a number r so that for each $V \in \mathcal{V}$ there are exactly r sets in $\mathcal{B}$ that contain V.

(ii) For each $V_1, V_2 \in \mathcal{V}$ with $V_1 \neq V_2$ there are exactly λ sets in $\mathcal{B}$ that contain V_1 and V_2.

The sets in $\mathcal{B}$ are called *blocks*.

We will only have use for BIBDs with $\lambda = 1$. These are sometimes called *Steiner systems*. Let $(\mathcal{V}, \mathcal{B})$ be a BIBD$(v, k, 1)$. In addition to v, k, λ and r, we also define the parameter $b := |\mathcal{B}|$. A simple counting argument shows that

$$r = \frac{v-1}{k-1} \quad \text{and} \quad b = \frac{v(v-1)}{k(k-1)}.$$

The *incidence matrix* of a BIBD is the $b \times v$ matrix A, with rows indexed by $\mathcal{B}$ and columns indexed by $\mathcal{V}$ given by

$$A(B,V) = \begin{cases} 1 & V \in B \\ 0 & \text{else.} \end{cases}$$

Proposition 3.2. *A $b \times v$ matrix A with entries in $\{0,1\}$ is the incidence matrix of a BIBD(v,k,λ) if and only if the following two conditions hold:*

(i) $A^\top A = (r-\lambda)I + \lambda J$.

(ii) $A\mathbb{1} = k\mathbb{1}$.

Example 3.3. The matrices

$$\begin{bmatrix} 1 & \cdot & \cdot & 1 \\ \cdot & 1 & 1 & \cdot \\ 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 \\ 1 & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & 1 & 1 \\ \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & 1 & \cdot & 1 \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \\ 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & 1 & \cdot \end{bmatrix}$$

(where 0 is replaced with $\cdot$) are the adjacency matrices of a BIBD$(4,2,1)$, a BIBD$(7,3,1)$, and a BIBD$(9,3,1)$, respectively.

Definition 3.4. Let $(\mathcal{V},\mathcal{B})$ be a BIBD(v,k,λ). A subset of $\mathcal{B}$ that forms a partition of $\mathcal{V}$ is called a *parallel class*. If there is a partition of $\mathcal{B}$ into parallel classes, then we say that the BIBD is *resolvable* and we refer to it as an RBIBD(v,k,λ).

Note that the BIBD$(9,3,1)$ in Example 3.3 is resolvable, and hence an RBIBD$(9,3,1)$.

Our construction requires the existence of two BIBDs. For a given $k\in\mathbb{N}$ we need an RBIBD$(k^2,k,1)$ and a BIBD$(k,2,1)$. The RBIBD$(k^2,k,1)$ will be used in the next section to construct a two-distace tight frame Ψ, then in the final section we will use a BIBD$(k,2,1)$ to construct embedding operators which we will use to embed several copies of Ψ in order to build the final frame. Fortunately, a BIBD$(v,2,1)$ can be constructed by taking $\mathcal{V}=\{1,\ldots,v\}$ and letting $\mathcal{B}$ be the set of all two element subsets of $\mathcal{V}$. The RBIBD$(k^2,k,1)$ is more difficult to construct.

4. AFFINE PLANES AND THE MATRIX Ψ

An RBIBD$(k^2,k,1)$ is equivalent to an affine plane of order k. Indeed, if k is a power of a prime, then we can take $\mathcal{V}$ to be the two-dimensional vector space over the field of order k, and let $\mathcal{B}$ be the collection of affine lines in $\mathcal{V}$. In this case, the parallel classes are the collections of all lines with a given slope. Unfortunately, there are no known BIBDs with parameters $(k^2,k,1)$ for k which is not a prime power.

For the remainder of this section we will assume $(\mathcal{V},\mathcal{B})$ is a RBIBD$(k^2,k,1)$. We may assume without loss of generality that $\mathcal{V}=\{1,\ldots,k^2\}$. Set $\mathcal{B}=\{B_1,\ldots,B_b\}$, and let

$$\mathcal{B}_1=\{B_1,\ldots,B_k\},\ \mathcal{B}_2=\{B_{k+1},\ldots,B_{2k}\},\ldots,\ \mathcal{B}_{k+1}=\{B_{k^2+1},\ldots,B_{k^2+k}\}$$

be the parallel classes. By permuting the points in $\mathcal{V}$ we may assume

$$B_1=\{1,\ldots,k\},\ B_2=\{k+1,\ldots,2k\},\ldots,\ B_k=\{k^2-k+1,\ldots,k^2\}$$

and

$$B_{k+1}=\{1,k+1,2k+1,\ldots,k^2-k+1\}\quad\text{and}\quad B_{k+1+j}=B_{k+1}+j\quad\text{for } j=1,2,\ldots,k-1.$$

Hence, the incidence matrix has the form

$$A=\begin{bmatrix} I_k\otimes\mathbb{1}_k^\top\\ \mathbb{1}_k^\top\otimes I_k\\ X\end{bmatrix}$$

where I_k is the $k\times k$ identity matrix, and $\mathbb{1}_k$ is the $k\times 1$ all-ones vector. Consider X as a $(k-1)\times k$ block matrix, and let $X_{i,j}$ denote the $k\times k$ matrix in row i, column j. The submatrix $[X_{i,1}\ X_{i,2}\ \cdots\ X_{i,k}]$ consists of the rows indexed by the parallel class $\mathcal{B}_{j+2}$. Hence, the rows of this submatrix are orthogonal. Moreover, the rows of $X_{i,j}$ are orthogonal (in fact, they have disjoint support) for each i,j, that is, $X_{i,j}X_{i,j}^\top$ is a diagonal matrix. This implies that all of the entries on the diagonal of $X_{i,j}^\top X_{i,j}$ are in $\{0,1\}$. From Proposition 3.2 we have

$$kI_{k^2}+J_{k^2}=A^\top A=\begin{bmatrix} I_k\otimes\mathbb{1}_k & \mathbb{1}_k\otimes I_k & X^\top\end{bmatrix}\begin{bmatrix} I_k\otimes\mathbb{1}_k^\top\\ \mathbb{1}_k^\top\otimes I_k\\ X\end{bmatrix}=(I_k\otimes J_k)+(J_k\otimes I_k)+X^\top X. \tag{4.1}$$

From this we deduce that the jth $k\times k$ diagonal block in $X^\top X$ is $(k-1)I_k$, that is,

$$\sum_{i=1}^{k-1}X_{i,j}^\top X_{i,j}=(k-1)I.$$

Note that each $X_{i,j}$ has entries in $\{0,1\}$, and hence each $X_{i,j}^\top X_{i,j}$ must be a diagonal matrix. We already deduced that the diagonal entries of $X_{i,j}^\top X_{i,j}$ are in $\{0,1\}$, thus $X_{i,j}^\top X_{i,j} = I$ for each i,j, that is, $X_{i,j}$ is a permutation matrix.

Again, from (4.1) we see that for $i \neq j$, the (i,j)th $k \times k$ block in $X^\top X$ is $J_k - I_k$, and hence

$$\sum_{n=1}^{k-1} X_{n,i}^\top X_{n,j} = J_k - I_k.$$

Since $X_{n,i}^\top X_{n,j}$ is a permutation for each n, we conclude that $X_{n,i}^\top X_{n,j}$ has zero diagonal for each n. Fix i and consider the sum

$$Z = \sum_{n=1}^{k} X_{i,n}.$$

Assume some entry of Z is greater than 1, then there is a n_0 such that the same entry in X_{i,n_0} is 1. Multiplying by $X_{i,n_0}^\top$ we obtain

$$X_{i,n_0}^\top Z = \sum_{n=1}^{k} X_{i,n_0}^\top X_{i,n} = I_k + \sum_{n \neq n_0} X_{i,n_0}^\top X_{i,n}.$$

Every term in the sum on the right has zero diagonal, and hence $X_{i,n_0}^\top Z$ has constant 1 on the diagonal. Multiplying by X_{i,n_0} we deduce that Z has a 1 in every entry where X_{i,n_0} is equal to 1. This contradiction shows that every entry of Z is either 0 or 1. Since Z is the sum of k permutation matrices of size $k \times k$, we conclude that Z is the all-ones matrix. Hence we have

$$\sum_{n=1}^{k} X_{i,n} = J_k \quad \text{for all } i. \tag{4.2}$$

By a similar argument we also obtain

$$\sum_{n=1}^{k} X_{i,n} X_{j,n}^\top = J_k \quad \text{for all } i \neq j. \tag{4.3}$$

From (4.3) and the previous observation that each $X_{i,j}$ is a permutation matrix we can conclude that $XX^\top = kI_{k^2-k} + J_{k^2-k} - (I_{k-1} \otimes J_k)$.

Let S be an ETF with k vectors in $\mathbb{R}^{k-1}$ such that $S^\top S = kI_k - J_k$. This can be constructed as follows: Let $R = \mathbb{1}_k^\top$, and let S be a $(k-1) \times k$ matrix so that the rows of R and S form an equal norm orthogonal basis, then S is the desired ETF. It follows that $SS^\top = kI_{k-1}$.

Next, we claim that $\Psi := (I_{k-1} \otimes S)X$ is a two-distance tight frame. First we calculate

$$\begin{aligned}
\Psi\Psi^\top &= (I_{k-1} \otimes S)X\big((I_{k-1} \otimes S)X\big)^\top = (I_{k-1} \otimes S)XX^\top(I_{k-1} \otimes S^\top) \\
&= (I_{k-1} \otimes S)\left(kI_{k^2-k} + J_{k^2-k} - (I_{k-1} \otimes J_k)\right)(I_{k-1} \otimes S^\top) \\
&= k(I_{k-1} \otimes S)(I_{k-1} \otimes S^\top) + (I_{k-1} \otimes S)(J_{k-1} \otimes J_k)(I_{k-1} \otimes S^\top) - (I_{k-1} \otimes S)(I_{k-1} \otimes J_k)(I_{k-1} \otimes S^\top) \\
&= (I_{k-1} \otimes kI_{k-1}) + 0 - (I_{k-1} \otimes SJ_kS^\top) = k^2 I_{k-1}
\end{aligned}$$

which shows that Ψ is a tight frame.

Note that Ψ is a $(k-1)\times k$ block matrix with (i,j)th block equal to $SX_{i,j}$. Hence, the (i,j)th block of $\Psi^\top\Psi$ is given by

$$\begin{aligned}\sum_{n=1}^{k-1} X_{n,i}^\top S^\top S X_{n,j} &= \sum_{n=1}^{k-1} X_{n,i}^\top (kI_k - J_k) X_{n,j} = k\sum_{n=1}^{k-1} X_{n,i}^\top X_{n,j} - \sum_{n=1}^{k-1} X_{n,i}^\top J_k X_{n,j} \\ &= k\sum_{n=1}^{k-1} X_{n,i}^\top X_{n,j} - (k-1)J_k \\ &= \begin{cases} k(J_k - I_k) - (k-1)J_k = J_k - kI_k & i \neq j \\ k(k-1)I_k - (k-1)J_k & i = j, \end{cases}\end{aligned}$$

and hence Ψ is a two-distance set with inner products 1 and $-(k-1)$ and squared norms $(k-1)^2$.

Example 4.1. If A is the incidence matrix of the RBIBD$(9,3,1)$ from Example 3.3, then X is the last 6 rows of A, and

$$\Psi = \begin{bmatrix} \sqrt{2} & -\sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & & & \\ 0 & \sqrt{\frac{3}{2}} & -\sqrt{\frac{3}{2}} & & & \\ & & & \sqrt{2} & -\sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} \\ & & & 0 & \sqrt{\frac{3}{2}} & -\sqrt{\frac{3}{2}} \end{bmatrix} \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & 1 & \cdot \end{bmatrix}$$

is a two-distance tight frame with inner products 1 and -2 and squared norms 4. If we take $a=1$ and apply Theorem 2.3, then we obtain an SRG$(9,4,1,2)$.

5. A NEW CONSTRUCTION

Now that we have constructed the two-distance tight frame Ψ, the goal is to use a BIBD$(k,2,1)$ to construct k embedding operators $\{E_v\}_{v=1}^k$ such that $\Psi = [E_1\Psi\ E_2\Psi\ \cdots\ E_k\Psi]$ is a two-distance tight frame. Let B be the incidence matrix of a BIBD$(k,2,1)$ with one of the 1's in each row replaced with -1. The columns of $B\otimes \mathbb{1}_{k-1}$ are indexed by $\mathcal{V}$. Hence, for each $V\in\mathcal{V}$ we construct a $\frac{(k-2)^2(k-1)}{2}\times 2(k-1)$ matrix E_V so that each column of E_V has exactly one non-zero entry, and the sum of the columns of E_V is the column of $B\otimes \mathbb{1}_{k-1}$ with index V.

Example 5.1. Consider the BIBD$(3,2,1)$ with incidence matrix $J_3 - I_3$. In this case one example of a matrix B is given by

$$B = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix},$$

and one set of embedding operators would be

$$E_1 = \begin{bmatrix} 0&0&0&0\\0&0&0&0\\1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&-1&0\\0&0&0&-1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} -1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\\0&0&0&0\\0&0&0&0 \end{bmatrix}.$$

Now, take Ψ given in Example 4.1. We claim that the matrix $\Phi = [E_1\Psi\ E_2\Psi\ E_3\Psi]$ is a 27 element two-distance tight frame for $\mathbb{R}^6$ with inner products 1 and -2 and squared norms 4. If we take $a=1$ and apply Theorem 2.3, then we obtain an SRG$(27,16,10,8)$.

This construction gives us our main theorem

Theorem 5.2. *If there exists a BIBD*$(k^2, k, 1)$*, then there exists a two-distance tight frame* $\Phi = \{\varphi_i\}_{i=1}^N$ *for* $\mathbb{R}^d$ *with* $N = k^3$ *and* $d = \frac{k(k-1)^2}{2}$ *with inner products* 1 *and* $-(k-1)$ *and squared norms* $(k-1)^2$*. Consequently, there exists an SRG*$(k^3, (k-1)^2(k+1), (k-2)(k^2-1)+2, (k-2)(k^2-1))$.

Inspecting Brouwer's table of strongly regular graphs[2] it does not appear that this theorem gives any SRGs with parameters for which a graph was not known to exist. In particular, it appears that graphs with parameters as in Theorem 5.2 are known to exist when k is a prime power. These are exactly the k for which a BIBD$(k^2, k, 1)$ is known to exist, and hence they are the only k for which we can apply Theorem 5.2. On the bright side, several of the graphs on the table, for example SRG$(512, 441, 380, 378)$, are only known to exist by a construction using a complicated object such as a generalized quadrangle. By contrast, the only object required in the present construction is an affine plane. It should also be noted that the graph from Theorem 5.2 with $k = 6$ is known to exist. Meanwhile, it is known that no BIBD$(36, 6, 1)$ exists.

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