

EXPERIMENTING WITH SYMPLECTIC HYPERGEOMETRIC MONODROMY GROUPS

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ABSTRACT. We present new computational results for symplectic monodromy groups of hypergeometric differential equations. In particular, we compute the arithmetic closure of each group, sometimes justifying arithmeticity. The results are obtained by extending our earlier algorithms for Zariski dense groups, based on the strong approximation and congruence subgroup properties.

1. INTRODUCTION

This paper is a sequel to [8], which treated symplectic monodromy groups of hypergeometric differential equations as a test case. Deciding arithmeticity of such a group in its Zariski closure is a basic problem (see [20, Section 3.5] and [3, p. 326]). More generally, one asks whether the group is arithmetic, or whether it is *thin*, i.e., Zariski dense but not arithmetic in the ambient algebraic group. This problem has received much attention. It was solved completely for monodromy groups associated with Calabi–Yau manifolds [4, 21, 23], which are 4-dimensional symplectic linear groups over $\mathbb{Q}$. Note also the results of [12], demonstrating thinness of certain orthogonal hypergeometric monodromy groups.

Our approach to all questions emphasizes computer-aided experimentation. We compute the arithmetic closure $\text{cl}(H)$ of a dense group H , the ‘closest’ arithmetic overgroup of H (specifically, we compute the index of $\text{cl}(H)$, and the level of the maximal principal congruence subgroup that $\text{cl}(H)$ contains). Then $\text{cl}(H)$ is used to investigate H . Sometimes we are able to prove that H is arithmetic. Moreover, we produce comprehensive extra information about all symplectic hypergeometric monodromy groups in the degrees considered.

Our methods are based on the strong approximation and congruence subgroup properties for the symplectic group. In Section 2, we extend algorithms developed in [8] for dense subgroups of $\text{Sp}(n, \mathbb{Z})$ to accept dense subgroups of $\text{Sp}(n, \mathbb{Q})$. Section 3 provides relevant background on hypergeometric groups, and details of

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our experimental strategy. Output for all dense hypergeometric monodromy subgroups of the symplectic group of degree 4 is tabulated in Section 4. We discuss this data in light of work by other authors, noting new proofs of arithmeticity and new index calculations. As further illustration, we give sample output for groups of degree 6.

We set down some notation. Input groups for all algorithms are finitely generated. Let $H = \langle S \rangle$ where $S = \{g_1, \dots, g_r\} \subseteq \mathrm{GL}(n, \mathbb{Q})$. The subring of $\mathbb{Q}$ generated by the entries of the g_i and g_i^{-1} will be denoted R . Thus $R = \frac{1}{\mu}\mathbb{Z}$ for a positive integer μ . If m is coprime to μ then the congruence homomorphism φ_m induced by natural surjection $\mathbb{Z} \rightarrow \mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ maps $\mathrm{GL}(n, R)$ into $\mathrm{GL}(n, \mathbb{Z}_m)$.

Throughout, $\mathbb{F}$ is a field and 1_m is the $m \times m$ identity matrix. Let V be the $\mathbb{F}$ -vector space of dimension $n = 2s > 2$, and let Φ be the matrix of a non-degenerate skew-symmetric bilinear form on V with respect to a basis of V . The full symplectic group in $\mathrm{GL}(n, \mathbb{F})$ preserving Φ is denoted $\mathrm{Sp}(\Phi, \mathbb{F})$. If $D \subseteq \mathbb{F}$ is a unital subring then $\mathrm{Sp}(\Phi, D) := \mathrm{Sp}(\Phi, \mathbb{F}) \cap \mathrm{GL}(n, D)$. We write $\mathrm{Sp}(n, D)$ instead of $\mathrm{Sp}(\Phi, D)$ if

$$\Phi = J_n := \begin{pmatrix} 0_s & 1_s \\ -1_s & 0_s \end{pmatrix}.$$

Since $\mathrm{Sp}(\Phi, \mathbb{F})$ and $\mathrm{Sp}(n, \mathbb{F})$ are $\mathrm{GL}(n, \mathbb{F})$ -conjugate, often it suffices to deal with the latter rather than the former group. The shorthand Sp_n stands for the symplectic group when $\mathbb{F}$ and Φ are unimportant.

2. COMPUTING WITH DENSE SUBGROUPS OF SYMPLECTIC GROUPS

In this section we establish the theoretical foundation for our algorithms.

2.1. Strong approximation and computing. Let H be a finitely generated dense subgroup of $\mathrm{Sp}(n, \mathbb{Q})$. The strong approximation theorem guarantees that H surjects onto $\mathrm{Sp}(n, p)$ for almost all primes $p \in \mathbb{Z}$ [17, Corollary 3, Window 9]. Let $\Pi(H)$ be the (finite) set of primes p such that $p \nmid \mu$ and $\varphi_p(H) \neq \mathrm{Sp}(n, p)$. Below we outline how to compute $\Pi(H)$.

In [10] we developed a method to compute the set of primes p such that $\varphi_p(H) \neq \mathrm{SL}(n, p)$ for dense $H \leq \mathrm{SL}(n, \mathbb{Q})$. This relies on testing irreducibility of the adjoint module of H , and the classification of maximal subgroups of $\mathrm{SL}(n, p)$. Something similar could be done for dense subgroups of $\mathrm{Sp}(n, \mathbb{Q})$.

For a dense subgroup H of $\mathrm{SL}(n, \mathbb{Z})$ or $\mathrm{Sp}(n, \mathbb{Z})$, another way to compute $\Pi(H)$ is described in [8, Section 3] (see also [9, Section 2.5]). Here we must know an explicit transvection in H (recall that a transvection $\tau \in \mathrm{GL}(n, \mathbb{F})$ is a unipotent element such that $1_n - \tau$ has rank 1). While an arbitrary dense subgroup of Sp_n may not contain a transvection, the groups in our experiments do.

Proposition 2.1. *Suppose that H is a finitely generated subgroup of $\mathrm{Sp}(n, \mathbb{Q})$ containing a transvection τ . Then H is dense if and only if $\langle \tau \rangle^H$ is absolutely irreducible.*

Proof. The proof of [8, Proposition 3.7] for $H \leq \mathrm{Sp}(n, \mathbb{Z})$ remains valid for $H \leq \mathrm{Sp}(n, R)$. $\square$

Having identified a transvection τ in $H \leq \mathrm{Sp}(n, \mathbb{Q})$, Proposition 2.1 allows us to apply the procedure `IsDense`(H, τ) from [8, Section 3.2] to test density of H . Given dense H , we compute $\Pi(H)$ using `PrimesForDense`(H, τ) from [8, Section 3.2] as follows. Let $\{A_1, \dots, A_{n^2}\}$ be a basis of the enveloping algebra $\langle N \rangle_{\mathbb{Q}}$, where $N = \langle \tau \rangle^H$ and the A_i are words in S . We can find a finite set Π_1 of primes such that the $\varphi_p(A_i)$ are linearly independent and $\varphi_p(1_n - \tau) \neq 0$ for any prime $p \notin \Pi_1$. That is, if $p \notin \Pi_1$ then $\varphi_p(N)$ is absolutely irreducible and contains the transvection $\varphi_p(\tau)$; so $\varphi_p(H) = \mathrm{Sp}(n, p)$ by [8, Theorem 3.2]. Thus $\Pi(H) \subseteq \Pi_1$. We obtain $\Pi(H)$ after checking whether $\varphi_p(H) = \mathrm{Sp}(n, p)$ for each $p \in \Pi_1$. This last step uses recognition algorithms for matrix groups over finite fields [18].

2.2. Integrality and computing the $\mathbb{Z}$ -intercept. Some of our algorithms require us to compute the ‘ $\mathbb{Z}$ -points’ $H_{\mathbb{Z}} := H \cap \mathrm{GL}(n, \mathbb{Z})$ of input $H \leq \mathrm{GL}(n, \mathbb{Q})$. This is possible by the next result.

Lemma 2.2 ([7, Lemma 5.1]). *For a finitely generated subgroup H of $\mathrm{GL}(n, \mathbb{Q})$, the following are equivalent:*

- H is integral, i.e., H is conjugate to a subgroup of $\mathrm{GL}(n, \mathbb{Z})$,
- $|H : H_{\mathbb{Z}}|$ is finite,
- there exists a positive integer d such that dH consists of $\mathbb{Z}$ -matrices.

In [7, Section 5] we explain how to find d if $|H : H_{\mathbb{Z}}|$ is finite. The procedure `IntegralIntercept`(S, d) from [6] then computes a generating set of $H_{\mathbb{Z}}$. However, its practicality is limited. In our experiments, we calculated a transversal of $H_{\mathbb{Z}}$ in H using an orbit algorithm for the multiplication action by H , starting with 1_n . Suppose that g is an image so obtained. We test whether $gh^{-1} \in \mathrm{GL}(n, \mathbb{Z})$ for each known orbit element h . If this happens for some h then g lies in the same coset of $H_{\mathbb{Z}}$ (and will yield a Schreier generator of $H_{\mathbb{Z}}$). If no such h exists then g is a representative of a new coset.

We avail of the following reduction when $|H : H_{\mathbb{Z}}|$ is large. Let σ be an integer whose prime divisors divide the denominators of entries in elements of H ; so $\frac{1}{\sigma}\mathbb{Z} \subseteq R$. Then $H_{\mathbb{Z}} \leq K \leq H$ where $K = H \cap \mathrm{Sp}(n, \frac{1}{\sigma}\mathbb{Z})$. Membership in K is tested by inspection of matrix denominators. We thus divide the transversal length into

two factors, first calculating a transversal of K in H , and then a transversal of $H_{\mathbb{Z}}$ in K .

A potential complication is too many Schreier generators for $H_{\mathbb{Z}}$. Rather than keeping them all, we randomly select about 300 subproducts of Schreier generators for each transversal step (cf. [1]). Conceivably we may not then compute all of $H_{\mathbb{Z}}$, but merely a proper subgroup. At the end we therefore verify, by a calculation in the congruence image modulo the level of $H_{\mathbb{Z}}$ (see Section 2.3), that all Schreier generators lie in the subgroup generated by the chosen set.

2.3. The congruence subgroup property and computing. Suppose that $H \leq \mathrm{Sp}(n, \mathbb{Q})$ is arithmetic, i.e., commensurable with $\mathrm{Sp}(n, \mathbb{Z})$. The congruence subgroup property holds for $\mathrm{Sp}(n, \mathbb{Z})$, so that $H_{\mathbb{Z}}$ contains a principal congruence subgroup $\ker \varphi_r \cap \mathrm{Sp}(n, \mathbb{Z})$ for some modulus $r > 1$. The *level* of H , denoted $M(H)$, is the modulus of the unique maximal principal congruence subgroup in $H_{\mathbb{Z}}$.

Now suppose that H is a finitely generated dense subgroup of $\mathrm{Sp}(n, \mathbb{Z})$. The *arithmetic closure* $\mathrm{cl}(H)$ of H in $\mathrm{Sp}(n, \mathbb{Z})$ is the intersection of all arithmetic subgroups of $\mathrm{Sp}(n, \mathbb{Z})$ containing H (see [8, Section 3.3]). If $H \leq \mathrm{Sp}(n, \mathbb{Q})$ is not necessarily arithmetic, but $H_{\mathbb{Z}}$ is dense, then we set $M(H) = M(\mathrm{cl}(H_{\mathbb{Z}}))$.

The level is a key component of our algorithms for computing with dense $H \leq \mathrm{Sp}(n, \mathbb{Z})$. If $|\mathrm{Sp}(n, \mathbb{Z}) : \mathrm{cl}(H)|$ is not too large, then we may test arithmeticity of H by coset enumeration [15, Chapter 5].

The algorithm `LevelMaxPCS` from [8] returns $M(H)$ for input dense $H \leq \mathrm{Sp}(n, \mathbb{Z})$ and $\Pi(H)$. In practice, for $H \leq \mathrm{Sp}(n, \mathbb{Q})$ such that $H_{\mathbb{Z}}$ is dense, we use `LevelMaxPCS`($H_{\mathbb{Z}}, \Pi(H_{\mathbb{Z}})$); by definition this returns $M(H)$. Certainly $\Pi(H) \subseteq \Pi(H_{\mathbb{Z}})$, but these sets need not coincide. For example, $p \in \Pi(H_{\mathbb{Z}})$ could divide μ .

3. HYPERGEOMETRIC GROUPS

3.1. Background. We adhere mainly to the conventions of [3].

Let $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$, where $a_j, b_k \in \mathbb{C}^\times$ and $a_j \neq b_k$ for $1 \leq j, k \leq n$. A subgroup of $\mathrm{GL}(n, \mathbb{C})$ generated by elements h_∞, h_0 such that $\det(t1_n - h_\infty) = \prod_{j=1}^n (t - a_j)$ and $\det(t1_n - h_0^{-1}) = \prod_{j=1}^n (t - b_j)$ is called a *hypergeometric group*, and denoted $H(a, b)$. It is absolutely irreducible by [3, Proposition 3.3]. The element $h_1 := (h_0 h_\infty)^{-1}$ of $H(a, b)$ is a reflection, i.e., $h_1 - 1_n$ has rank 1.

If $a_j = \exp(2\pi i \alpha_j)$ and $b_j = \exp(2\pi i \beta_j)$ for $\alpha_j, \beta_j \in \mathbb{C}$, then $H(a, b)$ is the monodromy group of a hypergeometric differential equation [3, Proposition 3.2].

Theorem 3.1 ([3, Theorem 3.5]). *For a_j, b_k as above, let*

$$f(t) = \prod_{j=1}^n (t - a_j) = t^n + A_1 t^{n-1} + \cdots + A_n$$

and

$$g(t) = \prod_{j=1}^n (t - b_j) = t^n + B_1 t^{n-1} + \cdots + B_n.$$

Further, let

$$A = \begin{pmatrix} 0 & \cdots & 0 & -A_n \\ 1 & \cdots & 0 & -A_{n-1} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & -A_1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \cdots & 0 & -B_n \\ 1 & \cdots & 0 & -B_{n-1} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & -B_1 \end{pmatrix}.$$

Then $h_\infty = A$, $h_0 = B^{-1}$ generate a hypergeometric group $H(a, b)$ for $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$. Any hypergeometric group with the same a, b is $\mathrm{GL}(n, \mathbb{C})$ -conjugate to this one.

We are concerned with $H(a, b)$ that are

- (i) symplectic,
- (ii) dense in Sp_n ,
- (iii) integral.

There are only finitely many $\mathrm{GL}(n, \mathbb{Q})$ -conjugacy classes of such $H(a, b)$. By [3, Proposition 6.1], $H(a, b)$ is symplectic if and only if $\{a_1, \dots, a_n\} = \{a_1^{-1}, \dots, a_n^{-1}\}$, $\{b_1, \dots, b_n\} = \{b_1^{-1}, \dots, b_n^{-1}\}$, and $\delta := \det(h_1) = 1$ (whence h_1 is a transvection). We remark that $H(a, b)$ need not be dense in Sp_n (by, e.g., [3, Theorem 6.5]). Additionally, $H(a, b)$ is integral if and only if the a_j and b_k are roots of unity. Hence the characteristic polynomials $f(t)$, $g(t)$ of A, B should be products of coprime cyclotomic polynomials. Under these conditions, $H(a, b) \leq \mathrm{Sp}(\Phi, \mathbb{Z})$ for some Φ . Since $H(a, b)$ is absolutely irreducible, Φ is unique up to a scalar multiple.

3.2. Strategy. Assuming that $H(a, b) \leq \mathrm{GL}(n, \mathbb{Q})$ satisfy the requirements (i), (ii), (iii) of Section 3.1, we proceed as follows.

- (I) We list all pairs $f(t), g(t)$ of polynomials of degree n , each of which is the product of coprime cyclotomic polynomials, and such that $\delta = 1$ (for h_∞, h_0 as in Theorem 3.1). Non-dense $H(a, b)$ are excluded by running $\mathrm{IsDense}(H(a, b), h_1)$.
- (II) The matrix Φ of a symplectic form fixed by $H(a, b)$ may be interpreted as a homomorphism between the natural module of $H(a, b)$ and its dual. We use MeatAxe techniques [15, Section 7.5.2] to compute Φ . Next, $g \in \mathrm{GL}(n, \mathbb{Q})$ such that $gJ_n g^\top = \Phi$ is found by simple linear algebra. Then $L = L(a, b) := g^{-1}H(a, b)g \leq \mathrm{Sp}(n, \mathbb{Q})$. (We seek a copy of $H(a, b)$ that

preserves the standard form because it is more convenient for computing; e.g., we have a presentation of $\mathrm{Sp}(n, \mathbb{Z})$ but not of $\mathrm{Sp}(\Phi, \mathbb{Z})$.) Note that $h := g^{-1}h_1g$ is a transvection in L .

Since Φ is not strictly unique, and g can vary by factors stabilizing the form, L depends on choices made. These might also impact $|L : L_{\mathbb{Z}}|$, which we want to keep small for efficiency purposes and to avoid large increases in the number of Schreier generators (remember that $|L : L_{\mathbb{Z}}| < \infty$ is finite by Lemma 2.2). Our code therefore uses heuristics to determine candidates for Φ and g . It calculates $L = \langle T \rangle$ and $\bar{k} = \mathrm{lcm}\{k \mid l^k \in L_{\mathbb{Z}} \forall l \in T\}$ (as a stand-in for $|L : L_{\mathbb{Z}}|$). Then g is chosen so that $\bar{k}$ is minimal.

(III) We compute $L_{\mathbb{Z}}$ (see Section 2.2). Although perhaps $h \notin L_{\mathbb{Z}}$, we can always find a transvection $\lambda = h^k \in L_{\mathbb{Z}}$ for some k .

(IV) We compute

$$\begin{aligned} \Pi(L_{\mathbb{Z}}) &= \text{PrimesForDense}(L_{\mathbb{Z}}, \lambda) \\ \text{LevelMaxPCS}(L_{\mathbb{Z}}, \Pi(L_{\mathbb{Z}})) \\ |\mathrm{Sp}(n, \mathbb{Z}) : \mathrm{cl}(L_{\mathbb{Z}})|. \end{aligned}$$

(V) When $|\mathrm{Sp}(n, \mathbb{Z}) : \mathrm{cl}(L_{\mathbb{Z}})|$ is sufficiently small, we express the generators of $L_{\mathbb{Z}}$ as words in generators of $\mathrm{Sp}(n, \mathbb{Z})$ [16], and try to find $|\mathrm{Sp}(n, \mathbb{Z}) : L_{\mathbb{Z}}|$ by coset enumeration. If this succeeds, i.e., confirms that the indices of $L_{\mathbb{Z}}$ and $\mathrm{cl}(L_{\mathbb{Z}})$ are equal, then we have proved that $L_{\mathbb{Z}}$ and thereby $H(a, b)$ are arithmetic (cf. [19, Theorem 4.1, p. 204]).

As the cost of coset enumeration is bounded below by the index, we restricted our attempts to groups with (presumed) indices less than 10^7 . If the index was expected to be in the range $10^7, \dots, 10^{14}$, then we tried to find an intermediate subgroup $L_{\mathbb{Z}} < U < \mathrm{Sp}(n, \mathbb{Z})$ such that $|U : L_{\mathbb{Z}}| \leq 10^7$. Enumeration was undertaken with a presentation for U found by Reidemeister–Schreier rewriting [15, Chapter 5]. Suitable U are generated by $L_{\mathbb{Z}}$ together with congruence subgroups in $\mathrm{Sp}(n, \mathbb{Z})$ of level dividing the level of $\mathrm{cl}(L_{\mathbb{Z}})$.

By [23, Theorem 1.1], if the leading coefficient of $f(t) - g(t)$ has absolute value at most 2 then $H(a, b)$ is arithmetic in $\mathrm{Sp}(\Phi, \mathbb{Z})$. At least in degree 4, we proved arithmeticity (and computed the level and index) whenever the criterion from [23] applies, and occasionally when it does not. Unfortunately, we lack a method for proving non-arithmeticity if coset enumeration fails.

4. EXPERIMENTAL RESULTS

Our algorithms have been implemented in **GAP** [13]. In this section, we present the complete results of various experiments for $n = 4$ (Table 2), and a sample

for $n = 6$ (Table 3). The results for all 916 groups of degree 6 are available at <https://www.math.colostate.edu/~hulpke/paper/hypergeom6.pdf>.

A group with $Nr \leq 60$ in Table 2 has the same number in [23, Table 1], while $Nr = m \geq 100$ matches number $m - 100$ in [23, Table 2]. The index and arithmeticity questions for these groups were studied previously in [14, 22]. Table 1 gives a correspondence between our numbering and the notation used in those two papers. ‘ $S_x y$ ’ denotes the group in Table x of [22] with line label y . ‘ $HS(d, k)$ ’ denotes the group labeled (d, k) in [14].

1	$HS(1, 2)$	118	$S_{12}(6)$	136	$S_3 7$
101	$S_2 1(1), HS(16, 8)$	119	$S_2 8(2)$	137	$S_4 4$
102	$S_1 1, HS(9, 6)$	120	$S_1 7(1)$	138	$S_4 5$
103	$S_2 2, HS(12, 7)$	121	$S_1 10(4)$	139	$S_3 8$
104	$S_1 2, HS(4, 4)$	122	$S_1 9(3)$	140	$S_4 6$
105	$S_2 3, HS(8, 6)$	123	$S_2 12(6)$	141	$S_4 7(1)$
106	$S_1 3, HS(6, 5)$	124	$S_2 10(4)$	142	$S_3 9(1)$
107	$S_2 4, HS(5, 5)$	125	$S_2 13(7)$	143	$S_4 8(2)$
108	$S_2 5, HS(4, 5)$	126	$S_4 1$	144	$S_3 10(2)$
109	$S_1 4, HS(3, 4)$	127	$S_3 2$	145	$S_4 11(6)$
110	$S_1 5, HS(2, 3)$	128	$S_3 3$	146	$S_3 14(7)$
111	$S_2 6, HS(2, 4)$	129	$S_3 4(4)$	147	$S_3 12(5)$
112	$S_1 6, HS(1, 3)$	130	$S_3 5$	148	$S_4 10(5)$
113	$S_2 7, HS(1, 4)$	131	$S_3 6$	149	$S_3 13(6)$
114	$S_2 11(5)$	132	$S_4 2$	150	$S_4 9(4)$
115	$S_2 9(3)$	133	$S_3 1$	151	$S_3 15(8)$
116	$S_1 8(2)$	134	$S_4 3(3)$		
117	$S_1 11(5)$	135	$S_3 11(3)$		

TABLE 1. Label correspondences

The column ‘Polynomials’ in Table 2 lists $f(t), g(t)$ as in (I) of Section 3.2 (the Nr entries in Table 3 derive from the listing of these polynomials). If $\mu > 1$ then its prime divisors are given in column ‘Mu’. ‘Int’ is $|L : L_{\mathbb{Z}}|$. ‘iLevel’ and ‘iIndex’ are level and index of $\text{cl}(L_{\mathbb{Z}})$ in $\text{Sp}(n, \mathbb{Z})$, respectively. ‘Coeff’ is the absolute value of the leading coefficient of $f(t) - g(t)$. The column ‘Enum’ records whether coset enumeration succeeded. We reiterate that if an enumeration terminates (signified by a tick $\checkmark$) then it returns $|\text{Sp}(n, \mathbb{Z}) : L_{\mathbb{Z}}|$, and the input group is arithmetic. A dash means that coset enumeration failed to terminate; of course, this does not prove that the group is not arithmetic. If the group is thin by an argument of [4] (see [22, Table 2]), so that coset enumeration is sure to fail, then a letter T is given.

In cases where large $|\mathrm{Sp}(n, \mathbb{Z}) : \mathrm{cl}(L_{\mathbb{Z}})|$ implies that coset enumeration is unlikely to succeed, we put a cross. Indices in degree 4 were small enough to attempt coset enumeration for all groups that are not known to be thin. The size of many indices in degree 6 dissuades any attempt at coset enumeration.

We discuss some test groups of interest. Table 2 shows that the groups Nr = 104, 109 are arithmetic; the question is open in [23, Table 3] (rows 5 and 10) but proved in [22, Table 1] (rows 2 and 4).

Our method proves arithmeticity for all groups in Tables 1 and 3 of [22] (which are listed, respectively proved, to be arithmetic there), apart from Nr = 102, 120, 122, 128, 135 (which all have large index, greater than $4 \cdot 10^8$).

The arithmetic groups Nr = 126 and 141 in Table 2 are No.s 1 and 7(1) in [22, Table 4], for which arithmeticity or thinness was unknown. Furthermore, our calculations show that the groups with Nr = 137, 138, 148, 150 would have rather small index if they were arithmetic. Failure of coset enumeration therefore seems to suggest that, unless we have been unlucky, these groups are more likely to be thin than not.

If the polynomial pair includes $(t - 1)^4$, then a special base change (also featuring in [14]) was used, to render the group in $\mathrm{Sp}(4, \mathbb{Z})$. This comprises the basis defined in [11, (2.2)], followed by the two base changes indicated in Remark 1 and Equation (9) of [5]. Thus, it becomes possible to compare our results with those of [14]. We see that $|\mathrm{Sp}(n, \mathbb{Z}) : L|$ is the same for Nr = 1, 104, 109, 110, 112; i.e., the groups $(d, k) = (1, 2), (4, 4), (3, 4), (2, 3), (1, 3)$ in [14, Theorem 4.3]. Hofmann and van Straten were not able to compute the index of $(d, k) = (6, 5)$, whereas we could do so for the corresponding group Nr = 106, confirming their estimate. Although we were unable to prove arithmeticity of the group $(d, k) = (9, 6)$, Nr = 102, the index $2^8 3^{14} 5^2$ that we computed is slightly better than the estimate in [14].

Let H_n, G_n be $H(a, b)$ with $f(t) = (t - 1)^n$ and $g(t) = (t^{n+1} - 1)/(t - 1)$, $(t^{n+1} + 1)/(t + 1)$, respectively. The arithmeticity problem for H_n and G_n was posed at the ‘Workshop on Thin Groups and Super Approximation’, Institute for Advanced Study, Princeton, March 2016. If $n = 4$ then H_n is thin [4]; see row 107 in Table 2, or row 8 in [8, Table 3] for $\mathrm{cl}(H_4)$. The group G_4 is arithmetic [23, Corollary 1.4] (row 1 in [8, Table 3] and row 112 in Table 2).

In degree 6 we proved arithmeticity for a smaller number of the groups not covered by [23, Theorem 1.1]; two notable exceptions are rows 468 and 534 of Table 3. We have not yet solved the arithmeticity problem for G_6 or H_6 . However, the level and index of their arithmetic closures are stated in rows 774 and 838 of Table 3.

Postscript. During revisions of our paper, we became aware of the preprint [2]. Bajpai et al. settle arithmeticity of many symplectic hypergeometric groups in degree 6, using an adaptation of [23]. We tried their method in degrees 4 and 6, and note that sometimes it succeeds when our method fails; but also vice versa.

Nr	Polynomials	μ	Int	iLevel	iIndex	Coeff	Enum
1	$t^4-4t^3+6t^2-4t+1$ $t^4-2t^3+3t^2-2t+1$	1	1	2	2·5	2	✓
2	$(t-1)^2(t+1)^2$ $t^4+2t^3+3t^2+2t+1$	3	2^2	$2\cdot 3^2$	$2^7 3^4 5^2$	2	✓
3	$(t-1)^2(t+1)^2$ $(t^2+1)(t^2+t+1)$	2, 3	3^2	$2^4 3^2$	$2^{10} 3^3 5^2$	1	✓
4	$(t-1)^2(t+1)^2$ $t^4+t^3+t^2+t+1$	5	5	$2\cdot 5^2$	$2^5 3^2 5\cdot 13$	1	✓
5	$t^4-2t^3+3t^2-2t+1$ $(t-1)^2(t+1)^2$	3	2^2	$2\cdot 3^2$	$2^7 3^4 5^2$	2	✓
6	$(t^2-t+1)(t^2+1)$ $(t-1)^2(t+1)^2$	2, 3	3^2	$2^4 3^2$	$2^{10} 3^3 5^2$	1	✓
7	$t^4-t^3+t^2-t+1$ $(t-1)^2(t+1)^2$	5	5	$2\cdot 5^2$	$2^4 3^2 5\cdot 13$	1	✓
8	$t^4+2t^3+3t^2+2t+1$ $t^4+4t^3+6t^2+4t+1$	1	1	2	2·5	2	✓
9	$(t-1)^2(t^2+t+1)$ t^4+2t^2+1	2	2·3	2^4	$2^6 3^2 5$	1	✓
10	$(t-1)^2(t^2+t+1)$ $t^4+t^3+t^2+t+1$	2, 5	$2^3 3\cdot 5^2$	$2^3 5^2$	$2^8 3^3 5^2 13$	2	✓
11	$t^4-2t^3+3t^2-2t+1$ $(t-1)^2(t^2+t+1)$	2	3^2	2^4	$2^8 3^2 5$	1	✓
12	$(t-1)^2(t^2+t+1)$ $(t+1)^2(t^2-t+1)$	2	2	2^4	$2^{11} 3\cdot 5$	2	✓
13	$(t-1)^2(t^2+t+1)$ $(t^2-t+1)(t^2+1)$	1	1	2^2	$2^6 3\cdot 5$	2	✓
14	$(t-1)^2(t^2+t+1)$ t^4+1	2	2	2^3	$2^2 3\cdot 5$	1	✓
15	$(t-1)^2(t^2+t+1)$ $t^4-t^3+t^2-t+1$	1	1	2	2·3	1	✓
16	$(t-1)^2(t^2+t+1)$ t^4-t^2+1	2	3	2^3	$2^3 3\cdot 5$	1	✓
17	t^4+2t^2+1 $t^4+2t^3+3t^2+2t+1$	1	1	2	2·5	2	✓
18	$(t+1)^2(t^2+1)$ $t^4+2t^3+3t^2+2t+1$	1	1	2	2·5	1	✓
19	$t^4+t^3+t^2+t+1$ $t^4+2t^3+3t^2+2t+1$	1	1	1	1	1	✓

Nr	Polynomials	μ	Int	iLevel	iIndex	Coeff	Enum
20	$(t+1)^2(t^2-t+1)$ $t^4+2t^3+3t^2+2t+1$	2	3^2	2^4	$2^7 3^2 5$	1	✓
21	t^4+1 $t^4+2t^3+3t^2+2t+1$	1	1	2	$2 \cdot 5$	2	✓
22	t^4-t^2+1 $t^4+2t^3+3t^2+2t+1$	1	1	2^2	$2^6 3 \cdot 5$	2	✓
23	$t^4+t^3+t^2+t+1$ $(t+1)^2(t^2+t+1)$	1	1	2	$2 \cdot 3$	2	✓
24	$(t-1)^2(t^2+1)$ $t^4-2t^3+3t^2-2t+1$	1	1	2	$2 \cdot 5$	1	✓
25	$(t-1)^2(t^2+1)$ $(t^2-t+1)(t^2+t+1)$	3	3	$2 \cdot 3^2$	$2^5 3 \cdot 5^2$	2	✓
26	$(t-1)^2(t^2+1)$ t^4+1	1	1	2^2	$2^7 3^2 5$	2	✓
27	$(t-1)^2(t^2+1)$ $t^4-t^3+t^2-t+1$	1	1	2	$2 \cdot 3$	1	✓
28	$(t-1)^2(t^2+1)$ t^4-t^2+1	3	2	$2 \cdot 3^2$	$2^5 3 \cdot 5^2$	2	✓
29	t^4+2t^2+1 $t^4+t^3+t^2+t+1$	1	1	2	$2 \cdot 3$	1	✓
30	$t^4-2t^3+3t^2-2t+1$ t^4+2t^2+1	1	1	2	$2 \cdot 5$	2	✓
31	t^4+2t^2+1 $(t+1)^2(t^2-t+1)$	2	$2 \cdot 3$	2^4	$2^6 3^2 5$	1	✓
32	$t^4-t^3+t^2-t+1$ t^4+2t^2+1	1	1	2	$2 \cdot 3$	1	✓
33	$t^4+t^3+t^2+t+1$ $(t+1)^2(t^2+1)$	1	1	2	$2 \cdot 3$	1	✓
34	$(t^2-t+1)(t^2+t+1)$ $(t+1)^2(t^2+1)$	3	3	$2 \cdot 3^2$	$2^5 3 \cdot 5^2$	2	✓
35	t^4+1 $(t+1)^2(t^2+1)$	1	1	2^2	$2^7 3^2 5$	2	✓
36	t^4-t^2+1 $(t+1)^2(t^2+1)$	3	2	$2 \cdot 3^2$	$2^5 3 \cdot 5^2$	2	✓
37	$t^4+t^3+t^2+t+1$ $(t^2+1)(t^2+t+1)$	1	1	2	$2 \cdot 3$	1	✓
38	$(t+1)^2(t^2-t+1)$ $(t^2+1)(t^2+t+1)$	1	1	2^2	$2^6 3 \cdot 5$	2	✓

Nr	Polynomials	μ	Int	iLevel	iIndex	Coeff	Enum
39	t^4+1 $(t^2+1)(t^2+t+1)$	1	1	2^3	$2^2 3 \cdot 5$	1	✓
40	$t^4-t^3+t^2-t+1$ $(t^2+1)(t^2+t+1)$	1	1	2	$2 \cdot 3$	2	✓
41	t^4-t^2+1 $(t^2+1)(t^2+t+1)$	2, 3	$2 \cdot 3$	$2^3 3^2$	$2^6 3^2 5^2$	1	✓
42	$(t+1)^2(t^2-t+1)$ $t^4+t^3+t^2+t+1$	1	1	2	$2 \cdot 3$	1	✓
43	$(t^2-t+1)(t^2+t+1)$ $t^4+t^3+t^2+t+1$	1	1	1	1	1	✓
44	$(t^2-t+1)(t^2+1)$ $t^4+t^3+t^2+t+1$	1	1	2	$2 \cdot 3$	2	✓
45	t^4+1 $t^4+t^3+t^2+t+1$	1	1	2	$2 \cdot 3$	1	✓
46	$t^4-t^3+t^2-t+1$ $t^4+t^3+t^2+t+1$	1	1	2^2	$2^9 3^2$	2	✓
47	t^4-t^2+1 $t^4+t^3+t^2+t+1$	1	1	1	1	1	✓
48	$(t-1)^2(t^2-t+1)$ $t^4-t^3+t^2-t+1$	1	1	2	$2 \cdot 3$	2	✓
49	$t^4-2t^3+3t^2-2t+1$ t^4+1	1	1	2	$2 \cdot 5$	2	✓
50	$t^4-2t^3+3t^2-2t+1$ $t^4-t^3+t^2-t+1$	1	1	1	1	1	✓
51	$t^4-2t^3+3t^2-2t+1$ t^4-t^2+1	1	1	2^2	$2^6 3 \cdot 5$	2	✓
52	t^4+1 $(t+1)^2(t^2-t+1)$	2	2	2^3	$2^2 3 \cdot 5$	1	✓
53	$t^4-t^3+t^2-t+1$ $(t+1)^2(t^2-t+1)$	5	5	$2 \cdot 5^2$	$2^4 3^2 5 \cdot 13$	2	✓
54	t^4-t^2+1 $(t+1)^2(t^2-t+1)$	2	3	2^3	$2^3 3 \cdot 5$	1	✓
55	$t^4-t^3+t^2-t+1$ $(t^2-t+1)(t^2+t+1)$	1	1	1	1	1	✓
56	$(t^2-t+1)(t^2+1)$ t^4+1	1	1	2^3	$2^2 3 \cdot 5$	1	✓
57	$t^4-t^3+t^2-t+1$ $(t^2-t+1)(t^2+1)$	1	1	2	$2 \cdot 3$	1	✓

Nr	Polynomials	μ	Int	iLevel	iIndex	Coeff	Enum
58	$(t^2-t+1)(t^2+1)$ t^4-t^2+1	2, 3	2·3	$2^3 3^2$	$2^6 3^2 5^2$	1	✓
59	$t^4-t^3+t^2-t+1$ t^4+1	1	1	2	2·3	1	✓
60	$t^4-t^3+t^2-t+1$ t^4-t^2+1	1	1	1	1	1	✓
101	$t^4-4t^3+6t^2-4t+1$ $t^4+4t^3+6t^2+4t+1$	1	1	2^{10}	$2^{40} 3^2 5$	8	T
102	$t^4-4t^3+6t^2-4t+1$ $t^4+2t^3+3t^2+2t+1$	1	1	$2 \cdot 3^5$	$2^8 3^{14} 5^2$	6	—
103	$t^4-4t^3+6t^2-4t+1$ $(t+1)^2(t^2+t+1)$	1	1	$2^5 3^2$	$2^{17} 3^6 5^2$	7	T
104	$t^4-4t^3+6t^2-4t+1$ t^4+2t^2+1	1	1	2^6	$2^{20} 3^2 5$	4	✓
105	$t^4-4t^3+6t^2-4t+1$ $(t+1)^2(t^2+1)$	1	1	2^7	$2^{24} 3^2 5$	6	T
106	$t^4-4t^3+6t^2-4t+1$ $(t^2+1)(t^2+t+1)$	1	1	$2^3 3^2$	$2^{10} 3^6 5^2$	5	✓
107	$t^4-4t^3+6t^2-4t+1$ $t^4+t^3+t^2+t+1$	1	1	$2 \cdot 5^3$	$2^8 3^3 5^8 13$	5	T
108	$t^4-4t^3+6t^2-4t+1$ $(t+1)^2(t^2-t+1)$	1	1	2^5	$2^{13} 3 \cdot 5$	5	T
109	$t^4-4t^3+6t^2-4t+1$ $(t^2-t+1)(t^2+t+1)$	1	1	$2^2 3^2$	$2^9 3^5 5^2$	4	✓
110	$t^4-4t^3+6t^2-4t+1$ $(t^2-t+1)(t^2+1)$	1	1	2^3	$2^6 3 \cdot 5$	3	✓
111	$t^4-4t^3+6t^2-4t+1$ t^4+1	1	1	2^4	$2^{11} 3^2 5$	4	T
112	$t^4-4t^3+6t^2-4t+1$ $t^4-t^3+t^2-t+1$	1	1	2	2·3	3	✓
113	$t^4-4t^3+6t^2-4t+1$ t^4-t^2+1	1	1	2^2	$2^5 5$	4	T
114	$(t-1)^2(t^2+t+1)$ $t^4+4t^3+6t^2+4t+1$	1	1	2^5	$2^{13} 3 \cdot 5$	5	T
115	$(t-1)^2(t^2+1)$ $t^4+4t^3+6t^2+4t+1$	2	2^3	2^8	$2^{27} 3^2 5$	6	T
116	t^4+2t^2+1 $t^4+4t^3+6t^2+4t+1$	1	1	2^4	$2^{20} 3^2 5$	4	✓

Nr	Polynomials	μ	Int	iLevel	iIndex	Coeff	Enum
117	$(t^2+1)(t^2+t+1)$ $t^4+4t^3+6t^2+4t+1$	2	2·3	2^4	$2^7 3^2 5$	3	✓
118	$t^4+t^3+t^2+t+1$ $t^4+4t^3+6t^2+4t+1$	1	1	2	2·3	3	✓
119	$(t-1)^2(t^2-t+1)$ $t^4+4t^3+6t^2+4t+1$	3	2^2	$2^5 3^3$	$2^{19} 3^6 5^2$	7	T
120	$t^4-2t^3+3t^2-2t+1$ $t^4+4t^3+6t^2+4t+1$	1	1	$2 \cdot 3^4$	$2^8 3^{14} 5^2$	6	—
121	$(t^2-t+1)(t^2+t+1)$ $t^4+4t^3+6t^2+4t+1$	3	$2^2 3$	$2^2 3^3$	$2^{11} 3^6 5^2$	4	✓
122	$(t^2-t+1)(t^2+1)$ $t^4+4t^3+6t^2+4t+1$	2, 3	$2^3 3$	$2^4 3^3$	$2^{13} 3^7 5^2$	5	—
123	t^4+1 $t^4+4t^3+6t^2+4t+1$	1	1	2^3	$2^{11} 3^2 5$	4	T
124	$t^4-t^3+t^2-t+1$ $t^4+4t^3+6t^2+4t+1$	1	1	$2 \cdot 5^2$	$2^7 3^3 5^8 13$	5	T
125	t^4-t^2+1 $t^4+4t^3+6t^2+4t+1$	1	1	2^2	$2^5 5$	4	T
126	$(t-1)^2(t^2+t+1)$ $(t+1)^2(t^2+1)$	2	$2^2 3$	2^6	$2^{11} 3^2 5$	3	✓
127	$(t-1)^2(t^2+1)$ $t^4+2t^3+3t^2+2t+1$	3	2^2	$2 \cdot 3^2$	$2^7 3^4 5^2$	4	✓
128	$(t-1)^2(t^2-t+1)$ $t^4+2t^3+3t^2+2t+1$	2, 3	$2^2 3^2$	$2^4 3^3$	$2^{13} 3^7 5^2$	5	—
129	$t^4-2t^3+3t^2-2t+1$ $t^4+2t^3+3t^2+2t+1$	1	1	2^4	$2^{16} 3 \cdot 5$	4	✓
130	$(t^2-t+1)(t^2+1)$ $t^4+2t^3+3t^2+2t+1$	2	3^2	2^4	$2^7 3^2 5$	3	✓
131	$t^4-t^3+t^2-t+1$ $t^4+2t^3+3t^2+2t+1$	1	1	1	1	3	✓
132	$(t-1)^2(t^2+1)$ $(t+1)^2(t^2+t+1)$	2, 3	2·3	$2^5 3^2$	$2^{14} 3^2 5^2$	5	—
133	t^4+2t^2+1 $(t+1)^2(t^2+t+1)$	2	2·3	$2^4 3$	$2^7 3^4 5^2$	3	✓
134	$(t-1)^2(t^2-t+1)$ $(t+1)^2(t^2+t+1)$	2	2	$2^4 3^2$	$2^{18} 3^8 5^2$	6	—
135	$t^4-2t^3+3t^2-2t+1$ $(t+1)^2(t^2+t+1)$	2, 3	$2^2 3^2$	$2^4 3^2$	$2^{13} 3^7 5^2$	5	—

Nr	Polynomials	μ	Int	iLevel	iIndex	Coeff	Enum
136	$(t+1)^2(t^2+t+1)$ $(t^2-t+1)(t^2+1)$	3	3	$2^2 3^2$	$2^{10} 3^3 5^2$	4	✓
137	$(t+1)^2(t^2+t+1)$ t^4+1	1	1	$2^2 3$	$2^2 3^3 5^2$	3	—
138	$t^4-t^3+t^2-t+1$ $(t+1)^2(t^2+t+1)$	5	5	$2 \cdot 5^2$	$2^4 3^2 5 \cdot 13$	4	—
139	t^4-t^2+1 $(t+1)^2(t^2+t+1)$	2	3	$2^2 3$	$2^3 3^3 5^2$	3	✓
140	$(t-1)^2(t^2+1)$ $t^4+t^3+t^2+t+1$	2, 5	$2^3 3 \cdot 5^2$	$2^3 5^2$	$2^8 3^3 5^2 13$	3	—
141	$(t-1)^2(t^2+1)$ $(t+1)^2(t^2-t+1)$	2	3	2^3	$2^9 3^2 5$	3	✓
142	$(t-1)^2(t^2-t+1)$ t^4+2t^2+1	2	2·3	$2^4 3$	$2^7 3^4 5^2$	3	✓
143	$(t-1)^2(t^2-t+1)$ $(t+1)^2(t^2+1)$	3	3	$2^3 3^2$	$2^{13} 3^2 5^2$	5	—
144	$t^4-2t^3+3t^2-2t+1$ $(t+1)^2(t^2+1)$	3	2^2	$2 \cdot 3^2$	$2^7 3^4 5^2$	4	✓
145	$t^4-t^3+t^2-t+1$ $(t+1)^2(t^2+1)$	2, 5	$2^3 3 \cdot 5$	$2^2 5$	$2^7 3^3 5 \cdot 13$	3	—
146	$(t-1)^2(t^2-t+1)$ $(t^2+1)(t^2+t+1)$	3	3	$2^2 3^2$	$2^{10} 3^3 5^2$	4	✓
147	$t^4-2t^3+3t^2-2t+1$ $(t^2+1)(t^2+t+1)$	2	3^2	2^4	$2^8 3^2 5$	3	✓
148	$(t-1)^2(t^2-t+1)$ $t^4+t^3+t^2+t+1$	5	5	$2 \cdot 5^2$	$2^5 3^2 5 \cdot 13$	4	—
149	$t^4-2t^3+3t^2-2t+1$ $t^4+t^3+t^2+t+1$	1	1	1	1	3	✓
150	$(t-1)^2(t^2-t+1)$ t^4+1	2	2	$2^3 3$	$2^3 3^3 5^2$	3	—
151	$(t-1)^2(t^2-t+1)$ t^4-t^2+1	2	3	$2^3 3$	$2^3 3^3 5^2$	3	✓

Table 2: Degree 4

Nr	Polynomials	Mu	Int	iLevel	iIndex	Coeff	Enum
158	$(t^2-t+1)(t^2+1)^2$ $(t^2+t+1)(t^4-t^3+t^2-t+1)$	2	3	2^3	$2^4 3^3 7$	1	✓
162	$(t-1)^2(t+1)^2(t^2+1)$ t^6-t^3+1	3	3	$2 \cdot 3^2$	$2^4 3 \cdot 7^2 13$	1	—
167	$(t-1)^2(t+1)^2(t^2+1)$ $(t^2+t+1)(t^4-t^2+1)$	3	2^2	$2 \cdot 3$	$2^9 3^3 5 \cdot 7^2 13$	1	×
390	$t^6-t^5+t^4-t^3+t^2-t+1$ $(t+1)^2(t^4-t^3+t^2-t+1)$	2, 7	$2^6 3 \cdot 5 \cdot 7^2$	$2^3 7^2$	$2^{14} 3^5 5 \cdot 7^2 19 \cdot 43$	2	×
394	$(t-1)^2(t^2-t+1)(t^2+1)$ $t^6-t^5+t^4-t^3+t^2-t+1$	2	$2^3 3 \cdot 5 \cdot 7$	2^3	$2^8 3^3 5 \cdot 7$	2	✓
437	$t^6+t^5+t^4+t^3+t^2+t+1$ $(t+1)^2(t^2+t+1)^2$	1	1	2	$2^5 3^2$	3	—
468	$t^6-t^5+t^4-t^3+t^2-t+1$ $(t^2+t+1)(t^4+t^3+t^2+t+1)$	1	1	1	1	3	✓
534	$(t^2-t+1)(t^4-t^3+t^2-t+1)$ $t^6+t^5+t^4+t^3+t^2+t+1$	1	1	1	1	3	✓
774	$t^6-6t^5+15t^4-20t^3+15t^2-6t+1$ $t^6-t^5+t^4-t^3+t^2-t+1$	1	1	2	$2^2 3^2$	5	—
819	$t^6-6t^5+15t^4-20t^3+15t^2-6t+1$ t^6-t^3+1	1	1	$2 \cdot 3$	$2^3 3^5 5 \cdot 7^2 13$	6	—
838	$t^6-6t^5+15t^4-20t^3+15t^2-6t+1$ $t^6+t^5+t^4+t^3+t^2+t+1$	1	1	$2 \cdot 7^2$	$2^{15} 3^6 5^2 7^{22} 19 \cdot 43$	7	×

Table 3: Degree 6

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