NOTIONS OF DIRICHLET PROBLEM FOR FUNCTIONS OF LEAST GRADIENT IN METRIC MEASURE SPACES

RIIKKA KORTE, PANU LAHTI, XINING LI, NAGESWARI SHANMUGALINGAM

Abstract. We study two notions of Dirichlet problem associated with BV energy minimizers (also called functions of least gradient) in bounded domains in metric measure spaces whose measure is doubling and supports a $(1,1)$-Poincaré inequality. Since one of the two notions is not amenable to the direct method of the calculus of variations, we construct, based on an approach of [23, 29], solutions by considering the Dirichlet problem for p-harmonic functions, $p > 1$, and letting $p \to 1$. Tools developed and used in this paper include the *inner* perimeter measure of a domain.

Contents

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1. Introduction

Existence, uniqueness, continuity, and stability of solutions to the Dirichlet problem for p-harmonic functions in metric measure space setting is now reasonably well understood when $1 < p < \infty$. The corresponding problem for $p = 1$, that is, finding a BV function of least gradient in the given domain, with prescribed trace on the boundary, is not well understood. Part of the problem is that without additional curvature restrictions for the boundary of the given domain, solutions to the Dirichlet problem, where the *trace* of the BV function is prescribed, are known to not always exist. Thus alternate notions of Dirichlet problem for the least gradient functions need to be explored. Based on the notion of Dirichlet problem set forth in [16], in [17] a notion of Dirichlet problem was proposed ([17] considers the area functional, but the results are easily applicable to the total variation functional). It was shown in [17] that for a wide class of domains in metric measure spaces equipped with a doubling measure supporting a $(1, 1)$-Poincaré inequality, solutions always exist if the boundary data are themselves given by a BV function. The notion proposed there required extension of the BV solution to the exterior of the domain of the problem.

In this paper we discuss an alternate notion of the Dirichlet problem for least gradient functions that does not require extension of the BV solution to the complement of the domain of interest. The boundary data is given by a fixed Lipschitz function. However, unlike in [17], the direct method of the calculus of variations does not yield existence of solutions for this notion of the Dirichlet problem. Thus an alternate method of verifying existence needs to be adopted. In [23, Theorem 3.1] it was shown, using the tools of viscosity solutions, that the limit of a sequence of p-harmonic functions in a Euclidean domain, as $p \to 1$, must be a function of least gradient. In the recent paper [29] it was shown that such a limit function, again in the Euclidean setting, satisfies the notion of Dirichlet problem considered in this paper. The key tool used in [29] is the divergence theorem. In our setting of metric measure spaces we do not have access to the divergence theorem nor notions of viscosity solutions. We instead employ a careful study of *inner trace* of BV functions for a class of domains.

We start by showing that if there is a sequence u_{p_k} of p_k-harmonic functions with $(p_k)_k$ a monotone decreasing sequence of real numbers larger than 1 such that $\lim_k p_k = 1$, and u_{p_k} converges to u in L^1, then the limit function u is a function of least gradient, see Theorem 3.3. In the case of p-energy with $p > 1$, there is no ambiguity in the sense in which we want to fix the boundary values of the function, if the boundary values are themselves restrictions of Sobolev functions. Note that Lipschitz functions are a priori in the Sobolev class $N^{1,p}$ for each

$1 \leq p < \infty$. However, when $p = 1$ and the solutions are merely functions of bounded variation, it is not clear what notion of the Dirichlet problem is the correct one.

In this paper, we propose two ways of defining solutions to the Dirichlet problem: the first one, described in Definition 4.1(B), is based on minimizing the BV-energy in the closure of the domain. In the second one, given in Definition 4.1(T), extension of solutions to the complement of the domain is not required, but the energy being minimized includes the integral of the jump in the inner trace of the BV function (in comparison with the boundary data) measured with respect to the interior perimeter of the domain.

The drawback of the first approach is that the structure of the underlying space close to the boundary but outside the domain also affects the minimization problem. This phenomenon occurs already in weighted Euclidean spaces; see the discussion following Definition 4.1. On the other hand, the advantage of the first approach is that the energy being minimized is lower semicontinuous with respect to L^1-convergence, and hence existence of solutions can be proven using the direct method of the calculus of variations. In the Euclidean setting, Dirichlet problems related to minimizing convex functionals with linear growth have been studied in [7], and the notion of Dirichlet problem considered there is also equivalent to the notion given by Definition 4.1(B) here. The second approach given in Definition 4.1(T) avoids the impact of the part of the complement of the domain that is near the boundary of the domain, but the drawback is that proving the existence of solutions using the direct method of the calculus of variations is not possible. In the setting of metric measure spaces considered here, we do not even have the tools of divergence or Green's theorem, and hence our proof is more involved.

One benefit of the proof we provide here is that the results hold even in a wider class of Euclidean domains; the standard theory from [29] only consider smooth domains, while [7] considers Euclidean Lipschitz domains.

The structure of this paper is as follows. In Section 2 we explain the notation and definitions of concepts used in this paper. In Section 3 we show that functions that arise as L^1-limits of p-harmonic functions are functions of least gradient, see Theorem 3.3. The focus of the fourth section is to describe the two notions of solution to the Dirichlet problem, see Definition 4.1, while the fifth section gives a way of finding good Lipschitz approximations of BV functions via discrete convolutions. Such discrete convolutions are used in Section 6 to compare the inner perimeter measure $P_+(\Omega, \cdot)$ of the bounded domain Ω with its perimeter measure $P(\Omega, \cdot)$, see Theorem 6.9.

In Section 7, we show that the least gradient functions, obtained as L^1-limits of p-harmonic functions that are solutions to the Dirichlet

problem with the fixed Lipschitz boundary data, are necessarily solutions to the Dirichlet problem defined in Definition 4.1(T) with the same Lipschitz boundary data. This result is Theorem 7.7. For this result, we need some additional assumptions on Ω. More precisely, we need to assume that Ω is of finite perimeter and that at $\mathcal{H}$-a.e. boundary point of Ω the complement of Ω has positive density.

The focus of Section 8 is to show that in addition to perturbing the BV energy to the L^p-energy (via p-harmonic functions), if we also perturb the domain by approximating the domain from outside, then the corresponding p-harmonic solutions have a subsequence that converges to a solution to the Dirichlet problem as given in Definition 4.1(B). While the problem (T) is associated with approximating the domain from inside, the results of Section 8 show that the problem (B) is associated with approximating the domain from outside; see Theorem 8.3. It should be noted that the restrictions placed on the domain in relation to problem (T) as in Section 7 are not needed in Section 8. Finally, in Section 9 we consider alternate notions of functions of least gradient, and show that all these notions coincide. For the convenience of the reader, in the appendix we provide a proof of the fact that the inner perimeter measure $P_+(\Omega, \cdot)$ as considered in Definition 2.23 is indeed a Radon measure.

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2. Preliminaries

Throughout this paper we assume that (X, d, μ) is a complete metric space equipped with a Borel regular outer measure μ that satisfies a doubling property and supports a $(1, 1)$-Poincaré inequality (see definitions below). We assume that X consists of at least 2 points. The doubling property means that there exists a constant $C_d \geq 1$ such that

$$0 < \mu(B(x, 2r)) \leq C_d \mu(B(x, r)) < \infty$$

for every ball $B(x, r) \subset X$. Given a ball $B = B(x, r)$ and $\tau > 0$, we denote by τB the ball $B(x, \tau r)$. In a metric space, a ball does not necessarily have a unique center and radius, but whenever we use the above abbreviation we will consider balls whose center and radii have been pre-specified.

In general, $C \geq 1$ will denote a generic constant whose particular value is not important for the purposes of this paper, and might differ between each occurrence. When we want to specify that a constant C depends on the parameters $a, b, \ldots$, we write $C = C(a, b, \ldots)$. Unless otherwise specified, all constants only depend on the doubling constant C_d and the constants C_P, λ associated with the Poincaré inequality defined below.

A complete metric space with a doubling measure is proper, that is, closed and bounded sets are compact. Since X is proper, for any open set $\Omega \subset X$ we define $\mathrm{Lip}_{\mathrm{loc}}(\Omega)$ to be the space of functions that are Lipschitz in every $\Omega' \Subset \Omega$. Here $\Omega' \Subset \Omega$ means that Ω' is open and that $\overline{\Omega'}$ is a compact subset of Ω. We define other local spaces similarly.

For any set $A \subset X$, and $0 < R < \infty$, the restricted spherical Hausdorff content of codimension 1 is defined by

$$\mathcal{H}_R(A) = \inf \left\{ \sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i} : A \subset \bigcup_{i=1}^{\infty} B(x_i, r_i),\ r_i \leq R \right\}.$$

The codimension 1 Hausdorff measure of a set $A \subset X$ is

$$\mathcal{H}(A) = \lim_{R \to 0} \mathcal{H}_R(A).$$

The codimension 1 Minkowski content of a set $A \subset X$ is defined for any positive Radon measure ν by

$$\nu^+(A) := \liminf_{R \to 0} \frac{\nu\left(\bigcup_{x \in A} B(x, R)\right)}{2R}. \tag{2.1}$$

Definition 2.2. The measure theoretic boundary $\partial^* E$ of a set $E \subset X$ is the set of all points $x \in X$ at which both E and its complement have positive upper density, i.e.

$$\limsup_{r \to 0^+} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} > 0 \quad \text{and} \quad \limsup_{r \to 0^+} \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} > 0.$$

The measure theoretic interior I_E is the set of all points $x \in X$ for which

$$\lim_{r \to 0^+} \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} = 0,$$

and the measure theoretic exterior O_E is the set of all points $x \in X$ for which

$$\lim_{r \to 0^+} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} = 0.$$

Observe that $\partial^* E = X \setminus (I_E \cup O_E)$. Note that when E is open, $E \subset I_E$. See the discussion following (2.13) for more on the relationship between the measure theoretic boundary and the perimeter measure.

A curve is a rectifiable continuous mapping from a compact interval into X.

Definition 2.3. A nonnegative Borel function g on X is an *upper gradient* of an extended real-valued function u on X if for all curves γ on X, we have

$$|u(x) - u(y)| \le \int_\gamma g\,ds, \tag{2.4}$$

where x and y are the end points of γ. We interpret $|u(x)-u(y)| = \infty$ whenever at least one of $|u(x)|$, $|u(y)|$ is infinite.

By replacing X with a set $A \subset X$ and considering curves γ in A, we can talk about a function g being an upper gradient of u in A. Upper gradients were originally introduced in [21].

We define the local Lipschitz constant of a locally Lipschitz function $u \in \operatorname{Lip}_{\mathrm{loc}}(X)$ by

$$\operatorname{Lip} u(x) := \limsup_{r\to 0^+} \sup_{y\in B(x,r)\setminus\{x\}} \frac{|u(y)-u(x)|}{d(y,x)}. \tag{2.5}$$

Then $\operatorname{Lip} u$ is an upper gradient of u, see e.g. [12, Proposition 1.11].

It is easy to check that if $u, v \in \operatorname{Lip}_{\mathrm{loc}}(X)$ and $\alpha, \beta \ge 0$, then we have the subadditivity

$$\operatorname{Lip}(\alpha u + \beta v)(x) \le \alpha \operatorname{Lip} u(x) + \beta \operatorname{Lip} v(x) \quad \text{for every } x \in X. \tag{2.6}$$

Let Γ be a family of curves, and let $1 \le p < \infty$. The p-modulus of Γ is defined by

$$\operatorname{Mod}_p(\Gamma) := \inf \int_X \rho^p\,d\mu$$

where the infimum is taken over all nonnegative Borel functions ρ such that $\int_\gamma \rho\,ds \ge 1$ for every $\gamma \in \Gamma$. If a property fails only for a curve family with p-modulus zero, we say that it holds for p-almost every (a.e.) curve.

Definition 2.7. If g is a nonnegative μ-measurable function on X and (2.4) holds for p-almost every curve, then g is a *p-weak upper gradient* of u. It is known that if u has an upper gradient $g \in L^p_{\mathrm{loc}}(\Omega)$ in Ω, then there exists a minimal p-weak upper gradient of u in Ω, which we always denote by g_u, satisfying $g_u(x) \le g(x)$ for μ-a.e. $x \in \Omega$, for any p-weak upper gradient $g \in L^p_{\mathrm{loc}}(\Omega)$ of u in Ω, see [8, Theorem 2.25].

Remark 2.8. Note that a priori the minimal p-weak upper gradient g_u of u may depend on p. However, if u has a minimal q-weak upper gradient g_0 in Ω with $1 \le q < p$, then $g_0 \le g_u$ μ-a.e. in Ω because a p-weak upper gradient of u is automatically a q-weak upper gradient of u. Also, a minimal p-weak upper gradient in Ω is also a minimal p-weak upper gradient in any open $U \subset \Omega$.

From the results in [12] (see [22] for further exposition on this) it follows that when the measure μ on X is doubling and supports a $(1,1)$-Poincaré inequality, the minimal p-weak upper gradient of a locally Lipschitz function u on Ω is $\operatorname{Lip} u$ for all $1 < p < \infty$.

We consider the following norm

$$\|u\|_{N^{1,p}(X)} := \|u\|_{L^p(X)} + \inf_g \|g\|_{L^p(X)},$$

with the infimum taken over all upper gradients g of u.

Definition 2.9. The substitute for the Sobolev space $W^{1,p}(\mathbb{R}^n)$ in the metric setting is the following Newton-Sobolev space

$$N^{1,p}(X) := \{u : \|u\|_{N^{1,p}(X)} < \infty\}/\sim,$$

where the equivalence relation $\sim$ is given by $u \sim v$ if and only if

$$\|u - v\|_{N^{1,p}(X)} = 0.$$

Similarly, we can define $N^{1,p}(\Omega)$ for any open set $\Omega \subset X$. For more on Newton-Sobolev spaces, we refer to [34, 22, 8].

The p-capacity of a set $A \subset X$ is given by

$$\operatorname{Cap}_p(A) := \inf \|u\|_{N^{1,p}(X)},$$

where the infimum is taken over all functions $u \in N^{1,p}(X)$ such that $u \geq 1$ in A.

Remark 2.10. When μ is doubling and supports a $(1,p)$-Poincaré inequality, then Lipschitz functions are dense in $N^{1,p}(X)$. When X is complete and μ is doubling, even if X does not support a $(1,p)$-Poincaré inequality Lipschitz functions are still dense in $N^{1,p}(X)$; this follows from the deep results in [5].

Next we recall the definition and basic properties of functions of bounded variation on metric spaces, see [30]. See also e.g. [4, 15, 16, 35] for the classical theory in the Euclidean setting. For $u \in L^1_{\mathrm{loc}}(X)$, we define the total variation of u on X to be

$$\|Du\|(X) := \inf\Big\{ \liminf_{i\to\infty} \int_X g_{u_i}\, d\mu : u_i \in \operatorname{Lip}_{\mathrm{loc}}(X), u_i \to u \text{ in } L^1_{\mathrm{loc}}(X)\Big\},$$

where each g_{u_i} is the minimal 1-weak upper gradient of u_i. Note that instead of merely requiring $u_i \to u$ in $L^1_{\mathrm{loc}}(X)$ we could require $u_i - u \to 0$ in $L^1(X)$. It turns out that even with this stricter definition, the norm $\|Du\|(X)$ does not change; see Lemma 5.5. Note also that by [2, Theorem 1.1] and Remark 2.8, we can replace g_{u_i} by the minimal p-weak upper gradient $\operatorname{Lip} u_i$, for $p > 1$.

We say that a function $u \in L^1(X)$ is *of bounded variation*, and denote $u \in \mathrm{BV}(X)$, if $\|Du\|(X) < \infty$. A μ-measurable set $E \subset X$ is said to be of *finite perimeter* if $\|D\chi_E\|(X) < \infty$. The perimeter of E in X is also denoted by

$$P(E, X) := \|D\chi_E\|(X).$$

By replacing X with an open set $U \subset X$ in the definition of the total variation, we can define $\|Du\|(U)$. The BV norm is given by

$$\|u\|_{\mathrm{BV}(U)} := \|u\|_{L^1(U)} + \|Du\|(U).$$

It was shown in [30, Theorem 3.4] that for $u \in \mathrm{BV}(X)$, $\|Du\|$ is the restriction to the class of open sets of a finite Radon measure defined on the class of all subsets of X. This outer measure is obtained from the map $U \mapsto \|Du\|(U)$ on open sets $U \subset X$ via the standard Carathéodory construction. Thus, for an arbitrary set $A \subset X$,

$$\|Du\|(A) := \inf\bigl\{\|Du\|(U) : U \text{ open}, A \subset U\bigr\}.$$

Similarly, if $u \in L^1_{\mathrm{loc}}(U)$ with $\|Du\|(U) < \infty$, then $\|Du\|(\cdot)$ is a finite Radon measure on U.

For any Borel sets $E_1, E_2 \subset X$, we have by [30, Proposition 4.7]

$$P(E_1 \cap E_2, X) + P(E_1 \cup E_2, X) \le P(E_1, X) + P(E_2, X).$$

The proof works equally well for μ-measurable $E_1, E_2 \subset X$ and with X replaced by any open set, and then by approximating an arbitrary set $A \subset X$ from the outside by open sets we obtain

$$P(E_1 \cap E_2, A) + P(E_1 \cup E_2, A) \le P(E_1, A) + P(E_2, A). \tag{2.11}$$

We have the following coarea formula from [30, Proposition 4.2]: if $F \subset X$ is a Borel set and $u \in \mathrm{BV}(X)$, then

$$\|Du\|(F) = \int_{-\infty}^{\infty} P(\{u > t\}, F)\, dt. \tag{2.12}$$

In particular, the map $t \mapsto P(\{u > t\}, F)$ is Lebesgue measurable on $\mathbb{R}$.

We assume that X supports a $(1,1)$-Poincaré inequality, meaning that there are constants $C_P > 0$ and $\lambda \ge 1$ such that for every ball $B(x,r)$, for every locally integrable function u on X, and for every upper gradient g of u, we have

$$\fint_{B(x,r)} |u - u_{B(x,r)}|\, d\mu \le C_P r \fint_{B(x,\lambda r)} g\, d\mu,$$

where

$$u_{B(x,r)} := \fint_{B(x,r)} u\, d\mu := \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u\, d\mu.$$

Given a set $E \subset X$ of finite perimeter, for $\mathcal{H}$-a.e. $x \in \partial^* E$ we have

$$\gamma \le \liminf_{r \to 0^+} \frac{\mu(E \cap B(x,r))}{\mu(B(x,r))} \le \limsup_{r \to 0^+} \frac{\mu(E \cap B(x,r))}{\mu(B(x,r))} \le 1 - \gamma, \tag{2.13}$$

where $\gamma \in (0, 1/2]$ only depends on the doubling constant and the constants in the Poincaré inequality, see [1, Theorem 5.4]. We denote the set of all such points by $\Sigma_\gamma E$.

For any open set $\Omega \subset X$, any μ-measurable set $E \subset X$ with $P(E, \Omega) < \infty$, and any Borel set $A \subset \Omega$, we know that

$$\|D\chi_E\|(A) = \int_{\partial^* E \cap A} \theta_E\, d\mathcal{H}, \tag{2.14}$$

where $\theta_E\colon \Omega\cap\partial^*E \to [\alpha, C_d]$, with $\alpha = \alpha(C_d, C_P, \lambda) > 0$, see [1, Theorem 5.3] and [6, Theorem 4.6].

The *jump set* of $u \in \mathrm{BV}(X)$ is the set

$$S_u := \{x \in X : u^\wedge(x) < u^\vee(x)\},$$

where $u^\wedge(x)$ and $u^\vee(x)$ are the lower and upper approximate limits of u defined respectively by

$$u^\wedge(x) := \sup\left\{t \in \overline{\mathbb{R}} : \lim_{r\to 0^+} \frac{\mu(B(x,r)\cap\{u<t\})}{\mu(B(x,r))} = 0\right\} \tag{2.15}$$

and

$$u^\vee(x) := \inf\left\{t \in \overline{\mathbb{R}} : \lim_{r\to 0^+} \frac{\mu(B(x,r)\cap\{u>t\})}{\mu(B(x,r))} = 0\right\}. \tag{2.16}$$

By [6, Theorem 5.3], the variation measure of a BV function can be decomposed into the absolutely continuous and singular part, and the latter into the Cantor and jump part, as follows. Given an open set $\Omega \subset X$ and $u \in \mathrm{BV}(\Omega)$, we have for any Borel set $A \subset \Omega$

$$\begin{aligned}\|Du\|(A) &= \|Du\|^a(A) + \|Du\|^s(A)\\ &= \|Du\|^a(A) + \|Du\|^c(A) + \|Du\|^j(A)\\ &= \int_A a\,d\mu + \|Du\|^c(A) + \int_{A\cap S_u}\int_{u^\wedge(x)}^{u^\vee(x)} \theta_{\{u>t\}}(x)\,dt\,d\mathcal{H}(x),\end{aligned} \tag{2.17}$$

where $a \in L^1(\Omega)$ is the density of the absolutely continuous part and the functions $\theta_{\{u>t\}}$ are as in (2.14).

Definition 2.18. Let $\Omega \subset X$ be a μ-measurable set and let u be a μ-measurable function on Ω. Let N_Ω be the collection of all points $x \in \partial\Omega$ for which there is some $r > 0$ with $\mu(B(x,r)\cap\Omega) = 0$. A function $T_+u\colon \partial\Omega \setminus N_\Omega \to \mathbb{R}$ is the interior trace of u if for $\mathcal{H}$-a.e. $x \in \partial\Omega$ we have

$$\lim_{r\to 0^+} \fint_{\Omega\cap B(x,r)} |u - T_+u(x)|\,d\mu = 0.$$

Note that if Ω is an open set, then N_Ω is empty. Furthermore, we have $N_{X\setminus\Omega} \subset \partial\Omega \setminus \partial^*\Omega$.

Definition 2.19. Given an open set $U \subset X$, the family $\mathrm{BV}_c(U)$ is the collection of all functions $u \in \mathrm{BV}(X)$ whose support is a compact subset of U. By $\mathrm{BV}_0(U)$ we mean the collection of all functions $u \in \mathrm{BV}(U)$ for which T_+u exists and $T_+u = 0$ $\mathcal{H}$-a.e. in ∂U.

Definition 2.20. Given an open set $\Omega \subset X$ and an open set $U \subset X$, we define

$$P_+(\Omega, U) := \inf\left\{\liminf_{i\to\infty} \int_U g_{\Psi_i}\,d\mu\right\},$$

where each g_{Ψ_i} is the minimal 1-weak upper gradient of Ψ_i in U, and where the infimum is taken over all sequences $(\Psi_i) \subset \mathrm{Lip}_{\mathrm{loc}}(U)$ such that $\Psi_i - \chi_\Omega \to 0$ in $L^1(U)$ and $\Psi_i = 0$ in $U \setminus \Omega$ for each $i \in \mathbb{N}$.

Furthermore, for any $A \subset X$ we let

$$P_+(\Omega, A) := \inf \{P_+(\Omega, U) : U \text{ open}, A \subset U\}.$$

In the Appendix we show that if $P_+(\Omega, X) < \infty$, then $P_+(\Omega, \cdot)$ is a Radon measure on X, which we call the *inner perimeter measure* of Ω.

Note that $P(\Omega, A) \le P_+(\Omega, A)$ for any $A \subset X$. We will show in Section 6 that the two quantities $P(\Omega, X)$ and $P_+(\Omega, X)$ are in fact comparable when Ω is open and bounded and satisfies the *exterior measure density* condition

$$\limsup_{r \to 0^+} \frac{\mu(B(x,r) \setminus \Omega)}{\mu(B(x,r))} > 0 \quad \text{for } \mathcal{H}\text{-a.e. } x \in \partial\Omega. \tag{2.21}$$

Definition 2.22. Let $1 < p < \infty$ and let $\Omega \subset X$ be a nonempty bounded open set with $\mathrm{Cap}_p(X \setminus \Omega) > 0$. A function $u \in N^{1,p}(\Omega)$ is said to be p-harmonic in Ω if whenever $\phi \in N^{1,p}(X)$ with $\phi = 0$ in $X \setminus \Omega$, we have

$$\int_\Omega g_u^p \, d\mu \le \int_\Omega g_{u+\phi}^p \, d\mu.$$

Given $f \in N^{1,p}(X)$, we say that a function u is a p-harmonic solution to the Dirichlet problem in Ω with boundary data f if $u \in N^{1,p}(X)$, u is p-harmonic in Ω, and $u = f$ in $X \setminus \Omega$.

The direct method of the calculus of variation yields existence of p-harmonic solutions to the Dirichlet problem ($p > 1$); see [33, 8] for this fact and for more on p-harmonic functions. If $f \colon \partial\Omega \to \mathbb{R}$ is a Lipschitz function and Ω is bounded, we can extend f to a boundedly supported Lipschitz function on X; such a function is necessarily in $N^{1,p}(X)$ for all $p \ge 1$. Thus we can also talk about solutions to the Dirichlet problem with Lipschitz boundary data $f \colon \partial\Omega \to \mathbb{R}$. In this paper we will always assume that the boundary data is a boundedly supported Lipschitz function on X.

We will often assume that $\mathrm{Cap}_1(X \setminus \Omega) > 0$, because then $\mathrm{Cap}_p(X \setminus \Omega) > 0$ for all $p > 1$. This follows from the fact that if $\mathrm{Cap}_p(X \setminus \Omega) = 0$, then $\|\chi_{X\setminus\Omega}\|_{N^{1,p}(X)} = 0$ by [8, Proposition 1.61], and so $\|\chi_{X\setminus\Omega}\|_{N^{1,1}(X)} = 0$ by Remark 2.8.

Definition 2.23. Let $\Omega \subset X$ be a an open set. We say that a function $u \in \mathrm{BV}(\Omega)$ is a function of least gradient in Ω if whenever $\phi \in \mathrm{BV}_c(\Omega)$, we have

$$\|Du\|(\Omega) \le \|D(u+\phi)\|(\Omega).$$

The principal objects of study in this paper are functions of least gradient as defined above.

3. Convergence to a function of least gradient

In this section we show that if there is an L^1-convergent sequence (u_p) of p-harmonic functions with $p \to 1^+$, then the limit is a function of least gradient. In this section, g_{u_p} always denotes the minimal p-weak upper gradient of $u_p \in N^{1,p}(X)$ on X. If g_p is the minimal 1-weak upper gradient of u_p on X, then for any open set $U \subset X$, by the fact that locally Lipschitz functions are dense in $N^{1,1}(U)$ (see [8, Theorem 5.47]) and by Remark 2.8, we have

$$\|Du_p\|(U) \le \int_U g_p \, d\mu \le \int_U g_{u_p} \, d\mu. \tag{3.1}$$

For a Lipschitz function f, g_f will denote the minimal p-weak upper gradient of f for any $p > 1$. Observe from Remark 2.8 that g_f is indeed independent of the choice of p.

First we note that while we do not know whether a sequence of p-harmonic functions is L^1-convergent as $p \to 1^+$, a convergent subsequence always exists.

Lemma 3.2. *Let $\Omega \subset X$ be a nonempty bounded open set with $\operatorname{Cap}_1(X \setminus \Omega) > 0$, and let $f \in \operatorname{Lip}(X)$ be boundedly supported. For each $p > 1$, let $u_p \in N^{1,p}(X)$ be a p-harmonic function in Ω such that $u_p|_{X\setminus\Omega} = f$. Then there exists a sequence $p_k \to 1^+$ such that $u_{p_k} \to u$ in $L^1(X)$ as $k \to \infty$ for some $u \in \operatorname{BV}(X)$.*

Proof. By the maximum principle for the Dirichlet problem for p-harmonic functions, $\|u_p\|_{L^\infty(X)} \le \|f\|_{L^\infty(X)}$, and so for all $p > 1$

$$\begin{aligned} \|u_p\|_{L^1(X)} &\le \|u_p\|_{L^\infty(X)}\mu(\Omega) + \|f\|_{L^1(X\setminus\Omega)} \\ &\le \|f\|_{L^\infty(X)}\mu(\Omega) + \|f\|_{L^1(X)} < \infty. \end{aligned}$$

Let L be the global Lipschitz constant of f. Then

$$\begin{aligned} \int_\Omega g_{u_p} \, d\mu \le \left(\int_\Omega g_{u_p}^p \, d\mu \right)^{1/p} \mu(\Omega)^{1-1/p} &\le \left(\int_\Omega g_f^p \, d\mu \right)^{1/p} \mu(\Omega)^{1-1/p} \\ &\le L\mu(\Omega)^{1-1/p}. \end{aligned}$$

On the other hand,

$$\int_{X\setminus\Omega} g_{u_p} \, d\mu = \int_{X\setminus\Omega} g_f \, d\mu,$$

see [8, Lemma 2.19]. Thus by (3.1),

$$\|Du_p\|(X) \le L\mu(\Omega)^{1-1/p} + \int_{X\setminus\Omega} g_f \, d\mu.$$

We conclude that the sequence $(u_p)_p$ is a bounded sequence in $\operatorname{BV}(X)$, and so by the compact embedding given in [30, Theorem 3.7], a subsequence converges in $L^1_{\mathrm{loc}}(X)$ and hence in $L^1(X)$ to some function

$u \in L^1(X)$, and by the lower semicontinuity of the total variation, we have $u \in \mathrm{BV}(X)$. □

Theorem 3.3. *Let $\Omega \subset X$ be a nonempty bounded open set with $\mathrm{Cap}_1(X \setminus \Omega) > 0$, and let $f \in \mathrm{Lip}(X)$ be boundedly supported. For each $p > 1$ let $u_p \in N^{1,p}(X)$ be a p-harmonic function in Ω such that $u_p|_{X\setminus\Omega} = f$. Suppose that $(u_p)_{p>1}$ is a sequence of such p-harmonic functions and that $u_p \to u$ in $L^1(X)$ as $p \to 1^+$. Then u is a function of least gradient in Ω.*

Proof. By the proof of Lemma 3.2, we have $u \in \mathrm{BV}(X)$. Let $\psi \in \mathrm{BV}_c(\Omega)$ and $K := \mathrm{spt}(\psi)$. Clearly

$$\int_\Omega g_{u_p}^p \, d\mu \le \int_\Omega g_f^p \, d\mu \le L^p \mu(\Omega),$$

where L is the global Lipschitz constant of f, and therefore $(g_{u_p}^p)_{1<p<2}$ is uniformly bounded in $L^1(\Omega)$. Consequently, there exists a subsequence, still written as $(g_{u_p}^p)_{p>1}$, and a positive Radon measure of finite mass ν on Ω such that

$$g_{u_p}^p \, d\mu \to d\nu \quad \text{weakly* in } \Omega \text{ as } p \to 1^+.$$

We now choose $\widetilde{K} \Subset \Omega$ such that $K \subset \widetilde{K}$ and $\nu(\partial\widetilde{K}) = 0$. For small enough $\varepsilon > 0$,

$$\widetilde{K}^\varepsilon := \bigcup_{x\in\widetilde{K}} B(x,\varepsilon) \Subset \Omega.$$

We fix $\eta \in \mathrm{Lip}(X)$ such that $0 \le \eta \le 1$,

$$\eta = 1 \text{ in } \widetilde{K}, \quad \eta = 0 \text{ in } X \setminus \widetilde{K}^{\varepsilon/2}, \quad \text{and} \quad g_\eta \le 2/\varepsilon.$$

As $u + \psi \in \mathrm{BV}(\widetilde{K}^\varepsilon)$, there exists a sequence $(\Psi_k) \subset \mathrm{Lip}_{\mathrm{loc}}(\widetilde{K}^\varepsilon)$ such that $\Psi_k \to u + \psi$ in $L^1(\widetilde{K}^\varepsilon)$ and

$$\|D(u+\psi)\|(\widetilde{K}^\varepsilon) = \lim_{k\to\infty} \int_{\widetilde{K}^\varepsilon} g_{\Psi_k} \, d\mu, \tag{3.4}$$

where g_{Ψ_k} is the minimal p-weak upper gradient of Ψ_k in $\widetilde{K}^\varepsilon$, for $p > 1$, see the discussion on page 7. We set

$$\psi_{k,p} := \eta\Psi_k + (1-\eta)u_p.$$

Then $\psi_{k,p} = u_p$ in $X \setminus \widetilde{K}^{\varepsilon/2}$ and $\psi_{k,p} = \Psi_k$ in $\widetilde{K}$. By the Leibniz rule given in [8, Lemma 2.18],

$$\begin{aligned} g_{\psi_{k,p}} &\le g_{\Psi_k}\eta + g_{u_p}(1-\eta) + g_\eta|\Psi_k - u_p| \\ &\le g_{\Psi_k}\chi_{\widetilde{K}^{\varepsilon/2}} + g_{u_p}\chi_{X\setminus\widetilde{K}} + (2/\varepsilon)|\Psi_k - u_p|\chi_{\widetilde{K}^{\varepsilon/2}\setminus\widetilde{K}}. \end{aligned}$$

Since u_p is p-harmonic, we have

$$\begin{aligned}\left(\int_{\widetilde{K}^\varepsilon} g_{u_p}^p \, d\mu\right)^{1/p} &\leq \left(\int_{\widetilde{K}^\varepsilon} g_{\psi_{k,p}}^p \, d\mu\right)^{1/p} \\ &\leq \left(\int_{\widetilde{K}^{\varepsilon/2}} g_{\Psi_k}^p \, d\mu\right)^{1/p} + \left(\int_{\widetilde{K}^\varepsilon \setminus \widetilde{K}} g_{u_p}^p \, d\mu\right)^{1/p} \\ &\qquad + \frac{2}{\varepsilon}\left(\int_{\widetilde{K}^{\varepsilon/2}\setminus\widetilde{K}} |\Psi_k - u_p|^p \, d\mu\right)^{1/p}.\end{aligned}$$

Therefore, by lower semicontinuity and (3.1) we get

$$\begin{aligned}\|Du\|(\widetilde{K}^\varepsilon) &\leq \liminf_{p\to 1^+} \|Du_p\|(\widetilde{K}^\varepsilon) \\ &\leq \liminf_{p\to 1^+} \int_{\widetilde{K}^\varepsilon} g_{u_p} \, d\mu \\ &\leq \liminf_{p\to 1^+} \mu(\widetilde{K}^\varepsilon)^{1-1/p}\left(\int_{\widetilde{K}^\varepsilon} g_{u_p}^p \, d\mu\right)^{1/p} \\ &\leq \int_{\widetilde{K}^{\varepsilon/2}} g_{\Psi_k} \, d\mu + \limsup_{p\to 1^+}\left(\int_{\widetilde{K}^\varepsilon\setminus\widetilde{K}} g_{u_p}^p \, d\mu\right)^{1/p} \\ &\qquad + \frac{2}{\varepsilon}\int_{\widetilde{K}^{\varepsilon/2}\setminus\widetilde{K}} |\Psi_k - u| \, d\mu,\end{aligned}$$

which in turn leads to

$$\|Du\|(\widetilde{K}) \leq \int_{\widetilde{K}^{\varepsilon/2}} g_{\Psi_k} \, d\mu + \nu(\overline{\widetilde{K}^\varepsilon} \setminus \widetilde{K}) + \frac{2}{\varepsilon}\int_{\widetilde{K}^{\varepsilon/2}\setminus\widetilde{K}} |\Psi_k - u| \, d\mu.$$

Letting $k \to \infty$, we get by (3.4)

$$\|Du\|(\widetilde{K}) \leq \|D(u+\psi)\|(\widetilde{K}^\varepsilon) + \nu(\overline{\widetilde{K}^\varepsilon} \setminus \widetilde{K}).$$

Then letting $\varepsilon \to 0$, by the fact that $\nu(\partial\widetilde{K}) = 0$ we get

$$\|Du\|(\widetilde{K}) \leq \|D(u+\psi)\|(\widetilde{K}).$$

The claim follows from this. □

4. Definitions of the Dirichlet problem for $p = 1$

The focus of this paper is to show that the limit of p-harmonic functions with Lipschitz boundary data f, as $p \to 1^+$, solves a reasonable notion of a Dirichlet problem with boundary data f. The issue is to give such a notion. In the case of the p-energy, there is no ambiguity in the sense in which we want to fix the boundary values of the function, if the boundary values are themselves restrictions of Newton-Sobolev functions. In the case $p = 1$, we propose the following two ways of defining solutions to the Dirichlet problem.

Definition 4.1. Let $\Omega \subset X$ be a nonempty bounded open set with $\operatorname{Cap}_1(X \setminus \Omega) > 0$, and let $f \in \operatorname{Lip}(X)$ be boundedly supported. We say that a function u is a solution to the Dirichlet problem for functions of least gradient with boundary data f in the sense of (B) (respectively in the sense of (T)) if it is a solution to the following minimization problem:

(B) Minimize $\|Dv\|(\overline{\Omega})$ over all functions $v \in \operatorname{BV}(X)$ with $v = f$ on $X \setminus \overline{\Omega}$,

(T) Minimize $\|Dv\|(\Omega) + \int_{\partial\Omega} |T_+ v - f|(x)\, dP_+(\Omega, x)$ over all functions $v \in \operatorname{BV}(\Omega)$.

Note that in definition (T), we need to make extra assumptions on Ω to ensure that the boundary integral is well defined. In both definitions, the solution is allowed to have jumps on the boundary of Ω. In definition (B), this is taken into account by including the variation measure from the boundary $\partial\Omega$ as well. The advantage of this approach is that its energy is more straightforward to calculate, and we need fewer assumptions on Ω. The drawback is that contrary to the formulation (T), the structure of the underlying space X close to the boundary but outside Ω also affects the minimization problem. For instance, let X be the Euclidean space $\mathbb{R}^n$ equipped with the Euclidean metric, and let Ω be the unit ball centered at the origin. Let $\alpha \in (0, 1]$ and equip X with the measure

$$d\mu_\alpha := (\chi_\Omega + \alpha \chi_{\mathbb{R}^n \setminus \Omega})\, d\mathcal{L}^n,$$

where $\mathcal{L}^n$ is the n-dimensional Lebesgue measure. It can be shown that for $u \in \operatorname{BV}(X)$, $\|Du\|(\overline{\Omega}) = \|D_{\mathrm{Euc}} u\|(\Omega) + \alpha \|D_{\mathrm{Euc}} u\|(\partial\Omega)$, where $\|D_{\mathrm{Euc}} u\|$ is the total variation with respect to $\mathcal{L}^n$. Similarly, in this setting we have $P_+(\Omega, X) = 2\pi$ but $P(\Omega, X) = 2\alpha\pi$.

5. Discrete convolutions

A tool that is commonly used in analysis on metric spaces is the *discrete convolution*. Given any open set $U \subset X$ and a scale $R > 0$, we can choose a *Whitney-type covering* $\{B_j = B(x_j, r_j)\}_{j=1}^\infty$ of U such that (see e.g. [9, Theorem 3.1])

(1) for each $j \in \mathbb{N}$,

$$r_j = \min\left\{ \frac{\operatorname{dist}(x_j, X \setminus U)}{40\lambda},\, R \right\},$$

(2) for each $k \in \mathbb{N}$, the ball $10\lambda B_k$ intersects at most $C_0 = C_0(C_d, \lambda)$ balls $10\lambda B_j$ (that is, a bounded overlap property holds),

(3) if $10\lambda B_j$ intersects $10\lambda B_k$, then $r_j \le 2r_k$.

Given such a covering of U, we can take a partition of unity $\{\phi_j\}_{j=1}^\infty$ subordinate to the covering, such that $0 \le \phi_j \le 1$, each ϕ_j is a C/r_j-Lipschitz function, and $\operatorname{supp}(\phi_j) \subset 2B_j$ for each $j \in \mathbb{N}$ (see e.g. [9,

Theorem 3.4]). Finally, we can define the *discrete convolution* v of any $u \in L^1_{\mathrm{loc}}(U)$ with respect to the Whitney-type covering by

$$v := \sum_{j=1}^{\infty} u_{5B_j} \phi_j.$$

In general, $v \in \mathrm{Lip}_{\mathrm{loc}}(U)$, and hence $v \in L^1_{\mathrm{loc}}(U)$.

Let v be the discrete convolution of $u \in L^1_{\mathrm{loc}}(U)$ with $\|Du\|(U) < \infty$, with respect to a Whitney-type covering $\{B_j\}_{j\in\mathbb{N}}$ of U at scale R. Then v has a local Lipschitz constant

$$\operatorname{Lip} v \le C_{\mathrm{lip}} \sum_{j=1}^{\infty} \chi_{B_j} \frac{\|Du\|(10\lambda B_j)}{\mu(B_j)}, \tag{5.1}$$

with C_{lip} depending only on the doubling constant of the measure and the constants in the Poincaré inequality, see e.g. the proof of [26, Proposition 4.1]. From this it follows by the bounded overlap property (2) that

$$\int_U \operatorname{Lip} v \, d\mu \le C_0 C_{\mathrm{lip}} \|Du\|(U). \tag{5.2}$$

Moreover (noting that v depends on the scale R),

$$\|v - u\|_{L^1(U)} \to 0 \quad \text{as} \quad R \to 0, \tag{5.3}$$

see the proof of [26, Proposition 4.1]; note that u does not need to be in $L^1(U)$, only in $L^1_{\mathrm{loc}}(U)$.

Now let (v_i) be a sequence of discrete convolutions of $u \in \mathrm{BV}_{\mathrm{loc}}(U)$ with respect to Whitney-type coverings at scales $R_i \searrow 0$. According to [26, Proposition 4.1], we have for some constant $\widetilde{\gamma} \in (0, 1/2]$

$$\begin{aligned} (1-\widetilde{\gamma})u^{\wedge}(y) + \widetilde{\gamma} u^{\vee}(y) &\le \liminf_{i\to\infty} v_i(y) \\ &\le \limsup_{i\to\infty} v_i(y) \le \widetilde{\gamma} u^{\wedge}(y) + (1-\widetilde{\gamma}) u^{\vee}(y) \end{aligned} \tag{5.4}$$

for $\mathcal{H}$-a.e. $y \in U$; recall the definitions of the lower and upper approximate limits from (2.15) and (2.16).

By applying discrete convolutions, we can show that in the definition of the total variation, we can replace convergence in $L^1_{\mathrm{loc}}(\Omega)$ with convergence in $L^1(\Omega)$.

Lemma 5.5. *Let $\Omega \subset X$ be an open set and let $u \in L^1_{\mathrm{loc}}(\Omega)$ with $\|Du\|(\Omega) < \infty$. Then there exists a sequence of functions $(w_i) \subset \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ with $w_i - u \to 0$ in $L^1(\Omega)$ and $\int_\Omega g_{w_i} \, d\mu \to \|Du\|(\Omega)$, where each g_{w_i} is the minimal 1-weak upper gradient of w_i.*

Note that we cannot write $w_i \to u$ in $L^1(\Omega)$, if we do not have $u \in L^1(\Omega)$.

Proof. For every $\delta > 0$, let

$$\Omega_\delta := \{y \in \Omega : \operatorname{dist}(y, X \setminus \Omega) > \delta\}.$$

Fix $\varepsilon > 0$ and $x \in X$, and choose $\delta \in (0,1)$ such that

$$\|Du\|(\Omega \setminus (\Omega_\delta \cap B(x, 1/\delta))) < \varepsilon.$$

Let

$$\eta(y) := \max\left\{0, 1 - \frac{4}{\delta}\operatorname{dist}(y, \Omega_{\delta/2} \cap B(x, 2/\delta))\right\},$$

which is a $4/\delta$-Lipschitz function.

Let each $v_i \in \operatorname{Lip}_{\mathrm{loc}}(\Omega)$ be a discrete convolution of u in Ω, at scale $1/i$. From the definition of the total variation we get a sequence of functions $u_i \in \operatorname{Lip}_{\mathrm{loc}}(\Omega)$ with $u_i \to u$ in $L^1_{\mathrm{loc}}(\Omega)$ and

$$\int_\Omega g_{u_i}\, d\mu \to \|Du\|(\Omega).$$

Now define

$$w_i := \eta u_i + (1-\eta)v_i,$$

so that $w_i - u \to 0$ in $L^1(\Omega)$ by (5.3), and by the Leibniz rule of [8, Lemma 2.18],

$$g_{w_i} \le g_{u_i}\eta + g_{v_i}(1-\eta) + g_\eta |u_i - v_i|.$$

Here g_{w_i}, g_{u_i}, g_{v_i}, and g_η all denote minimal 1-weak upper gradients. Since $g_\eta = 0$ outside $\Omega_{\delta/4} \cap B(x, 4/\delta) \Subset \Omega$, we have $g_\eta |u_i - v_i| \to 0$ in $L^1(\Omega)$, and by also using (5.1), we get

$$\begin{aligned}\limsup_{i\to\infty} \int_\Omega g_{w_i}\, d\mu &\le \limsup_{i\to\infty} \int_\Omega g_{u_i}\, d\mu + \limsup_{i\to\infty} \int_{\Omega\setminus(\Omega_{\delta/2}\cap B(x,2/\delta))} g_{v_i}\, d\mu \\ &\le \|Du\|(\Omega) + C\|Du\|(\Omega \setminus (\Omega_\delta \cap B(x,1/\delta))) \\ &\le \|Du\|(\Omega) + C\varepsilon.\end{aligned}$$

By a diagonalization argument, where we also let $\varepsilon \to 0$ (and hence $\delta \to 0$), we complete the proof. □

6. Comparability of P_+ and P

Recall the definition of $P_+(\Omega, \cdot)$ from Definition 2.20. As shown by the example found in the discussion following Definition 4.1, $P_+(\Omega, \cdot)$ does not necessarily agree with $P(\Omega, \cdot)$. In light of this, the current section aims to compare $P_+(\Omega, \cdot)$ and $P(\Omega, \cdot)$. The main result of this section is Theorem 6.9.

An analog of $P_+(\Omega, X)$ was studied in [32], where it was shown that for certain open sets $\Omega \subset \mathbb{R}^n$, one has $P_+(\Omega, \mathbb{R}^n) = P(\Omega, \mathbb{R}^n)$. More precisely, it was shown that in the Euclidean setting, if an open set $\Omega \subset \mathbb{R}^n$ satisfies $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial^*\Omega) = 0$, then it is possible to find open sets $\Omega_i \Subset \Omega$ with $\Omega = \bigcup_{i\in\mathbb{N}} \Omega_i$ and $\mathcal{H}^{n-1}(\partial\Omega_i) \to P(\Omega, \mathbb{R}^n)$, where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. We obtain in Corollary 6.11 a weak analog of this result. In fact, our corollary is applicable to a wider class of Euclidean domains than the result of [32],

since we can permit the part of the boundary in which Ω is "thin" to be very large.

In the following lemma, we essentially follow an argument that can be found e.g. in [31, p. 67].

Lemma 6.1. *Let $K \subset X$ be compact, and let $\alpha \in [0,1)$ and $\varepsilon > 0$. Take a sequence $(v_i) \subset C(K)$ with $0 \le v_i \le 1$ for every $i \in \mathbb{N}$, and*

$$\limsup_{i\to\infty} v_i(x) \le \alpha$$

for every $x \in K$. Then there exists a convex combination of v_i, denoted by $\widehat{v}$, such that $\widehat{v}(x) \le \alpha + \varepsilon$ for every $x \in K$.

Proof. We have

$$\lim_{i\to\infty} \max\{v_i(x), \alpha\} = \alpha$$

for every $x \in K$. Note that the functions $\max\{v_i(x), \alpha\}$, and the constant function α, are continuous and take values between 0 and 1. Thus for any signed Radon measure ν on K we have by Lebesgue's dominated convergence theorem that

$$\int_K \max\{v_i(x), \alpha\}\, d\nu \to \int_K \alpha\, d\nu.$$

Since K is compact, we have $C(K) = C_c(K)$ and then by the Riesz representation theorem we conclude that $\max\{v_i(x), \alpha\} \to \alpha$ weakly in the space $C(K)$. By Mazur's lemma, see [31, Theorem 3.13], we can find convex combinations of the functions $w_i := \max\{v_i, \alpha\}$, denoted by $\widehat{w}_i$, which converge strongly in the space $C(K)$ to α. In other words, $\widehat{w}_i \to \alpha$ *uniformly* in K. Thus for a sufficiently large choice of $i \in \mathbb{N}$, we have $\widehat{w}_i(x) \le \alpha + \varepsilon$ for all $x \in K$. With $\widehat{w}_i = \sum_{j=1}^N \lambda_{i,j} w_j$ for some $N \in \mathbb{N}$ and the appropriate choice of numbers $\lambda_{i,j} \in [0,1]$ such that $\sum_{j=1}^N \lambda_{i,j} = 1$, we set $\widehat{v} = \sum_{j=1}^N \lambda_{i,j} v_j$. □

Proposition 6.2. *Let $\Omega, U \subset X$ be open sets with $P(\Omega, U) < \infty$, and suppose that there exists $A \subset \partial\Omega \cap U$ with $\mathcal{H}(A) < \infty$ such that*

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r) \setminus \Omega)}{\mu(B(x,r))} > 0 \tag{6.3}$$

for every $x \in \partial\Omega \cap U \setminus A$. Let $U' \Subset U$. Then there exists a sequence $(w_k) \subset \mathrm{Lip}_{\mathrm{loc}}(U)$ such that for each $k \in \mathbb{N}$, $w_k = 0$ in $U' \setminus \Omega$, $w_k \to \chi_\Omega$ in $L^1(U)$ and

$$\limsup_{k\to\infty} \int_U \mathrm{Lip}\, w_k\, d\mu \le C_{in}(P(\Omega, U) + \mathcal{H}(A))$$

for a constant $C_{in} = C_{in}(C_d, C_P, \lambda)$.

Remark 6.4. If $X = \mathbb{R}^2$ (unweighted) and Ω is the slit disk

$$\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \setminus [-1, 0] \times \{0\},$$

and U is the unit disk, we have $P(\Omega, U) = 0$ and the set A can be taken to be the slit. If we add countably many slits, see Example 6.14 below, we still have $P(\Omega, U) = 0$ but $\mathcal{H}(A) = \infty$ and the conclusion of the proposition becomes meaningless.

Proof of Proposition 6.2. Let each v_i, $i \in \mathbb{N}$, be the discrete convolution of χ_Ω with respect to a Whitney-type covering of U at scale $1/i$. We can add to the set A the $\mathcal{H}$-negligible set where (5.4) fails with $u = \chi_\Omega$. Fix $\varepsilon > 0$. We can pick balls $B(x_j, s_j)$ intersecting A with $s_j \le \varepsilon$,

$$A \subset \bigcup_{j\in\mathbb{N}} B(x_j, s_j),$$

and

$$\sum_{j\in\mathbb{N}} \frac{\mu(B(x_j, s_j))}{s_j} \le \mathcal{H}(A) + \varepsilon.$$

Furthermore, we can choose radii $r_j \in [s_j, 2s_j]$ such that

$$P(B(x_j, r_j), X) \le C\frac{\mu(B(x_j, r_j))}{r_j}$$

for each $j \in \mathbb{N}$, see [24, Lemma 6.2].

For brevity, let us write $B_j := B(x_j, r_j)$, $j \in \mathbb{N}$. Then by the subadditivity (2.11) and the lower semicontinuity of perimeter, we have

$$\begin{aligned} P\left(\Omega \setminus \bigcup_{j\in\mathbb{N}} B_j, U\right) &\le P(\Omega, U) + \sum_{j\in\mathbb{N}} P(B_j, X) \\ &\le P(\Omega, U) + C\sum_{j\in\mathbb{N}} \frac{\mu(B_j)}{r_j} \\ &\le P(\Omega, U) + C\mathcal{H}(A) + C\varepsilon. \end{aligned} \tag{6.5}$$

Let each $\breve{v}_i$, $i \in \mathbb{N}$, be the discrete convolution of $\chi_{\Omega\setminus\bigcup_{j\in\mathbb{N}} B_j}$ with respect to the same Whitney-type covering of U at scale $1/i$ used also in defining the functions v_i. By the properties (5.2) and (5.3) of discrete convolutions, we have $\breve{v}_i - \chi_{\Omega\setminus\bigcup_{j\in\mathbb{N}} B_j} \to 0$ in $L^1(U)$ and

$$\|\operatorname{Lip} \breve{v}_i\|_{L^1(U)} \le C_0 C_{\mathrm{lip}} P\left(\Omega \setminus \bigcup_{j\in\mathbb{N}} B_j, U\right) \tag{6.6}$$

for each $i \in \mathbb{N}$. Note that $\breve{v}_i(x) \le v_i(x)$ for every $x \in U$. Thus for every $x \in \partial\Omega \cap U \setminus A$ we have by (5.4)

$$\limsup_{i\to\infty} \breve{v}_i(x) \le \limsup_{i\to\infty} v_i(x) \le \widetilde{\gamma}\chi_\Omega^\wedge(x) + (1 - \widetilde{\gamma})\chi_\Omega^\vee(x) \le 1 - \widetilde{\gamma};$$

note that $\chi_\Omega^\wedge(x) = 0$ for every $x \in \partial\Omega \cap U \setminus A$ by (6.3). Moreover, $\lim_{i\to\infty} \breve{v}_i(x) = 0$ for every $x \in A$, since $\chi_{\Omega\setminus\bigcup_{j\in\mathbb{N}} B_j} = 0$ in a neighborhood of every $x \in A$. Note that $\overline{U'} \setminus \Omega$ is a compact set. Using Lemma 6.1, we find for every $i \in \mathbb{N}$ a convex combination of the functions $\{\breve{v}_k\}_{k=i}^\infty$, denoted by $\widehat{v}_i$, such that

$$\widehat{v}_i(x) \le 1 - \widetilde{\gamma}/2 \qquad \text{for every } x \in \overline{U'} \setminus \Omega. \tag{6.7}$$

Clearly we still have $\widehat{v}_i \in \mathrm{Lip}_{\mathrm{loc}}(U)$ with $\widehat{v}_i - \chi_{\Omega\setminus\bigcup_{j\in\mathbb{N}} B_j} \to 0$ in $L^1(U)$, and by (6.6) and the subadditivity (2.6),

$$\|\operatorname{Lip}\widehat{v}_i\|_{L^1(U)} \le C_0 C_{\mathrm{lip}} P\left(\Omega \setminus \bigcup_{j\in\mathbb{N}} B_j, U\right). \tag{6.8}$$

Next, let

$$\breve{w}_i := \frac{\max\{0, \widehat{v}_i - 1 + \widetilde{\gamma}/2\}}{\widetilde{\gamma}/2}, \qquad i \in \mathbb{N}.$$

Then by (6.7), $\breve{w}_i = 0$ in $\overline{U'} \setminus \Omega$. Again, we still have $\breve{w}_i \in \mathrm{Lip}_{\mathrm{loc}}(U)$ with $\breve{w}_i - \chi_{\Omega\setminus\bigcup_{j\in\mathbb{N}} B_j} \to 0$ in $L^1(U)$, and by (6.5) and (6.8),

$$\begin{aligned}\|\operatorname{Lip}\breve{w}_i\|_{L^1(U)} &\le 2\widetilde{\gamma}^{-1} C_0 C_{\mathrm{lip}} P\left(\Omega \setminus \bigcup_{j\in\mathbb{N}} B_j, U\right)\\ &\le C(P(\Omega, U) + \mathcal{H}(A) + \varepsilon).\end{aligned}$$

We can do the above for each $\varepsilon = 1/k$, $k \in \mathbb{N}$. Denote $\Omega_k := \Omega \setminus \bigcup_{j\in\mathbb{N}} B_j$, with the balls B_j picked corresponding to the choice $\varepsilon = 1/k$. Thus we obtain sequences $\breve{w}_{k,i}$ with $\breve{w}_{k,i} - \chi_{\Omega_k} \to 0$ in $L^1(U)$ as $i \to \infty$. Then for each $k \in \mathbb{N}$ we can pick a sufficiently large $i_k \ge k$ such that

$$\|\breve{w}_{k,i_k} - \chi_{\Omega_k}\|_{L^1(U)} \le 1/k,$$

$$\|\operatorname{Lip}\breve{w}_{k,i_k}\|_{L^1(U)} \le C(P(\Omega, U) + \mathcal{H}(A) + 1/k),$$

and $\breve{w}_{k,i_k} = 0$ in $U' \setminus \Omega$. Since furthermore $\chi_{\Omega_k} - \chi_\Omega \to 0$ in $L^1(U)$ as $k \to \infty$, we have $\breve{w}_{k,i_k} - \chi_\Omega \to 0$ in $L^1(U)$. Finally, we can define $w_k := \breve{w}_{k,i_k}$, $k \in \mathbb{N}$. □

Note that we always have $P(\Omega, U) \le P_+(\Omega, U)$, since the definition of the latter involves a more restricted class of approximating functions. Now we can show the following.

Theorem 6.9. *Let $\Omega \subset X$ be a bounded open set with $P(\Omega, X) < \infty$, and suppose that*

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r) \setminus \Omega)}{\mu(B(x,r))} > 0$$

for $\mathcal{H}$-a.e. $x \in \partial\Omega$. Then $P_+(\Omega, X) < \infty$, and for any open set $U \subset X$, we have $P(\Omega, U) \le P_+(\Omega, U) \le CP(\Omega, U)$.

Proof. Take a bounded open set $U' \subset X$ with $\Omega \Subset U'$. Let $w_k \in \mathrm{Lip}_{\mathrm{loc}}(X)$ be the sequence given by Proposition 6.2 with the choice $U = X$ (note that now $\mathcal{H}(A) = 0$). Then $w_k = 0$ in $U' \setminus \Omega$, and in fact from the proof of Proposition 6.2 it is easy to see that $w_k = 0$ in $X \setminus \Omega$. Then by the definition of $P_+(\Omega, \cdot)$ and the fact that the local Lipschitz constant is an upper gradient,

$$P_+(\Omega, X) \le \liminf_{k\to\infty} \int_X g_{w_k}\, d\mu \le \liminf_{k\to\infty} \int_X \mathrm{Lip}\, w_k\, d\mu \le CP(\Omega, X) < \infty.$$

Therefore $P_+(\Omega, \cdot)$ is a Radon measure on X, see Appendix. For an open set $U \subset X$, the first inequality of the second claim is clear. To prove the second inequality, fix $\varepsilon > 0$. For some $U' \Subset U$ we have $P_+(\Omega, U) \le P_+(\Omega, U') + \varepsilon$. Let $(w_k) \subset \mathrm{Lip}_{\mathrm{loc}}(U)$ be the sequence given by Proposition 6.2. Then

$$\begin{aligned} P_+(\Omega, U) \le P_+(\Omega, U') + \varepsilon &\le \liminf_{k\to\infty} \int_{U'} g_{w_k}\, d\mu + \varepsilon \\ &\le \liminf_{k\to\infty} \int_{U'} \mathrm{Lip}\, w_k\, d\mu + \varepsilon \\ &\le CP(\Omega, U) + \varepsilon. \end{aligned}$$

By letting $\varepsilon \to 0$, we obtain the result. □

To conclude this section, we prove two corollaries of Proposition 6.2 that will not be needed in the sequel, but may be of independent interest. First we need a lemma.

Lemma 6.10. *For any $w \in \mathrm{Lip}_c(X)$,*

$$\int_{-\infty}^{\infty} \mathcal{H}(\partial\{w > t\})\, dt \le C_{co} \int_X \mathrm{Lip}\, w\, d\mu,$$

where C_{co} only depends on the doubling constant of the measure.

Proof. By [27, Proposition 3.5] (which is based on [11]) the following coarea inequality holds: for any $w \in \mathrm{Lip}_c(X)$,

$$\int_{-\infty}^{\infty} \mu^+(\partial\{w > t\})\, dt \le \int_X \mathrm{Lip}\, w\, d\mu.$$

Since $\mathcal{H}(A) \le C_d^3 \mu^+(A)$ for any $A \subset X$ (see e.g. [27, Proposition 3.12]), we obtain the result. □

Corollary 6.11. *Let $\Omega \subset X$ be a bounded open set with $P(\Omega, X) < \infty$, and suppose that there exists $A \subset \partial\Omega$ with $\mathcal{H}(A) < \infty$ such that*

$$\limsup_{r\to 0^+} \frac{\mu(B(x, r) \setminus \Omega)}{\mu(B(x, r))} > 0$$

for every $x \in \partial\Omega \setminus A$. Then there exists a sequence of open sets $\Omega_j \Subset \Omega$ with $\chi_{\Omega_j}(x) \to 1$ for every $x \in \Omega$ and

$$\mathcal{H}(\partial\Omega_j) \le C(P(\Omega, X) + \mathcal{H}(A))$$

for each $j \in \mathbb{N}$.

Proof. Choose an open set U' with $\Omega \Subset U' \Subset X$. Apply Proposition 6.2 with $U = X$ to obtain a sequence $w_k \in \operatorname{Lip}_{\mathrm{loc}}(X)$ with $w_k \to \chi_\Omega$ in $L^1(X)$,

$$\int_X \operatorname{Lip} w_k \, d\mu \le C_{in}(P(\Omega, X) + \mathcal{H}(A)),$$

and $w_k = 0$ in $U' \setminus \Omega$. From the proof of Proposition 6.2 it is easy to see that in fact $w_k = 0$ in $X \setminus \Omega$, so that $w_k \in \operatorname{Lip}_c(X)$ for each $k \in \mathbb{N}$. From the proof of Proposition 6.2 we can also see that $w_k(x) \to 1$ for every $x \in \Omega$, so that for any $t \in (0,1)$, $\chi_{\{w_k > t\}}(x) \to 1$ for every $x \in \Omega$. By Lemma 6.10,

$$\int_0^1 \mathcal{H}(\partial\{w_k > t\})\, dt \le C_{co} \int_X \operatorname{Lip} w_k \, d\mu \le C_{co} C_{in}(P(\Omega, X) + \mathcal{H}(A))$$

for all $k \in \mathbb{N}$. Thus for any fixed $k \in \mathbb{N}$ we find a set $T_k \subset (0,1)$ with $\mathcal{L}^1(T_k) \ge 1/2$ such that for all $t \in T_k$,

$$\mathcal{H}(\partial\{w_k > t\}) \le 2C_{co}C_{in}(P(\Omega, X) + \mathcal{H}(A)).$$

Now, if for every $t \in (0,1)$ there were an index $N_t \in \mathbb{N}$ such that $t \notin T_k$ for all $k \ge N_t$, then by the Lebesgue dominated convergence theorem we would have

$$\int_0^1 \chi_{T_k} \, d\mathcal{L}^1 \to 0,$$

which is a contradiction. Thus there exists $t \in (0,1)$ such that for some subsequence k_j, we have $t \in T_{k_j}$ for all $j \in \mathbb{N}$.

Thus we can define $\Omega_j := \{w_{k_j} > t\}$. □

We know the following fact about the extension of sets of finite perimeter: if $\Omega \subset X$ is an open set with $\mathcal{H}(\partial\Omega) < \infty$ and $E \subset \Omega$ is a μ-measurable set with $P(E, \Omega) < \infty$, then $P(E, X) < \infty$ and in fact

$$P(E, X) \le P(E, \Omega) + C\mathcal{H}(\partial\Omega), \tag{6.12}$$

see [24, Proposition 6.3]. Now we can show a partially more general result.

Corollary 6.13. *Let $\Omega \subset X$ be a bounded open set with $P(\Omega, X) < \infty$, and suppose that there exists $A \subset \partial\Omega$ with $\mathcal{H}(A) < \infty$ such that*

$$\limsup_{r \to 0^+} \frac{\mu(B(x,r) \setminus \Omega)}{\mu(B(x,r))} > 0$$

for every $x \in \partial\Omega \setminus A$. Let $E \subset \Omega$ be a μ-measurable set with $P(E, \Omega) < \infty$. Then $P(E, X) < \infty$.

Proof. Take the sequence of sets $\Omega_j \Subset \Omega$ given by Corollary 6.11. By Lebesgue's dominated convergence theorem, we have $\mu(\Omega \setminus \Omega_j) \to 0$, and so by the lower semicontinuity of perimeter and (6.12), we have

$$\begin{aligned} P(E,X) &\le \liminf_{j\to\infty} P(E\cap\Omega_j, X) \\ &\le \liminf_{j\to\infty} (P(E,\Omega_j) + C\mathcal{H}(\partial\Omega_j)) \\ &\le P(E,\Omega) + C(P(\Omega,X) + \mathcal{H}(A)) < \infty. \end{aligned}$$

□

Example 6.14. Without the requirement of the measure density condition for $X \setminus \Omega$ given in the hypothesis of the above corollary, the conclusion of the corollary fails. For example, with $\mathbb{D} \subset \mathbb{R}^2$ the unit disk in $X = \mathbb{R}^2 = \mathbb{C}$ centered at 0, set $\theta_n = \sum_{j=1}^n \frac{\pi}{2^j}$ and let

$$\Omega := \mathbb{D} \setminus \{z \in \mathbb{C} : \operatorname{Arg}(z) = \theta_n \text{ for some } n \in \mathbb{N}\}.$$

Then $P(\Omega, X) = P(\mathbb{D}, \mathbb{R}^2) < \infty$. Now with

$$E = \bigcup_{n\in\mathbb{N}} \{z \in \mathbb{D} : \theta_{2n} < \operatorname{Arg}(z) < \theta_{2n+1}\},$$

we see that $P(E,\Omega) = 0$, but as $\mathcal{H}(\partial^* E) = \infty$ (note that $\mathcal{H}$ is now comparable to the one-dimensional Hausdorff measure), it follows that $P(E,X) = P(E,\mathbb{R}^2) = \infty$.

7. Dirichlet problem (T): trace definition

In this section we consider the Dirichlet problem (T) given in Definition 4.1. We show that the limit of p-harmonic functions with boundary data f is a solution to this problem.

In the Euclidean setting, it is known that if a bounded domain Ω has a Lipschitz boundary, the trace operator $T_+ \colon \mathrm{BV}(\Omega) \to L^1(\partial\Omega, \mathcal{H})$ is continuous under *strict convergence*, see e.g. [4, Theorem 3.88]. In the following proposition we give a generalization of this fact to the metric setting.

Proposition 7.1. *Let $\Omega \subset X$ be an open set such that the trace operator $T_+ \colon \mathrm{BV}(\Omega) \to L^1(\partial\Omega, \mathcal{H})$ is linear and bounded. Let $u \in \mathrm{BV}(\Omega)$, and let $u_k \in \mathrm{BV}(\Omega)$, $k \in \mathbb{N}$, such that*

$$u_k \to u \ \text{ in } L^1(\Omega) \quad \text{and} \quad \|Du_k\|(\Omega) \to \|Du\|(\Omega).$$

Then $T_+ u_k \to T_+ u$ in $L^1(\partial\Omega, \mathcal{H})$.

Proof. For $t > 0$, let

$$\Omega_t := \{x \in \Omega : \operatorname{dist}(x, X \setminus \Omega) > t\}.$$

Fix $\varepsilon > 0$. Choose $\eta \in \operatorname{Lip}_c(\Omega)$ such that $0 \le \eta \le 1$, $\eta = 1$ in $\Omega_{2\varepsilon}$, $\eta = 0$ in $X \setminus \Omega_\varepsilon$, and $g_\eta \le 1/\varepsilon$. Let

$$v_k := \eta u + (1-\eta)u_k, \quad k \in \mathbb{N}.$$

Note that by lower semicontinuity of the total variation with respect to L^1-convergence,

$$\|Du\|(\Omega_{2\varepsilon}) \le \liminf_{k\to\infty} \|Du_k\|(\Omega_{2\varepsilon}).$$

Since also $\|Du_k\|(\Omega) \to \|Du\|(\Omega)$ by assumption, necessarily

$$\|Du\|(\Omega \setminus \Omega_{2\varepsilon}) \ge \limsup_{k\to\infty} \|Du_k\|(\Omega \setminus \Omega_{2\varepsilon}). \tag{7.2}$$

We have $v_k - u = (1-\eta)(u_k - u)$, and so by the Leibniz rule from [19, Lemma 3.2], and (2.11),

$$\begin{aligned}
&\limsup_{k\to\infty} \|D(v_k - u)\|(\Omega) \\
&\quad\le \limsup_{k\to\infty} \|D(u_k - u)\|(\Omega \setminus \Omega_{2\varepsilon}) + \limsup_{k\to\infty} \int_{\Omega_\varepsilon \setminus \Omega_{2\varepsilon}} g_\eta |u_k - u|\, d\mu \\
&\quad\le \limsup_{k\to\infty} \|Du_k\|(\Omega \setminus \Omega_{2\varepsilon}) + \|Du\|(\Omega \setminus \Omega_{2\varepsilon}) + 0 \\
&\quad\le 2\|Du\|(\Omega \setminus \Omega_{2\varepsilon})
\end{aligned}$$

by (7.2). Moreover, $\lim_{k\to\infty} \|v_k - u\|_{L^1(\Omega)} = 0$, so in total

$$\limsup_{k\to\infty} \|v_k - u\|_{\mathrm{BV}(\Omega)} \le 2\|Du\|(\Omega \setminus \Omega_{2\varepsilon}).$$

Since T_+ is assumed to be linear and bounded, for some constant $C_\Omega > 0$ and for any $v \in \mathrm{BV}(\Omega)$ we have

$$\int_{\partial\Omega} |T_+ v|\, d\mathcal{H} \le C_\Omega \|v\|_{\mathrm{BV}(\Omega)}.$$

Note that $T_+ v_k = T_+ u_k$ since $v_k = u_k$ in a neighborhood of $\partial\Omega$. Thus

$$\begin{aligned}
\limsup_{k\to\infty} \int_{\partial\Omega} |T_+ u_k - T_+ u|\, d\mathcal{H} &= \limsup_{k\to\infty} \int_{\partial\Omega} |T_+ v_k - T_+ u|\, d\mathcal{H} \\
&\le \limsup_{k\to\infty} C_\Omega \|v_k - u\|_{\mathrm{BV}(\Omega)} \\
&\le 2C_\Omega \|Du\|(\Omega \setminus \Omega_{2\varepsilon}).
\end{aligned}$$

By letting $\varepsilon \to 0$, we obtain the result. □

Lemma 7.3. *Let $\Omega \subset X$ be a bounded open set such that Ω satisfies the exterior measure density condition* (2.21)*, Ω supports a $(1,1)$-Poincaré inequality, and there is a constant $C \ge 1$ such that whenever $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\Omega)$, we have*

$$\mu(B(x,r) \cap \Omega) \ge \frac{\mu(B(x,r))}{C}.$$

Assume also that for all $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\Omega)$,

$$\mathcal{H}(\Omega \cap B(x,r)) \le C \frac{\mu(B(x,r))}{r}.$$

Let $f \in \operatorname{Lip}(X)$ be boundedly supported, and let $u \in \mathrm{BV}(\Omega)$. Then there exists a sequence $(\psi_k) \subset \operatorname{Lip}(X)$ converging to u in $L^1(\Omega)$ such that $\psi_k = f$ in $X \setminus \Omega$ and

$$\limsup_{k\to\infty} \|D\psi_k\|(\Omega) \le \|Du\|(\Omega) + \int_{\partial^*\Omega} |T_+u - f|\, dP_+(\Omega, \cdot).$$

Remark 7.4. Note that some requirement similar to the exterior measure density condition in the above lemma is needed, for without such a requirement we cannot talk about the trace T_+u of a function $u \in \mathrm{BV}(\Omega)$. This difficulty is illustrated by the example of the slit disk, see [28, Example 3.2].

Proof. The assumptions on Ω guarantee that the trace operator $T_+\colon \mathrm{BV}(\Omega) \to L^1(\partial\Omega, \mathcal{H})$ is linear and bounded, see [28, Theorem 5.5]. The assumptions also together imply that $\mathcal{H}(\partial\Omega \setminus \partial^*\Omega) = 0$.

Let $(\eta_m) \subset \operatorname{Lip}_{\mathrm{loc}}(X)$ such that $0 \le \eta_m \le 1$ on X, $\eta_m = 0$ on $X \setminus \Omega$, $\eta_m \to \chi_\Omega$ in $L^1(X)$, and

$$P_+(\Omega, X) = \lim_{m\to\infty} \int_\Omega g_{\eta_m}\, d\mu.$$

Clearly we have in fact $\eta_m \in \operatorname{Lip}(X)$ for every $m \in \mathbb{N}$. It is straightforward to check that then also $g_{\eta_m}\, d\mu \to dP_+(\Omega, \cdot)$ weakly* in the sense of measures on X. Since Ω supports a $(1,1)$-Poincaré inequality, Lipschitz functions are dense in $N^{1,1}(\Omega)$, see [8, Theorem 5.1]. It follows that there exists a sequence $(\phi_k) \subset \operatorname{Lip}(\Omega)$ such that $\phi_k \to u$ in $L^1(\Omega)$ and

$$\lim_{k\to\infty} \int_\Omega g_{\phi_k}\, d\mu = \|Du\|(\Omega).$$

By lower semicontinuity of the total variation with respect to L^1-convergence, necessarily also

$$\lim_{k\to\infty} \|D\phi_k\|(\Omega) = \|Du\|(\Omega).$$

Now we set

$$\psi_{k,m} := \eta_m \phi_k + (1 - \eta_m) f.$$

Then $\psi_{k,m} \in \operatorname{Lip}(X)$ and

$$\psi_{k,m} \to u \quad \text{in } L^1(\Omega)$$

as $m \to \infty$ and then $k \to \infty$. Furthermore, $\psi_{k,m} = f$ on $X \setminus \Omega$. By the Leibniz rule of [8, Lemma 2.18],

$$g_{\psi_{k,m}} \le g_{\phi_k}\eta_m + g_f(1 - \eta_m) + g_{\eta_m}|\phi_k - f|.$$

Here $g_{\psi_{k,m}}$, g_{ϕ_k}, g_f, and g_{η_m} all denote minimal 1-weak upper gradients. It follows that

$$\int_\Omega g_{\psi_{k,m}}\, d\mu \le \int_\Omega g_{\phi_k}\, d\mu + \int_\Omega g_f(1 - \eta_m)\, d\mu + \int_\Omega g_{\eta_m}|\phi_k - f|\, d\mu.$$

As f is a Lipschitz function and $\eta_m \to 1$ in $L^1(\Omega)$, we have

$$\lim_{m\to\infty} \int_\Omega g_f(1-\eta_m)\,d\mu = 0.$$

Note that the Lipschitz functions ϕ_k have Lipschitz extensions to X, which we still denote by ϕ_k, and that necessarily $T_+\phi_k = \phi_k$ on $\partial\Omega$. Since $g_{\eta_m}\,d\mu \to dP_+(\Omega,\cdot)$ weakly* in the sense of measures,

$$\lim_{m\to\infty} \int_\Omega |\phi_k - f| g_{\eta_m}\,d\mu = \int_{\partial\Omega} |T_+\phi_k - f|\,dP_+(\Omega,\cdot).$$

It follows from Lemma 7.1 that $T_+\phi_k \to T_+u$ in $L^1(\partial\Omega,\mathcal{H})$, and then by (2.14) and Theorem 6.9, also $T_+\phi_k \to T_+u$ in $L^1(\partial\Omega, P_+(\Omega,\cdot))$. Thus, recalling also that $\mathcal{H}(\partial\Omega\setminus\partial^*\Omega)=0$,

$$\lim_{k\to\infty}\lim_{m\to\infty} \int_\Omega |\phi_k - f| g_{\eta_m}\,d\mu = \int_{\partial^*\Omega} |T_+u - f|\,dP_+(\Omega,\cdot),$$

and now we can choose a diagonal sequence $\{\psi_{k,m_k}\}_k$ to satisfy the conclusion of the lemma. □

In what follows, we denote by T_-u the outer trace (if it exists) of a BV function $u \in \mathrm{BV}(X)$, namely, T_-u is the interior trace of u considered with respect to $X\setminus\Omega$ as given in Definition 2.18. We will only need the following proposition for the case where $u = f$ on $X\setminus\Omega$ for some Lipschitz function f; in this case, we always have $T_-u = f$ on $\partial\Omega\setminus N_{X\setminus\Omega}$, in particular, $T_-u = f$ on $\partial^*\Omega$.

Proposition 7.5. *Let $\Omega\subset X$ be a μ-measurable set with $P(\Omega,X)<\infty$ and let $u\in \mathrm{BV}(X)$ such that for $\mathcal{H}$-almost every $x\in\partial^*\Omega$, $T_+u(x)$ and $T_-u(x)$ exist. Then*

$$\|Du\|(X) = \|Du\|(X\setminus\partial^*\Omega) + \int_{\partial^*\Omega} |T_+u - T_-u|\,dP(\Omega,\cdot).$$

Proof. We only need to prove that

$$\|Du\|(\partial^*\Omega) = \int_{\partial^*\Omega} |T_+u - T_-u|\,dP(\Omega,\cdot).$$

By [6, Theorem 5.3], we have $\|Du\|^c(\partial^*\Omega)=0$, and then by the decomposition (2.17),

$$\|Du\|(\partial^*\Omega) = \int_{\partial^*\Omega}\int_{u^\wedge(x)}^{u^\vee(x)} \theta_{\{u>t\}}(x)\,dt\,d\mathcal{H}(x).$$

It is fairly easy to check that $\{u^\wedge(x),u^\vee(x)\} = \{T_-u(x),T_+u(x)\}$, whenever both traces exist. This is also proved in [18, Proposition 5.8(v)]. Suppose that $u^\wedge(x)=T_-u(x)$ and $u^\vee(x)=T_+u(x)$, the other case being analogous. In the proof of [18, Proposition 5.8(v)] it is also shown that

$$\lim_{r\to0^+} \frac{\mu(B(x,r)\cap(\{u>t\}\Delta\Omega))}{\mu(B(x,r))} = 0$$

for all $t \in (u^\wedge(x), u^\vee(x))$. We also have $x \in \partial^*\{u > t\}$ for all $t \in (u^\wedge(x), u^\vee(x))$. According to [6, Proposition 6.2], we have $\theta_{\{u>t\}}(x) = \theta_\Omega(x)$ for $\mathcal{H}$-almost every such x. Hence we have

$$\begin{aligned}
&\int_{\partial^*\Omega} \int_{u^\wedge(x)}^{u^\vee(x)} \theta_{\{u>t\}}(x)\, dt\, d\mathcal{H}(x) \\
&= \int_{\partial^*\Omega} \int_{-\infty}^{\infty} \chi_{\{(u^\wedge(x),u^\vee(x))\}}(t)\theta_{\{u>t\}}(x)\, dt\, d\mathcal{H}(x) \\
&= \int_{-\infty}^{\infty} \int_{\partial^*\Omega} \chi_{\{(-\infty,t)\}}(u^\wedge(x))\chi_{\{(t,\infty)\}}(u^\vee(x))\theta_{\{u>t\}}(x)\, d\mathcal{H}(x)\, dt \\
&= \int_{-\infty}^{\infty} \int_{\partial^*\Omega} \chi_{\{(-\infty,t)\}}(u^\wedge(x))\chi_{\{(t,\infty)\}}(u^\vee(x))\theta_\Omega(x)\, d\mathcal{H}(x)\, dt \\
&= \int_{\partial^*\Omega} \int_{-\infty}^{\infty} \chi_{\{(u^\wedge(x),u^\vee(x))\}}(t)\, dt\, \theta_\Omega(x)\, d\mathcal{H}(x) \\
&= \int_{\partial^*\Omega} (u^\vee(x) - u^\wedge(x))\theta_\Omega(x)\, d\mathcal{H}(x) \\
&= \int_{\partial^*\Omega} (u^\vee - u^\wedge)\, dP(\Omega, \cdot) \\
&= \int_{\partial^*\Omega} |T_+u - T_-u|\, dP(\Omega, \cdot).
\end{aligned}$$

□

For μ-measurable $\Omega \subset X$ and any $\kappa > 0$, define the weighted measure

$$d\mu_\kappa := (\chi_\Omega + \kappa\, \chi_{X\setminus\Omega})\, d\mu. \tag{7.6}$$

Consider then the space (X, d, μ_κ). It is easy to show that this is still a complete metric space such that μ_κ is doubling and supports a $(1,1)$-Poincaré inequality. We use the subscript κ to signify that a perimeter or some other quantity is taken with respect to the measure μ_κ.

Theorem 7.7. *Let $\Omega \subset X$ be a nonempty bounded open set of finite perimeter such that $\operatorname{Cap}_1(X \setminus \Omega) > 0$, Ω satisfies the exterior measure density condition (2.21), and Ω supports a $(1,1)$-Poincaré inequality. Suppose also that there is a constant $C \geq 1$ such that whenever $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\Omega)$, we have*

$$\mu(B(x,r) \cap \Omega) \geq \frac{\mu(B(x,r))}{C}.$$

Finally, assume that for all $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\Omega)$,

$$\mathcal{H}(\Omega \cap B(x,r)) \leq C\, \frac{\mu(B(x,r))}{r}.$$

Let $f \in \operatorname{Lip}(X)$ be boundedly supported. For each $p > 1$ let u_p be a p-harmonic function in Ω such that $u_p|_{X\setminus\Omega} = f$. Suppose that $(u_p)_{p>1}$ is a sequence of such p-harmonic functions and that $u_p \to u$ in $L^1(\Omega)$

as $p \to 1^+$. Then u is a solution to the minimization problem (T) of Definition 4.1.

Beginning of the proof of Theorem 7.7. Note that by combining the exterior measure density condition (2.21) and (2.13), we obtain that

$$\liminf_{r\to 0^+} \frac{\mu(B(x,r)\setminus\Omega)}{\mu(B(x,r))} \geq \gamma \qquad \text{for } \mathcal{H}\text{-a.e. } x \in \partial\Omega. \tag{7.8}$$

Note also again that the assumptions on Ω guarantee that the trace operator $T_+\colon \mathrm{BV}(\Omega) \to L^1(\partial\Omega,\mathcal{H})$ is linear and bounded, see [28, Theorem 5.5].

Let $v \in \mathrm{BV}(\Omega)$. By combining (2.14) and Theorem 6.9, we know that $P_+(\Omega,\cdot)$ is concentrated on $\partial^*\Omega$. Thus we need to show that

$$\|Dv\|(\Omega)+\int_{\partial^*\Omega} |T_+v-f|\,dP_+(\Omega,\cdot) \leq \|Dv\|(\Omega)+\int_{\partial^*\Omega} |T_+v-f|\,dP_+(\Omega,\cdot).$$

By Lemma 7.3, there is a sequence $(\psi_k) \subset \mathrm{Lip}(X)$ with $\psi_k = f$ in $X\setminus\Omega$ such that

$$\limsup_{k\to\infty} \int_\Omega g_{\psi_k}\,d\mu \leq \|Dv\|(\Omega) + \int_{\partial^*\Omega} |T_+v - f|\,dP_+(\Omega,\cdot).$$

Observe that each ψ_k can act as a test function for testing the p-harmonicity of u_p. Therefore by (3.1)

$$\|Du_p\|(\Omega) \leq \mu(\Omega)^{1-1/p}\left(\int_\Omega g_{u_p}^p\,d\mu\right)^{1/p} \leq \mu(\Omega)^{1-1/p}\left(\int_\Omega g_{\psi_k}^p\,d\mu\right)^{1/p}.$$

Letting $p \to 1^+$, we see that

$$\limsup_{p\to 1^+} \|Du_p\|(\Omega) \leq \int_\Omega g_{\psi_k}\,d\mu.$$

Therefore by now letting $k\to\infty$, we have

$$\limsup_{p\to 1^+} \|Du_p\|(\Omega) \leq \|Dv\|(\Omega) + \int_{\partial^*\Omega} |T_+v - f|\,dP_+(\Omega,\cdot). \tag{7.9}$$

Thus we need to prove that

$$\limsup_{p\to 1^+} \|Du_p\|(\Omega) \geq \|Du\|(\Omega) + \int_{\partial^*\Omega} |T_+u - f|\,dP_+(\Omega,\cdot) \tag{7.10}$$

in order to complete the proof.

Recall the definitions of O_Ω and I_Ω from Definition 2.2. By the exterior measure density condition (2.21), we know that $\mathcal{H}(\partial\Omega\cap I_\Omega) = 0$. Recall the definition of (X,d,μ_κ) from (7.6). We note that $\|D_\kappa u\|$ is absolutely continuous with respect to $\mathcal{H}$, which follows from the BV coarea formula (2.12) and (2.14). Thus $\|D_\kappa u\|(I_\Omega\setminus\Omega) = 0$. Note also

that since $u = f$ on $X \setminus \Omega$, $\partial^*\{u - f > t\} \cap O_\Omega = \emptyset$ for all $t \in \mathbb{R}$, and so by the coarea formula and (2.14)

$$\begin{aligned}\|D_\kappa(u-f)\|(O_\Omega) &= \int_{-\infty}^{\infty} P_\kappa(\partial^*\{u-f>t\}, O_\Omega)\,dt \\ &\le C(C_d,\kappa)\int_{-\infty}^{\infty} \mathcal{H}_\kappa(\partial^*\{u-f>t\}\cap O_\Omega)\,dt \\ &= 0.\end{aligned}$$

Then by the lower semicontinuity of the total variation and Proposition 7.5, we have

$$\begin{aligned}\liminf_{p\to 1^+} \|D_\kappa u_p\|(X) &\ge \|D_\kappa u\|(X) \\ &= \|D_\kappa u\|(I_\Omega) + \|D_\kappa u\|(O_\Omega) + \|D_\kappa u\|(\partial^*\Omega) \\ &= \|D_\kappa u\|(\Omega) + \|D_\kappa u\|(O_\Omega) + \int_{\partial^*\Omega} |T_+u - f|\,dP_\kappa(\Omega,\cdot) \\ &= \|Du\|(\Omega) + \|D_\kappa f\|(O_\Omega) + \int_{\partial^*\Omega} |T_+u - f|\,dP_\kappa(\Omega,\cdot).\end{aligned}$$

Similarly, on the left-hand side we have

$$\begin{aligned}\|D_\kappa u_p\|(X) &= \|Du_p\|(\Omega) + \|D_\kappa u_p\|(\partial^*\Omega) + \|D_\kappa u_p\|(O_\Omega) \\ &= \|Du_p\|(\Omega) + \|D_\kappa f\|(O_\Omega);\end{aligned}$$

note that $\|D_\kappa u_p\|(\partial^*\Omega) = 0$ since $\mu(\partial^*\Omega) = 0$. In total, we have

$$\liminf_{p\to 1^+} \|Du_p\|(\Omega) \ge \|Du\|(\Omega) + \int_{\partial^*\Omega} |T_+u - f|\,dP_\kappa(\Omega,\cdot). \tag{7.11}$$

The inequality (7.10) will follow from the above inequality if we know that

$$\lim_{\kappa\to\infty} \int_{\partial^*\Omega} |T_+u - f|\,dP_\kappa(\Omega,\cdot) = \int_{\partial^*\Omega} |T_+u - f|\,dP_+(\Omega,\cdot). \tag{7.12}$$

This is the focus of the rest of this section, and we will complete the proof at the end of the section. □

We will need the following approximation of a set of finite perimeter by "regular" sets. This is inspired by a similar result in [3], but note that we use a somewhat different, "two-sided" definition of the Minkowski content, as given in (2.1). First recall that by [27, Proposition 3.5] (which is based on [11]) the following coarea inequality holds: for any $w \in \mathrm{Lip}_c(X)$,

$$\int_{-\infty}^{\infty} \nu^+(\partial\{w>t\})\,dt \le \int_X \mathrm{Lip}\, w\,d\nu,$$

where ν is any positive Radon measure. From this it follows in a straightforward manner that for any $w \in \mathrm{Lip}_{\mathrm{loc}}(X)$,

$$\int_{-\infty}^{\infty} \mu^+(\partial\{w > t\})\, dt \le \int_X \mathrm{Lip}\, w\, d\mu. \tag{7.13}$$

Lemma 7.14. *Let $E \subset X$ be a set of finite perimeter. Fix $0 < \delta < 1$. Then there exists a sequence of open sets of finite perimeter $E_i \subset X$ with $\chi_{E_i} - \chi_E \to 0$ in $L^1(X)$, $\mu(\partial E_i) = 0$ for each $i \in \mathbb{N}$,*

$$\limsup_{i\to\infty} P(E_i, X) \le (1-\delta)^{-1} P(E, X),$$

and

$$\limsup_{i\to\infty} \mu^+(\partial E_i) \le \frac{CP(E,X)}{\delta}.$$

Proof. By Lemma 5.5, we can pick a sequence $(v_i) \subset \mathrm{Lip}_{\mathrm{loc}}(X)$ with $v_i - \chi_E \to 0$ in $L^1(X)$ and $\int_X g_{v_i}\, d\mu \to P(E, X)$, where each g_{v_i} is the minimal 1-weak upper gradient of v_i. We may also choose the functions so that $v_i \ge 0$. Furthermore, $\mathrm{Lip}\, v_i \le C g_{v_i}$ μ-almost everywhere, see [12, Proposition 4.26] or [22, Proposition 13.5.2]. According to the coarea formula for BV functions, see (2.12), for every $i \in \mathbb{N}$ we have

$$\int_0^1 P(\{v_i > t\}, X)\, dt \le \int_X g_{v_i}\, d\mu.$$

Now by Chebyshev's inequality,

$$\mathcal{L}^1\left(\left\{t \in [0,1] :\ P(\{v_i > t\}, X) > (1-\delta)^{-1} \int_X g_{v_i}\, d\mu\right\}\right) \le 1 - \delta;$$

note that this holds also if $\int_X g_{v_i}\, d\mu = 0$, as then $P(\{v_i > t\}, X) = 0$ for a.e. $t \in [0,1]$. Therefore there is a measurable set $A_i \subset [\delta/4, 1-\delta/4]$ with $\mathcal{L}^1(A_i) \ge \delta/2$ and

$$P(\{v_i > t\}, X) \le \frac{1}{1-\delta} \int_X g_{v_i}\, d\mu$$

for all $t \in A_i$. Moreover, since the sets $\partial\{v_i > t\} \subset \{v_i = t\}$ are disjoint for distinct values of t, we have $\mu(\partial\{v_i > t\}) = 0$ for a.e. $t \in [0,1]$. By the version of the coarea formula found in (7.13), we have

$$\int_0^1 \mu^+(\partial\{v_i > t\})\, dt \le \int_X \mathrm{Lip}\, v_i\, d\mu.$$

Thus for each $i \in \mathbb{N}$, there exists $t_i \in A_i$ with

$$\mu^+(\partial\{v_i > t_i\}) \le \frac{2}{\delta} \int_X \mathrm{Lip}\, v_i\, d\mu \le \frac{C}{\delta} \int_X g_{v_i}\, d\mu$$

and $\mu(\partial\{v_i > t_i\}) = 0$. Define $E_i := \{v_i > t_i\}$. Then

$$\limsup_{i\to\infty} P(E_i, X) \le (1-\delta)^{-1} P(E, X),$$

and

$$\limsup_{i\to\infty} \mu^+(\partial E_i) \le \frac{C}{\delta} P(E, X).$$

Now we need to show that $\chi_{\{v_i>t_i\}} - \chi_E \to 0$ in $L^1(X)$. Note that for any $t \in [\delta/4, 1-\delta/4]$, for any x such that

$$x \in \{v_i > t\} \cap E \quad \text{or} \quad x \in X \setminus (\{v_i > t\} \cup E),$$

we have

$$|\chi_{\{v_i>t\}}(x) - \chi_E(x)| = 0 \le 4|v_i(x) - \chi_E(x)|/\delta.$$

If $x \in \{v_i > t\}$ but $x \notin E$, then

$$|\chi_{\{v_i>t\}}(x) - \chi_E(x)| = 1 \le v_i(x)/t \le 4|v_i(x) - \chi_E(x)|/\delta.$$

On the other hand, if $x \notin \{v_i > t\}$ but $x \in E$, then $v_i(x) \le t \le 1-\delta/4$ and it follows that

$$|\chi_{\{v_i>t\}}(x) - \chi_E(x)| = 1 \le 4|v_i(x) - \chi_E(x)|/\delta.$$

Thus

$$\int_X |\chi_{\{v_i>t_i\}} - \chi_E| \, d\mu \le \frac{4}{\delta} \int_X |v_i - \chi_E| \, d\mu \to 0$$

so that $\chi_{\{v_i>t_i\}} - \chi_E \to 0$ in $L^1(X)$ (even though t_i depends on i). □

The reason for utilizing the Minkowski content is that it scales nicely according to the parameter κ in μ_κ, in the following sense.

Lemma 7.15. *Let $\Omega \subset X$ be μ-measurable, let $A \subset X \setminus \Omega$, let $\beta > 0$, and suppose that there is some $R > 0$ for which*

$$\inf_{0<r<R} \frac{\mu(B(x,r) \setminus \Omega)}{\mu(B(x,r))} \ge \frac{\beta}{2} \tag{7.16}$$

for every $x \in A$. Then we have

$$C\mu_\kappa^+(A) \ge \kappa\beta\mu^+(A).$$

Proof. For any $x \in A$ and radii $r \in (0, R)$,

$$\mu_\kappa(B(x,r)) \ge \mu_\kappa(B(x,r) \setminus \Omega) = \kappa\mu(B(x,r) \setminus \Omega) \ge \frac{\kappa\beta}{2}\mu(B(x,r)).$$

Fix $0 < r < R/5$ and consider the collection of balls $\{B(x,r)\}_{x\in A}$. By the 5-covering theorem we can pick a countable collection of disjoint balls $B(x_j, r)$ such that the balls $B(x_j, 5r)$ cover $\bigcup_{x\in A} B(x,r)$. We

have

$$\begin{aligned}\frac{\mu\left(\bigcup_{x\in A} B(x,r)\right)}{2r} &\le \sum_{j\in\mathbb{N}} \frac{\mu(B(x_j,5r))}{2r} \le C_d^3 \sum_{j\in\mathbb{N}} \frac{\mu(B(x_j,r))}{2r} \\ &\le \frac{2C_d^3}{\kappa\beta} \sum_{j\in\mathbb{N}} \frac{\mu_\kappa(B(x_j,r))}{2r} \\ &\le \frac{2C_d^3}{\kappa\beta} \frac{\mu_\kappa\left(\bigcup_{x\in A} B(x,r)\right)}{2r}.\end{aligned}$$

By taking the limit infimum as $r \to 0$ on both sides, we obtain

$$\mu^+(A) \le \frac{2C_d^3}{\kappa\beta}\mu_\kappa^+(A).$$

□

Moreover, we have the following simple estimate for the Minkowski content and Hausdorff measure that we will need in the proof of Proposition 7.19. The estimate can be proved by a simple covering argument, see [27, Proposition 3.12].

Lemma 7.17. *For any $A \subset X$, we have $\mathcal{H}(A) \le C_d^3 \mu^+(A)$.*

It is less clear that this estimate would hold if we used a "one-sided" definition of the Minkowski content, as for example in [3].

Lemma 7.18. *Let $\Omega \subset X$ be an open set. If $K_i \subset \Omega$, $i \in \mathbb{N}$, are compact sets with $\chi_{K_i} - \chi_\Omega \to 0$ in $L^1(X)$, then*

$$P_+(\Omega, X) \le \liminf_{i\to\infty} P(K_i, X).$$

Proof. By [20, Lemma 2.6], for each $i \in \mathbb{N}$ we can find a function $v_i \in \mathrm{Lip}_c(\Omega)$ such that $\|v_i - \chi_\Omega\|_{L^1(\Omega)} < 1/i$ and

$$\int_\Omega g_{v_i}\, d\mu \le P(K_i, \Omega) + 1/i.$$

The conclusion follows by the definition of $P_+(\Omega, \cdot)$. □

Proposition 7.19. *Let $\Omega \subset X$ be a bounded open set with $P(\Omega, X) < \infty$, and assume that for some constant $\beta > 0$, we have*

$$\liminf_{r\to 0^+} \frac{\mu(B(x,r) \setminus \Omega)}{\mu(B(x,r))} \ge \beta \tag{7.20}$$

for $\mathcal{H}$-a.e. $x \in \partial\Omega$. Then we have

$$P_+(\Omega, X) = \lim_{\kappa\to\infty} P_\kappa(\Omega, X).$$

Proof. By Theorem 6.9 we have $P_+(\Omega, X) < \infty$. Note that for any $\kappa > 0$ we have $P_+(\Omega, X) \geq P_\kappa(\Omega, X)$, so only the other inequality needs to be proved. Fix $0 < \delta < 1$, and fix $\kappa > 0$. By Lemma 7.14 we can find a sequence of open sets $\Omega_i \subset X$ of finite perimeter such that $\chi_{\Omega_i} \to \chi_\Omega$ in $L^1(X)$, $\mu(\partial\Omega_i) = 0$ for all $i \in \mathbb{N}$,

$$\limsup_{i\to\infty} P_\kappa(\Omega_i, X) \leq (1-\delta)^{-1} P_\kappa(\Omega, X), \tag{7.21}$$

and

$$\limsup_{i\to\infty} \mu_\kappa^+(\partial\Omega_i) \leq \frac{C}{\delta} P_\kappa(\Omega, X) \leq \frac{C}{\delta} P_+(\Omega, X). \tag{7.22}$$

In the following, we will repeatedly use the measure property and the subadditivity property (2.11) of sets of finite perimeter. Since $\chi_{\Omega_i} \to \chi_\Omega$ in $L^1(X)$, it follows that $\chi_{\Omega\cup\Omega_i} \to \chi_\Omega$ in $L^1(X)$ as well. By the lower semicontinuity of perimeter and the fact that the perimeter of a set is concentrated on its measure theoretic boundary, we estimate

$$\begin{aligned}
P(\Omega, X) &\leq \liminf_{i\to\infty} P(\Omega \cup \Omega_i, X) \\
&= \liminf_{i\to\infty} P(\Omega \cup \Omega_i, X \setminus (I_{\Omega_i} \cup I_\Omega)) \\
&\leq \liminf_{i\to\infty} \Big(P(\Omega, X \setminus (I_{\Omega_i} \cup I_\Omega)) + P(\Omega_i, X \setminus (I_{\Omega_i} \cup I_\Omega)) \Big) \\
&\leq \liminf_{i\to\infty} \Big(P(\Omega, X \setminus I_{\Omega_i}) + P(\Omega_i, X \setminus I_\Omega) \Big).
\end{aligned}$$

It follows that

$$P(\Omega, I_{\Omega_i}) \leq P(\Omega_i, X \setminus \Omega) + \varepsilon_i, \tag{7.23}$$

where $\varepsilon_i \to 0$ as $i \to \infty$. For any sets $A, B \subset X$, we have

$$\partial^*(A \cap B) \subset (\partial^* A \cap \partial^* B) \cup (\partial^* A \cap I_B) \cup (\partial^* B \cap I_A).$$

Thus we have

$$\begin{aligned}
\partial^*(\Omega_i \setminus \Omega) &\subset (\partial^*\Omega_i \cap \partial^*\Omega) \cup (\partial^*\Omega_i \cap O_\Omega) \cup (\partial^*\Omega \cap I_{\Omega_i}) \\
&\subset (\partial^*\Omega_i \setminus \Omega) \cup (\partial^*\Omega \cap I_{\Omega_i}).
\end{aligned}$$

By (2.11), $P(\Omega_i \setminus \Omega, X) < \infty$, and then by using (2.14), we obtain

$$\begin{aligned}
P(\Omega_i \setminus \Omega, X) &\leq C\mathcal{H}(\partial^*(\Omega_i \setminus \Omega)) \\
&\leq C\mathcal{H}(\partial^*\Omega_i \setminus \Omega) + C\mathcal{H}(\partial^*\Omega \cap I_{\Omega_i}) \\
&\leq CP(\Omega_i, X \setminus \Omega) + CP(\Omega, I_{\Omega_i}).
\end{aligned}$$

Combining this with (7.23), we obtain that for all $i \in \mathbb{N}$

$$P(\Omega_i \setminus \Omega, X) \leq CP(\Omega_i, X \setminus \Omega) + C\varepsilon_i. \tag{7.24}$$

Note that $\mathcal{H}|_{\partial\Omega_i}$ is a Borel measure of finite mass, since by Lemma 7.17,

$$\mathcal{H}(\partial\Omega_i) \leq C\mu^+(\partial\Omega_i) \leq C\mu_\kappa^+(\partial\Omega_i) < \infty.$$

Note that for any fixed $r > 0$, the map

$$x \mapsto \frac{\mu(B(x,r) \setminus \Omega)}{\mu(B(x,r))}$$

is a Borel map as the ratio of two lower semicontinuous functions. Hence for each $\tau > 0$ the function f_τ given by

$$f_\tau(x) = \inf_{r \in \mathbb{Q} \cap (0,\tau)} \frac{\mu(B(x,r) \setminus \Omega)}{\mu(B(x,r))}$$

is also Borel measurable, and so is

$$f_\infty(x) := \liminf_{r \to 0^+} \frac{\mu(B(x,r) \setminus \Omega)}{\mu(B(x,r))} = \lim_{\tau \to 0^+} f_\tau(x).$$

So for each $i \in \mathbb{N}$, by Egorov's theorem we can choose a set $A_i \subset \partial\Omega_i$ with $\mathcal{H}(\partial\Omega_i \setminus A_i) < \varepsilon_i$ and such that $f_\tau \to f_\infty$ uniformly in A_i. Thus (7.16) is satisfied for $A = A_i \setminus \Omega$ and some $R > 0$. By Lemma 7.17 and Lemma 7.15, we get

$$\begin{aligned} \mathcal{H}(\partial\Omega_i \setminus \Omega) &\leq \mathcal{H}(A_i \setminus \Omega) + \varepsilon_i \\ &\leq C\mu^+(A_i \setminus \Omega) + \varepsilon_i \\ &\leq \frac{C}{\kappa\beta}\mu^+_\kappa(A_i \setminus \Omega) + \varepsilon_i \\ &\leq \frac{C}{\kappa\beta}\mu^+_\kappa(\partial\Omega_i) + \varepsilon_i. \end{aligned} \tag{7.25}$$

Then by (2.14)

$$P(\Omega_i, X \setminus \Omega) \leq C\mathcal{H}(\partial\Omega_i \setminus \Omega) \leq \frac{C}{\kappa\beta}\mu^+_\kappa(\partial\Omega_i) + C\varepsilon_i,$$

so by combining with (7.24), we have

$$P(\Omega_i \setminus \Omega, X) \leq \frac{C}{\kappa\beta}\mu^+_\kappa(\partial\Omega_i) + C\varepsilon_i.$$

Recall that $\mu(\overline{\Omega_i} \setminus \Omega_i) = \mu(\partial\Omega_i) = 0$ for all $i \in \mathbb{N}$. Thus by (7.22),

$$\begin{aligned} \limsup_{i\to\infty} P(\overline{\Omega_i} \setminus \Omega, X) &= \limsup_{i\to\infty} P(\Omega_i \setminus \Omega, X) \\ &\leq \limsup_{i\to\infty} \frac{C}{\kappa\beta}\mu^+_\kappa(\partial\Omega_i) \\ &\leq \frac{C}{\kappa\beta\delta}P_+(\Omega, X). \end{aligned} \tag{7.26}$$

For $A \subset X$, we set

$$A^\beta := \left\{ x \in X : \limsup_{r \to 0^+} \frac{\mu(B(x,r) \cap A)}{\mu(B(x,r))} \geq \beta \right\}.$$

Let us denote by $D \subset \partial\Omega$ the $\mathcal{H}$-negligible set where (7.20) fails. Note that $\mu(X) > 0$, and so we can assume that $\mu(\Omega_i \setminus \Omega) < \mu(X)/2$ for all $i \in \mathbb{N}$. Now by the boxing inequality, see [25, Remark 3.3(1)], we can

find a collection of balls $\{B(x_j^i, r_j^i)\}_{j\in\mathbb{N}}$ covering $(\overline{\Omega_i}\setminus\Omega)^\beta$ such that the balls $B(x_j^i, r_j^i/5)$ are disjoint,

$$\frac{\mu(B(x_j^i, r_j^i/5)\cap\overline{\Omega_i}\setminus\Omega)}{\mu(B(x_j^i, r_j^i/5))}\geq\frac{\beta}{C}, \tag{7.27}$$

and

$$\sum_{j\in\mathbb{N}}\frac{\mu(B(x_j^i, r_j^i))}{r_j^i}\leq C\beta^{-1}P(\overline{\Omega_i}\setminus\Omega, X).$$

Note that in [25] it is assumed that $\mu(X)=\infty$, but the condition $\mu(\Omega_i\setminus\Omega)<\mu(X)/2$ is sufficient for the proof to work. Then by (7.26),

$$\limsup_{i\to\infty}\sum_{j\in\mathbb{N}}\frac{\mu(B(x_j^i, r_j^i))}{r_j^i}\leq\frac{C}{\kappa\beta^2\delta}P_+(\Omega, X). \tag{7.28}$$

Note that if $x\in I_{\Omega_i}\setminus(\Omega\cup D)$, then $x\in(\overline{\Omega_i}\setminus\Omega)^\beta$. Thus we have

$$(\overline{\Omega_i}\setminus\Omega)\setminus(\overline{\Omega_i}\setminus\Omega)^\beta\subset D\cup\partial\Omega_i\setminus\Omega.$$

But by (7.25),

$$\mathcal{H}(D\cup\partial\Omega_i\setminus\Omega)=\mathcal{H}(\partial\Omega_i\setminus\Omega)\leq\frac{C}{\kappa\beta}\mu_\kappa^+(\partial\Omega_i)+\varepsilon_i.$$

Thus we can pick another collection $\{B(y_k^i, s_k^i)\}_{k\in\mathbb{N}}$ of balls covering $(\overline{\Omega_i}\setminus\Omega)\setminus(\overline{\Omega_i}\setminus\Omega)^\beta$ with $s_k^i\leq 1/i$ and

$$\sum_{k\in\mathbb{N}}\frac{\mu(B(y_k^i, s_k^i))}{s_k^i}\leq\frac{C}{\kappa\beta}\mu_\kappa^+(\partial\Omega_i)+2\varepsilon_i,$$

and so by (7.22),

$$\limsup_{i\to\infty}\sum_{k\in\mathbb{N}}\frac{\mu(B(y_k^i, s_k^i))}{s_k^i}\leq\frac{C}{\kappa\beta\delta}P_+(\Omega, X). \tag{7.29}$$

Note that the collections $\{B(x_j^i, r_j^i)\}_{j\in\mathbb{N}}$ and $\{B(y_k^i, s_k^i)\}_{k\in\mathbb{N}}$ together cover all of $\overline{\Omega_i}\setminus\Omega$. By [24, Lemma 6.2] we can pick radii $\widetilde{r}_j^i\in[r_j^i, 2r_j^i]$ such that

$$\frac{1}{C}P(B(x_j^i, \widetilde{r}_j^i), X)\leq\frac{\mu(B(x_j^i, \widetilde{r}_j^i))}{\widetilde{r}_j^i}\leq C_d\frac{\mu(B(x_j^i, r_j^i))}{r_j^i},$$

and similarly we find radii $\widetilde{s}_k^i\in[s_k^i, 2s_k^i]$. Define for each $i\in\mathbb{N}$

$$K_i:=\overline{\Omega_i}\setminus\left(\bigcup_{j\in\mathbb{N}}B(x_j^i, \widetilde{r}_j^i)\cup\bigcup_{k\in\mathbb{N}}B(y_k^i, \widetilde{s}_k^i)\right).$$

Note that these are closed sets contained in Ω, and thus compact. By (7.28) and (7.29), we have

$$\begin{aligned}
&\limsup_{i\to\infty} P(K_i, X)\\
&\quad\le \limsup_{i\to\infty}\left(P(\overline{\Omega}_i, X) + \sum_{j\in\mathbb{N}} P(B(x_j^i, \widetilde{r}_j^i), X) + \sum_{k\in\mathbb{N}} P(B(y_k^i, \widetilde{s}_k^i), X)\right)\\
&\quad\le \limsup_{i\to\infty} P_\kappa(\overline{\Omega}_i, X) + \frac{C}{\kappa\beta^2\delta} P_+(\Omega, X) + \frac{C}{\kappa\beta\delta} P_+(\Omega, X)\\
&\quad= \limsup_{i\to\infty} P_\kappa(\Omega_i, X) + \frac{C}{\kappa\beta^2\delta} P_+(\Omega, X) + \frac{C}{\kappa\beta\delta} P_+(\Omega, X)\\
&\quad\le (1-\delta)^{-1} P_\kappa(\Omega, X) + \frac{C}{\kappa\beta^2\delta} P_+(\Omega, X),
\end{aligned}$$

where we used (7.21) in the last step (and absorbed a factor 2 into the constant C). By (7.27) we have

$$\begin{aligned}
\|\chi_{K_i} - \chi_\Omega\|_{L^1(X)} &\le \sum_{j\in\mathbb{N}} \mu(B(x_j^i, \widetilde{r}_j^i)) + \sum_{k\in\mathbb{N}} \mu(B(y_k^i, \widetilde{s}_k^i))\\
&\le C\sum_{j\in\mathbb{N}} \mu(B(x_j^i, r_j^i/5)) + C_d \sum_{k\in\mathbb{N}} \mu(B(y_k^i, s_k^i))\\
&\le \frac{C}{\beta}\sum_{j\in\mathbb{N}} \mu(B(x_j^i, r_j^i/5)\cap \overline{\Omega}_i \setminus \Omega) + \frac{C_d}{i}\sum_{k\in\mathbb{N}} \frac{\mu(B(y_k^i, s_k^i))}{s_k^i}\\
&\le \frac{C\mu(\overline{\Omega}_i \setminus \Omega)}{\beta} + \frac{C_d}{i}\sum_{k\in\mathbb{N}} \frac{\mu(B(y_k^i, s_k^i))}{s_k^i},
\end{aligned}$$

since the balls $B(x_j^i, r_j^i/5)$ are disjoint. Now by the fact that $\chi_{\Omega_i} \to \chi_\Omega$ in $L^1(X)$ and (7.29), we obtain $\chi_{K_i} \to \chi_\Omega$ in $L^1(X)$. Thus by Lemma 7.18

$$\begin{aligned}
P_+(\Omega, X) &\le \liminf_{i\to\infty} P(K_i, X)\\
&\le (1-\delta)^{-1} P_\kappa(\Omega, X) + \frac{C}{\kappa\beta^2\delta} P_+(\Omega, X).
\end{aligned}$$

Letting $\kappa \to \infty$ and then $\delta \to 0$, we obtain the result. □

Corollary 7.30. *Under the assumptions of Proposition 7.19, for any Borel set $D \subset X$ we have*

$$P_+(\Omega, D) = \lim_{\kappa\to\infty} P_\kappa(\Omega, D).$$

Proof. For any Borel set $A \subset X$ and any $\kappa > 0$, we have $P_+(\Omega, A) \ge P_\kappa(\Omega, A)$. Thus by the measure property of $P_+(\Omega, \cdot)$ proved in the

Appendix, and by Proposition 7.19,

$$\begin{aligned} P_+(\Omega, D) &= P_+(\Omega, X) - P_+(\Omega, X \setminus D) \\ &= \lim_{\kappa\to\infty} P_\kappa(\Omega, X) - P_+(\Omega, X \setminus D) \\ &\le \limsup_{\kappa\to\infty} \big(P_\kappa(\Omega, X) - P_\kappa(\Omega, X \setminus D)\big) \\ &= \lim_{\kappa\to\infty} P_\kappa(\Omega, D). \end{aligned}$$

□

End of the proof of Theorem 7.7. Note that by (7.8), Ω satisfies the assumptions of Proposition 7.19 (and thus Corollary 7.30) with $\beta = \kappa$. By Cavalieri's principle, Corollary 7.30, and Lebesgue's monotone convergence theorem, we have

$$\begin{aligned} &\int_{\partial^*\Omega} |T_+u - f|\, dP_\kappa(\Omega, \cdot) \\ &\qquad = \int_0^\infty P_\kappa(\Omega, \partial^*\Omega \cap \{|T_+u - f| > t\})\, dt \\ &\qquad \to \int_0^\infty P_+(\Omega, \partial^*\Omega \cap \{|T_+u - f| > t\})\, dt \qquad \text{as } \kappa \to \infty \\ &\qquad = \int_{\partial^*\Omega} |T_+u - f|\, dP_+(\Omega, \cdot). \end{aligned}$$

This proves (7.12), thus completing the proof of Theorem 7.7.

□

Corollary 7.31. *Let $\Omega \subset X$ satisfy the assumptions of Theorem 7.7, and let $f \in \mathrm{Lip}(X)$ be boundedly supported. Then the minimization problem (T) of Definition 4.1 has a solution.*

Proof. For every $p > 1$, there exists a p-harmonic function u_p in Ω such that $u_p|_{X\setminus\Omega} = f$. Then the result follows by combining Lemma 3.2 and Theorem 7.7. □

8. Dirichlet Problems (T), (B) and perturbation of the domain

From the definition of $P_+(\Omega, \cdot)$ it is clear that Problem (T) of Definition 4.1 is associated with approximating the bounded open set Ω from inside. Moreover, if for each $k \in \mathbb{N}$ we have $\Omega_k \Subset \Omega$ such that $\Omega = \bigcup_{k\in\mathbb{N}} \Omega_k$, and v_{p_k} is the p_k-harmonic solution to the Dirichlet problem on Ω_k with boundary data f, then under reasonable hypotheses on Ω we have $v_{p_k} \to u$ with u a solution to Problem (T) in Ω with boundary data f. Indeed, suppose that for each $p > 1$ there are constants $C_p \ge 1$ and $\beta_p > 0$ such that Ω satisfies the condition that whenever u_p

is a p-harmonic solution to the Dirichlet problem on Ω with boundary data f we have

$$\operatorname*{osc}_{\overline{\Omega}\cap B(x,\rho)} u_p \le \operatorname*{osc}_{\partial\Omega\cap B(x,r)} f + C_p\left(\frac{\rho}{r}\right)^{\beta_p} \tag{8.1}$$

for all $x \in \partial\Omega$ and $0 < \rho < r/2$. Let p_k be any sequence as obtained in Lemma 3.2. Take a sequence $\varepsilon_k \to 0$ as $k \to \infty$, and let $L > 0$ such that f is L-Lipschitz continuous. For each $k \in \mathbb{N}$ we can fix $0 < r_k < \operatorname{diam}(\Omega)/2$ such that $4Lr_k < \varepsilon_k$. Then by (8.1), whenever $x \in \partial\Omega$ and $0 < \rho < r_k/2$ and $y \in B(x,\rho)\cap\Omega$,

$$\begin{aligned} |u_{p_k}(y) - f(y)| &\le |u_{p_k}(y) - u_{p_k}(x)| + |f(x) - f(y)| \\ &\le 2Lr_k + C_{p_k}\left(\frac{\rho}{r_k}\right)^{\beta_{p_k}} + Lr_k. \end{aligned}$$

We can then choose $0 < \rho_k < r_k/2$ such that

$$C_{p_k}\left(\frac{\rho_k}{r_k}\right)^{\beta_{p_k}} < Lr_k,$$

in which case for $y \in B(x,\rho_k)\cap\Omega$ we have

$$|u_{p_k}(y) - f(y)| \le 4Lr_k < \varepsilon_k.$$

Now if we choose $\Omega_k \Subset \Omega$ such that $0 < \operatorname{dist}(\partial\Omega_k, \partial\Omega) < \rho_k$, we have by the comparison principle for p-harmonic functions (see [8]) that $|v_k - u_{p_k}| < \varepsilon_k$ on Ω_k. It then follows that $v_k \to u$ in $L^1(X)$ as $k \to \infty$, where we know from the previous section that u satisfies the Dirichlet problem (T) on Ω with boundary data f. Examples of domains where (8.1) hold include the domains whose complements are uniformly 1-fat, see [10]; in particular, domains whose complements are porous satisfy this requirement.

In contrast to problem (T), the Dirichlet problem (B) of Definition 4.1 is associated with approximation of Ω from *outside*, as we will see next.

Let $\Omega \subset X$ be a nonempty bounded open set with $X \setminus \overline{\Omega} \neq \emptyset$. Let $\Omega_k \subset X$, $k \in \mathbb{N}$, be a sequence of bounded open sets such that $\overline{\Omega} = \bigcap_{k\in\mathbb{N}} \Omega_k$ and $\Omega_{k+1} \Subset \Omega_k$ for each $k \in \mathbb{N}$. Note that since $X\setminus\overline{\Omega} \neq \emptyset$ is open, we have $\mu(X\setminus\overline{\Omega}) > 0$. Thus also $\operatorname{Cap}_1(X\setminus\Omega_k) \ge \mu(X\setminus\Omega_k) > 0$ for all sufficiently large k, and so we may as well assume that this is true for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, fix a decreasing sequence $(p_{k,m})_m$ such that $p_{k,m} > 1$ and

$$\lim_{m\to\infty} p_{k,m} = 1.$$

Let $f \in \operatorname{Lip}(X)$ be boundedly supported, and let $u_{k,m} \in N^{1,p_{k,m}}(X)$ be the $p_{k,m}$-harmonic function solving the Dirichlet problem on Ω_k with boundary data f. According to Lemma 3.2, by passing to a subsequence of $(p_{k,m})_m$ (not relabeled), we find a function $u_k \in \operatorname{BV}(X)$

with $u_k = f$ on $X \setminus \Omega_k$ such that $u_{k,m} \to u_k$ in $L^1(X)$ as $m \to \infty$. We have

$$\begin{aligned}\int_{\Omega_k} g_{u_{k,m}}\,d\mu &\le \left(\int_{\Omega_k} g_{u_{k,m}}^{p_{k,m}}\,d\mu\right)^{1/p_{k,m}} \mu(\Omega_k)^{1-1/p_{k,m}}\\ &\le \left(\int_{\Omega_k} g_f^{p_{k,m}}\,d\mu\right)^{1/p_{k,m}} \mu(\Omega_k)^{1-1/p_{k,m}}.\end{aligned}$$

As in Section 3, $g_{u_{k,m}}$ always denotes the minimal p-weak upper gradient of $u_{k,m}$, and for a Lipschitz function f, g_f denotes the minimal p-weak upper gradient of f for any $p > 1$. By (3.1),

$$\|Du_{k,m}\|(X) \le \int_X g_{u_{k,m}}\,d\mu = \int_{\Omega_k} g_{u_{k,m}}\,d\mu + \int_{X\setminus\Omega_k} g_f\,d\mu.$$

By the lower semicontinuity of the total variation,

$$\|Du_k\|(X) \le \liminf_{m\to\infty} \|Du_{k,m}\|(X) \le \int_X g_f\,d\mu.$$

We have $|f| \le M$ on X for some $M \ge 0$. By the comparison principle, we also have $|u_{k,m}| \le M$, and then $|u_k| \le M$. Thus for all $k \in \mathbb{N}$,

$$\|u_k\|_{L^1(X)} \le \|u_k\|_{L^\infty(X)}\mu(\Omega_k) + \|f\|_{L^1(X\setminus\Omega_k)} \le M\mu(\Omega_1) + \|f\|_{L^1(X)} < \infty.$$

Then by the compact embedding given in [30, Theorem 3.7], by passing to a subsequence of k (not relabeled), we obtain $u_k \to u$ in $L^1(X)$ as $k \to \infty$, for $u \in \mathrm{BV}(X)$. By passing to a further subsequence (not relabeled), we can assume that

$$\int_X |u_k - u|\,d\mu < 1/k. \tag{8.2}$$

Theorem 8.3. *Let $\Omega \subset X$ be a nonempty bounded open set such that $X \setminus \overline{\Omega} \ne \emptyset$, and suppose that $f \in \mathrm{Lip}(X)$ is boundedly supported. Let u be the function constructed above. Then u solves Problem (B) of Definition 4.1.*

Note that in this section we do not need Ω to satisfy the extra conditions imposed in Section 7.

Proof. Take a test function $\psi \in \mathrm{BV}(X)$ such that $\psi = 0$ on $X \setminus \overline{\Omega}$. We can choose a sequence $(\Psi_j) \subset \mathrm{Lip}_{\mathrm{loc}}(X)$ such that $\Psi_j \to u + \psi$ in $L^1(X)$ as $j \to \infty$ and

$$\lim_{j\to\infty} \int_X g_{\Psi_j}\,d\mu = \|D(u+\psi)\|(X),$$

where g_{Ψ_j} is the minimal p-weak upper gradient of Ψ_j in X, for any $p > 1$, see the discussion on page 7. Then also $g_{\Psi_j}\,d\mu \to d\|D(u+\psi)\|$

weakly* in the sense of measures (see e.g. [18, Proposition 3.8]), so that for each $k \in \mathbb{N}$,

$$\begin{aligned}\limsup_{j\to\infty} \int_{\Omega_k} g_{\Psi_j}\, d\mu &\le \|D(u+\psi)\|(\overline{\Omega_k}) \\ &= \|D(u+\psi)\|(\overline{\Omega}) + \|D(u+\psi)\|(\overline{\Omega_k}\setminus\overline{\Omega}) \\ &= \|D(u+\psi)\|(\overline{\Omega}) + \|Df\|(\overline{\Omega_k}\setminus\overline{\Omega})\end{aligned} \tag{8.4}$$

since $u+\psi = f$ in $\overline{\Omega_k}\setminus\overline{\Omega}$. For each $k \in \mathbb{N}$, let $\varepsilon_k := \operatorname{dist}(\overline{\Omega}, X\setminus\Omega_k) > 0$, and let $\eta_{\varepsilon_k} \in \operatorname{Lip}_c(X)$ with $0 \le \eta_{\varepsilon_k} \le 1$, $\eta_{\varepsilon_k} = 1$ in $\overline{\Omega}$, $\eta_{\varepsilon_k} = 0$ in $X\setminus\Omega_k$, and $g_{\eta_{\varepsilon_k}} \le 1/\varepsilon_k$. We set

$$\psi_{j,k} := \eta_{\varepsilon_k}\Psi_j + (1-\eta_{\varepsilon_k})f.$$

By the Leibniz rule of [8, Lemma 2.18],

$$\begin{aligned}g_{\psi_{j,k}} &\le g_{\Psi_j}\eta_{\varepsilon_k} + g_f(1-\eta_{\varepsilon_k}) + g_{\eta_{\varepsilon_k}}|\Psi_j - f| \\ &\le g_{\Psi_j}\chi_{\Omega_k} + g_f\chi_{\Omega_k\setminus\overline{\Omega}} + (1/\varepsilon_k)|\Psi_j - f|\chi_{\Omega_k\setminus\overline{\Omega}}.\end{aligned} \tag{8.5}$$

Note that this function agrees with f in $X\setminus\Omega_k$.

As noted above, $|u_k| \le M$ for all $k \in \mathbb{N}$, and so $|u| \le M$. As truncation decreases total variation, we can also assume that $|u+\psi| \le M$ and that the approximating functions Ψ_j also satisfy $|\Psi_j| \le M$. Then we have (assuming $p_{k,m} < 2$ and $M \ge 1$)

$$\begin{aligned}\int_{X\setminus\overline{\Omega}} |\Psi_j - f|^{p_{k,m}}\, d\mu &\le 2^{p_{k,m}-1}M^{p_{k,m}-1}\int_{X\setminus\overline{\Omega}} |\Psi_j - f|\, d\mu \\ &\le 2M\int_{X\setminus\overline{\Omega}} |\Psi_j - f|\, d\mu.\end{aligned}$$

For each $k \in \mathbb{N}$, by (8.4) we can choose $j_k \in \mathbb{N}$ large enough so that

$$\int_{\Omega_k} g_{\Psi_{j_k}}\, d\mu \le \|D(u+\psi)\|(\overline{\Omega}) + \|Df\|(\overline{\Omega_k}\setminus\overline{\Omega}) + 1/k \tag{8.6}$$

and

$$\int_{X\setminus\overline{\Omega}} |\Psi_{j_k} - f|\, d\mu \le \frac{\varepsilon_k^2}{2M},$$

and then it follows that for all m

$$\int_{X\setminus\overline{\Omega}} |\Psi_{j_k} - f|^{p_{k,m}}\, d\mu \le \varepsilon_k^2. \tag{8.7}$$

Then we choose $m_k \in \mathbb{N}$ large enough so that $1 < p_{k,m_k} < 1 + 1/k$,

$$\int_X |u_{k,m_k} - u_k|\, d\mu < 1/k, \tag{8.8}$$

and

$$\left(\int_{\Omega_k} g_{\Psi_{j_k}}^{p_{k,m_k}}\, d\mu\right)^{1/p_{k,m_k}} \le \int_{\Omega_k} g_{\Psi_{j_k}}\, d\mu + 1/k.$$

It then follows by (8.6) that

$$\left(\int_{\Omega_k} g_{\Psi_{j_k}}^{p_{k,m_k}}\,d\mu\right)^{1/p_{k,m_k}} \le \|D(u+\psi)\|(\overline{\Omega}) + \|Df\|(\overline{\Omega_k}\setminus\overline{\Omega}) + 2/k. \quad (8.9)$$

Combining (8.8) with (8.2), we have $u_{k,m_k} \to u$ in $L^1(X)$ as $k\to\infty$. By lower semicontinuity of the total variation, Hölder's inequality, and the fact that $\psi_{j_k,k}$ can be used to test the $p_{k,m}$-harmonicity of u_{k,m_k}, we get

$$\begin{aligned}
\|Du\|(\overline{\Omega}) &\le \liminf_{k\to\infty}\int_{\Omega_k} g_{u_{k,m_k}}\,d\mu\\
&\le \liminf_{k\to\infty}\left(\int_{\Omega_k} g_{u_{k,m_k}}^{p_{k,m_k}}\,d\mu\right)^{1/p_{k,m_k}}\\
&\le \liminf_{k\to\infty}\left(\int_{\Omega_k} g_{\psi_{j_k,k}}^{p_{k,m_k}}\,d\mu\right)^{1/p_{k,m_k}}\\
&\le \limsup_{k\to\infty}\left(\int_{\Omega_k} g_{\Psi_{j_k}}^{p_{k,m_k}}\,d\mu\right)^{1/p_{k,m_k}} + \limsup_{k\to\infty}\left(\int_{\Omega_k\setminus\overline{\Omega}} g_f^{p_{k,m_k}}\,d\mu\right)^{1/p_{k,m_k}}\\
&\quad + \limsup_{k\to\infty}\frac{1}{\varepsilon_k}\left(\int_{\Omega_k\setminus\overline{\Omega}} |\Psi_{j_k}-f|^{p_{k,m_k}}\,d\mu\right)^{1/p_{k,m_k}}\\
&\le \|D(u+\psi)\|(\overline{\Omega}) + 0 + \liminf_{k\to\infty}\varepsilon_k^{2/p_{k,m_k}-1}\\
&= \|D(u+\psi)\|(\overline{\Omega}).
\end{aligned}$$

In the above, we used (8.5) to arrive at the fourth inequality, and (8.9), (8.7) in obtaining the penultimate inequality. □

9. Alternate definitions of functions of least gradient

In this section we consider possible definitions of what it means for a function $u\in \mathrm{BV}(\Omega)$ to be of least gradient in an open set $\Omega\subset X$. This is not to be conflated with the notions of solutions to the Dirichlet problems studied in the previous sections, as such solutions must in addition satisfy a boundary condition.

Recall that by $\mathrm{BV}_c(\Omega)$ we mean the collection of functions $\psi\in \mathrm{BV}(\Omega)$ such that $\operatorname{supp}(\psi)\subset\Omega$, and by $\mathrm{BV}_0(\Omega)$ we mean the collection of functions $\psi\in\mathrm{BV}(\Omega)$ for which $T_+\psi$ exists and $T_+\psi = 0$ $\mathcal{H}$-a.e. in $\partial\Omega$.

In addition to the class $\mathrm{BV}_0(\Omega)$, in this section we also consider a larger class of test functions, the class $wk-\mathrm{BV}_0(\Omega)$ of functions $\psi\in\mathrm{BV}(\Omega)$ such that

$$\lim_{r\to0^+}\frac{1}{\mu(B(x,r))}\int_{B(x,r)\cap\Omega}|\psi|\,d\mu = 0$$

for $\mathcal{H}$-a.e. $x\in\partial\Omega$.

Note that $\mathrm{BV}_0(\Omega) \subset wk-\mathrm{BV}_0(\Omega)$. If $P(\Omega, X) < \infty$, recall that $\mathcal{H}(\partial^*\Omega \setminus \Sigma_\gamma\Omega) = 0$, where $\Sigma_\gamma\Omega$ was defined after (2.13). Note that if $f \in wk-\mathrm{BV}_0(\Omega)$, then for $\mathcal{H}$-a.e. $x \in (I_\Omega \cap \partial\Omega) \cup \Sigma_\gamma\Omega$ we have

$$\lim_{r\to 0^+} \frac{1}{\mu(B(x,r)\cap\Omega)} \int_{B(x,r)\cap\Omega} |u|\, d\mu = 0,$$

that is, $T_+u(x) = 0$. The philosophy here is that the trace of test functions in O_Ω, from the point of view of solving a Dirichlet problem for BV functions, should not affect the solution to the problem.

Recall that $\|Du\|^s$ denotes the singular part of the variation measure with respect to μ.

Definition 9.1. We have the following alternative notions of functions of least gradient. Let $u \in \mathrm{BV}(\Omega)$.

(1) We say that u is of *least gradient* in Ω if whenever $\psi \in \mathrm{BV}_c(\Omega)$, we have $\|Du\|(\Omega) \le \|D(u+\psi)\|(\Omega)$.
(2) We say that u is *globally of least gradient* in Ω if whenever $\psi \in \mathrm{BV}_0(\Omega)$, we have $\|Du\|(\Omega) \le \|D(u+\psi)\|(\Omega)$.
(3) We say that u is *globally of least gradient in the wider sense* in Ω if whenever $\psi \in wk-\mathrm{BV}_0(\Omega)$, we have $\|Du\|(\Omega) \le \|D(u+\psi)\|(\Omega)$.
(4) We say that u is of *least gradient in the sense of Anzellotti* in Ω if whenever $\psi \in wk-\mathrm{BV}_0(\Omega)$ with $\|D\psi\|^s \ll \|Du\|^s$, we have $\|Du\|(\Omega) \le \|D(u+\psi)\|(\Omega)$.
(5) We say that u is of *least gradient relative to Lipschitz functions* in Ω if whenever $\varphi \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ with $\varphi - u \in wk-\mathrm{BV}_0(\Omega)$, we have $\|Du\|(\Omega) \le \|D\varphi\|(\Omega)$.

Lemma 9.2. *Let $\Omega \subset X$ be an open set with $\mathcal{H}(\partial\Omega) < \infty$. Let u be of least gradient relative to Lipschitz functions in Ω. Then u is globally of least gradient in the wider sense in Ω.*

Proof. Let $\psi \in wk-\mathrm{BV}_0(\Omega)$. By [28, Corollary 6.8], we find a sequence $(\varphi_j) \subset \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ such that $\varphi_j - (u+\psi) \in wk-\mathrm{BV}_0(\Omega)$, $\varphi_j \to u+\psi$ in $L^1(\Omega)$, and $\|D\varphi_j\|(\Omega) \to \|D(u+\psi)\|(\Omega)$ as $j\to\infty$. Then also $\varphi_j - u \in wk-\mathrm{BV}_0(\Omega)$. Thus by the fact that u is of least gradient relative to Lipschitz functions,

$$\|Du\|(\Omega) \le \lim_{j\to\infty} \|D\varphi_j\|(\Omega) = \|D(u+\psi)\|(\Omega).$$

□

Proposition 9.3. *Let $\Omega \subset X$ be an open set with $\mathcal{H}(\partial\Omega) < \infty$, and let $u \in \mathrm{BV}(\Omega)$. Then the alternative definitions (1)-(5) of u being of least gradient in Ω are equivalent.*

Proof.

- (1) $\implies$ (3): This follows from the fact that for any $\psi \in wk-\mathrm{BV}_0(\Omega)$ we find a sequence of functions $\psi_k \in \mathrm{BV}_c(\Omega)$ such that $\psi_k \to \psi$ in $\mathrm{BV}(\Omega)$, by [28, Theorem 6.10].
- (3) $\implies$ (2) $\implies$ (1): These implications are trivial.
- (3) $\implies$ (4): This is trivial.
- (4) $\implies$ (5): Let $\varphi \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ with $\varphi - u \in wk - \mathrm{BV}_0(\Omega)$. Then clearly $\|D(\varphi - u)\|^s = \|Du\|^s$, so we have
$$\|Du\|(\Omega) \le \|D(u + (\varphi - u))\|(\Omega) = \|D\varphi\|(\Omega).$$
- (5) $\implies$ (3): This is Lemma 9.2.

□

Consider again the Dirichlet problem (T) of Definition 4.1. We say that $u \in \mathrm{BV}(\Omega)$ solves the Dirichlet problem (T) relative to Lipschitz functions if
$$\|Du\|(\Omega) + \int_{\partial^*\Omega} |T_+u - f|\, dP_+(\Omega, x) \le \|Dv\|(\Omega)$$
for all $v \in \mathrm{Lip}(X)$ with $v = f$ in $X \setminus \Omega$. Note that the boundary term vanishes for v.

Proposition 9.4. *Let $\Omega \subset X$ satisfy the assumptions of Lemma 7.3, and let $f \in \mathrm{Lip}(X)$ be boundedly supported. If $u \in \mathrm{BV}(\Omega)$ solves the Dirichlet problem (T) relative to Lipschitz functions, then u solves the Dirichlet problem (T).*

Proof. Pick a sequence of functions $\psi_k \in \mathrm{Lip}(X)$ given by Lemma 7.3. Then
$$\begin{aligned}\|Du\|(\Omega) + \int_{\partial^*\Omega} |T_+u - f|\, dP_+(\Omega, \cdot) &\le \lim_{k\to\infty} \|D\psi_k\|(\Omega) \\ &= \|Dv\|(\Omega) + \int_{\partial^*\Omega} |T_+v - f|\, dP_+(\Omega, \cdot).\end{aligned}$$

□

Appendix: Proof that $P_+(\Omega, \cdot)$ is a Radon measure

Let $\Omega \subset X$ be an open set with $P_+(\Omega, X) < \infty$. In showing that $P_+(\Omega, \cdot)$ is a Radon measure on X, we rely on the following theorem, due to De Giorgi and Letta [13, Theorem 5.1(3,5)], whose proof can also be found in e.g. [4, Theorem 1.53].

Theorem A.1. *Let ν be a function defined on the open sets of X taking values in $[0, \infty]$ such that $\nu(\emptyset) = 0$, $\nu(U_1) \le \nu(U_2)$ for any open sets $U_1 \subset U_2$, and such that the following properties hold:*

(1) *If $U_1, U_2 \subset X$ are open sets, then $\nu(U_1 \cup U_2) \le \nu(U_1) + \nu(U_2)$.*

(2) *If $U_1, U_2 \subset X$ are disjoint open sets, then*

$$\nu(U_1 \cup U_2) \geq \nu(U_1) + \nu(U_2).$$

(3) *For any open set $U \subset X$, $\nu(U) = \sup\{\nu(V),\ V \text{ open}, V \Subset U\}$.*

Then the extension of ν to all sets in X, defined by

$$\nu(A) := \inf\{\nu(U) : U \text{ open}, U \supset A\}, \quad A \subset X,$$

is a Borel outer measure.

Clearly we have $P_+(\Omega, \emptyset) = 0$ and $P_+(\Omega, U_1) \leq P_+(\Omega, U_2)$ for any open sets $U_1 \subset U_2$. We also note that if $U_1, U_2 \subset X$ are two disjoint open sets, then

$$P_+(\Omega, U_1 \cup U_2) = P_+(\Omega, U_1) + P_+(\Omega, U_2), \tag{A.2}$$

verifying property (2). Next we prove two "pasting" lemmas, which are analogs of [30, Lemma 3.3]. In this section, g_u always denotes the minimal 1-weak upper gradient of u.

Lemma A.3. *Let $U_1, U_2 \subset X$ be open sets such that U_1 is bounded and $\partial U_1 \cap \partial U_2 = \emptyset$. Then there exists a function $\eta \in \mathrm{Lip}_c(X)$ with $0 \leq \eta \leq 1$ such that whenever $u \in \mathrm{Lip}_{\mathrm{loc}}(U_1)$ and $v \in \mathrm{Lip}_{\mathrm{loc}}(U_2)$, the function $w := \eta u + (1-\eta)v \in \mathrm{Lip}_{\mathrm{loc}}(U_1 \cup U_2)$ satisfies*

$$\int_{U_1 \cup U_2} g_w \, d\mu \leq \int_{U_1} g_u \, d\mu + \int_{U_2} g_v \, d\mu + C(U_1, U_2) \int_{U_1 \cap U_2} |u - v| \, d\mu,$$

and whenever $h \in L^1_{\mathrm{loc}}(U_1 \cup U_2)$,

$$\int_{U_1 \cup U_2} |w - h| \, d\mu \leq \int_{U_1} |u - h| \, d\mu + \int_{U_2} |v - h| \, d\mu,$$

where $C(U_1, U_2)$ depends solely on $\mathrm{dist}(\partial U_1, \partial U_2)$ and is independent of u, v.

Proof. Let η be a Lipschitz map from $U_1 \cup U_2$ to $[0,1]$ such that $\eta = 1$ on $U_1 \setminus U_2$ and $\eta = 0$ on $U_2 \setminus U_1$. Then the desired results follow from the Leibniz rule [8, Lemma 2.18]. □

Lemma A.4. *Let $U_1' \Subset U_1$ and $U_2' \Subset U_2$ be open sets. Then there exists a function $\eta \in \mathrm{Lip}_c(U_1)$ with $0 \leq \eta \leq 1$ such that whenever $u \in \mathrm{Lip}_{\mathrm{loc}}(U_1)$ and $v \in \mathrm{Lip}_{\mathrm{loc}}(U_2)$, the function $w := \eta u + (1-\eta)v \in \mathrm{Lip}(U_1' \cup U_2')$ satisfies*

$$\int_{U_1' \cup U_2'} g_w \, d\mu \leq \int_{U_1} g_u \, d\mu + \int_{U_2} g_v \, d\mu + C(U_1', U_1) \int_{U_1 \cap U_2} |u - v| \, d\mu.$$

Proof. Let $\eta \in \mathrm{Lip}_c(X)$ such that $0 \leq \eta \leq 1$, $\eta = 1$ in U_1', and $\eta = 0$ in $X \setminus U_1$. Then the desired result again follows from the Leibniz rule. □

For an open set $W \subset X$ and $\delta > 0$, let

$$W_\delta := \{x \in W : \operatorname{dist}(x, X \setminus W) > \delta\}.$$

Lemma A.5. *Let $W \subset X$ be open and $0 < \delta_1 < \delta_2$. Then*

$$P_+(\Omega, W) \le P_+(\Omega, W_{\delta_1}) + P_+(\Omega, W \setminus \overline{W}_{\delta_2}).$$

Proof. We set $U_1 = W_{\delta_1}$ and $U_2 = W \setminus \overline{W}_{\delta_2}$. We can find sequences $(u_i) \subset \operatorname{Lip}_{\mathrm{loc}}(U_1)$ and $(v_i) \subset \operatorname{Lip}_{\mathrm{loc}}(U_2)$ such that u_i vanishes in $U_1 \setminus \Omega$, v_i vanishes in $U_2 \setminus \Omega$, $u_i - \chi_\Omega \to 0$ in $L^1(U_1)$, $v_i - \chi_\Omega \to 0$ in $L^1(U_2)$, and

$$\lim_{i\to\infty} \int_{U_1} g_{u_i}\, d\mu = P_+(\Omega, U_1), \qquad \lim_{i\to\infty} \int_{U_2} g_{v_i}\, d\mu = P_+(\Omega, U_2).$$

Applying Lemma A.3 with $u = u_i$ and $v = v_i$, we obtain functions $w_i \in \operatorname{Lip}_{\mathrm{loc}}(U_1 \cup U_2) = \operatorname{Lip}_{\mathrm{loc}}(W)$ such that $w_i = u_i$ in $U_1 \setminus U_2$, $w_i = v_i$ in $U_2 \setminus U_1$,

$$\int_W |w_i - \chi_\Omega|\, d\mu \le \int_{U_1} |u_i - \chi_\Omega|\, d\mu + \int_{U_2} |v_i - \chi_\Omega|\, d\mu,$$

and

$$\int_W g_{w_i}\, d\mu \le \int_{U_1} g_{u_i}\, d\mu + \int_{U_2} g_{v_i}\, d\mu + C(U_1, U_2) \int_{U_1\cap U_2} |u_i - v_i|\, d\mu.$$

Furthermore, by the construction of w_i, we see that w_i vanishes in $W\setminus\Omega$. From the first of the above two inequalities we see that $w_i - \chi_\Omega \to 0$ in $L^1(W)$, and hence by the second of the above two inequalities,

$$\begin{aligned} P_+(\Omega, W) &\le \liminf_{i\to\infty} \int_W g_{w_i}\, d\mu \\ &\le P_+(\Omega, U_1) + P_+(\Omega, U_2) + \limsup_{i\to\infty} \int_{U_1\cap U_2} |u_i - v_i|\, d\mu. \end{aligned}$$

Note that

$$\int_{U_1\cap U_2} |u_i - v_i|\, d\mu \le \int_{U_1\cap U_2} |u_i - \chi_\Omega| + |v_i - \chi_\Omega|\, d\mu \to 0$$

as $i \to \infty$. The desired conclusion now follows. □

Lemma A.6. *Let $W \subset X$ be an open set and let $\varepsilon > 0$. Then for some $\delta > 0$ we have*

$$P_+(\Omega, W \setminus \overline{W}_\delta) < \varepsilon.$$

Proof. Let (δ_k) be a strictly decreasing sequence of positive numbers such that $\lim_{k\to\infty} \delta_k = 0$. For integers $k \ge 2$, let $V_k := W_{\delta_{2k}} \setminus \overline{W}_{\delta_{2k-3}}$. Note then that $\{V_{2k}\}_{k\in\mathbb{N}}$ is a collection of pairwise disjoint open sets,

$\{V_{2k+1}\}_{k\in\mathbb{N}}$ is a collection of pairwise disjoint open sets, and $V_k \cap V_j$ is non-empty if and only if $|k-j| \leq 1$. From (A.2) we see that

$$\sum_{k=1}^{\infty} P_+(\Omega, V_{2k}) \leq P_+\left(\Omega, \bigcup_{k=1}^{\infty} V_{2k}\right) \leq P_+(\Omega, W) < \infty,$$

and so we can find $k_1 \in \mathbb{N}$ such that

$$\sum_{k=k_1}^{\infty} P_+(\Omega, V_{2k}) < \frac{\varepsilon}{2}.$$

Analogously, we can find $k_2 \in \mathbb{N}$ such that

$$\sum_{k=k_2}^{\infty} P_+(\Omega, V_{2k+1}) < \frac{\varepsilon}{2}.$$

Thus by choosing $k_\varepsilon = 2\max\{k_1, k_2\} + 1$, we obtain

$$\sum_{k=k_\varepsilon}^{\infty} P_+(\Omega, V_k) < \varepsilon. \tag{A.7}$$

For each $k \geq k_\varepsilon$ we can choose a sequence $(u_{k,i}) \subset \mathrm{Lip}_{\mathrm{loc}}(V_k)$ such that $u_{k,i}$ vanishes in $V_k \setminus \Omega$,

$$\int_{V_k} |u_{k,i} - \chi_\Omega| \, d\mu \leq 2^{-i-k} \min\left\{1, \frac{1}{C\left(\bigcup_{j=k_\varepsilon}^{k} V_j, V_{k+1}\right)}\right\},$$

and

$$\int_{V_k} g_{u_{k,i}} \, d\mu \leq P_+(\Omega, V_k) + 2^{-i-k}.$$

Fix $i \in \mathbb{N}$. We construct a function w_i inductively as follows. For $k = k_\varepsilon$ we set $w_{i,k} = u_{k,i}$. We apply Lemma A.3 with $U_1 = V_{k_\varepsilon}$, $U_2 = V_{k_\varepsilon+1}$, $u = w_{i,k_\varepsilon}$, and $v = u_{k+1,i}$ to obtain $w_{i,k_\varepsilon+1} \in \mathrm{Lip}_{\mathrm{loc}}(V_{k_\varepsilon} \cup V_{k_\varepsilon+1})$. Note that $w_{i,k_\varepsilon+1}$ vanishes in $(V_{k_\varepsilon} \cup V_{k_\varepsilon+1}) \setminus \Omega$,

$$\int_{V_{k_\varepsilon} \cup V_{k_\varepsilon+1}} |w_{i,k_\varepsilon+1} - \chi_\Omega| \, d\mu \leq 2^{-i}(2^{-k_\varepsilon} + 2^{-k_\varepsilon-1}),$$

and

$$\int_{V_{k_\varepsilon} \cup V_{k_\varepsilon+1}} g_{w_{i,k_\varepsilon+1}} \, d\mu \leq P_+(\Omega, V_{k_\varepsilon}) + P_+(\Omega, V_{k_\varepsilon+1}) + 2^{-i+1}(2^{-k_\varepsilon} + 2^{-k_\varepsilon-1}).$$

Now we inductively apply Lemma A.3 with $U_1 = \bigcup_{k=k_\varepsilon}^{\ell-1} V_k$ and $U_2 = V_\ell$, and $u = w_{i,\ell-1}$ and $v = u_{\ell,i}$, with $\ell > k_\varepsilon + 1$, to obtain a sequence of functions $(w_{i,\ell})_\ell$ with $w_{i,\ell} \in \mathrm{Lip}_{\mathrm{loc}}\left(\bigcup_{k=k_\varepsilon}^{\ell} V_k\right)$ such that $w_{i,\ell}$ vanishes in $\bigcup_{k=k_\varepsilon}^{\ell} V_k \setminus \Omega$,

$$\int_{\bigcup_{k=k_\varepsilon}^{\ell} V_k} |w_{i,\ell} - \chi_\Omega| \, d\mu < 2^{-i} \sum_{k=k_\varepsilon}^{\infty} 2^{-k},$$

and

$$\int_{\bigcup_{k=k_\varepsilon}^{\ell} V_k} g_{w_{i,\ell}}\,d\mu \le \sum_{k=k_\varepsilon}^{\infty} P_+(\Omega, V_k) + 2^{-i+1}\sum_{k=k_\varepsilon}^{\infty} 2^{-k}.$$

Note in addition that $w_{i,\ell} = w_{i,n+1}$ in V_n for $\ell \ge n+1$. It follows that $w_i := \lim_{\ell\to\infty} w_{i,\ell}$ exists, belongs to $\operatorname{Lip}_{\rm loc}\left(\bigcup_{k=k_\varepsilon}^{\infty} V_k\right)$, and vanishes in $\bigcup_{k=k_\varepsilon}^{\infty} V_k \setminus \Omega$. Moreover,

$$\int_{\bigcup_{k=k_\varepsilon}^{\infty} V_k} |w_i - \chi_\Omega|\,d\mu \le 2^{-(i+k_\varepsilon-1)},$$

and

$$\int_{\bigcup_{k=k_\varepsilon}^{\infty} V_k} g_{w_i}\,d\mu \le \sum_{k=k_\varepsilon}^{\infty} P_+(\Omega, V_k) + 2^{-(i+k_\varepsilon-2)}.$$

From the first of the above two inequalities it follows that $w_i - \chi_\Omega \to 0$ in $L^1\left(\bigcup_{k=k_\varepsilon}^{\infty} V_k\right)$, and so by the second of the above two inequalities,

$$P_+\left(\Omega, \bigcup_{k=k_\varepsilon}^{\infty} V_k\right) \le \liminf_{i\to\infty} \int_{\bigcup_{k=k_\varepsilon}^{\infty} V_k} g_{w_i}\,d\mu \le \sum_{k=k_\varepsilon}^{\infty} P_+(\Omega, V_k) < \varepsilon$$

by (A.7), as desired. Moreover, $\bigcup_{k=k_\varepsilon}^{\infty} V_k = W \setminus \overline{W}_\delta$ for $\delta := \delta_{2k_\varepsilon-3}$. □

By combining Lemma A.6 and Lemma A.5, we obtain property (3) of Theorem A.1, that is, for any open set $U \subset X$,

$$P_+(\Omega, U) = \sup\{P_+(\Omega, V),\ V \text{ open},\ V \Subset U\}. \tag{A.8}$$

Finally, we prove property (1) of Theorem A.1.

Lemma A.9. *Let $U_1, U_2 \subset X$ be open sets. Then*

$$P_+(\Omega, U_1 \cup U_2) \le P_+(\Omega, U_1) + P_+(\Omega, U_2).$$

Proof. Take $V \Subset U_1 \cup U_2$ and note that $V = U_1' \cup U_2'$ for some $U_1' \Subset U_1$ and $U_2' \Subset U_2$. We can find sequences $(u_i) \subset \operatorname{Lip}_{\rm loc}(U_1)$ and $(v_i) \subset \operatorname{Lip}_{\rm loc}(U_2)$ such that u_i vanishes in $U_1\setminus\Omega$, v_i vanishes in $U_2\setminus\Omega$, $u_i - \chi_\Omega \to 0$ in $L^1(U_1)$, $v_i - \chi_\Omega \to 0$ in $L^1(U_2)$, and

$$\lim_{i\to\infty}\int_{U_1} g_{u_i}\,d\mu = P_+(\Omega, U_1), \qquad \lim_{i\to\infty}\int_{U_2} g_{v_i}\,d\mu = P_+(\Omega, U_2).$$

By Lemma A.4, we then find functions $w_i \in \operatorname{Lip}_{\rm loc}(U_1' \cup U_2')$ satisfying $w_i \to \chi_\Omega$ in $L^1(U_1' \cup U_2')$ and

$$\int_{U_1'\cup U_2'} g_{w_i}\,d\mu \le \int_{U_1} g_{u_i}\,d\mu + \int_{U_2} g_{v_i}\,d\mu + C(U_1', U_1)\int_{U_1\cap U_2} |u_i - v_i|\,d\mu.$$

Note also that by the construction of w_i in Lemma A.4, w_i vanishes in $U_1' \cup U_2' \setminus \Omega$. Letting $i \to \infty$, we obtain

$$P_+(\Omega, U_1' \cup U_2') \le \liminf_{i\to\infty} \int_{U_1'\cup U_2'} g_{w_i}\,d\mu = P_+(\Omega, U_1) + P_+(\Omega, U_2).$$

By (A.8), we obtain the desired conclusion. □

Thus we have proved that $P_+(\Omega,\cdot)$ satisfies the conditions of Theorem A.1, so that $P_+(\Omega,\cdot)$ is a Borel outer measure. Borel regularity follows easily from the definition

$$P_+(\Omega,A) := \inf\{P_+(\Omega,U) : U \text{ open}, U \supset A\}, \quad A \subset X.$$

In conclusion, $P_+(\Omega,\cdot)$ is a Radon measure.

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Address:

R.K.: Department of Mathematics and Systems Analysis, Aalto University, P.O. Box 11100, FI-00076 Aalto, Finland.
E-mail: riikka.korte@aalto.fi

P.L.: Department of Mathematical Sciences, P.O. Box 210025, University of Cincinnati, Cincinnati, OH 45221–0025, U.S.A.
E-mail: lahtipk@ucmail.uc.edu

X.L.: Department of Mathematics, Sun Yat-sen University, Guangzhou 510275, China.
E-mail: lixining3@mail.sysu.edu.cn

N.S.: Department of Mathematical Sciences, P.O. Box 210025, University of Cincinnati, Cincinnati, OH 45221–0025, U.S.A.
E-mail: shanmun@uc.edu