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Equivalence of two BV classes of functions in metric spaces, and existence of a Semmes family of curves under a 1-Poincaré inequality

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Abstract

We consider two notions of functions of bounded variation in complete metric measure spaces, one due to Martio [M1, M2] and the other due to Miranda Jr. [Mi]. We show that these two notions coincide, if the measure is doubling and supports a 1-Poincaré inequality. In doing so, we also prove that if the measure is doubling and supports a 1-Poincaré inequality, then the metric space supports a *Semmes family of curves* structure.

Key words: AM-modulus, bounded variation, 1-Poincaré inequality, metric measure space, Semmes pencil of curves.

MSC classification: 26A45, 30L99, 31E05.

1 Introduction

Given $1 \le p < \infty$, a function u in $L^p(\mathbb{R}^n)$ is in $W^{1,p}(\mathbb{R}^n)$ if and only if u has an L^p-representative that is absolutely continuous on almost every non-constant compact rectifiable curve in $\mathbb{R}^n$ with derivative in $L^p(\mathbb{R}^n)$, see [Vä] for an in-depth discussion on this. Equivalently, $u \in W^{1,p}(\mathbb{R}^n)$ if and only if $u \in L^p(\mathbb{R}^n)$ and there is a non-negative Borel function $g \in L^p(\mathbb{R}^n)$ such that for all

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non-constant compact rectifiable curves γ in $\mathbb{R}^n$,

$$|u(\gamma(b)) - u(\gamma(a))| \le \int_a^b g \circ \gamma(t)|\gamma'(t)|\, dt. \tag{1}$$

On the other hand, the class of BV functions on $\mathbb{R}^n$ has a more complicated analog; there should be a sequence $f_k \in W^{1,1}(\mathbb{R}^n)$, with $f_k \to u$ in $L^1(\mathbb{R}^n)$ and function g_k associated with f_k as in the inequality above, such that $\liminf_{k\to\infty} \int_{\mathbb{R}^n} g_k\, dx$ is finite. Thus to verify that a function u belongs to the class $\mathrm{BV}(\mathbb{R}^n)$ we need a sequence of *pairs* of functions (f_k, g_k) satisfying (1), where f_k approximates u in $L^1(\mathbb{R}^n)$, whereas to define a function in $W^{1,1}(\mathbb{R}^n)$ we only need a single energy function g that satisfies (1).

The above complication carries through from $\mathbb{R}^n$ to more general metric measure spaces X, and so while we need only the energy function g in order to know that u is in the Sobolev class, to know that u is in the BV class we need both, the approximating sequence f_k as well as the corresponding energy functions g_k. To avoid this discrepancy, the recent work of Martio [M1, M2] proposed a new definition of BV functions in the Euclidean and general metric measure setting, denoted in the current paper by $\mathrm{BV}_{\mathrm{AM}}(X)$, see Definition 2.5. In this notion one needs a single sequence of "energy" functions g_k associated with the function u in a specific manner in order to determine whether $u \in \mathrm{BV}_{\mathrm{AM}}(X)$. The backbone of the construction of $\mathrm{BV}_{\mathrm{AM}}(X)$ is the notion of AM-modulus, and it appears that this modulus is better suited to the study of sets of finite perimeter than the standard 1-modulus. It is shown in [HMM2, Theorem 11] that Euclidean Borel sets E are of finite perimeter if and only if the AM-modulus of the collection of all curves that cross the measure-theoretic boundary $\partial_* E$ of E is finite; and in this case the perimeter measure of E is precisely the AM-modulus of that collection of curves. This is a variant of the Federer characterization of sets of finite perimeter. Federer proved that a measurable set $E \subset \mathbb{R}^n$ is of finite perimeter if and only if $\mathcal{H}^{n-1}(\partial_* E)$ is finite; a new, potential-theoretic proof of this characterization, valid even in the metric setting, can be found in [L].

The goal of this paper is to show that if the metric measure space X is of controlled geometry, that is, if X is complete, the measure μ is doubling and supports a 1-Poincaré inequality, then the notion of $\mathrm{BV}_{\mathrm{AM}}(X)$ from [M1, M2] gives the same function space as the BV class $\mathrm{BV}(X)$ as defined by Miranda Jr. in [Mi]. To do so we also prove that if μ is doubling, then X supports a 1-Poincaré inequality if and only if X supports a *Semmes family of curves* corresponding to each pair $x, y \in X$ of points, that is, there is a family Γ_{xy} of quasiconvex curves connecting x to y and a probability measure σ_{xy} on Γ_{xy} satisfying a Riesz-type inequality, see Definition 3.6 below. This auxiliary result is of independent interest. The notion of Semmes family of curves, first proposed in [Se] (where clearly it was not termed a "Semmes family"), is known to imply the support of a 1-Poincaré inequality, see the discussion in [He, page 29]. In this paper we show that the converse also holds true, that is, if the measure is doubling and supports a 1-Poincaré inequality, then it supports a Semmes family of curves structure. Thus, our paper also characterizes the support of a 1-Poincaré inequality (in doubling complete metric measure spaces) via the existence of a Semmes family of curves. A recent preprint [FO] gives another characterization of the support of a 1-Poincaré inequality in terms of the existence of normal 1-currents for each pair of points $x, y \in X$, in the sense of Ambrosio and Kirchcheim, such that the mass of the current is controlled by the Riesz measure R_{xy}, see (6) below. For the study comparing BV-AM spaces with BV classes of functions, a Semmes family of curves seems to be more useful.

The equality of $\mathrm{BV}(X)$ with $\mathrm{BV}_{\mathrm{AM}}(X)$ and the equivalence between the Semmes family of curves structure and the 1-Poincaré inequality form the two main results in this paper, see Theorem 3.10.

2 Two definitions of BV functions

In the rest of the paper, (X, d, μ) is a *metric measure space*, where (X, d) is a complete metric space and μ is a Borel measure. We denote by B an open ball in X and by λB the ball with the same center as B and radius λ times the radius of B. Recall that the measure μ is said to be *doubling* if there is a constant $C \geq 1$ such that $\mu(2B) \leq C\mu(B)$ for every ball B in X.

Given a compact interval $I \subset \mathbb{R}$, a *curve* $\gamma : I \to X$ is a continuous mapping. We only consider curves that are non-constant and rectifiable. A curve γ, connecting two points $x, y \in X$, is C-*quasiconvex* if its length is at most $C\, d(x, y)$.

2.1 p-Modulus and AM-modulus of a family of curves

Definition 2.1 Given a family Γ of curves in X, set $\mathcal{A}(\Gamma)$ to be the family of all Borel measurable functions $\rho : X \to [0, \infty]$ such that

$$\int_\gamma \rho\, ds \geq 1 \quad \text{for all} \quad \gamma \in \Gamma,$$

and set $\mathcal{A}_{seq}(\Gamma)$ to be the family of all sequences (ρ_i) of non-negative Borel measurable functions ρ_i on X such that

$$\liminf_{i\to\infty} \int_\gamma \rho_i\, ds \geq 1 \quad \text{for all} \quad \gamma \in \Gamma.$$

The integral $\int_\gamma \rho\, ds$ denotes the path-integral of γ against the arc-length re-parametrization of γ, see for example the description in [He]. We define the ∞-*modulus of* Γ by

$$\mathrm{Mod}_\infty(\Gamma) = \inf_{\rho\in\mathcal{A}(\Gamma)} \|\rho\|_{L^\infty(X)},$$

and for $1 \leq p < \infty$ the p-modulus of Γ is

$$\mathrm{Mod}_p(\Gamma) = \inf_{\rho\in\mathcal{A}(\Gamma)} \int_X \rho^p\, d\mu.$$

Following [M1, M2], we define the *approximate modulus* (AM-modulus) of Γ by

$$\mathrm{AM}(\Gamma) = \inf_{(\rho_i)\in\mathcal{A}_{seq}(\Gamma)} \left\{ \liminf_{i\to\infty} \int_X \rho_i\, d\mu \right\}.$$

The notion of $\mathrm{AM}_p(\Gamma)$ is defined analogously, with $\int_X \rho_i \, d\mu$ replaced by $\int_X \rho_i^p \, d\mu$. If a property holds for all except for a family Γ of curves with $\mathrm{Mod}_p \Gamma = 0$ (respectively with $\mathrm{AM}(\Gamma) = 0$), then we say that the property holds for *p-a.e. curve* (respectively for AM-*a.e. curve*).

Note that $\mathrm{AM}(\Gamma) \le \mathrm{Mod}_1(\Gamma)$. Thanks to Mazur's lemma, it is a trivial consequence of the reflexivity of $L^p(X)$ that $\mathrm{AM}_p(\Gamma) = \mathrm{Mod}_p(\Gamma)$ when $1 < p < \infty$, see [HMM, Theorem 1]. It is also easy to see that for any family of curves Γ we have $\mathrm{AM}_\infty(\Gamma) = \mathrm{Mod}_\infty(\Gamma)$. Indeed, let $\tau = \mathrm{AM}_\infty(\Gamma)$. If $\tau = \infty$ there is nothing to prove, so let us assume that $\tau < \infty$. By definition, there is a sequence of non-negative Borel functions $(g_i^\varepsilon) \in \mathcal{A}_{seq}(\Gamma)$ such that

$$\liminf_{i\to\infty} \|g_i^\varepsilon\|_{L^\infty(X)} < \tau + \varepsilon \quad \text{and} \quad \liminf_{i\to\infty} \int_\gamma g_i^\varepsilon \, ds \ge 1 \text{ for each } \gamma \in \Gamma.$$

Let $\rho_\varepsilon := \sup_i g_i^\varepsilon$. As $\rho_\varepsilon \ge g_i^\varepsilon$ for each $i \in \mathbb{N}$, it follows that

$$1 \le \liminf_{i\to\infty} \int_\gamma g_i^\varepsilon \, ds \le \int_\gamma \rho_\varepsilon \, ds,$$

and so $\mathrm{Mod}_\infty(\Gamma) \le \|\rho_\varepsilon\|_{L^\infty(X)} \le \tau + \varepsilon$ and the result follows.

Note that if every curve in Γ is contained in a fixed ball B, then

$$\mathrm{AM}(\Gamma) \le \mathrm{Mod}_1(\Gamma) \le \mu(B)^{1-1/p} \, \mathrm{Mod}_p(\Gamma)^{1/p} \le \mu(B) \, \mathrm{Mod}_\infty(\Gamma),$$

and therefore

$$\limsup_{p\to\infty} \left[\mathrm{Mod}_p(\Gamma)\right]^{1/p} \le \mathrm{Mod}_\infty(\Gamma).$$

The next example shows that it is possible to have $\mathrm{Mod}_1(\Gamma) = \infty$ but $\mathrm{AM}(\Gamma) = 1$. Further examples can be found in [HMM, Section 9]. The examples found there are families of curves that tangentially approach a smooth co-dimension one sub-manifold of $\mathbb{R}^n$.

Example 2.2 Let Γ be the collection of all rectifiable curves of length at most 1 in the plane, and start from the x-axis with $0 \le x \le 1$ and are parallel to the y-axis. Then there is no acceptable $\rho \in L^1(X)$ for computing $\mathrm{Mod}_1(\Gamma)$, and hence $\mathrm{Mod}_1(\Gamma) = \infty$. On the other hand, $\mathrm{AM}(\Gamma)$ is finite but positive. To see this, for each positive integer let $\rho_n = n \, \chi_{[0,1]\times[0,1/n]}$. Then $\int_\gamma \rho_n \, ds \ge 1$ whenever γ is in Γ with length at least $1/n$, and as every curve in Γ has positive length, we have that

$$\lim_{n\to\infty} \int_\gamma \rho_n \, ds \ge 1.$$

So the sequence (ρ_n) is admissible for Γ, and thus

$$\mathrm{AM}(\Gamma) \le \limsup_{n\to\infty} \int_{\mathbb{R}^2} \rho_n \, d\mathcal{L}^2 = \limsup_{n\to\infty} n \left(\frac{1}{n} \times 1\right) = 1.$$

To see that $\mathrm{AM}(\Gamma) > 0$, we consider the sub-family $\Gamma_{1/2}$ of all line segments in Γ with length $1/2$, and let $(\rho_i) \in \mathcal{A}_{seq}(\Gamma_{1/2})$. Then by Fubini's theorem, for each $i \in \mathbb{N}$ we have

$$\int_{\mathbb{R}^2} \rho_i \, d\mathcal{L}^2 \geq \int_0^1 \int_0^{1/2} \rho_i(x, y) \, dy \, dx = \int_0^1 \left(\int_0^{1/2} \rho_i(x, y) \, dy \right) dx.$$

Now by Fatou's lemma,

$$\liminf_{i \to \infty} \int_{\mathbb{R}^2} \rho_i \, d\mathcal{L}^2 \geq \int_0^1 \left(\liminf_{i \to \infty} \int_0^{1/2} \rho_i(x, y) \, dy \right) dx \geq 1.$$

It follows that

$$\mathrm{AM}(\Gamma) \geq \mathrm{AM}(\Gamma_{1/2}) \geq 1.$$

2.2 BV functions based on the notion of AM-modulus.

Definition 2.3 A nonnegative Borel function g on X is a 1-*weak upper gradient* of an extended real-valued function u on X if for 1-a.e. curve $\gamma : [a, b] \to X$,

$$|u(\gamma(a)) - u(\gamma(b))| \leq \int_\gamma g \, ds.$$

Given a function u that has a 1-weak upper gradient in $L^1(X)$, there is a *minimal* 1-weak upper gradient of u, denoted g_u, in the sense that whenever g is a 1-weak upper gradient of u, we have $g_u \leq g$ almost everywhere in X.

The following notion of BV functions on X is due to Miranda Jr. [Mi].

Definition 2.4 (BV functions) For $u \in L^1_{\mathrm{loc}}(X)$, we define the total variation of u as

$$\|Du\|(X) := \inf \left\{ \liminf_{i \to \infty} \inf_{g_{u_i}} \int_X g_{u_i} d\mu : u_i \in \mathrm{LIP}_{\mathrm{loc}}(X), u_i \to u \,\mathrm{in}\, L^1_{\mathrm{loc}}(X) \right\},$$

where the second infimum is over all 1-weak upper gradients g_{u_i} of u_i. We say that a function $u \in L^1_{\mathrm{loc}}(X)$ is of *bounded variation*, $u \in \mathrm{BV}(X)$ if $\|Du\|(X) < \infty$. A measurable set $E \subset X$ is said of *finite perimeter* if $\|D\chi_E\|(X) < \infty$.

The following definition of $\mathrm{BV}_{\mathrm{AM}}$ class is from [M1].

Definition 2.5 (BV-AM functions) A function $u \in L^1(X)$ is in the $\mathrm{BV}_{\mathrm{AM}}(X)$ class if there is a family Γ of rectifiable curves in X with $\mathrm{AM}(\Gamma) = 0$, and a sequence (g_i) of non-negative Borel measurable functions in $L^1(X)$ such that whenever $\gamma : [a, b] \to X$ is a non-constant compact rectifiable curve that does not belong to Γ, we have that

$$|u(\gamma(t)) - u(\gamma(s))| \leq \liminf_{i \to \infty} \int_{\gamma|_{[s,t]}} g_i \, ds \tag{2}$$

for $\mathcal{H}^1$-a.e. $s,t \in [a,b]$ with $s<t$, and

$$\liminf_{i\to\infty} \int_X g_i \, d\mu < \infty.$$

Such a sequence (g_i) is said to be a $\mathrm{BV}_{\mathrm{AM}}$-*upper bound* of u. We set

$$\|D_{\mathrm{AM}} u\|(X) := \inf_{(g_i)} \liminf_{i\to\infty} \int_X g_i \, d\mu,$$

where the infimum is over all $\mathrm{BV}_{\mathrm{AM}}$-upper bounds of u.

Notice that by [M2, Theorem 4.1], $\mathrm{BV}(X) \subseteq \mathrm{BV}_{\mathrm{AM}}(X)$. This also follows from the next lemma. The following lemma holds even if μ is not doubling or does not support a 1-Poincaré inequality.

Lemma 2.6 *Assume that $u \in \mathrm{BV}(X)$. Then there is a set $N \subset X$ with $\mu(N) = 0$ and a sequence (g_i) of non-negative Borel measurable functions in $L^1(X)$ such that whenever γ is a non-constant compact rectifiable curve with end-points $x, y \in X \setminus N$,*

$$|u(y) - u(x)| \le \liminf_{i\to\infty} \int_\gamma g_i \, ds$$

(*that is,* (2) *holds*) *and*

$$\liminf_{i\to\infty} \int_X g_i \, d\mu < \infty.$$

Note that the lemma gives a stronger control of u than allowed by the $\mathrm{BV}_{\mathrm{AM}}$-control. For functions in $\mathrm{BV}_{\mathrm{AM}}(X)$, we know that given a path γ there is a set N_γ with $\mathcal{H}^1(\gamma^{-1}(N_\gamma)) = 0$ so that whenever x, y lie in the trajectory of γ with $x, y \notin N_\gamma$, inequality (2) holds. Here we show that we can choose N_γ to be independent of γ and in addition with μ-measure zero.

Proof. Given $u \in \mathrm{BV}(X)$ there is a sequence $u_i \in \mathrm{LIP}_{\mathrm{loc}}(X)$ such that $u_i \to u$ in $L^1(X)$ and $\lim_{i\to\infty} \int_X g_i \, d\mu \le M < \infty$ for a choice of upper gradients g_i of u_i. By passing to a subsequence if necessary, we may also assume that $u_i \to u$ pointwise μ-a.e. in X. Let N be the set of all points $x \in X$ for which $\lim_{i\to\infty} u_i(x) \neq u(x)$. Then $\mu(N) = 0$. Let γ be a non-constant compact rectifiable curve in X with end points $x, y \in X \setminus N$. Then

$$|u(x) - u(y)| = \lim_{i\to\infty} |u_i(x) - u_i(y)| \le \liminf_{i\to\infty} \int_\gamma g_i \, ds.$$

$\square$

The main focus of this paper is to show that $\mathrm{BV}_{\mathrm{AM}}(X) = \mathrm{BV}(X)$ when the measure on X is doubling and supports a 1-Poincaré inequality.

2.3 The spaces $N^{1,1}(X)$ and $N^{1,1}_{\mathrm{AM}}(X)$

Let $\widetilde{N}^{1,1}(X,d,\mu)$, where $1 \le p < \infty$, be the class of all L^1-integrable Borel functions on X for which there exists a 1-weak upper gradient in $L^1(X)$. For $u \in \widetilde{N}^{1,1}(X,d,\mu)$ we define

$$\|u\|_{\widetilde{N}^{1,1}(X)} = \|u\|_{L^1(X)} + \inf_g \|g\|_{L^1(X)},$$

where the infimum is taken over all 1-weak upper gradients g of u. As usual, we can now define $N^{1,1}(X,d,\mu)$ equipped with the norm $\|u\|_{N^{1,1}(X)} = \|u\|_{\widetilde{N}^{1,1}(X)}$.

Once we have the new concept of AM-a.e. curve, it is natural to define an upper gradient and a Sobolev class related to this notion.

Definition 2.7 (Weak AM-upper gradient) A nonnegative Borel function g on X is a *weak* AM-*upper gradient* of u on X if $|u(\gamma(a)) - u(\gamma(b))| \le \int_\gamma g\, ds$ for AM-a.e. curve $\gamma : [a,b] \to X$.

Definition 2.8 ($N^{1,1}_{\mathrm{AM}}$ functions) Let $\widetilde{N}^{1,1}_{\mathrm{AM}}(X,d,\mu)$, be the class of all Borel functions in $L^1(X)$ for which there exists a weak AM-upper gradient in $L^1(X)$. For $u \in \widetilde{N}^{1,1}_{\mathrm{AM}}(X)$ we define

$$\|u\|_{\widetilde{N}^{1,1}_{\mathrm{AM}}(X)} = \|u\|_{L^1(X)} + \inf_g \|g\|_{L^1(X)},$$

where the infimum is taken over all weak AM-upper gradient g of u. We can now define $N^{1,1}_{\mathrm{AM}}(X)$ to be the class $\widetilde{N}^{1,1}_{\mathrm{AM}}(X,d,\mu)$, equipped with the norm $\|u\|_{N^{1,1}_{\mathrm{AM}}(X)} = \|u\|_{\widetilde{N}^{1,1}_{\mathrm{AM}}(X)}$.

The following lemma proves that the first definition implies the second one. In some sense, the first definition is related to the Sobolev class $N^{1,1}$ while the second is related to the BV class.

Lemma 2.9 *If a function u on X has g as a weak* AM-*upper gradient, then there exists a* $\mathrm{BV}_{\mathrm{AM}}$-*upper bound of u.*

Proof. Assume that

$$|u(\gamma(a)) - u(\gamma(b))| \le \int_\gamma g\, ds \tag{3}$$

for AM-a.e. curve $\gamma : [a,b] \to X$. Let Γ be the collection of curves for which (3) does not hold. By definition $\mathrm{AM}(\Gamma) = 0$ and so by [HMM, Theorem 7] there is a sequence of non-negative Borel functions $\widetilde{g}_i$ such that

$$\liminf_{i\to\infty} \|\widetilde{g}_i\|_{L^1} < \infty \quad \text{and} \quad \liminf_{i\to\infty} \int_\gamma \widetilde{g}_i\, ds = \infty \ \text{ for all } \gamma \in \Gamma.$$

Let Γ_0 be the collection of all non-constant compact rectifiable curves γ in X for which

$$\liminf_{i\to\infty} \int_\gamma \widetilde{g}_i\, ds = \infty;$$

then $\mathrm{AM}(\Gamma_0) = 0$. Observe that if γ is a non-constant compact rectifiable curve in X such that $\gamma \notin \Gamma_0$, then every sub-curve of γ also does not belong to Γ_0. Now, for each $\varepsilon > 0$ the sequence of functions $g_i = g + \varepsilon \widetilde{g}_i$ has the property that for $\gamma \notin \Gamma_0$,

$$|u(\gamma(a)) - u(\gamma(b))| \le \int_\gamma g\, ds \le \liminf_{i\to\infty} \int_\gamma (g + \varepsilon \widetilde{g}_i)\, ds,$$

and for $\gamma \in \Gamma_0$,

$$|u(\gamma(a)) - u(\gamma(b))| \le \infty = \int_\gamma (g + \varepsilon \widetilde{g}_i)\, ds,$$

so $|u(\gamma(a)) - u(\gamma(b))| \le \liminf_{i\to\infty} \int_\gamma g_i\, ds$ holds for every curve γ. $\square$

Note that we have more than just that the sequence (g_i) forms a $\mathrm{BV}_{\mathrm{AM}}$-upper bound of u; the inequality holds for *every* subcurve of γ, not merely for $\mathcal{H}^1$-almost every pair of points in the domain of γ.

From the above we know that for $1 < p < \infty$, $N^{1,1}(X) \subseteq N^{1,1}_{\mathrm{AM}}(X) \subsetneq \mathrm{BV}_{\mathrm{AM}}(X)$ and

$$\mathrm{LIP}^\infty(X) \subseteq N^{1,\infty}(X) \subseteq N^{1,p}(X) \subseteq N^{1,1}(X) \subsetneq \mathrm{BV}(X) \subseteq \mathrm{BV}_{\mathrm{AM}}(X).$$

In Section 3 we will show that if X supports a 1-Poincaré inequality then $\mathrm{BV}_{\mathrm{AM}}(X) = \mathrm{BV}(X)$ and that $N^{1,1}_{\mathrm{AM}}(X) = N^{1,1}(X)$.

Remark 2.10 For $u \in \mathrm{BV}_{\mathrm{AM}}(X)$ and a sequence (g_i) such that $\lim_{i\to\infty} \int_X g_i\, d\mu < \infty$, the sequence of measures $(g_i\, d\mu)$ is a bounded sequence. We can assume (by localizing the argument if need be) that X is compact as well. Then there is a subsequence, also denoted $(g_i\, d\mu)$, and a Radon measure ν on X such that the sequence of measures $(g_i\, d\mu)$ converges weakly* to $d\nu$ in X. As X is compact, we see that $\|D_{\mathrm{AM}} u\|(X) \le \nu(X)$.

3 Equivalence of BV and AM-BV classes under Poincaré inequality

The aim of this section is to show the equivalence of the functional spaces $\mathrm{BV}(X)$ and $\mathrm{BV}_{\mathrm{AM}}(X)$, under the additional hypothesis that the metric space supports a doubling measure and a 1-Poincaré inequality.

Definition 3.1 The metric measure space X supports a 1-*Poincaré inequality* if there are positive constants C, λ such that whenever B is a ball in X and g is an upper gradient of u,

$$\fint_B |u - u_B|\, d\mu \le C \operatorname{rad}(B) \fint_{\lambda B} g\, d\mu.$$

Here $u_B := \mu(B)^{-1} \int_B u\, d\mu = \fint_B u\, d\mu$ is the average of u on the ball B.

With the notion of $\mathrm{BV}_{\mathrm{AM}}$ class, one could even define a stronger version of 1-Poincaré inequality.

Definition 3.2 We say that X supports an AM-*Poincaré inequality* if there exist constants $C > 0$, $\lambda \geq 1$ such that for each measurable function u on X, each BV-upper bound (g_i) of u, and each ball $B \subset X$, we have

$$\fint_B |u - u_B| \, d\mu \leq C \,\mathrm{rad}(B) \liminf_{i\to\infty} \fint_{\lambda B} g_i \, d\mu.$$

This should imply that

$$\fint_B |u - u_B| \, d\mu \leq C \,\mathrm{rad}(B) \, \frac{\|D_{\mathrm{AM}} u\|(\lambda B)}{\mu(\lambda B)}.$$

On the other hand, notice that 1-Poincaré inequality implies

$$\fint_B |u - u_B| \, d\mu \leq C \,\mathrm{rad}(B) \frac{\|Du\|(\tau B)}{\mu(\tau B)}.$$

As a first step, in the following proposition we prove the equivalence of $\mathrm{BV}(X)$ and $\mathrm{BV}_{\mathrm{AM}}(X)$ under the hypotheses that the measure is doubling and the space supports an AM-Poincaré inequality. We will see in Theorem 3.10 that the support of an AM-Poincaré inequality is equivalent to the support of a 1-Poincaré inequality.

Proposition 3.3 *If X supports a* AM-*Poincaré inequality and μ is doubling, then the two classes* $\mathrm{BV}_{\mathrm{AM}}(X)$ *and* $\mathrm{BV}(X)$ *are equal, with comparable norms.*

Proof. Note first that $\mathrm{BV}(X) \subset \mathrm{BV}_{\mathrm{AM}}(X)$, see Lemma 2.6.

Now let us prove that if $u \in \mathrm{BV}_{\mathrm{AM}}(X)$, then $u \in \mathrm{BV}(X)$. By the doubling property of μ, for $\varepsilon > 0$ we can cover X by balls $B_i = B(x_i, \varepsilon)$ such that the balls $5\lambda B_i$ have bounded overlap. Let φ_i^ε be a partition of unity subordinate to the cover $2B_i$. For $u \in \mathrm{BV}_{\mathrm{AM}}(X)$ let

$$u_\varepsilon = \sum_i u_{B_i} \varphi_i^\varepsilon.$$

Recall that we have bounded overlap of the collection $5B_i$ with $X = \bigcup_j B_j$, μ is doubling, and that if $2B_i$ intersects B_j then $5B_j \supset 2B_i$. Then we have for $x \in B_j \subset X$,

$$\begin{aligned}
|u(x) - u_\varepsilon(x)| = \left| \sum_i [u_{B_i} - u(x)] \varphi_i^\varepsilon(x) \right| &\leq \sum_i |u_{B_i} - u(x)| \varphi_i^\varepsilon(x) \\
&\leq \sum_{i\,:\,x \in 2B_i} |u_{B_i} - u(x)| \\
&\leq \sum_{i\,:\,2B_i \cap B_j \neq \emptyset} |u_{B_i} - u(x)| \\
&\leq C\, C_D^3 \, |u_{5B_j} - u(x)|.
\end{aligned}$$

Therefore, by the AM-Poincaré inequality,

$$\begin{aligned}\int_X |u-u_\varepsilon|\,d\mu \le \sum_j \int_{B_j} |u-u_\varepsilon|\,d\mu &\le C\sum_j \int_{B_j} |u_{5B_j}-u|\,d\mu \\ &\le C\sum_j \int_{5B_j} |u-u_{5B_j}|\,d\mu \\ &\le C\varepsilon \sum_j \|D_{\mathrm{AM}}u\|(5\lambda B_j).\end{aligned}$$

Since $\|D_{\mathrm{AM}}u\|$ is a Radon measure ([M1, Theorem 3.4]) and $5\lambda B_j$ have bounded overlap, we have

$$\int_X |u-u_\varepsilon|\,d\mu \le C\varepsilon\,\|D_{\mathrm{AM}}u\|(X) \to 0 \text{ as } \varepsilon\to 0^+.$$

Thus $u_\varepsilon \to u$ in $L^1(X)$, and we also know from the definition of u_ε that each u_ε is locally Lipschitz and hence in $N^{1,1}_{loc}(X)$. Next, for $x, y \in B_j$,

$$\begin{aligned}|u_\varepsilon(x)-u_\varepsilon(y)| = \left|\sum_i u_{B_i}[\varphi_i^\varepsilon(x)-\varphi_i^\varepsilon(y)]\right| &= \left|\sum_i [u_{B_i}-u_{B_j}][\varphi_i^\varepsilon(x)-\varphi_i^\varepsilon(y)]\right| \\ &\le \frac{d(x,y)}{\varepsilon} \sum_{i\,:\,2B_i\cap B_j\neq\emptyset} |u_{B_i}-u_{B_j}| \\ &\le \frac{C\,d(x,y)}{\varepsilon} \sum_{i\,:\,2B_i\cap B_j\neq\emptyset} \fint_{5B_j} |u-u_{5B_j}|\,d\mu.\end{aligned}$$

It follows from the bounded overlap of $5B_i$ that

$$\mathrm{Lip} u_\varepsilon(x) \le \frac{C}{\varepsilon}\fint_{5B_j} |u-u_{5B_j}|\,d\mu$$

whenever $x\in B_j$. Integrating the above inequality over $X=\bigcup_j B_j$, we obtain

$$\begin{aligned}\int_X \mathrm{Lip} u_\varepsilon\,d\mu &\le \frac{C}{\varepsilon}\sum_j \mu(B_j)\fint_{5B_j} |u-u_{5B_j}|\,d\mu \\ &\le \frac{C}{\varepsilon}\sum_j \int_{5B_j} |u-u_{5B_j}|\,d\mu \\ &\le \frac{C}{\varepsilon}\sum_j \varepsilon\,\|D_{\mathrm{AM}}u\|(5\lambda B_j) \\ &\le C\,\|D_{\mathrm{AM}}u\|(X). \end{aligned}\tag{4}$$

Thus

$$\liminf_{\varepsilon\to 0^+}\int_X g_{u_\varepsilon}\,d\mu \le C\,\|D_{\mathrm{AM}}u\|(X) < \infty,$$

and as $u_\varepsilon \to u$ in $L^1(X)$, it follows that $u\in \mathrm{BV}(X)$ with $\|Du\|(X)\le C\,\|D_{\mathrm{AM}}u\|(X)$.

We also have $\|D_{\rm AM}u\|(X) \le \|Du\|(X)$, as we now show. Suppose now that $\|Du\|(X)$ is finite, and let $u_k \in \mathrm{BV}(X)$ be such that $u_k \to u$ in $L^1(X)$ and $\lim_{k\to\infty} \int_X g_{u_k}\, d\mu = \|Du\|(X)$. By passing to a subsequence if necessary, we may also assume that $u_k \to u$ pointwise almost everywhere in X as well. For each $k \in \mathbb{N}$ we choose an upper gradient g_k of u_k such that $\int_X g_k\, d\mu \le \int_X g_{u_k}\, d\mu + \varepsilon/2^k$. We set N to be the collection of all points $x \in X$ at which $u_k(x)$ does not converge to $u(x)$. Then $\mu(N) = 0$, and so the 1-modulus of the collection $\widehat{\Gamma}_N^+$ of non-constant compact rectifiable curves γ in X for which $\mathcal{H}(\gamma^{-1}(N)) > 0$ is zero. Using [He, (7.8)], we know that the collection Γ_N^+ of all non-constant compact rectifiable curves in X with a subcurve in $\widehat{\Gamma}_N^+$ is also of 1-modulus zero. Let γ be a non-constant compact rectifiable curve in X with $\gamma \notin \Gamma_N^+$. By re-parametrizing if necessary, we now assume that $\gamma : [a, b] \to X$ is arc-length parametrized; then $\mathcal{H}^1([a, b] \setminus \gamma^{-1}(N)) = 0$. For $s, t \in [a, b] \setminus \gamma^{-1}(N)$ with $s > t$ we have that

$$|u(\gamma(t)) - u(\gamma(s))| = \lim_{k\to\infty} |u_k(\gamma(t)) - u_k(\gamma(s))| \le \liminf_{k\to\infty} \int_{\gamma|_{[t,s]}} g_k\, ds \le \liminf_{k\to\infty} \int_\gamma g_k\, ds.$$

This verifies that (g_k) is a $\mathrm{BV}_{\rm AM}$-upper bound for u in the sense of Definition 2.5. $\square$

Proposition 3.4 *If X supports an* AM*-Poincaré inequality and μ is doubling, then $N^{1,1}_{\rm AM}(X) = N^{1,1}(X)$ with comparable norms.*

Proof. Note that $N^{1,1}(X) \subset N^{1,1}_{\rm AM}(X)$. Thus it suffices to prove the reverse inclusion. Let $u \in N^{1,1}_{\rm AM}(X)$, and let $g \in L^1(X)$ be a weak AM-upper gradient of u. Let Γ be the corresponding exceptional family; then $\mathrm{AM}(\Gamma) = 0$. Then by the proof of Lemma 2.9 there is a sequence (ρ_i) of non-negative Borel functions in $L^1(X)$ such that $\int_X \rho_i\, d\mu \le M < \infty$ for each $i \in \mathbb{N}$ and for each $\gamma \in \Gamma$ we have

$$\lim_{i\to\infty} \int_\gamma \rho_i\, ds = \infty.$$

Then for each $\varepsilon > 0$ we have that $(g + \varepsilon\rho_i)$ forms a $\mathrm{BV}_{\rm AM}$-upper bound of u, and so as X supports an AM-Poincaré inequality, whenever B is a ball in X we have

$$\fint_B |u - u_B|\, d\mu \le \frac{C\,\mathrm{rad}(B)}{\mu(B)} \liminf_{i\to\infty} \int_{\lambda B} [g + \varepsilon\rho_i]\, d\mu.$$

As before, by passing to a subsequence if necessary, we may assume that $\rho_i\, d\mu$ converges weakly to a Radon measure ν on X, and so the above turns into

$$\fint_B |u - u_B|\, d\mu \le C\,\mathrm{rad}(B) \left(\fint_{\lambda B} g\, d\mu + \varepsilon \frac{\nu(2\lambda B)}{\mu(2\lambda B)} \right).$$

Letting $\varepsilon \to 0$ we get

$$\fint_B |u - u_B|\, d\mu \le C\,\mathrm{rad}(B) \fint_{\lambda B} g\, d\mu.$$

We now know from Proposition 3.3 that $u \in \mathrm{BV}(X)$. Now an argument as in the proof of Proposition 3.3, up to and including (4), applied to open sets $U \subset X$ with $\mu(\partial U) = 0$, we obtain that

$$\|Du\|(U) \leq C \int_{\overline{U}} g\, d\mu = \int_U g\, d\mu.$$

Note that $g \in L^1(X)$, and hence for each $\eta > 0$ there is some $\varepsilon > 0$ such that whenever $K \subset X$ is measurable with $\mu(K) < \varepsilon$, we have $\int_K g\, d\mu < \eta$. Since whenever $E \subset X$ with $\mu(E) = 0$, for each $\varepsilon > 0$ we can find an open set $U_\varepsilon \supset E$ such that $\mu(U_\varepsilon) < \varepsilon$ and $\mu(\partial U_\varepsilon) = 0$, it follows that $\|Du\| \ll \mu$, and hence $u \in N^{1,1}(X)$ by [HKLL, Theorem 4.6]. $\square$

Note that if X does not support a 1-Poincaré inequality, we do not know the equivalence of $N^{1,1}(X)$ with $N^{1,1}_{\mathrm{AM}}(X)$. Similar difficulties show up in comparing other alternative notions of $N^{1,1}(X)$ as well, see for example [ADiM, Section 8]. We will prove in Theorem 3.10 that X supports a 1-Poincaré inequality if and only if it supports the a priori stronger AM-Poincaré inequality.

The key point in the above proof is that if $u \in \mathrm{BV}(X)$ and $\|Du\| \ll \mu$, then $u \in N^{1,1}(X)$; the validity of this point requires a doubling measure supporting a 1-Poincaré inequality. The following counterexample is from [ADiM, Example 7.4]. We do not have a counterexample for the statement "$\|Du\| \ll \mu$ implies $u \in N^{1,1}(X)$" in the case μ is doubling, but the measure μ in the following example is asymptotically doubling.

Example 3.5 Let $X = \mathbb{R}^2$ be equipped with the Euclidean metric and the measure $\mu = \mathcal{L}^2 + \mathcal{H}^1|_C$ where C is the boundary of the unit disk D in $\mathbb{R}^2$ centered at the origin. Let $u = \chi_D$. Then, by the approximations $f_\varepsilon(x) = (1 - \varepsilon^{-1}\mathrm{dist}(x, D))_+$ of u we see that $u \in \mathrm{BV}(X)$ with $\|Du\| \equiv \mathcal{H}^1|_C$. It follows that $\|Du\| \ll \mu$. However, $u \notin N^{1,1}(X)$: with Γ the collection of all line segments γ_x, $-1 < x < 1$, given by $\gamma_x : [-2, 2] \to X$ where $\gamma_x(t) = (x, t)$, we have that $u \circ \gamma_x$ is not absolutely continuous on $[-2, 2]$, and furthermore, $\mathrm{Mod}_1(\Gamma) > 0$.

The existence of a Semmes family of curves provides a key tool for the proof that the AM-Poincaré inequality and the standard 1-Poincaré inequality are equivalent, which in turn allows us to prove equivalence of the two classes $\mathrm{BV}(X)$ and $\mathrm{BV}_{\mathrm{AM}}(X)$ with just the assumption of a 1-Poincaré inequality in addition to the doubling property of μ. Thus we next prove that the existence of 1-Poincaré inequality in the doubling complete metric measure space X is equivalent to the existence of the following Semmes pencil of curves. See [FO] for a closely related characterization of the 1-Poincaré inequality in terms of 1-currents in the sense of Ambrosio and Kirchheim [AK].

If A is a Borel subset of X and γ is a rectifiable curve, we define $\ell(\gamma \cap A) := \mathcal{H}^1(\gamma \cap A)$.

Definition 3.6 ([Se, He]) A space X supports a *Semmes pencil of curves* if there exists a constant $C > 0$ such that for each pair of points $x, y \in X$ with $x \neq y$ there is a family Γ_{xy} of rectifiable curves in X equipped with a probability measure $d\sigma = d\sigma_{x,y}$ so that each $\gamma \in \Gamma_{xy}$ is a C-quasiconvex curve joining x to y, and for each Borel set $A \subset X$, the map $\gamma \mapsto \ell(\gamma \cap A)$ is σ-measurable and satisfies

$$\int_{\Gamma_{xy}} \ell(\gamma \cap A)\, d\sigma(\gamma) \leq C \int_{CB_{x,y} \cap A} R_{xy}(z)\, d\mu(z). \tag{5}$$

In the previous inequality, for $C > 0$, $CB_{x,y} := B(x, Cd(x,y)) \cup B(y, Cd(x,y))$ and

$$R_{xy}(z) := \frac{d(x,z)}{\mu(B(x, d(x,z)))} + \frac{d(y,z)}{\mu(B(y, d(y,z)))}. \tag{6}$$

We next show that if the measure on X is doubling and supports a 1-Poincaré inequality, then it supports a Semmes pencil of curves.

Denote

$$\Gamma_{xy}^C := \{(\gamma, I) \,:\; \text{ curve } \gamma : I \to X \text{ is 1-Lipschitz, } \gamma(0) = x, \gamma(\max(I)) = y\}, \tag{7}$$

where I are intervals contained in $[0, C\, d(x,y)]$ with left-hand end point 0. We equip Γ_{xy}^C with the following metric. The elements of Γ_{xy}^C can be identified with their graphs

$$\Gamma_\gamma = \{(t, \gamma(t)) \,:\, t \in I\} \subset \mathbb{R} \times X.$$

We define a metric on Γ_{xy}^C by setting

$$d(\gamma, \gamma') := d_H(\Gamma_\gamma, \Gamma_{\gamma'}),$$

where d_H is the Hausdorff metric. Thanks to the Arzela-Ascoli theorem, this metric makes Γ_{xy}^C into a complete and compact metric space because X is complete and doubling and hence closed bounded subsets of X are compact. For $f \in C(X)$, the functional $\Phi_f : \Gamma_{xy}^C \to \mathbb{R}$ given by

$$\Phi_f((\gamma, I)) := \int_I f \circ \gamma\, dt,$$

is continuous on Γ_{xy}^C.

We denote the Riesz measure by

$$d\overline{\mu}_{xy}^C(z) = R_{xy}\; d\mu|_{CB_{x,y}}.$$

Theorem 3.7 *If (X, d, μ) satisfies a 1-Poincaré inequality, then there exists $C \geq 1$ such that for every $x, y \in X$ with $x \neq y$, there exist a compact family of curves Γ_{xy} and a Radon probability measure α_{xy} on Γ_{xy} which constitutes a Semmes family of curves, i.e. for every Borel set A,*

$$\int_{\Gamma_{xy}^C} \int_\gamma \chi_A\, ds\, d\alpha_{xy}(\gamma) \leq C \int_{CB_{x,y} \cap A} R_{xy}(z)\, d\mu(z) = \overline{\mu}_{xy}^C(A).$$

The proof of the above statement could be derived by a careful application of the techniques in [AMS] combined with the modulus estimates of [K]. However, the method in [AMS] directly works only for $p > 1$, and some additional care is necessary for $p = 1$. Further, the following proof is somewhat more direct than theirs. Our proof is more in line with the approaches in [B, S] in combination with the estimates from [K] to construct probability measures on the space of curve fragments. The papers [B, S] employ the Rainwater lemma from [R2, Theorem 9.4.3]. However, we are able to avoid this lemma by directly using the Min-Max theorem [R2, Theorem 9.4], restated below for the reader's convenience.

Proposition 3.8 (Min-Max Theorem [R2, Theorem 9.4.2]) *Suppose that*

(i) G is a convex subset of some vector space,

(ii) K is a compact convex subset of some topological vector space, and

(iii) $F : G \times K \to \mathbb{R}$ satisfies

- *(a) $F(\cdot, y)$ is convex on G for every $y \in K$,*
- *(b) $F(x, \cdot)$ is concave and continuous on K for every $x \in G$.*

Then

$$\sup_{y \in K} \inf_{x \in G} F(x, y) = \inf_{x \in G} \sup_{y \in K} F(x, y).$$

Proof. [Proof of Theorem 3.7] Denote $d(x, y) = r$. By the 1-Poincaré inequality and [K, Theorem 2], there exists a C such that we have

$$\mathrm{Mod}_{1,\overline{\mu}_{xy}^C}(\Gamma_{xy}^C) = \inf \int_X \rho \, d\overline{\mu}_{xy}^C > \frac{1}{C},$$

where the infimum is over non-negative Borel functions ρ with $\int_\gamma \rho \geq 1$ for every $\gamma \in \Gamma_{xy}^C$. Note that the estimates in [K] give the modulus bound for the set of all rectifiable curves between x, y, but the collection of curves that are longer than $4C^2 d(x, y)$ has modulus less than $1/(2C)$, and can be excluded using the subadditivity of the modulus.

Another way of stating this estimate is that if f is a non-negative continuous function, and $\int_X f \, d\overline{\mu}_{xy}^C < \infty$, then for every $\epsilon > 0$ there exists a $\gamma \in \Gamma_{xy}^C$ such that

$$\int_\gamma f \ ds \leq (C + \epsilon) \int_X f \, d\overline{\mu}_{xy}^C,$$

for otherwise, $\frac{f}{(C+\epsilon) \int_X f \, d\overline{\mu}_{xy}^C}$ would be admissible with a too small a norm. In particular,

$$\inf_{(\gamma, I) \in \Gamma_{xy}^C} \int_\gamma f \, ds \leq C \int_X f \, d\overline{\mu}_{xy}^C. \tag{8}$$

Since f is continuous and Γ_{xy}^C is a compact family, the above infimum is a minimum. Parametrizing the curves γ by length we also get

$$\int_{(\gamma,I)\in\Gamma_{xy}^C}\int_I f\circ\gamma(t)\,dt\,d\beta(\gamma,I)\le C\int_X f\,d\overline{\mu}_{xy}^C, \tag{9}$$

where β is the Dirac measure on Γ_{xy}^C based at any of the optimal choices (γ, I) that achieves the infimum in (8).

Let K be the set of probability measures α on Γ_{xy}^C; thus K is a compact and convex set of measures with respect to weak* convergence. Set

$$G = \{\ f : X \to [0,1]\ \mid f \text{ is continuous }\} \subset C(X).$$

Here $C(X)$ is the set of all continuous functions equipped with the uniform topology and G is a closed convex subset thereof. Then define $F : G\times K \to \mathbb{R}$ by

$$F(f,\alpha) = C\int_X f\,d\overline{\mu}_{xy}^C - \int_{\Gamma_{xy}^C}\int_I f(\gamma(t))\,dt\,d\alpha(\gamma,I).$$

Clearly F is continuous in α, since $\Phi_f((\gamma,I)) = \int_I f(\gamma(t))\,dt$ is continuous in γ. Also, $F(\cdot,\alpha)$ is convex for every $\alpha\in K$, and $F(f,\cdot)$ is affine and *a fortiori* concave for any $f\in G$. Thus, we can apply Proposition 3.8 to obtain

$$\sup_{\alpha\in K}\inf_{f\in G} F(f,\alpha) = \inf_{f\in G}\sup_{\alpha\in K} F(f,\alpha).$$

Now, for $f \in G$, by estimate (8) we have $\sup_{\alpha\in K} F(f,\alpha)\ge 0$. Thus, we get

$$\sup_{\alpha\in K}\inf_{f\in G} F(f,\alpha)\ge 0.$$

In particular, for every $\epsilon > 0$ and every $f \in G$ there exists a $\alpha_\epsilon \in K$, such that

$$F(f,\alpha_\epsilon)\ge -\epsilon.$$

Since for each $f\in G$ the map $K\ni\alpha\mapsto F(f,\alpha)$ is continuous, we can extract a weakly convergent sequence $\alpha_{\epsilon_i}\rightharpoonup\alpha_{xy}\in K$ (with $\epsilon_i\to 0$ as $i\to\infty$), such that for every $f\in G$

$$F(f,\alpha_{xy})\ge 0.$$

Now, recalling the definition of F, for every $f \in G$,

$$\int_{\Gamma_{xy}^C}\int_I f(\gamma(t))\,dt\,d\alpha_{xy}(\gamma,I)\le C\int_X f\,d\overline{\mu}_{xy}^C.$$

Also, since the curves γ are 1-Lipschitz, it follows that $\int_\gamma f\ ds \le \int_I f(\gamma(t))\,dt$, and α_{xy} induces a measure (which we denote by the same symbol) on $\Gamma_{xy} = \{\gamma\ :\ (\gamma, I)\in\Gamma_{xy}^C \text{ for some } I\}$. With respect to this measure, we have for every $f\in C(X)$ with $0 \le f \le 1$ that

$$\int_{\Gamma_{xy}}\int_\gamma f\,ds\ d\alpha_{xy}(\gamma)\le C\int_X f\,d\overline{\mu}_{xy}^C.$$

By a limiting argument we obtain the same inequality for $f = \chi_A$ corresponding to Borel sets $A \subset X$, and thus the measure $\sigma_{xy} = \alpha_{xy}$, which is supported on the compact set Γ_{xy}, constitutes a Semmes family of curves in the sense of Definition 3.6, and the proof is complete. $\square$

Each Borel function in $L^1_{loc}(X)$ can be approximated by simple Borel functions. Hence it follows from (5) that

$$\int_{\Gamma_{xy}} \int_\gamma g\, ds\, d\sigma(\gamma) \le C \int_{CB_{x,y}} g(z) R_{xy}(z)\, d\mu(z), \tag{10}$$

for Borel functions $g : CB_{x,y} \to \mathbb{R}$. Doubling metric measure spaces supporting a Semmes pencil curves support a 1-Poincaré inequality (see e.g. the discussion following [Se, Definition 14.2.4]). In what follows we prove that they also support the AM-Poincaré inequality. Recall that

$$I_A(u)(x) = \int_A \frac{u(z) d(x,z)}{\mu(B(x, d(x,z)))}\, d\mu(z)$$

denotes the *Riesz potential* of a non-negative function u defined on X on a subset $A \subset X$.

Proposition 3.9 *If X supports a Semmes pencil of curves, then X supports the* AM-*Poincaré inequality.*

Proof. Let $u \in L^1_{loc}(X)$ and let (g_i) be a BV-upper bound of u, and let N be the collection of all points $x \in X$ for which

$$\limsup_{r \to 0^+} ⨍_{B(x,r)} |u - u(x)|\, d\mu > 0;$$

Then $\mu(N) = 0$. We focus on points $x, y \in X \setminus N$. Then for each $\varepsilon > 0$ we know that the sets

$$E_\varepsilon(x) := \{z \in X \,:\, |u(z) - u(x)| > \varepsilon\}, \qquad E_\varepsilon(y) = \{z \in X \,:\, |u(z) - u(y)| > \varepsilon\}$$

satisfy

$$\limsup_{r \to 0^+} \frac{\mu(B(x,r) \cap E_\varepsilon(x))}{\mu(B(x,r))} = 0 = \limsup_{r \to 0^+} \frac{\mu(B(y,r) \cap E_\varepsilon(y))}{\mu(B(y,r))}.$$

We can inductively choose a strictly decreasing sequence $r_i > 0$ such that $r_1 < d(x,y)/4$, $r_{i+1} < r_i/4$, and

$$\frac{\mu(B(x,r_i) \cap E_\varepsilon(x))}{\mu(B(x,r_i))} < \frac{2^{-i}}{2C_d}, \qquad \frac{\mu(B(y,r_i) \cap E_\varepsilon(y))}{\mu(B(y,r_i))} < \frac{2^{-i}}{2C_d}.$$

For each i let $\Gamma_i(x)$ denote the collection of all $\gamma \in \Gamma_{xy}$ such that

$$\mathcal{H}^1(\gamma^{-1}([B(x,r_i) \setminus B(x,r_i/2)] \setminus E_\varepsilon(x))) = 0,$$

and $\Gamma_i(y)$ the analogous family with y playing the role of x. By the fact that Γ_{xy} is a Semmes family and by the fact that μ is doubling, we have that

$$\frac{r_i}{2}\sigma_{xy}(\Gamma_i(x)) \le \int_{\Gamma_{xy}} \ell(\gamma \cap E_\varepsilon(x) \cap B(x,r_i) \setminus B(x,r_i/2))\, d\sigma_{xy}(\gamma) \le C_d \frac{r_i}{\mu(B(x,r_i))}\, \mu(E_\varepsilon(x) \cap B(x,r_i)),$$

and so by the choice of r_i we have

$$\sigma_{xy}(\Gamma_i(x)) \le 2^{-i}.$$

Hence for each positive integer n,

$$\sigma_{xy}\left(\bigcup_{i=n}^{\infty} \Gamma_i(x)\right) \le 2^{1-n},$$

and so with

$$\Gamma(x) = \bigcap_{n\in\mathbb{N}} \bigcup_{i=n}^{\infty} \Gamma_i(x),$$

we have that $\sigma_{xy}(\Gamma(x)) = 0$. Note that if $\gamma \in \Gamma_{xy} \setminus \Gamma(x)$, then whenever N_γ is a subset of the domain of γ with $\mathcal{H}^1(N_\gamma) = 0$, we can find a sequence of points $x_i \in \gamma \setminus [E_\varepsilon(x) \cup \gamma(N_\gamma)]$ such that $x_i \to x$ as $i \to \infty$. Let $\Gamma(y)$ be the analogous subfamily of curves with respect to the point y; then $\sigma_{xy}(\Gamma(x) \cup \Gamma(y)) = 0$. Let (g_i) be a $\mathrm{BV}_{\mathrm{AM}}$-upper bound for u. For $\gamma \in \Gamma_{xy} \setminus [\Gamma(x) \cup \Gamma(y)]$, we set N_γ to be the set of points in the domain of γ that forms the exceptional set in the condition (2), and we select the sequences x_i, y_i as above. Then we have that

$$|u(x) - u(y)| - 2\varepsilon \le \liminf_{i\to\infty} |u(x_i) - u(y_i)| \le \liminf_{i\to\infty} \int_\gamma g_i\, ds.$$

Therefore, for $x, y \in X \setminus N$ and for each $\gamma \in \Gamma_{xy} \setminus (\Gamma(x) \cup \Gamma(y))$, we have

$$|u(x) - u(y)| - 2\varepsilon \le \liminf_{i\to\infty} \int_\gamma g_i\, ds.$$

We then have by Fatou's lemma and (10) that

$$\begin{aligned}
|u(x) - u(y)| - 2\varepsilon \le &\int_{\Gamma_{xy}} \liminf_{i\to\infty} \int_\gamma g_i\, ds\, d\sigma_{xy}(\gamma) \\
\le &\liminf_{i\to\infty} \int_{\Gamma_{xy}} \int_\gamma g_i\, ds d\sigma_{xy}(\gamma) \\
\le &C \liminf_{i\to\infty} \int_{CB_{x,y}} g_i(z) R_{xy}(z)\, d\mu(z) \\
\le &\int_{CB_{x,y}} R_{xy}(z)\, d\nu(z) \\
\le &C(I_{CB_{x,y}}\nu(x) + I_{CB_{x,y}}\nu(y)),
\end{aligned}$$

where ν is the Radon measure as in Remark 2.10. The constant C in the above does not depend on ε; hence, by letting $\varepsilon \to 0^+$ we obtain

$$|u(x) - u(y)| \le C(I_{CB_{x,y}}\nu(x) + I_{CB_{x,y}}\nu(y))$$

whenever $x, y \in X \setminus N$. For $x, y \in B$ with R the radius of B, setting $B_i = B(x, 2^{-i}Cd(x,y))$ for $i \in \mathbb{N} \cup \{0\}$, we see that

$$\begin{aligned} I_{CB_{x,y}}\nu(x) \le \int_{B(x,Cd(x,y))} \frac{d(x,z)}{\mu(B(x,d(z,x)))}\, d\mu(z) &\le C \sum_{i=0}^{\infty} \frac{2^{-i}Cd(x,y)}{\mu(B_i)}\, \nu(B_i) \\ &\le C\, d(x,y)\, h_B(x) \sum_{i=0}^{\infty} 2^{-i}, \end{aligned}$$

where

$$h_B(x) = \sup_{0<r\le CR} \frac{\nu(B(x,r))}{\mu(B(x,r))}.$$

Thus h_B is a Hajłasz gradient of u in B, that is,

$$|u(x) - u(y)| \le Cd(x,y)[h_B(x) + h_B(y)]$$

for μ-a.e. $x, y \in B$, and we have the weak inequality

$$\mu(\{x \in B \,:\, h_B(x) > t\}) \le C\, \frac{\nu(B)}{t} \text{ for } t > 0.$$

Thus $h_B \in L^q(B)$ for $0 < q < 1$, and hence $u \in M^{1,q}(B)$ in the sense of [HajC], and so by [HajC, Corollary 8.9 of page 202] or by [KLS, Proposition 2.4], we know that

$$⨍_B |u - u_B|\, d\mu \le C\, R\, \frac{\nu(B)}{\mu(B)}.$$

The proof is then completed by taking a sequence of sequences $(g_i^j)_i$ that are $\mathrm{BV_{AM}}$-upper bound of u with corresponding measures ν_j such that $\lim_j \nu_j(2B) = \|D_{\mathrm{AM}}u\|(2B)$. □

From Proposition 3.9, Theorem 3.7, and Proposition 3.3 we have the following.

Theorem 3.10 *Let μ be a doubling measure on X. Then the following are equivalent:*

1. *X supports a 1-Poincaré inequality.*
2. *X supports a Semmes pencil of curves.*
3. *X supports an AM-Poincaré inequality.*

In any (and therefore all) of the above, we have $\mathrm{BV}(X) = \mathrm{BV_{AM}}(X)$ and $N^{1,1}(X) = N^{1,1}_{\mathrm{AM}}(X)$.

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