Modulus of families of sets of finite perimeter and quasiconformal maps between metric spaces of globally Q-bounded geometry

Rebekah Jones, Panu Lahti, Nageswari Shanmugalingam*

August 7, 2019

Dedicated to the memory of William P. Ziemer.

Abstract We generalize a result of Kelly [17] to the setting of Ahlfors Q-regular metric measure spaces supporting a 1-Poincaré inequality. It is shown that if X and Y are two Ahlfors Q-regular spaces supporting a 1-Poincaré inequality and $f : X \to Y$ is a quasiconformal mapping, then the $Q/(Q-1)$-modulus of the collection of measures $\mathcal{H}^{Q-1}\lfloor_{\Sigma E}$ corresponding to any collection of sets $E \subset X$ of finite perimeter is quasi-preserved by f. We also show that for $Q/(Q-1)$-modulus almost every ΣE, $f(E)$ is also of finite perimeter. Even in the standard Euclidean setting our results are more general than that of Kelly, and hence are new even in there.

Key words and phrases: finite perimeter, quasiconformal mapping, modulus of families of surfaces, Ahlfors regular, Poincaré inequality.

Mathematics Subject Classification (2010): Primary: 30L10; Secondary: 26B30, 31E05.

1 Introduction

While classification of domains via conformal mappings gives a rich theory in the setting of planar domains, domains in higher dimensional Euclidean spaces support no non-Möbius conformal maps. The most suitable geometric classification in that setting is given by quasiconformal mappings. A homeomorphism $f : \Omega \to \Omega'$ between two domains $\Omega, \Omega' \subset \mathbb{R}^n$ is quasiconformal if $f \in W^{1,n}_{loc}(\Omega; \Omega')$ and there is a constant $K \geq 1$ such that whenever $x \in \Omega$,

$$\limsup_{r \to 0^+} \frac{\sup_{y \in \overline{B}(x,r)} |f(y) - f(x)|}{\inf_{y \in \Omega \setminus B(x,r)} |f(y) - f(x)|} \leq K.$$

*The first and third authors' research was partially supported by the grants DMS #1500440 and DMS #1800161 of NSF (U.S.A.). Part of this research was conducted during the visit of the three authors to Linköping University in Spring 2017 and Spring 2018; the authors wish to thank that institution for its kind hospitality.

The theory of quasiconformal mappings was extended by Heinonen and Koskela in [13] to the setting of metric measure spaces, and in this non-smooth setting properties of quasiconformal mappings have been studied extensively, see for example [12, 13, 14, 23, 6, 18]. In this paper we continue this study by considering relationships between sets of finite perimeter and quasiconformal mappings in the spirit of [17].

The traditional perspective on quasiconformal mappings between Euclidean domains is that such a map is characterized by its ability to quasi-preserve the conformal modulus of families of rectifiable curves in the respective domains. Thus a homeomorphism $f : \Omega \to \Omega'$ for two domains $\Omega, \Omega' \subset \mathbb{R}^n$ is quasiconformal if there is a constant $C \geq 1$ such that whenever Γ is a family of non-constant compact rectifiable curves in Ω,

$$\frac{1}{C}\mathrm{Mod}_n(\Gamma) \leq \mathrm{Mod}_n(f\Gamma) \leq C\,\mathrm{Mod}_n(\Gamma).$$

Here $f\Gamma$ is the family of curves obtained as images of curves in Γ under f. An excellent discussion about Euclidean quasiconformal mappings can be found in [22], see also [17, Theorem 2.6.1]. A less well-known fact is that quasiconformal mappings between two domains $\Omega, \Omega' \subset \mathbb{R}^n$ quasi-preserve the $\frac{n}{n-1}$-modulus of certain families of surfaces obtained as "essential boundaries" of sets of finite perimeter. This result is due to Kelly [17, Theorem 6.6]. In [17] the families considered were the classes of sets $E \subset \Omega \subset \mathbb{R}^n$ of finite perimeter such that $\mathcal{H}^{n-1}(\partial E)$ is finite *and* satisfies a double-sided cone condition at *every* point in ∂E, see [17, Definition 6.1]. Kelly calls the boundary of such a set E a *surface*. Building upon the Federer theory of differential forms for sets of finite perimeter and the associated Gauss-Green theorem (see [8]), in [17] it is shown that if f is a quasiconformal mapping, then for $\mathrm{Mod}_{n/(n-1)}$-almost every surface, there is a change of variables formula, see [17, Theorem 4.7.1]. Moreover, there is a family Σ_0 of sets of finite perimeter in Ω with $\mathrm{Mod}_{n/(n-1)}(\Sigma_0) = 0$ such that whenever $\partial E \subset \Omega$ is a surface with $E \notin \Sigma_0$, then $f(E)$ is of finite perimeter in Ω' ([17, Theorem 6.3]). However, there is a gap in the proof of [17, Theorem 6.3], where the actual object studied is the reduced boundary of E, denoted $\beta(E)$ in [17, page 372], and this part of the boundary could be strictly smaller than the measure-theoretic boundary of E. According to Federer's characterization, a Euclicean set is of finite perimeter if and only if the $(n-1)$-dimensional Hausdorff measure of the measure-theoretic boundary of E is finite. In [17] it is shown that for almost every E, $\beta(f(E))$ has finite $(n-1)$-dimensional Hausdorff measure, and this is not sufficient to conclude that $f(E)$ is of finite perimeter.

The link between quasiconformal mappings and families of surfaces is natural also in light of the link between quasiconformal mappings and moduli of families of curves, for in the Euclidean setting it is known that there is a natural reciprocal connection between families of surfaces separating two compacta and families of curves connecting the two compacta, see [2]. This link was already portended in [1, Lemma 5] (in planar geometry, the separating "surfaces" are also curves). Motivated by the results in [17, 2], the goal of this paper is to prove a result similar to that of [17, Theorem 6.6] for quasiconformal mappings between two complete metric measure spaces equipped with an Ahlfors Q-regular (with $Q > 1$) measure and supporting a 1-Poincaré inequality in the sense of Heinonen and Koskela [13], and indeed, we consider families of sets from the collection of *all* sets of finite perimeter without the additional geometric constraints considered in [17] (see Theorem 1.1). To do so, we use the tools

developed in [13, 14] regarding first-order calculus on non-smooth spaces and the theory of BV functions first constructed in [21], together with the result from [18] that quasiconformal maps are characterized by quasi-preserving the measure density of measurable subsets of Ω. This latter result is itself a generalization of the work of Gehring and Kelly [10]. Unlike in the work of Kelly [17], the notions of the Gauss-Green theorem and differential forms are not available in the metric setting, and instead, we adapt the geometric measure theory tools developed in [6] in the metric setting to verify an analog of the change of variables formula for sets of finite perimeter in the non-smooth setting. The results of [6] are not directly applicable to our setting as neither the measure-theoretic boundary $\partial^* E$ (see Definition 2.8) nor the essential boundary ΣE of a set E of finite perimeter is a $(Q-1)$-set (that is, it is not Ahlfors $(Q-1)$-regular). Here ΣE is the subset of the boundary of E made up of points that see both E and its complement as having positive *lower density*, see Definition 2.12 below. Therefore in this paper we combine some of the techniques of [6] with the current technology on sets of finite perimeter to conduct a careful analysis of the images of ΣE. Note that for Euclidean sets of finite perimeter E where the reduced boundary $\beta(E)$ makes sense, we have $\beta(E) \subset \Sigma E \subset \partial^* E$. The recent paper [19] indicates that in the Euclidean setting as well as in more general metric measure space settings, the more useful boundary of a set is its essential boundary. In the setting of Ahlfors Q-regular metric measure spaces supporting a 1-Poincaré inequality, the result of [19] tells us that $\mathcal{H}^{Q-1}(\partial^* E)$ is finite *if and only if* $\mathcal{H}^{Q-1}(\Sigma E)$ is finite. Even in the Euclidean setting this improvement is new; we will take advantage of this new tool here and so focus on ΣE for measurable sets E.

In what follows, both X and Y are complete metric spaces equipped with an Ahlfors Q-regular measure for some $Q > 1$, $f : X \to Y$ a quasiconformal homeomorphism, $\mathcal{L}$ denotes the collection of measures $\mathcal{H}^{Q-1}\big\lfloor_{\Sigma E}$ corresponding to sets $E \subset X$ that are of finite perimeter in X, and $f\mathcal{L}$ is the corresponding collection of measures $\mathcal{H}^{Q-1}\big\lfloor_{\Sigma f(E)}$.

The quantities L_f and l_f represent the following:

$$L_f(x) = \limsup_{r\to 0^+} \frac{\sup_{y\in\overline{B}(x,r)} d_Y(f(x),f(y))}{r}, \quad l_f(x) = \liminf_{r\to 0^+} \frac{\inf_{y\in X\setminus B(x,r)} d_Y(f(x),f(y))}{r}, \tag{1.1}$$

with d_Y denoting the metric on the space Y, see Definition 3.1 below. Here, for $1 \le p < \infty$,

$$\operatorname{Mod}_p(\mathcal{L}) = \inf\left\{ \int_X \rho^p \, d\mu \,:\, \rho \text{ non-negative Borel with } \int_{\Sigma E} \rho \, d\mathcal{H}^{Q-1} \ge 1 \text{ for each } E \in \mathcal{L} \right\},$$

with $\mu \approx \mathcal{H}^Q$ denoting the measure on the space X, see Definition 2.4 below. Here $\mathcal{L}$ has a dual identity, one as a collection of sets $E \subset X$ of finite perimeter, and the other as the collection of measures $\mathcal{H}^{Q-1}\big\lfloor_{\Sigma E}$. In considering the quasiconformal images $f\mathcal{L}$, the family $f\mathcal{L}$ stands for both the collection $f(E)$, $E \in \mathcal{L}$, and also for the measures $\mathcal{H}^{Q-1}\big\lfloor_{\Sigma f(E)}$. Thanks to the result of [19] (see Theorem 4.2 below), we know that for *each* measurable set $F \subset Y$, $\mathcal{H}^{Q-1}\big\lfloor_{\Sigma F} = \mathcal{H}^{Q-1}\big\lfloor_{\partial^* F}$, and so the above definition is tailor-made for sets of finite perimeter. Whenever F is not of finite perimeter, $\mathcal{H}^{Q-1}\big\lfloor_{\partial^* F}(Y) = \infty$.

The following is the main result of this paper.

Theorem 1.1. *Let X, Y be two complete Ahlfors Q-regular metric spaces, $Q > 1$, that support a 1-Poincaré inequality, and let $f : X \to Y$ be a quasiconformal map. Then there*

exists $C > 0$ depending only on the quasiconformality constant of f and the Ahlfors regularity constants of X and Y such that for every collection $\mathcal{L}$ of bounded sets of finite perimeter measure in X we have that

$$\frac{1}{C}\operatorname{Mod}_{Q/(Q-1)}(f\mathcal{L}) \leq \operatorname{Mod}_{Q/(Q-1)}(\mathcal{L}) \leq C\operatorname{Mod}_{Q/(Q-1)}(f\mathcal{L}). \tag{1.2}$$

The above theorem gives new results even in the Euclidean setting, addressing the wider class of *all* sets of finite perimeter rather than just those that satisfy a cone property at each point of the topological boundary, with the topological boundary of finite Hausdorff $(n-1)$-dimensional measure as considered in [17, Definition 6.1].

In proving Theorem 1.1 we also show that for $\operatorname{Mod}_{Q/(Q-1)}$-almost every set $E \subset X$ of finite perimeter the pull-back measure under f of $\mathcal{H}^{Q-1}\lfloor_{\Sigma f(E)}$ is absolutely continuous with respect to $\mathcal{H}^{Q-1}\lfloor_{\Sigma E}$, with its Radon-Nikodym derivative estimated by $J_f^{(Q-1)/Q}$, see Lemma 5.4 and Proposition 5.2. We also address the question of whether images of sets of finite perimeter are of finite perimeter. There are examples of planar quasiconformal mappings that map the unit disk to the von Koch snowflake domain, and so map a set of finite perimeter to a set that is not of finite perimeter. Hence we cannot expect images of all sets of finite perimeter to be of finite perimeter, see also [6]. Recall from there also that both f and f^{-1} satisfy Lusin's condition N, that is, for sets $K \subset X$, $\mathcal{H}^Q(f(K)) = 0$ if and only if $\mathcal{H}^Q(K) = 0$, or equivalently, $f_\#\mathcal{H}^Q_Y \ll \mathcal{H}^Q_X \ll f_\#\mathcal{H}^Q_Y$ (see [14, Theorem 8.12]). Following [14], the Radon-Nikodym derivative of $f_\#\mathcal{H}^Q_Y$ with respect to $\mathcal{H}^Q_X$ is denoted by J_f. The quantities J_f, L^Q_f, and l^Q_f are comparable to each other almost everywhere in X, see Lemma 4.6 below.

Theorem 1.2. *Let X, Y be complete Ahlfors Q-regular metric spaces, $Q > 1$, that support a 1-Poincaré inequality, and let $f : X \to Y$ be a quasiconformal map. Then for $\operatorname{Mod}_{Q/(Q-1)}$-almost every bounded set $E \subset X$ of positive and finite perimeter in X, the set $f(E)$ is of finite perimeter in Y and the pull-back measure satisfies*

$$f_\#(\mathcal{H}^{Q-1}\lfloor_{\partial^* f(E)}) \ll \mathcal{H}^{Q-1}\lfloor_{\partial^* E} \ll f_\#(\mathcal{H}^{Q-1}\lfloor_{\partial^* f(E)}) \tag{1.3}$$

with Radon-Nikodym derivative $J_{f,E} \cong J_f^{(Q-1)/Q}$ where the comparison constant depends only on the quasiconformality constant of f and the Ahlfors regularity constants of X and Y.

It is well-established that quasiconformal maps are characterized by their quasipreservation of Q-modulus of families of curves. The above results indicate that quasiconformal maps also quasipreserve the $Q/(Q-1)$-modulus of families of sets of finite perimeter, and are not as discordant with the established theory of quasiconformal mappings as it might seem. The characterization of quasiconformal mappings by quasi-preservation of Q-modulus of families of curves is too strong; one only needs the quasi-preservation of Q-modulus of families of rectifiable curves that connect pairs of disjoint compact sets K, F. Such classes of curves are associated with the relative capacity $\operatorname{cap}_Q(K, F)$, and the super-level sets of the potential associated with this capacity give sets of finite perimeter whose perimeter sets separate K from F, and it is the families of such sets that connect quasiconformal maps to the BV theory. The converse of Theorem 1.1, characterizing quasiconformal maps as those homeomorphisms that satisfy (1.2), is proven in [16].

We now consider the implications of the above Theorem 1.2 for the Euclidean setting $\mathbb{R}^n$. An analog of the above theorem found in [17] is Theorem 6.3, but the proof of [17, Theorem 6.3] has a gap as explained above. However, in light of the restrictions placed on the sets E considered in [17], we know that those sets E satisfy $\partial E = \Sigma E$, and for such E it follows from Proposition 5.1(3) that except for a $\mathrm{Mod}_{n/(n-1)}$-null family, we know that $\mathcal{H}^{n-1}(f(\partial E)) < \infty$, and therefore [17, Theorem 6.3] follows from Theorem 1.2, that is, the gap in the proof found in [17] is filled by the above theorem.

The structure of the paper is as follows. In Section 2 we give the notations and definitions related to function spaces and measure-theoretic aspects of sets used in this paper. In Section 3 we give a brief background related to quasiconformal mappings between metric measure spaces, and in Section 4 we list the needed background results related to the concepts described in the previous two sections. Here we also give proofs and/or references to papers where the interested reader can find proofs of these results. We give a proof of Theorem 1.1 in Section 5, and the last section deals with the proof of Theorem 1.2. In the last section we also show that there are large families of sets of finite perimeter whose images under quasiconformal mappings are also of finite perimeter.

2 Notations and definitions

In this section we gather together the basic definitions we need in this paper. The definitions used here are extensions to the non-smooth setting of the natural notions in Euclidean setting discussed in the introduction. In this section, (X, d, μ) is a complete metric measure space with μ a Radon measure.

Given $x \in X$ and $r > 0$, we denote an open ball by $B(x, r) = \{y \in X : d(y, x) < r\}$. Given that in a metric space a ball, as a set, could have more than one radius and more than one center, we will consider a ball to be also equipped with a radius and center; thus two different balls might correspond to the same set. We then denote $\mathrm{rad}(B) := r$ as the pre-assigned radius of the ball B, and $aB := B(x, ar)$. If X is connected (as it must be in order to support a Poincaré inequality), and if $X \setminus B$ is non-empty, then $\mathrm{rad}(B) \leq \mathrm{diam}(B) \leq 2\,\mathrm{rad}(B)$.

Definition 2.1. Let $A \subset X$. Then for $d \geq 0$, the *d-dimensional Hausdorff measure* of A is given by

$$\mathcal{H}^d(A) = \lim_{r \to 0^+} \inf \left\{ \sum_{k \in I} \mathrm{rad}(B_k)^d \,\middle|\, A \subset \bigcup_{k \in I} B_k \text{ where } \mathrm{rad}(B_k) \leq r \text{ and } I \subset \mathbb{N} \right\}.$$

Definition 2.2. We say that (X, d, μ) is *Ahlfors Q-regular* if X has at least two points and there is a constant $C_A \geq 1$ such that whenever $x \in X$ and $0 < r < 2\mathrm{diam}(X)$, we have

$$\frac{r^Q}{C_A} \leq \mu(B(x, r)) \leq C_A\, r^Q.$$

As a consequence, we get

$$\frac{\mu(A)}{C_A} \leq \mathcal{H}^Q(A) \leq C_A\, \mu(A).$$

Given an open set $U \subset X$, we write $u \in L^1_{loc}(U)$ if $u \in L^1(V)$ for every open $V \Subset U$; this expression means that $\overline{V}$ is a compact subset of U. Other local spaces are defined analogously.

A *curve* is a continuous mapping from an interval into X, and a *rectifiable* curve is a curve with finite length.

Definition 2.3. Let Y be a metric space with metric d_Y. Given a function $u : X \to Y$, a Borel function $g_u : X \to [0, \infty]$ is said to be an *upper gradient of* u if for every compact rectifiable curve γ

$$d_Y(u(x), u(y)) \leq \int_\gamma g_u \, ds$$

where x and y are the endpoints of γ. Let $1 \leq p < \infty$. A function $f : X \to Y$ is said to be in $N^{1,p}_{loc}(X; Y)$ if $f \in L^p_{loc}(X; Y)$ and there is an upper gradient g of f such that $g \in L^p_{loc}(X)$. If $Y = \mathbb{R}$ and $f, g \in L^p(X)$ then we say that $f \in N^{1,p}(X)$.

We refer the reader to [14, 15] for the details regarding mappings in $N^{1,p}_{loc}(X; Y)$.

Definition 2.4. Let $\mathcal{M}$ be a collection of measures on X. Then the admissible class of $\mathcal{M}$, denoted $\mathcal{A}(\mathcal{M})$, is the set of all positive Borel functions $\rho : X \to [0, \infty]$ such that

$$\int_X \rho \, d\lambda \geq 1$$

for all $\lambda \in \mathcal{M}$. Then the *p-modulus* of the family $\mathcal{M}$ is given by

$$\operatorname{Mod}_p(\mathcal{M}) = \inf_{\rho \in \mathcal{A}(\mathcal{M})} \int_X \rho^p \, d\mu.$$

Mod_p is an outer measure on the class of all measures, see [9]. There are two types of collections of measures associated with quasiconformal maps. Given a collection Γ of curves in X, we set Γ to also denote the arc length measures restricted to each curve in Γ; for this collection of measures, the above notion of $\operatorname{Mod}_p(\Gamma)$ agrees with the standard notion of the p-modulus of the family Γ of curves from [22, 13, 15]. For a collection $\mathcal{L}$ of sets of finite perimeter in X, we consider the measure $\mathcal{H}^{Q-1}\lfloor_{\Sigma E}$ for each $E \in \mathcal{L}$; it is known that this measure is comparable to the perimeter measure associated with E as in Definition 2.10, see Theorem 4.1.

Definition 2.5. The *relative p-capacity* of two sets $E, F \subset X$ is given by

$$\operatorname{cap}_p(E, F) = \inf \int_X g_u^p \, d\mu$$

where the infimum is over all upper gradients g_u of all functions $u \in N^{1,Q}_{loc}(X)$ such that $u|_E \leq 0$ and $u|_F \geq 1$.

Definition 2.6. We say that the space X supports a *p-Poincaré inequality* if there exist constants $C_P > 0$ and $\lambda \geq 1$ such that for all open balls B in X, all measurable functions u on λB and all upper gradients g_u of u,

$$\fint_B |u - u_B| \, d\mu \leq C_P \operatorname{rad}(B) \left(\fint_{\lambda B} g_u^p \, d\mu \right)^{1/p}.$$

Here we denote the integral average of u over B by

$$u_B := \fint_B u \, d\mu := \frac{1}{\mu(B)} \int_B u \, d\mu.$$

One of the consequences of a space being complete and Ahlfors regular and supporting a Poincaré inequality is that such a metric space must necessarily be quasiconvex, that is, there is some constant $C_q \geq 1$ such that for every $x, y \in X$ there is a rectifiable curve γ with end points x, y and length $\ell(\gamma) \leq C_q \, d(x, y)$, see [11, Proposition 4.4] or [7, Theorem 4.32]. Thus a bi-Lipschitz change in the metric results in X being a geodesic space, that is, a quasiconvex space with the quasiconvexity constant $C_q = 1$. Notions such as Poincaré inequality, quasiconformality, upper gradients and functions of bounded variation (see below), and Hausdorff measure are quasi-invariant under a bi-Lipschitz change in the metric, hence we do not lose generality by assuming that X is a geodesic space. Geodesic spaces that support a Poincaré inequality do so even with $\lambda = 1$, see [11] or [7, Theorem 4.39].

Definition 2.7. For a measurable set $E \subset X$ and $x \in X$, we define the *upper density of* E at x by

$$\overline{D}(E, x) = \limsup_{r \to 0^+} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))}$$

and the *lower density of* E at x by

$$\underline{D}(E, x) = \liminf_{r \to 0^+} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))}.$$

Definition 2.8. For a set E, the *measure-theoretic boundary* is the set

$$\partial^* E = \{x \in X \, : \, \overline{D}(E, x) > 0 \text{ and } \overline{D}(X \setminus E, x) > 0\}.$$

Definition 2.9. For $u \in L^1_{\mathrm{loc}}(X)$, the total variation of u on an open set $U \subset X$ is given by

$$\|Du\|(U) = \inf \left\{ \liminf_{n \to \infty} \int_U g_{u_n} \, d\mu \, \middle| \, (u_n)_{n \in \mathbb{N}} \subset \mathrm{Lip}_{\mathrm{loc}}(U), u_n \to u \text{ in } L^1_{\mathrm{loc}}(U) \right\}.$$

In the above, g_{u_n} stands for an upper gradient of u_n in U (here we consider U to be the metric measure space with metric and measure inherited from X). We say u is of *bounded variation* on X (denoted $u \in BV(X)$) if $\|Du\|(X) < \infty$. We say that $u \in BV_{loc}(X)$ if $u \in BV(U)$ for each open set $U \Subset X$.

It is shown in [21] that $\|Du\|$ is a Radon measure for any $u \in BV_{loc}(X)$. We call $\|Du\|$ the variation measure of u.

Definition 2.10. A measurable set $E \subset X$ has *finite perimeter* if χ_E is of bounded variation on X. We call $\|D\chi_E\|$ the *perimeter measure* of E and we will denote it $P(E, \cdot)$.

Definition 2.11. We say that X supports a *relative isoperimetric inequality* if there exist constants $C_I > 0$ and $\lambda \geq 1$ such that for all balls $B(x, r)$ and for all measurable sets E, we have

$$\min\{\mu(B(x, r) \cap E), \mu(B(x, r) \setminus E)\} \leq C_I r P(E, B(x, \lambda r)).$$

Again, with X a geodesic space, we can choose $\lambda = 1$. We know that if X is Ahlfors regular and supports a 1-Poincaré inequality, then it supports a relative isoperimetric inequality, see for example [3, Theorem 4.3].

Definition 2.12. For $\beta > 0$, let

$$\Sigma_\beta E = \left\{ x \in \partial^* E \;\middle|\; \underline{D}(E, x) \geq \beta \text{ and } \underline{D}(X \setminus E, x) \geq \beta \right\},$$

and set

$$\Sigma E = \bigcup_{\beta \in (0,1)} \Sigma_\beta E.$$

See [3], [19] or Theorem 4.1 below for connections between $\Sigma_\beta E$, $\partial^* E$, and the perimeter measure $P(E, \cdot)$.

Standing assumptions on the metric spaces: Throughout this paper we will assume that both (X, d_X, μ_X) and (Y, d_Y, μ_Y) are complete metric spaces that are Ahlfors Q-regular for some $Q > 1$ and support a 1-Poincaré inequality. Often we will denote $\mu_X = \mu$. We will also, without loss of generality, assume that X and Y are geodesic spaces. We will use the letter C to denote various constants that depend, unless otherwise specified, only on the Ahlfors regularity constants and the Poincaré inequality constants of X, and the value of C could differ at each occurrence.

3 Quasiconformal mappings

In this section we gather together definitions related to the notion of quasiconformal mappings between two metric spaces. Here, (X, d_X, μ_Y) and (Y, d_Y, μ_Y) are complete metric measure spaces with μ_X, μ_Y Radon measures and $Q > 1$ such that $\mu_X \approx \mathcal{H}^Q = \mathcal{H}^Q_X$ and $\mu_Y \approx \mathcal{H}^Q = \mathcal{H}^Q_Y$ as in the standing assumptions from Section 2. Recall also the definitions of L_f and l_f from (1.1):

Definition 3.1. Define $L_f : X \to \mathbb{R}$ by

$$L_f(x) = \limsup_{r \to 0^+} \frac{L_f(x,r)}{r} \quad \text{where} \quad L_f(x,r) = \sup_{y \in \overline{B}(x,r)} d_Y(f(x), f(y)).$$

Similarly, define $l_f : X \to \mathbb{R}$ by

$$l_f(x) = \liminf_{r \to 0^+} \frac{l_f(x,r)}{r} \quad \text{where} \quad l_f(x,r) = \inf_{y \in X \setminus B(x,r)} d_Y(f(x), f(y)).$$

When f is a homeomorphism, we always have $l_f(x,r) \leq L_f(x,r)$. When f is a quasiconformal homeomorphism, there is a constant K_D such that $L_f(x,r) \leq K_D\, l_f(x,r)$, see for example [13] or (4.2) below.

There are different geometric notions of quasiconformal maps on metric spaces.

Definition 3.2. The homeomorphism f is *metric quasiconformal* if there is a constant $K_D \geq 1$ such that for all $x \in X$ we have

$$\limsup_{r \to 0^+} \frac{L_f(x,r)}{l_f(x,r)} \leq K_D.$$

When we need to emphasize the constant K_D we say that f is K_D-quasiconformal.

The map f is *geometric quasiconformal* if there is a constant $K \geq 1$ such that whenever Γ is a family of non-constant compact rectifiable curves in X, we have

$$\frac{1}{K}\mathrm{Mod}_Q(f\Gamma) \leq \mathrm{Mod}_Q(\Gamma) \leq K\,\mathrm{Mod}_Q(f\Gamma).$$

Definition 3.3. A homeomorphism $f : X \to Y$ is *quasisymmetric* if there is a homeomorphism $\eta : [0,\infty) \to [0,\infty)$ such that for every distinct triple of points $x, y, z \in X$ and $t > 0$,

$$\frac{d_X(x,y)}{d_X(x,z)} \leq t \implies \frac{d_Y(f(x),f(y))}{d_Y(f(x),f(z))} \leq \eta(t).$$

The notions of quasisymmetry, metric quasiconformality, and geometric quasiconformality are connected, see [13, 23] and Theorem 4.3 below.

Definition 3.4. A homeomorphism $f : X \to Y$ between Ahlfors Q-regular metric spaces satisfies *Lusin's condition* (N) if whenever $A \subset X$ is such that $\mathcal{H}^Q(A) = 0$ then $\mathcal{H}^Q(f(A)) = 0$. We say that f satisfies condition (N^{-1}) if its inverse satisfies condition (N).

Quasiconformal maps satisfy both Lusin's condition (N) and (N^{-1}), see Theorem 4.3 below.

Definition 3.5. If ν_Y is a Radon measure on Y and $f : X \to Y$ is a homeomorphism, then the *pull-back of the measure* ν_Y is the measure on X given by

$$f_\# \nu_Y(D) := \nu_Y(f(D))$$

whenever D is a Borel subset of X. Note that since f is a homeomorphism, $f_\#\nu_Y$ defines a Borel measure.

Definition 3.6. We define the (generalized) *Jacobian of* f at the point $x \in X$ as follows:

$$J_f(x) = \limsup_{r \to 0^+} \frac{\mathcal{H}^Q(f(B(x,r)))}{\mathcal{H}^Q(B(x,r))}.$$

Note that J_f is the Radon-Nikodym derivative of the pull-back measure $f_\#(\mathcal{H}^Q_Y)$ with respect to $\mathcal{H}^Q_X$ and $\mathcal{H}^Q$ is a doubling measure, and so the limit supremum in the definition of J_f is actually a limit at $\mathcal{H}^Q$-almost every x.

Definition 3.7. For a set $E \subset X$ of finite perimeter, should $f_\#(\mathcal{H}^{Q-1}\lfloor_{\Sigma f(E)}) \ll \mathcal{H}^{Q-1}\lfloor_{\Sigma E}$, we define the $(Q-1)$-*Jacobian of* f with respect to ΣE by

$$J_{f,E}(x) = \lim_{r \to 0^+} \frac{\mathcal{H}^{Q-1}\lfloor_{\Sigma f(E)}(f(B(x,r)))}{\mathcal{H}^{Q-1}\lfloor_{\Sigma E}(B(x,r))}.$$

Given that $\mathcal{H}^{Q-1}(\partial^* E \setminus \Sigma E) = 0$ (see Theorem 4.1), we can equivalently consider $J_{f,E}$ to be the Radon-Nikodym derivative on $\partial^* E$.

Further standing assumptions: In this paper, in addition to the standing assumptions listed at the end of Section 2, we will also assume that $f : X \to Y$ is a quasiconformal mapping.

4 Background results

In this section we will gather together some of the background results needed in the paper. Recall that we assume the 1-Poincaré inequality to hold with the scaling constant $\lambda = 1$.

Theorem 4.1 ([3, Theorem 5.3, Theorem 5.4], [5, Theorem 4.6])**.** *There exists $\gamma > 0$ (depending only on the Ahlfors regularity constant C_A of μ and the Poincaré inequality* **constant** *C_P) such that for any set of finite perimeter E, the perimeter measure $P(E, \cdot)$ is concentrated on $\Sigma_\gamma E$. Furthermore, $\mathcal{H}^{Q-1}(\partial^* E \setminus \Sigma_\gamma E) = 0$ and there exist constants $\tilde{\alpha} > 0$ and $C > 0$ (again depending only on C_A and C_P) and a Borel function $\Theta_E : X \to [\tilde{\alpha}, C]$ such that*

$$P(E, B(x,r)) = \int_{B(x,r)\cap \partial^* E} \Theta_E \, d\mathcal{H}^{Q-1} = \int_{B(x,r)\cap \Sigma_\gamma E} \Theta_E \, d\mathcal{H}^{Q-1},$$

for any $x \in X$ and $r > 0$. Consequently we have that

$$\tilde{\alpha}\, \mathcal{H}^{Q-1}(B(x,r) \cap \partial^* E) \leq P(E, B(x,r)) \leq C\, \mathcal{H}^{Q-1}(B(x,r) \cap \partial^* E) \tag{4.1}$$

and

$$\mathcal{H}^{Q-1}(\partial^* E \setminus \Sigma E) = 0.$$

The results of [3] did not need the measure to be Ahlfors regular, only that the measure be doubling, but as Ahlfors regularity is a stronger condition, the results of [3] hold here as well. The next theorem is a strengthening of Federer's characterization of sets of finite perimeter as those sets E with $\mathcal{H}^{Q-1}(\partial^* E) < \infty$. The Federer characterization in the metric setting can be found in [20].

Theorem 4.2 ([19, Theorem 1.1])**.** *There is some $0 < \beta \leq 1/2$, depending only on C_A and C_P, such that whenever $E \subset X$ is measurable, we have that E is of finite perimeter if and only if $\mathcal{H}^{Q-1}(\Sigma_\beta E) < \infty$.*

Now we turn to preliminary results related to quasiconformal mappings needed in the paper.

Theorem 4.3 ([14, Theorem 9.8])**.** *Let $f : X \to Y$ be a homeomorphism between metric spaces of locally Q-bounded geometry. Then the following conditions are quantitatively equivalent:*

1. *f is H-quasiconformal for some $H \geq 1$,*
2. *There is a homeomorphism η such that f is locally η-quasisymmetric,*

3. $f \in N^{1,Q}_{loc}(X : Y)$ *and* $L_f(x)^Q \leq K J_f(x)$ *for almost every* $x \in X$ *and a constant* $K > 0$,

4. *There is some* $L > 0$ *such that for all curve families* Γ *in* X,

$$L^{-1}\mathrm{Mod}_Q(\Gamma) \leq \mathrm{Mod}_Q(f\Gamma) \leq L\,\mathrm{Mod}_Q(\Gamma).$$

Furthermore, if any one of these conditions holds, then both f *and* f^{-1} *are quasiconformal and satisfy Lusin's condition (N).*

Remark 4.4. A metric space is said to be of locally Q-bounded geometry if X is separable, path connected, locally compact, locally uniformly Ahlfors Q-regular and satisfies the *Loewner condition* locally uniformly. Since the support of 1-Poincaré inequality implies the support of a Q-Poincaré inequality whenever $Q > 1$, and since for Ahlfors Q-regular spaces the Loewner condition is equivalent to the support of a Q-Poincaré inequality (as shown in [13]), we know that under our assumptions, X is locally (even globally) of Q-bounded geometry.

In fact, under our assumptions the quasiconformal mapping f is necessarily quasisymmetric.

Proposition 4.5. *The quasiconformal mapping* f *is quasisymmetric.*

Proof. Whenever $A \subset X$ is bounded, $\overline{A}$ is compact, and then since f is continuous, $f(\overline{A})$ is compact and thus bounded, that is, f maps bounded sets to bounded sets. Since $f : X \to Y$ is a homeomorphism, f^{-1} also maps bounded sets to bounded sets, and so X is bounded if and only if Y is bounded. Now if X is unbounded, by [13, Corollary 4.8, Theorem 5.7] we know that f is quasisymmetric; if X is bounded, we know this by [13, Theorem 4.9]. □

The next lemma follows also from Theorem 4.3(3) together with the chain rule applied to f and to f^{-1}, but as the proof is simple we provide it here for the convenience of the reader.

Lemma 4.6. *If* η *is the homeomorphism of quasisymmetry for* f*, then there exists* $C > 0$*, depending only on the Ahlfors regularity constant of the measures on* X *and* Y *and on* $\eta(1)$*, such that at every* $x \in X$*,*

$$\frac{L_f(x)^Q}{C} \leq J_f(x) \leq C\, L_f(x)^Q.$$

Proof. Fix $x \in X$, and choose $r > 0$ small enough so that $X \setminus \overline{B}(x, r)$ is non-empty. Let $w \in B(x, r)$ and $z \in X \setminus B(x, r)$. Then

$$\frac{d_Y(f(x), f(w))}{d_Y(f(x), f(z))} \leq \eta\left(\frac{d_X(x, w)}{d_X(x, z)}\right) \leq \eta(1)$$

because $d(x, w) \leq d(x, z)$. This holds for all $w \in B(x, r)$ and $z \in X \setminus B(x, r)$, and so

$$\frac{L_f(x, r)}{l_f(x, r)} \leq \eta(1). \tag{4.2}$$

Now for $y \in Y \setminus f(B(x,r))$ we have $d_X(f^{-1}(y),x) \geq r$, and so

$$d_Y(y,f(x)) \geq l_f(x,r) \geq \frac{L_f(x,r)}{\eta(1)}.$$

It follows that $y \notin B(f(x), L_f(x,r)/\eta(1))$. Hence, $B(f(x), L_f(x,r)/\eta(1)) \subset f(B(x,r))$. Now at every $x \in X$, $J_f(x)$ is given by

$$\begin{aligned} J_f(x) = \limsup_{r\to 0^+} \frac{\mathcal{H}^Q(f(B(x,r)))}{\mathcal{H}^Q(B(x,r))} &\geq \limsup_{r\to 0^+} \frac{\mathcal{H}^Q\left(B\left(f(x), \frac{L_f(x,r)}{\eta(1)}\right)\right)}{\mathcal{H}^Q(B(x,r))} \\ &\geq \limsup_{r\to 0^+} \frac{\frac{1}{C_A}\left(\frac{L_f(x,r)}{\eta(1)}\right)^Q}{C_A r^Q} \\ &= \limsup_{r\to 0^+} \frac{1}{C_A^2\,\eta(1)^Q}\left(\frac{L_f(x,r)}{r}\right)^Q \\ &= \frac{1}{C_A^2\,\eta(1)^Q} L_f(x)^Q, \end{aligned}$$

and

$$J_f(x) \leq \limsup_{r\to 0^+} \frac{\mathcal{H}^Q(B(f(x), L_f(x,r)))}{\mathcal{H}^Q(B(x,r))} \leq C_A^2\, L_f(x)^Q.$$

Letting $C = C_A^2 \max\{\eta(1)^Q, 1\}$, the conclusion follows. □

Lemma 4.7 ([6, Lemma 2.7])**.** *For every ball $B(f(x),s) \subset Y$ there exists $r > 0$ such that*

$$B(f(x),s) \subset f(B(x,r)) \subset f(B(x,10r)) \subset B(f(x), \eta(10)s).$$

Furthermore, if f^{-1} is uniformly continuous with modulus of continuity $\omega(\cdot)$, then we can choose $r \leq \omega(s)$.

Proof. Since Y is proper and f^{-1} is continuous, there exists $r > 0$ such that the first inclusion holds. Let $r = \inf\{r' \mid f(B(x,r')) \supset B(f(x),s)\}$; since f is a homeomorphism, $B(f(x),s) \subset f(B(x,r))$. Then for any $0 < c < 1$, there exists a point $z_c \in f^{-1}(B(f(x),s)) \setminus B(x,cr)$. Then $f(z_c) \in B(f(x),s) \setminus f(B(x,cr))$ which implies that $d_Y(f(x), f(z_c) \leq s$ and $d_X(x,z_c) \geq cr$. Let $w \in B(x,10r)$. Now f is quasisymmetric by Proposition 4.5, with an associated homeomorphism η, and so

$$d_Y(f(x),f(w)) \leq \eta\left(\frac{d_X(x,w)}{d_X(x,z_c)}\right) d_Y(f(x),f(z_c)) \leq \eta\left(\frac{10}{c}\right) s.$$

Letting c tend to 1, we get that $d_Y(f(x),f(w)) \leq \eta(10)s$. Thus the last inclusion holds. Now if $\omega(t)$ is a modulus of continuity of f^{-1}, then $r \leq \omega(s)$ since we chose r minimally. □

Recall that $\Sigma E = \bigcup_{\beta\in(0,1)} \Sigma_\beta E$, see Definition 2.12.

Lemma 4.8. *Let $E \subset X$ be measurable. For each $\beta \in (0,1)$ there exists $\beta_0 \in (0,1)$ such that $\Sigma_\beta f(E) \subset f(\Sigma_{\beta_0} E)$. Consequently, $\Sigma f(E) = f(\Sigma E)$. Also, $\partial^* f(E) = f(\partial^* E)$.*

Proof. Let $x \in X$. By [18, Theorem 6.2] we know that for all sufficiently small balls B_1 centered at x, we have for some $a, b > 0$

$$\frac{\mu_Y(f(E) \cap B_2)}{\mu_Y(B_2)} \leq b \left(\frac{\mu_X(E \cap B_1)}{\mu_X(B_1)} \right)^a, \tag{4.3}$$

where B_2 denotes the largest open ball in $f(B_1)$ with center $f(x)$.

Suppose $\beta \in (0,1)$ and $f(x) \in \Sigma_\beta f(E)$, that is, both $\underline{D}(f(E), f(x))$ and $\underline{D}(Y \setminus f(E), f(x))$ are at least as large as β. As the radius of B_1 converges to 0, so does the radius of B_2, and so it follows from (4.3) that

$$\beta_0 := \left(\frac{\beta}{b} \right)^{1/a} \leq \underline{D}(E, x).$$

By using the fact that $\underline{D}(Y \setminus f(E), f(x)) \geq \beta$, and (4.3) with E replaced by $X \setminus E$, we get also $\beta_0 \leq \underline{D}(X \setminus E, x)$. Thus $x \in \Sigma_{\beta_0} E$, and so we have proved that $\Sigma_\beta f(E) \subset f(\Sigma_{\beta_0} E)$. It follows that $\Sigma f(E) \subset f(\Sigma E)$. Since f^{-1} is also quasiconformal, we also get

$$\Sigma f^{-1}(f(E)) \subset f^{-1}(\Sigma f(E))$$

and so $f(\Sigma E) \subset \Sigma f(E)$. We therefore have $f(\Sigma E) = \Sigma f(E)$.

Next suppose that $f(x) \in \partial^* f(E)$. It follows from (4.3) that

$$0 < \left(\frac{\overline{D}(f(E), f(x))}{b} \right)^{1/a} \leq \overline{D}(E, x).$$

From the fact that $\overline{D}(Y \setminus f(E), f(x)) > 0$, and (4.3) with E replaced by $X \setminus E$, we get also $0 < \overline{D}(X \setminus E, x)$. Thus $x \in \partial^* E$, and so we have proved that $\partial^* f(E) \subset f(\partial^* E)$. Since f^{-1} is also quasiconformal, we also get

$$\partial^* f^{-1}(f(E)) \subset f^{-1}(\partial^* f(E))$$

and so $f(\partial^* E) \subset \partial^* f(E)$, whence we conclude that $f(\partial^* E) = \partial^* f(E)$. □

Lemma 4.9. *There exists $\alpha > 0$ such that for every $E \subset X$ of finite perimeter and for $\mathcal{H}^{Q-1}$-almost every $x \in \partial^* E$,*

$$\liminf_{r \to 0^+} \frac{P(E, B(x,r))}{r^{Q-1}} > \alpha.$$

Moreover, if $0 < \beta < 1$, then there is some $\alpha(\beta) > 0$ such that whenever $E \subset X$ is a measurable set and $x \in \Sigma_\beta E$, we have

$$\liminf_{r \to 0^+} \frac{P(E, B(x,r))}{r^{Q-1}} > \alpha(\beta).$$

Proof. By Ahlfors Q-regularity we know that $r^{-1}\mu(B(x,r))$ is comparable to r^{Q-1}. Recall that C_I is the constant from the isoperimetric inequality and C_A is the Ahlfors regularity constant. By Theorem 4.1, there exists $\gamma > 0$ such that for $\mathcal{H}^{Q-1}$-almost every $x \in \partial^* E$,

$$\gamma \leq \liminf_{r \to 0^+} \frac{\mu(B(x,r) \cap E)}{\mu(B(x,r))} \quad \text{and} \quad \gamma \leq \liminf_{r \to 0^+} \frac{\mu(B(x,r) \setminus E)}{\mu(B(x,r))}.$$

Then by the relative isoperimetric inequality,

$$\begin{aligned}\gamma &\leq \liminf_{r\to 0^+} \frac{\min\{\mu(B(x,r)\cap E), \mu(B(x,r)\setminus E)\}}{\mu(B(x,r))} \\ &\leq \liminf_{r\to 0^+} \frac{rC_I P(E, B(x,r))}{\mu(B(x,r))} \\ &\leq \liminf_{r\to 0^+} \frac{C_I C_A P(E, B(x,r))}{r^{Q-1}}.\end{aligned}$$

Letting $\alpha = \frac{\gamma}{2C_I C_A}$ concludes the proof of the first part of the lemma. The second part of the lemma is proved in the same way as the first part, with

$$\alpha(\beta) := \frac{\beta}{2C_I C_A}. \qquad \square$$

Definition 4.10. For $\alpha > 0$, define

$$\partial^\alpha E := \Big\{ x \in \partial^* E \;\Big|\; \liminf_{r\to 0^+} \frac{P(E, B(x,r))}{r^{Q-1}} > \alpha \Big\}.$$

Remark 4.11. By Lemma 4.9 above, we have that for each $0 < \beta < 1$,

$$\Sigma_\beta E \subset \partial^{\alpha(\beta)} E.$$

We need the following "continuity from below" for families of measures, with the families not necessarily measurable with respect to the outer measure Mod_p.

Lemma 4.12 (Ziemer's lemma [6, Lemma 3.1(3)])**.** *Let $\{\mathcal{L}_i\}_{i\in\mathbb{N}}$ be a sequence of families of measures in X such that for each i, $\mathcal{L}_i \subset \mathcal{L}_{i+1}$. Then for $1 < p < \infty$,*

$$\mathrm{Mod}_p\Big(\bigcup_{i\in\mathbb{N}} \mathcal{L}_i\Big) = \lim_{i\to\infty} \mathrm{Mod}_p(\mathcal{L}_i).$$

5 Proof of Theorem 1.1

In this section we prove Theorem 1.1. To do so, we adapt the tools given in [6] to study boundaries of sets of finite perimeter, which are *not* Ahlfors regular in general. The adaptation of the tools to this setting is given in Proposition 5.1.

We remind the reader of the standard assumptions set forth at the end of Sections 2 and 3. Recall also the definition of $\partial^\alpha E$ from Definition 4.10. As noted in the introduction, with a slight abuse in notation, we will use $\mathcal{L}$ to denote both the family of sets E of finite perimeter and the family of measures $\mathcal{H}^{Q-1}\lfloor_{\Sigma E}$.

If $\mathcal{L}$ contains a set E with $P(E, X) = 0$, i.e. $\mathcal{H}^{Q-1}(\partial^* E) = 0$, then $\mathrm{Mod}_{Q/(Q-1)}(\mathcal{L}) = \infty$ as there can be no admissible test function ρ for the class $\mathcal{L}$. In addition, by the fact that X supports a 1-Poincaré inequality we will also have that $\mathcal{H}^Q(E) = 0$ or $\mathcal{H}^Q(X\setminus E) = 0$. It then follows that $\mathcal{H}^Q(f(E)) = 0$ or $\mathcal{H}^Q(f(X\setminus E)) = \mathcal{H}^Q(Y \setminus f(E)) = 0$, whence it follows that $f(E)$ is of finite perimeter with $P(f(E), Y) = 0$. We conclude that $\mathrm{Mod}_{Q/(Q-1)}(f\mathcal{L}) = \infty$. In this case Theorem 1.1 trivially holds true. So in the proof of the next proposition (and in the next section) we will assume that every $E \in \mathcal{L}$ satisfies $0 < P(E, X) < \infty$.

Proposition 5.1. *Let $\mathcal{L}$ denote the given collection of bounded sets $E \subset X$ of finite perimeter measure. Then the following hold true:*

1. *for each $\alpha > 0$ and $\mathrm{Mod}_{Q/(Q-1)}$-almost every $E \in \mathcal{L}$ we have $\mathcal{H}^{Q-1}(f(\partial^\alpha E)) < \infty$,*
2. *if $U_0 \subset X$ with $\mathcal{H}^Q(f(U_0)) = 0$ and $\alpha > 0$, then we have that $\mathcal{H}^{Q-1}(f(\partial^\alpha E \cap U_0)) = 0$ for $\mathrm{Mod}_{Q/(Q-1)}$-almost every $E \in \mathcal{L}$,*
3. *if $U_0 \subset X$ with $\mathcal{H}^Q(f(U_0)) = 0$, then with $\mathcal{L}_{bad}$ the collection of all $E \in \mathcal{L}$ with $\mathcal{H}^{Q-1}(\Sigma f(E) \cap f(U_0)) > 0$, we have $\mathrm{Mod}_{Q/(Q-1)}(\mathcal{L}_{bad}) = 0 = \mathrm{Mod}_{Q/(Q-1)}(f\mathcal{L}_{bad})$.*

Proof. Recall from Lemma 4.8 that $f(\partial^* E) = \partial^* f(E)$ and $f(\Sigma E) = \Sigma f(E)$.

Let $U \subset X$ be a bounded measurable set. Then since f is a homeomorphism and closed and bounded subsets of X are compact, $f(U)$ is also bounded and f and f^{-1} are uniformly continuous on the sets U and $f(U)$, respectively. Let $\omega(\cdot)$ be a modulus of continuity for f^{-1} on $f(U)$. Let $\varepsilon > 0$. By the definition of Hausdorff measure, there exist $y_i \in f(U)$ and $0 < s_i < \varepsilon$ such that $\{B(y_i, s_i)\}_{i\in\mathbb{N}}$ covers $f(U)$ and $\sum_{i\in\mathbb{N}} s_i^Q \leq (\mathcal{H}^Q(f(U)) + \varepsilon)$. By Lemma 4.7, for every $x_i := f^{-1}(y_i)$, there exists $0 < r_i \leq \omega(\varepsilon)$ such that

$$B(y_i, s_i) \subset f(B(x_i, r_i)) \subset f(B(x_i, 10r_i)) \subset B(y_i, \eta(10)s_i).$$

Set $B_i = B(x_i, r_i)$ and $B_i' = B(y_i, s_i)$ and define $g : X \to \mathbb{R}$ by

$$g(x) = \sup_{i\in\mathbb{N}} \left(\frac{s_i}{r_i}\right)^{Q-1} \chi_{2B_i}(x).$$

Fix $\alpha > 0$ and define

$$\partial^\alpha_\delta E = \left\{ x \in \partial^* E \;\middle|\; \frac{P(E, B(x,r))}{r^{Q-1}} > \alpha \text{ for all } 0 < r \leq \delta \right\}$$

and note that $\partial^\alpha E = \bigcup_{k\in\mathbb{N}} \partial^\alpha_{1/k} E$. For each $M > 0$, define

$$\mathcal{L}^M_{U,\varepsilon} = \left\{ E \in \mathcal{L} \;\middle|\; \mathcal{H}^{Q-1}_{\varepsilon\eta(10)}(f(\partial^\alpha_{\omega(\varepsilon)} E \cap U)) > M \right\}. \tag{5.1}$$

We want to show that $C\eta(10)^{Q-1}[\alpha M]^{-1}\, g$ is admissible for $\mathcal{L}^M_{U,\varepsilon}$. Let $E \in \mathcal{L}^M_{U,\varepsilon}$ and set $I_E = \{i \in \mathbb{N} \,\big|\, B_i \cap (\partial^\alpha_{\omega(\varepsilon)} E \cap U) \neq \emptyset\}$. Note that $\partial^\alpha_{\omega(\varepsilon)} E \cap U \subset \bigcup_{i\in I_E} 2B_i$. Then by the 5-Covering Lemma, there exists $J_E \subset I_E$ such that $\{2B_j\}_{j\in J_E}$ is pairwise disjoint and $\partial^\alpha_{\omega(\varepsilon)} E \cap U \subset \bigcup_{j\in J_E} 10B_j$. Since $B_j \cap \partial^\alpha_{\omega(\varepsilon)} E \cap U \neq \emptyset$ for each $j \in J_E$, there exists $z_j \in B_j \cap \partial^\alpha_{\omega(\varepsilon)} E \cap U$. So $B(z_j, r_j) \subset 2B_j$. Moreover, $z_j \in \partial^\alpha_{\omega(\varepsilon)} E$ and $r_j \leq \omega(\varepsilon)$ imply that $r_j^{-(Q-1)} P(E, B(z_j, r_j)) > \alpha$, and hence $P(E, 2B_j) \geq P(E, B(z_j, r_j)) > \alpha r_j^{Q-1}$. Then by the

pairwise disjointness property of $\{2B_j\}_{j\in J_E}$,

$$\begin{aligned}\int_{\Sigma E} g\, d\mathcal{H}^{Q-1} \geq \int_{\Sigma E} \Big(\sup_{j\in J_E} \left(\frac{s_j}{r_j}\right)^{Q-1} \chi_{2B_j} \Big)\, d\mathcal{H}^{Q-1} &= \int_{\Sigma E} \Big(\sum_{j\in J_E} \left(\frac{s_j}{r_j}\right)^{Q-1} \chi_{2B_j} \Big)\, d\mathcal{H}^{Q-1} \\ &\geq \frac{1}{C} \sum_{j\in J_E} \left(\frac{s_j}{r_j}\right)^{Q-1} P(E, 2B_j) \quad \text{by (4.1)} \\ &\geq \frac{1}{C} \sum_{j\in J_E} \left(\frac{s_j}{r_j}\right)^{Q-1} \alpha\, r_j^{Q-1} \\ &= \frac{\alpha}{C} \sum_{j\in J_E} s_j^{Q-1}.\end{aligned}$$

Thus we have that

$$\int_{\Sigma E} g\, d\mathcal{H}^{Q-1} \geq \frac{\alpha}{C} \sum_{j\in J_E} s_j^{Q-1}. \tag{5.2}$$

Because $U \cap \partial^\alpha_{\omega(\varepsilon)} E \subset \bigcup_{j\in J_E} 10B_j$,

$$f(\partial^\alpha_{\omega(\varepsilon)} E \cap U) \subset f\Big(\bigcup_{j\in J_E} 10B_j \Big) \subset \bigcup_{j\in J_E} B(y_j, \eta(10)s_j).$$

Since $E \in \mathcal{L}^M_{U,\varepsilon}$, we have that $\mathcal{H}^{Q-1}_{\varepsilon\eta(10)}(f(\partial^\alpha_{\omega(\varepsilon)} E \cap U)) > M$. As $s_j < \varepsilon$, $\bigcup_{j\in J_E} \eta(10) B'_j$ is an admissible cover for computing $\mathcal{H}^{Q-1}_{\varepsilon\eta(10)}(f(\partial^\alpha_{\omega(\varepsilon)} E \cap U))$, and hence

$$M < \mathcal{H}^{Q-1}_{\varepsilon\eta(10)}(f(\partial^\alpha_{\omega(\varepsilon)} E \cap U)) \leq \sum_{j\in J_E} (\eta(10)s_j)^{Q-1} = \eta(10)^{Q-1} \sum_{j\in J_E} s_j^{Q-1}.$$

So

$$\frac{M}{\eta(10)^{Q-1}} < \sum_{j\in J_E} s_j^{Q-1}. \tag{5.3}$$

Combining (5.2) and (5.3), we get

$$\int_{\Sigma E} g\, d\mathcal{H}^{Q-1} \geq \frac{\alpha}{C} \left(\frac{M}{\eta(10)^{Q-1}} \right).$$

Therefore

$$\frac{C\,\eta(10)^{Q-1}}{\alpha\, M} g$$

is admissible for $\mathcal{L}^M_{U,\varepsilon}$. Setting

$$C_M = \frac{C\,\eta(10)^{Q-1}}{\alpha\, M},$$

where C is the constant from (4.1), we obtain

$$\begin{aligned}
\mathrm{Mod}_{Q/(Q-1)}(\mathcal{L}^M_{U,\varepsilon}) &\le C_A \int_X (C_M g)^{Q/(Q-1)}\, d\mathcal{H}^Q \\
&= C_M^{Q/(Q-1)} C_A \int_X \sup_{i\in\mathbb{N}} \left(\left(\frac{s_i}{r_i}\right)^{Q-1} \chi_{2B_i} \right)^{Q/(Q-1)} d\mathcal{H}^Q \\
&\le C_M^{Q/(Q-1)} C_A \int_X \left(\sum_{i\in\mathbb{N}} \left(\frac{s_i}{r_i}\right)^{Q} \chi_{2B_i} \right) d\mathcal{H}^Q \\
&\le C_M^{Q/(Q-1)} C_A \sum_{i\in\mathbb{N}} \left(\frac{s_i}{r_i}\right)^{Q} \mathcal{H}^Q(2B_i) \\
&\le C_M^{Q/(Q-1)} C_A \sum_{i\in\mathbb{N}} \left(\frac{s_i}{r_i}\right)^{Q} C_A\,(2r_i)^Q \\
&= C_M^{Q/(Q-1)} C_A^2 2^Q \sum_{i\in\mathbb{N}} s_i^Q \\
&\le C_M^{Q/(Q-1)} C_A^2 2^Q (\mathcal{H}^Q(f(U)) + \varepsilon).
\end{aligned}$$

Recalling the definition of $\mathcal{L}^M_{U,\epsilon}$ from (5.1), set $\mathcal{L}^M_U = \bigcup_{k\in\mathbb{N}} \mathcal{L}^M_{U,1/k}$. Then

$$\mathcal{L}^M_U = \{E \in \mathcal{L} \,|\, \mathcal{H}^{Q-1}(f(\partial^\alpha E \cap U)) > M\}. \tag{5.4}$$

Note that if $m < k$ then $\mathcal{L}^M_{U,1/m} \subset \mathcal{L}^M_{U,1/k}$. Applying Lemma 4.12, we get that

$$\begin{aligned}
\mathrm{Mod}_{Q/(Q-1)}(\mathcal{L}^M_U) = \lim_{k\to\infty} \mathrm{Mod}_{Q/(Q-1)}(\mathcal{L}^M_{U,1/k}) &\le \lim_{k\to\infty} C_M^{Q/(Q-1)} C_A^2 2^Q \Big(\mathcal{H}^Q(f(U)) + \frac{1}{k}\Big) \\
&= C_M^{Q/(Q-1)} C_A^2 2^Q \mathcal{H}^Q(f(U)).
\end{aligned}$$

To summarize, we have

$$\mathrm{Mod}_{Q/(Q-1)}(\mathcal{L}^M_U) \le C_* \frac{\mathcal{H}^Q(f(U))}{M^{Q/(Q-1)}}, \text{ where } C_* = 2^Q C_A^2 \left(\frac{C\eta(10)^{Q-1}}{\alpha}\right)^{Q/(Q-1)}. \tag{5.5}$$

Recall that $\mathrm{Mod}_{Q/(Q-1)}$ is an outer measure on the family of all curves in X, see [9, Theorem 1]. We will make use of the corresponding monotonicity and subadditivity properties in the following.

Proof of Claim *1*: Set $\mathcal{L}^\infty_U = \{E \in \mathcal{L} \,|\, \mathcal{H}^{Q-1}(f(\partial^\alpha E \cap U)) = \infty\}$. By the monotonicity property of $\mathrm{Mod}_{Q/(Q-1)}$, when $M_1 > M_2$ we have $\mathrm{Mod}_{Q/(Q-1)}(\mathcal{L}^{M_1}_U) \le \mathrm{Mod}_{Q/(Q-1)}(\mathcal{L}^{M_2}_U)$, and so $\lim_{M\to\infty} \mathrm{Mod}_{Q/(Q-1)}(\mathcal{L}^M_U)$ exists. Note that for each $M > 0$ we have $\mathcal{L}^M_U \supset \mathcal{L}^\infty_U$, and so

$$\mathrm{Mod}_{Q/(Q-1)}(\mathcal{L}^\infty_U) \le \lim_{M\to\infty} \mathrm{Mod}_{Q/(Q-1)}(\mathcal{L}^M_U) \le \lim_{M\to\infty} C_* \frac{\mathcal{H}^{Q/(Q-1)}(f(U))}{M^{Q/(Q-1)}} = 0.$$

Fixing $x_0 \in X$ and considering $U_i = B(x_0, i)$ for each $i \in \mathbb{N}$, we set $\mathcal{L}^\infty = \bigcup_{i\in\mathbb{N}} \mathcal{L}^\infty_{U_i}$. Then as the sets in $\mathcal{L}$ are bounded, $\mathcal{L}^\infty = \{E \in \mathcal{L} \,|\, \mathcal{H}^{Q-1}(f(\partial^\alpha E)) = \infty\}$ and

$$\mathrm{Mod}_{Q/(Q-1)}(\mathcal{L}^\infty) = \mathrm{Mod}_{Q/(Q-1)}\Big(\bigcup_{i\in\mathbb{N}} \mathcal{L}^\infty_{U_i}\Big) \le \sum_{i\in\mathbb{N}} \mathrm{Mod}_{Q/(Q-1)}(\mathcal{L}^\infty_{U_i}) = 0.$$

Therefore the set for which $\mathcal{H}^{Q-1}(f(\partial^\alpha E))$ is not finite has $Q/(Q-1)$-modulus zero. This proves Claim *1*.

Proof of Claim *2*: Let $U_0 \subset X$ such that $\mathcal{H}^Q(f(U_0)) = 0$. Then for any bounded $U \subset U_0$, $\mathcal{H}^Q(f(U)) = 0$. Recall $\mathcal{L}^M_U$ from (5.4) for $M > 0$, and let $\mathcal{L}^+_U = \bigcup_{m\in\mathbb{N}} \mathcal{L}^{1/m}_U$. Then

$$\mathcal{L}^+_U = \{E \in \mathcal{L} \,|\, \mathcal{H}^{Q-1}(f(\partial^\alpha E \cap U)) > 0\},$$

and by (5.5) and by the subadditivity property of modulus,

$$\mathrm{Mod}_{Q/(Q-1)}\left(\mathcal{L}^+_U\right) = \mathrm{Mod}_{Q/(Q-1)}(\bigcup_{m\in\mathbb{N}} \mathcal{L}^{1/m}_U) \le \sum_{m\in\mathbb{N}} \mathrm{Mod}_{Q/(Q-1)}(\mathcal{L}^{1/m}_U) = 0.$$

Define $V_i = B(x_0, i) \cap U_0$ and set $\mathcal{L}^+ = \bigcup_{i\in\mathbb{N}} \mathcal{L}^+_{V_i}$. Then

$$\mathcal{L}^+ = \{E \in \mathcal{L} \,|\, \mathcal{H}^{Q-1}(f(\partial^\alpha E \cap U_0)) > 0\},$$

and

$$\mathrm{Mod}_{Q/(Q-1)}(\mathcal{L}^+) \le \sum_{i\in\mathbb{N}} \mathrm{Mod}_{Q/(Q-1)}(\mathcal{L}^+_{V_i}) = 0.$$

Proof of Claim *3*: Again, suppose $\mathcal{H}^Q(f(U_0)) = 0$. By Lemma 4.9, for any $0 < \beta < 1$ there exists $\alpha(\beta) > 0$ such that $\Sigma_\beta E \subset \partial^{\alpha(\beta)} E$. Then by Claim *2*, for $\mathrm{Mod}_{Q/(Q-1)}$-almost every E

$$\begin{aligned}
\mathcal{H}^{Q-1}(f(\Sigma E \cap U_0)) = \mathcal{H}^{Q-1}\left(\bigcup_{\beta\in(0,1)} f(\Sigma_\beta E \cap U_0)\right) &\le \mathcal{H}^{Q-1}\left(\bigcup_{\beta\in(0,1)\cap\mathbb{Q}} f(\partial^{\alpha(\beta)} E \cap U_0)\right) \\
&\le \sum_{\beta\in(0,1)\cap\mathbb{Q}} \mathcal{H}^{Q-1}(f(\partial^{\alpha(\beta)} E \cap U_0)) \\
&= 0.
\end{aligned}$$

Finally, by considering the function $\infty\chi_{f(U_0)}$, we see that $\mathrm{Mod}_{Q/(Q-1)}(f\mathcal{L}_{bad}) = 0$. □

Proposition 5.2. *Let $\mathcal{L}$ denote the given collection of bounded sets of finite perimeter in X. For $\mathrm{Mod}_{Q/(Q-1)}$-almost every $E \in \mathcal{L}$ we have*

$$f_\#(\mathcal{H}^{Q-1}\lfloor_{\Sigma f(E)}) \ll \mathcal{H}^{Q-1}\lfloor_{\partial^* E} = \mathcal{H}^{Q-1}\lfloor_{\Sigma E}.$$

Proof. The last equality follows from Theorem 4.1. To prove the absolute continuity, first we set

$$P = \{x \in X \,|\, L_f(x) \in \{0, \infty\}\}.$$

For $n, m \in \mathbb{N}$, set

$$A_{n,m} = \left\{ x \in X \;\middle|\; \frac{L_f(x,r)}{r} \leq m \text{ for } 0 < r < \frac{1}{n} \right\}. \tag{5.6}$$

Then $\bigcup_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} A_{n,m} \supset X \setminus P$. If $x, y \in A_{n,m}$ such that $d_X(x,y) < 1/n$, then

$$d_Y(f(x), f(y)) \leq m\, d_X(x,y), \tag{5.7}$$

that is, f is locally m-Lipschitz on $A_{n,m}$. Note from Lemma 4.6 that $J_f(x) = 0$ for every x for which $L_f(x) = 0$. So by the fact that f and f^{-1} satisfy condition (N) (from Theorem 4.3), we know that the zero set of J_f and hence the zero set of L_f is a null set. As $J_f \in L^1_{loc}(X)$, we know that $\mathcal{H}^Q(P) = 0$ and thus $\mathcal{H}^Q(f(P)) = 0$. So by Proposition 5.1(3), for $\mathrm{Mod}_{Q/(Q-1)}$-almost every $E \in \mathcal{L}$ we have $\mathcal{H}^{Q-1}(f(\Sigma E \cap P)) = 0$. For such $E \in \mathcal{L}$, fix $N \subset \Sigma E$ such that $\mathcal{H}^{Q-1}(N) = 0$. Then for each $m, n \in \mathbb{N}$, by the local m-Lipschitz property of f on $A_{n,m}$, we have

$$\mathcal{H}^{Q-1}\left(f\big(N \cap A_{n,m}\big)\right) \leq m^{Q-1}\, \mathcal{H}^{Q-1}\left(N \cap A_{n,m}\right) = 0.$$

Thus

$$\mathcal{H}^{Q-1}(f(N \setminus P)) \leq \mathcal{H}^{Q-1}\left(f\big(N \cap \bigcup_{n,m\in\mathbb{N}} A_{n,m}\big)\right) \leq \sum_{n,m\in\mathbb{N}} m^{Q-1}\, \mathcal{H}^{Q-1}\left(N \cap A_{n,m}\right) = 0,$$

and therefore as $N \subset \Sigma E$, we have

$$\mathcal{H}^{Q-1}(f(N)) \leq \mathcal{H}^{Q-1}(f(N \setminus P)) + \mathcal{H}^{Q-1}(f(\Sigma E \cap P)) = 0.$$

Combining this with the fact that $\Sigma f(E) = f(\Sigma E)$ (see Lemma 4.8) completes the proof. □

Recall the definition of $J_{f,E}$ from Definition 3.7, and note that it is a function on ΣE.

Proposition 5.3. *For* $\mathrm{Mod}_{Q/(Q-1)}$*-almost every* $E \in \mathcal{L}$ *we have* $\mathcal{H}^{Q-1}(\Sigma f(E)) < \infty$ *and* $f(E)$ *is of finite perimeter.*

Proof. By Lemma 4.9 there exists $\alpha_0 > 0$ such that $\mathcal{H}^{Q-1}(\Sigma E \setminus \partial^{\alpha_0} E) = 0$ for every $E \in \mathcal{L}$. Then by Proposition 5.2 we know that $\mathcal{H}^{Q-1}(\Sigma f(E) \setminus f(\partial^{\alpha_0} E)) = 0$ for $\mathrm{Mod}_{Q/(Q-1)}$-almost every $E \in \mathcal{L}$. Finally, by Proposition 5.1(1) we have $\mathcal{H}^{Q-1}(f(\partial^{\alpha_0} E)) < \infty$ after eliminating a further family of $\mathrm{Mod}_{Q/(Q-1)}$-zero from $\mathcal{L}$.

The last claim now follows from Theorem 4.2. □

Lemma 5.4. *There exists* $C > 0$ *such that for* $\mathrm{Mod}_{Q/(Q-1)}$*-almost every* $E \in \mathcal{L}$*,*

$$J_{f,E}(x) \leq C J_f(x)^{(Q-1)/Q}$$

for $\mathcal{H}^{Q-1}\lfloor_{\Sigma E}$*-almost every* x*.*

Proof. From Proposition 5.2 we know that for $\mathrm{Mod}_{Q/(Q-1)}$-almost every E, $f_{\#}\mathcal{H}^{Q-1}\lfloor_{\Sigma f(E)}$ is absolutely continuous with respect to $\mathcal{H}^{Q-1}\lfloor_{\partial^* E} = \mathcal{H}^{Q-1}\lfloor_{\Sigma E}$. Furthermore, from Proposition 5.3 we know that $\mathcal{H}^{Q-1}(\Sigma f(E)) < \infty$ for $\mathrm{Mod}_{Q/(Q-1)}$-almost every E. We focus only on such $E \in \mathcal{L}$ now.

Note that L_f is a locally integrable function on X, so by Lusin's Theorem, for each $k \in \mathbb{N}$ there is some open set $U_k \subset X$ such that $\mu(U_k) \leq 2^{-k}$ and $L_f|_{X \setminus U_k}$ is continuous. By enlarging U_k if necessary, we can also assume that for each $k \in \mathbb{N}$, $P \subset U_k$. Let $U = \bigcap_{k \in \mathbb{N}} U_k$. Then $\mu(U) = 0$. Let $\mathcal{L}_0 = \{E \in \mathcal{L} : \mathcal{H}^{Q-1}(U \cap \Sigma E) > 0\}$. Then $\infty \chi_U$ is admissible for computing $\operatorname{Mod}_{Q/(Q-1)}(\mathcal{L}_0)$, and so $\operatorname{Mod}_{Q/(Q-1)}(\mathcal{L}_0) \leq \int_X \infty \chi_U \, d\mu = 0$. We ignore such E as well.

Observe that for every $x \in X \setminus U$ we have $L_f(x) > 0$. For $n \in \mathbb{N}$ we set

$$A_n = \{x \in \Sigma E : \, L_f(x, r)/r \leq 2L_f(x) \text{ for } 0 < r < 1/n\}.$$

It is not difficult to show that A_n is a Borel set by writing it as an intersection of Borel sets, one for each rational $r \in (0, 1/n)$. Now by Proposition 5.2 and the Radon-Nikodym Theorem,

$$\int_{\Sigma E} J_{f,E} \, d\mathcal{H}^{Q-1} = \mathcal{H}^{Q-1}(f(\Sigma E)).$$

Since $\mathcal{H}^{Q-1}(f(\Sigma E)) = \mathcal{H}^{Q-1}(\Sigma f(E))$ is finite, $J_{f,E} \in L^1(\Sigma E, \mathcal{H}^{Q-1})$. For $k, n \in \mathbb{N}$ let E_n^k denote the collection of all $x \in \Sigma E$ that are not Lebesgue points of $\chi_{A_n \setminus U_k} J_{f,E}$. Then for each $k \in \mathbb{N}$ we have $\mathcal{H}^{Q-1}(\bigcup_{n \in \mathbb{N}} E_n^k) = 0$. For $x \in \Sigma E \setminus \bigcap_{k \in \mathbb{N}} \left(U_k \cup \bigcup_{n \in \mathbb{N}} E_n^k\right)$, there is some $k \in \mathbb{N}$ and $n \in \mathbb{N}$ such that $x \in A_n$ but $x \notin E_n^k \cup U_k$. Therefore

$$\begin{aligned} J_{f,E}(x) &= \lim_{r \to 0^+} \fint_{B(x,r) \cap \Sigma E} \chi_{A_n \setminus U_k} \, J_{f,E} \, d\mathcal{H}^{Q-1} \\ &= \lim_{r \to 0^+} \frac{\int_{B(x,r) \cap A_n \setminus U_k} J_{f,E} \, d\mathcal{H}^{Q-1}}{\mathcal{H}^{Q-1}(B(x,r) \cap \Sigma E)} \\ &= \lim_{r \to 0^+} \frac{f_\# \mathcal{H}^{Q-1}(B(x,r) \cap A_n \setminus U_k)}{\mathcal{H}^{Q-1}(B(x,r) \cap \Sigma E)}. \end{aligned} \tag{5.8}$$

From an argument similar to (5.7) we know that f is $2 \sup_{B(x,r) \setminus U_k} L_f$–Lipschitz continuous on $B(x,r) \cap A_n \setminus U_k$ when $0 < r < 1/2n$, and so

$$\begin{aligned} f_\# \mathcal{H}^{Q-1}(B(x,r) \cap A_n \setminus U_k) &= \mathcal{H}^{Q-1}(f(B(x,r) \cap A_n \setminus U_k)) \\ &\leq \left[2 \sup_{B(x,r) \setminus U_k} L_f\right]^{Q-1} \mathcal{H}^{Q-1}(B(x,r) \cap A_n \setminus U_k) \\ &\leq \left[2 \sup_{B(x,r) \setminus U_k} L_f\right]^{Q-1} \mathcal{H}^{Q-1}(B(x,r) \cap \Sigma E), \end{aligned}$$

and so by (5.8) and the continuity of L_f in $X \setminus U_k$,

$$J_{f,E}(x) \leq \limsup_{r \to 0^+} \left[2 \sup_{B(x,r) \setminus U_k} L_f\right]^{Q-1} = 2^{Q-1} \, L_f(x)^{Q-1}.$$

Now by applying Lemma 4.6, we obtain

$$J_{f,E}(x) \leq C J_f(x)^{(Q-1)/Q}.$$

This holds for all $x \in \Sigma E \setminus (\bigcap_{k\in\mathbb{N}}(U_k \cup \bigcup_{n\in\mathbb{N}} E_n^k))$. As $E \notin \mathcal{L}_0$ and $\mathcal{H}^{Q-1}(\bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} E_n^k) = 0$, we have

$$\mathcal{H}^{Q-1}\left(\Sigma E \cap \bigcap_{k\in\mathbb{N}} \left(U_k \cup \bigcup_{n\in\mathbb{N}} E_n^k\right)\right) = 0,$$

and so the claim holds. □

Proof of Theorem 1.1. Let $\mathcal{L}_0$ be the set of $E \in \mathcal{L}$ for which the conclusion of either Proposition 5.2 or Lemma 5.4 fails. Then by those results, we know that $\mathrm{Mod}_{Q/(Q-1)}(\mathcal{L}_0) = 0$. Let $E \in \mathcal{L} \setminus \mathcal{L}_0$.

From Proposition 5.2 and Lemma 4.8 we know that the pull-back measure $f_\#(\mathcal{H}^{Q-1}\lfloor_{\Sigma f(E)}) = f_\#(\mathcal{H}^{Q-1}\lfloor_{f(\Sigma E)})$ is absolutely continuous with respect to $\mathcal{H}^{Q-1}\lfloor_{\Sigma E}$. Therefore, whenever φ is a nonnegative Borel function on Y, we have

$$\int_{f(\Sigma E)} \varphi \, d\mathcal{H}^{Q-1} = \int_{\Sigma E} \varphi \circ f \, J_{f,E} \, d\mathcal{H}^{Q-1}.$$

From Lemma 5.4 we know that the corresponding Radon-Nikodym derivative is dominated by $C_0 \, J_f^{(Q-1)/Q}$. Suppose $\rho : Y \to [0,\infty]$ is admissible for $f\mathcal{L}$. Define $\widetilde{\rho} : X \to [0,\infty]$ by

$$\widetilde{\rho} = C_0 \, (\rho \circ f) J_f{}^{(Q-1)/Q}.$$

Then $\widetilde{\rho}$ is admissible for calculating $\mathrm{Mod}_{Q/(Q-1)}(\mathcal{L} \setminus \mathcal{L}_0)$ since by Proposition 5.2 and the change of variables formula,

$$\begin{aligned}\int_{\Sigma E} \widetilde{\rho} \, d\mathcal{H}^{Q-1} = \int_{\Sigma E} C_0 \, (\rho \circ f) J_f{}^{(Q-1)/Q} \, d\mathcal{H}^{Q-1} &\geq \int_{\Sigma E} (\rho \circ f) J_{f,E} \, d\mathcal{H}^{Q-1} \\ &= \int_{f(\Sigma E)} \rho \, d\mathcal{H}^{Q-1} \\ &= \int_{\Sigma f(E)} \rho \, d\mathcal{H}^{Q-1} \geq 1.\end{aligned}$$

It follows that

$$\begin{aligned}\mathrm{Mod}_{Q/(Q-1)}(\mathcal{L}) = \mathrm{Mod}_{Q/(Q-1)}(\mathcal{L} \setminus \mathcal{L}_0) &\leq C_A \int_X \widetilde{\rho}^{Q/(Q-1)} \, d\mathcal{H}^Q \\ &= C_A \int_X \left(C_0 \, (\rho \circ f) J_f{}^{(Q-1)/Q}\right)^{Q/(Q-1)} \, d\mathcal{H}^Q \\ &\leq C \int_X (\rho \circ f)^{Q/(Q-1)} J_f \, d\mathcal{H}^Q \\ &= C \int_Y \rho^{Q/(Q-1)} \, d\mathcal{H}^Q.\end{aligned}$$

Taking the infimum over admissible ρ, we obtain

$$\frac{1}{C}\mathrm{Mod}_{Q/(Q-1)}(\mathcal{L}) \leq \mathrm{Mod}_{Q/(Q-1)}(f\mathcal{L}).$$

We cannot directly apply the argument to f^{-1} to obtain a similar inequality for $f\mathcal{L}$ as we do not know that that family consists solely of sets of finite perimeter. However, note that if $\mathcal{L}^*$ is the set of all $E \in \mathcal{L}$ such that $f(E)$ is not of finite perimeter, then from Theorem 4.2 we know that $\mathcal{H}^{Q-1}(\Sigma f(E)) = \infty$. Thus fixing $y_0 \in Y$ and setting the function $h : Y \to [0, \infty)$ to be

$$h(y) := \sum_{i \in \mathbb{N}} \frac{2^{-i}}{\mathcal{H}^Q(B(y_0, i))^{(Q-1)/Q}} \chi_{B(y_0,i)},$$

we see that for all $E \in \mathcal{L}^*$ there is $i_E \in \mathbb{N}$ such that

$$\int_{\Sigma f(E)} h \, d\mathcal{H}^{Q-1} \geq \frac{2^{-i_E}}{\mathcal{H}^Q(B(y_0, i_E))^{(Q-1)/Q}} \mathcal{H}^{Q-1}(\Sigma f(E)) = \infty,$$

and so h is admissible for $f(\mathcal{L}^*)$. Moreover,

$$\left(\int_Y h^{Q/(Q-1)} \, d\mathcal{H}^Q \right)^{(Q-1)/Q} \leq \sum_{i \in \mathbb{N}} 2^{-i} < \infty,$$

and so $\mathrm{Mod}_{Q/(Q-1)}(f(\mathcal{L}^*)) = 0$. We also know from Proposition 5.3 that $\mathrm{Mod}_{Q/(Q-1)}(\mathcal{L}^*) = 0$. We can now apply the argument given for the first inequality to f^{-1} to obtain a similar inequality for $f(\mathcal{L} \setminus \mathcal{L}^*)$, and then the above argument with h shows that we obtain the second inequality stated in the theorem. □

6 Proof of Theorem 1.2

In this section we focus on the proof of Theorem 1.2. The proof uses the tools developed in the previous section.

Proof of Theorem 1.2. Let $\mathcal{L}$ be the collection of all bounded sets $E \subset X$ of finite perimeter. From Proposition 5.2, we know that for $\mathrm{Mod}_{Q/(Q-1)}$-almost every $E \in \mathcal{L}$,

$$f_\#(\mathcal{H}^{Q-1}\lfloor_{\Sigma f(E)}) \ll \mathcal{H}^{Q-1}\lfloor_{\partial^* E}.$$

Furthermore, by Proposition 5.3, we know that for $\mathrm{Mod}_{Q/(Q-1)}$-almost every such E, $f(E)$ is of finite perimeter, and hence $\mathcal{H}^{Q-1}(\partial^* f(E) \setminus \Sigma f(E)) = 0$. It follows that

$$f_\#(\mathcal{H}^{Q-1}\lfloor_{\partial^* f(E)}) = f_\#(\mathcal{H}^{Q-1}\lfloor_{\Sigma f(E)}) \ll \mathcal{H}^{Q-1}\lfloor_{\partial^* E}.$$

Let $\mathcal{L}^*$ be the collection of all $E \in \mathcal{L}$ for which $f(E)$ is not of finite perimeter; then we know from Theorem 1.1 that

$$\mathrm{Mod}_{Q/(Q-1)}(\mathcal{L}^*) = 0 = \mathrm{Mod}_{Q/(Q-1)}(f(\mathcal{L}^*)).$$

Applying the above argument to the family $f(\mathcal{L} \setminus \mathcal{L}^*)$, we also obtain that for $\mathrm{Mod}_{Q/(Q-1)}$-almost every $f(E) \in f(\mathcal{L} \setminus \mathcal{L}^*)$,

$$\mathcal{H}^{Q-1}\lfloor_{\partial^* E} = \mathcal{H}^{Q-1}\lfloor_{\Sigma E} \ll f_\#(\mathcal{H}^{Q-1}\lfloor_{\partial^* f(E)}).$$

Finally, note that the collection of all E for which $f(E) \in f(\mathcal{L}^*)$ is the collection $\mathcal{L}^*$, which also satisfies $\operatorname{Mod}_{Q/(Q-1)}(\mathcal{L}^*) = 0$. Moreover, the collection $\mathcal{L}^{**}$ of all $E \in \mathcal{L} \setminus \mathcal{L}^*$ for which the above absolute continuity fails satisfies $\operatorname{Mod}_{Q/(Q-1)}(f(\mathcal{L}^{**})) = 0$ and hence $\operatorname{Mod}_{Q/(Q-1)}(\mathcal{L}^{**}) = 0$ by Theorem 1.1.

An application of Lemma 5.4 tells us that $J_{f,E} \leq C J_f^{(Q-1)/Q}\, \mathcal{H}^{Q-1}\lfloor_{\Sigma E}$-almost everywhere for $\operatorname{Mod}_{Q/(Q-1)}$-almost every $E \in \mathcal{L}$. Applying this lemma to f^{-1} also gives

$$J_{f,E}^{-1} = J_{f^{-1},f(E)} \circ f \leq C[J_{f^{-1}} \circ f]^{(Q-1)/Q} = C J_f^{(1-Q)/Q}$$

$\mathcal{H}^{Q-1}\lfloor_{\Sigma f(E)}$-almost everywhere for $\operatorname{Mod}_{Q/(Q-1)}$-almost every such $f(E)$. An application of Theorem 1.1 concludes the proof. □

Remark 6.1. We now complete the discussion in this paper by considering the reasonableness of the two main theorems of this paper. The results would not be useful if the collection of all sets of finite perimeter was $\operatorname{Mod}_{Q/(Q-1)}$-null. However, there is a large family of sets of finite perimeter whose quasiconformal images are also sets of finite perimeter. Indeed, thanks to the BV co-area formula (see [21, Proposition 4.2]) and the fact that $N^{1,Q}(X) \subset BV_{loc}(X)$, we know that if $u \in N^{1,Q}(X)$ is compactly supported, then for $\mathcal{H}^1$-almost every $t \in \mathbb{R}$ we have that the super-level set $E_t := \{u > t\}$ is of finite perimeter in X. Here, by $N^{1,Q}(X)$ we mean the function class $N^{1,Q}(X;\mathbb{R})$ from Definition 2.3. By [14, Theorem 9.10] we know that $u \circ f^{-1} \in N^{1,Q}(Y)$ since f is quasiconformal. Therefore for $\mathcal{H}^1$-almost every $t \in \mathbb{R}$ we have that $f(E_t) = \{u \circ f^{-1} > t\}$ is also of finite perimeter, and so the collection of all $t \in \mathbb{R}$ for which either E_t is not of finite perimeter or $f(E_t)$ is not of finite perimeter is of $\mathcal{H}^1$-measure zero. Hence there are plenty of sets of finite perimeter in X whose image under f is of finite perimeter in Y. The remaining part of this section is devoted to making concrete the notion of "plenty", see Remark 6.4.

Proposition 6.2. *Let $u \in N^{1,Q}(X)$ be compactly supported such that $\int_X g_u^Q \, d\mu > 0$, where g_u is the minimal Q-weak upper gradient of u (*see e.g. [14, Section 6]*), and let $\mathcal{L}$ be the collection of all sets $E_t = \{x \in X : u(x) > t\}$, $t \in \mathbb{R}$, for which $0 < P(E_t, X) < \infty$. Then $\operatorname{Mod}_{Q/(Q-1)}(\mathcal{L}) > 0$.*

Proof. Since $u \in N^{1,Q}(X)$ has compact support, as explained above we have that for almost every $t \in \mathbb{R}$ the set E_t is of finite perimeter. By employing truncation of u and by adding a constant to u if necessary, we may assume without loss of generality that $0 \leq u \leq 1$ on X, and by the monotonicity of $\operatorname{Mod}_{Q/(Q-1)}$ we may replace $\mathcal{L}$ with the collection of all E_t with $0 < P(E_t, X) < \infty$, $0 < t < 1$. If $P(E_t, X) = 0$ for almost every $t \in [0,1]$, then by the 1-Poincaré inequality we know that E_t is either almost all of X or is of measure zero, whence we would have u is constant, violating that $\int_X g_u^Q \, d\mu > 0$. Therefore $\mathcal{L}$ has many sets, one for each t in a positive $\mathcal{H}^1$-measure subset of $[0,1]$.

Let ρ be admissible for $\mathcal{L}$. Then for every $t \in [0,1]$ for which $E_t \in \mathcal{L}$,

$$\int_{\Sigma E_t} \rho \, dP(E_t, \cdot) \geq 1.$$

Integrating over $[0,1]$ and applying the co-area formula and Hölder's inequality, we obtain

$$\begin{aligned}\mathcal{H}^1(\{t\in[0,1]\,|\,E_t\in\mathcal{L}\}) &\le \int_0^1 \int_{\Sigma E_t} \rho\, dP(E_t,\cdot)\, dt \\ &= \int_X \rho\, d\|Du\| \\ &\le C\int_X \rho\, g_u\, d\mathcal{H}^Q \\ &\le C\left(\int_X \rho^{Q/(Q-1)}\, d\mathcal{H}^Q\right)^{1-\frac{1}{Q}} \left(\int_X g_u^Q\, d\mathcal{H}^Q\right)^{\frac{1}{Q}}.\end{aligned}$$

Taking the infimum over all such ρ gives

$$0 < \frac{\mathcal{H}^1(\{t\in[0,1]\,|\,E_t\in\mathcal{L}\})^{Q/(Q-1)}}{C_A C^{Q/(Q-1)}\left(\int_X g_u^Q\, d\mathcal{H}^Q\right)^{1/(Q-1)}} \le \mathrm{Mod}_{Q/(Q-1)}(\mathcal{L}). \tag{6.1}$$

□

Suppose $\int_X g_u^Q\, d\mu > 0$ and that $0 \le u \le 1$ on X. If there is some $0 < t_0 < 1$ for which $P(E_{t_0}, X) = 0$, then either $u > t_0$ almost everywhere on X, or else $u \le t_0$ almost everywhere on X. In the former case we have $P(E_t, X) = 0$ for all $0 < t \le t_0$, and in the latter case we have $P(E_t, X) = 0$ for all $t_0 \le t < 1$. Thus the set D of all $t \in (0,1)$ for which $0 < P(E_t, X) < \infty$ is a full-measure subset of a subinterval of $[0,1]$.

Remark 6.3. If $t_1 < t_2$ and $\int_{\{t_1<u<t_2\}} g_u^Q\, d\mathcal{H}^Q > 0$, then for $\mathcal{L}(t_1,t_2)$, which consists of all $E_t \in \mathcal{L}$ as in the above proposition for $t_1 \le t \le t_2$, we have

$$0 < \frac{\mathcal{H}^1(\{t_1 < t < t_2 :\, E_t\in\mathcal{L}\})}{C} \le \mathrm{Mod}_{\frac{Q}{Q-1}}(\mathcal{L}(t_1,t_2))^{\frac{Q-1}{Q}} \left(\int_{\{t_1<u<t_2\}} g_u^Q\, d\mathcal{H}^Q\right)^{\frac{1}{Q}}.$$

Remark 6.4. As above, we consider the family of super-level sets E_t of a given compactly supported function $u \in N^{1,Q}(X)$. Then, with $f^{-1}: Y \to X$ quasiconformal, we must have $u \circ f^{-1} \in N^{1,Q}(Y)$, and so for almost every $t \in \mathbb{R}$ we have that both E_t and $f(E_t)$ are of finite perimeter. Moreover, as noted at the beginning of Section 5, $P(E_t, X) = 0$ if and only if $P(f(E_t), Y) = 0$.

Now the proof of the inequality (6.1) tells us that the $Q/(Q-1)$-modulus of the family of all E_t for which $P(E_t, X) < \infty$ and $P(f(E_t), Y) < \infty$ is positive. Indeed, if $\mathcal{L}_0$ is the collection of all E_t for which $0 < P(E_t, X) < \infty$ but $P(f(E_t), Y) = \infty$, then whenever ρ_0 is admissible for computing $\mathrm{Mod}_{Q/(Q-1)}(\mathcal{L} \setminus \mathcal{L}_0)$, we then have $1 \le \int_{\Sigma E_t} \rho\, dP(E_t,\cdot)$ for almost every $t \in [0,1]$ with $E_t \in \mathcal{L}$; thus the computation that derives (6.1) also gives the validity of (6.1) with the role of ρ played by ρ_0. That is, $\mathrm{Mod}_{Q/(Q-1)}(\mathcal{L} \setminus \mathcal{L}_0) > 0$. Therefore there are plenty of sets of positive and finite perimeter in X whose image under f is also of positive and finite perimeter; that is, the collection $\mathcal{L}_u$ of all E_t for which $0 < P(E_t, X) < \infty$ and $P(f(E_t), Y) < \infty$ (and hence $0 < P(f(E_t), Y) < \infty$) satisfies $\mathrm{Mod}_{Q/(Q-1)}(\mathcal{L}_u) > 0$ and $\mathrm{Mod}_{Q/(Q-1)}(f(\mathcal{L}_u)) > 0$ provided $\int_X g_u^Q\, d\mathcal{H}^Q > 0$.

References

[1] L. V. Ahlfors and A. Beurling, *Conformal invariants and function-theoretic null-sets*, Acta Math. **83**, (1950). 101–129.

[2] H. Aikawa and M. Ohtsuka, *Extremal length of vector measures*, Ann. Acad. Sci. Fenn. Math. **24** (1999), no. 1, 61–88.

[3] L. Ambrosio, *Fine properties of sets of finite perimeter in doubling metric measure spaces*, Calculus of variations, nonsmooth analysis and related topics. Set-Valued Anal. **10** (2002), no. 2–3, 111–128.

[4] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems.*, Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[5] L. Ambrosio, M. Miranda, Jr., and D. Pallara, *Special functions of bounded variation in doubling metric measure spaces*, Calculus of variations: topics from the mathematical heritage of E. De Giorgi, 1–45, Quad. Mat., **14**, Dept. Math., Seconda Univ. Napoli, Caserta, 2004.

[6] C. Bishop, H. Hakobyan, and M. Williams, *Quasisymmetric dimension distortion of Ahlfors regular subsets of a metric space*, Geom. Funct. Anal. Vol. **26** (2016), 379–421.

[7] A. Björn and J. Björn, *Nonlinear potential theory on metric spaces*, EMS Tracts in Mathematics, 17. European Mathematical Society (EMS), Zürich, 2011, xii+403 pp.

[8] H. Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften **153**, Springer-Verlag New York Inc., New York 1969, xiv+676 pp.

[9] B. Fuglede, *Extremal length and functional completion*, Acta Math. **98** (1957), 171–219.

[10] F. Gehring and J. C. Kelly, *Quasi-conformal mappings and Lebesgue density*, Discontinuous groups and Riemann surfaces (Proc. Conf., Univ. Maryland, College Park, Md., 1973), Ann. of Math. Studies **79**, Princeton Univ. Press, Princeton, N.J., 1974, pp. 171–179.

[11] P. Hajłasz and P. Koskela, *Sobolev met Poincare*, Mem. Amer. Math. Soc. **145** (2000), no. 688, x+101 pp.

[12] J. Heinonen and P. Koskela, *Definitions of quasiconformality*, Invent. Math. **120** (1995), no. 1, 61–79.

[13] J. Heinonen and P. Koskela, *Quasiconformal maps in metric spaces with controlled geometry*, Acta Math. **181** (1998), no. 1, 1–61.

[14] J. Heinonen, P. Koskela, N. Shanmugalingam, and J. Tyson, *Sobolev classes of Banach space-valued functions and quasiconformal mappings*, J. Anal. Math. **85** (2001), 87–139.

[15] J. Heinonen, P. Koskela, N. Shanmugalingam, and J. Tyson, *Sobolev spaces on metric measure spaces: An approach based on upper gradients*, New Mathematical Monographs, 27. Cambridge University Press, Cambridge, 2015, xii+434 pp.

[16] R. Jones and P. Lahti, *Duality of moduli and quasiconformal mappings in metric spaces*, preprint (2019), https://arxiv.org/abs/1905.02873

[17] J. C. Kelly, *Quasiconformal mappings and sets of finite perimeter*, Trans. Amer. Math. Soc. **180** (1973), 367–387.

[18] R. Korte, N. Marola, and N. Shanmugalingam, *Quasiconformality, homeomorphisms between metric measure spaces preserving quasiminimizers, and uniform density property*, Ark. Mat. **50** (2012), 111–134.

[19] P. Lahti, *A new Federer-type characterization of sets of finite perimeter in metric spaces*, preprint (2019), https://arxiv.org/pdf/1906.03125.pdf

[20] P. Lahti, *Federer's characterization of sets of finite perimeter in metric spaces*, to appear in Analysis & PDE. https://arxiv.org/pdf/1804.11216.pdf

[21] M. Miranda, Jr., *Functions of bounded variation on "good" metric spaces*, J. Math. Pures Appl. (9) **82** (2003), no. 8, 975–1004.

[22] J. Väisälä, *Lectures on n-dimensional quasiconformal mappings*, Lecture Notes in Mathematics, **229** Springer-Verlag, Berlin-New York, 1971, xiv+144 pp.

[23] M. Williams, *Geometric and analytic quasiconformality in metric measure spaces*, Proc. Amer. Math. Soc. **140** (2012), no. 4, 1251–1266.

Address:

R.J. and N.S.: Department of Mathematical Sciences,
P. O. Box 210025, University of Cincinnati, Cincinnati, OH 45221-0025, U.S.A.

E-mail: jones3rh@mail.uc.edu, shanmun@uc.edu

P.L. University of Jyväskylä, Department of Mathematics and Statistics,
P.O. Box 35, FI-40014 University of Jyväskylä, Finland

E-mail: panu.lahti@aalto.fi