

THE KALMAN CONDITION FOR THE BOUNDARY
CONTROLLABILITY OF COUPLED 1-D WAVE EQUATIONS

SERGEI AVDONIN AND JEFF PARK

Dept. of Mathematics and Statistics
University of Alaska Fairbanks
Fairbanks, AK 99775, USA

LUZ DE TERESA

Instituto de Matemáticas
Universidad Nacional Autónoma de México
Circuito Exterior, G. U. 04510 D.F., México

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ABSTRACT. The focus of this paper is the exact controllability of a system of N one-dimensional coupled wave equations when the control is exerted on a part of the boundary by means of one control. We give a *Kalman condition* (necessary and sufficient) and give a description of the attainable set. In general, this set is not optimal, but can be refined under certain conditions.

1. Statement of the problem and main results. This work is devoted to the study of the controllability properties of the following hyperbolic system

$$\begin{cases} u_{tt} - u_{xx} + Au = 0, & \text{in } Q = (0, \pi) \times (0, T), \\ u(0, t) = bf(t), \quad u(\pi, t) = 0 & \text{for } t \in (0, T), \\ u(x, 0) = u^0(x), \quad u_t(x, 0) = u^1(x) & \text{for } x \in (0, \pi), \end{cases} \quad (1)$$

where $T > 0$ is given, $A \in \mathcal{L}(\mathbb{R}^N)$ is a given matrix referred to as the coupling matrix, b a given vector from \mathbb{R}^N and $f \in L^2(0, T)$ is a control function to be determined which acts on the system by means of the Dirichlet boundary condition at the point $x = 0$. The initial data (u^0, u^1) will belong to a Hilbert space \mathcal{H} , which is to be specified in our main result. Our goal is to give necessary and sufficient conditions for the exact controllability of System (1) and the space \mathcal{H} where this can be done.

We recall that System (1) is exactly controllable in \mathcal{H} at time T if, for every initial and final data $(u^0, u^1), (z^0, z^1)$, both in \mathcal{H} , there exists a control $f \in L^2(0, T)$ such that the solution of System (1) corresponding to (u^0, u^1, f) satisfies

$$u(x, T) = z^0(x), \quad u_t(x, T) = z^1(x). \quad (2)$$

Due to the linearity and time reversibility of System (1), this is equivalent to exact controllability from zero at time T . In other words, System (1) is exactly

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controllable if for every final state $(z^0, z^1) \in \mathcal{H}$, there exists a control $f \in L^2(0, T)$ such that the solution u to System (1) corresponding to f satisfies (2) and

$$u(x, 0) = 0 = u_t(x, 0). \quad (3)$$

Indeed, given exact controllability from zero at time T with controls $f, g \in L^2(0, T)$ and their corresponding solutions $u^f(x, t)$ and $u^g(x, t)$ satisfying

$$\begin{aligned} u^f(x, 0) &= u_t^f(x, 0) = 0, \\ u^f(x, T) &= u^0(x), \quad u_t^f(x, T) = u^1(x), \\ u^g(x, 0) &= u_t^g(x, 0) = 0, \\ u^g(x, T) &= z^0(x), \quad u_t^g(x, T) = z^1(x). \end{aligned}$$

We can define the control $h \in L^2(0, T)$ by $h(t) = f(T - t) + g(t)$. Then the corresponding solution $u(x, t)$ has the form $u(x, t) = u^f(x, T - t) + u^g(x, t)$ and satisfies

$$\begin{aligned} u(x, 0) &= u^0(x), \quad u_t(x, 0) = u^1(x), \\ u(x, T) &= z^0(x), \quad u_t(x, T) = z^1(x). \end{aligned}$$

For this reason, we will assume that $u^0 \equiv 0, u^1 \equiv 0$.

As of now, the controllability properties of System (1) are well known in the scalar case, i.e. when $N = 1$ (see for example [12]). When $N = 1$ and $b \neq 0$, System (1) is exactly controllable in $\mathcal{H} = L^2(0, \pi) \times H^{-1}(0, \pi)$ if $T \geq T_0 = 2\pi$.

Most of the known controllability results of (1) are in the case of two coupled equations: see [5, 16, 11], but the results are for a particular coupling matrix A . In the d -dimensional situation, that is, for a system of coupled wave equations in a domain $\Omega \subset \mathbb{R}^d$, Alabau-Boussouria and collaborators have obtained several results in the case of two equations with the Laplacian plus additional zero order terms and particular coupling matrices (see e.g. [1, 2, 3] and the references therein).

On the other hand, controllability properties of linear ordinary differential systems are well understood. In particular, we have the famous Kalman rank condition (see for example [13] Chapter 2, p.35). That is, if $N, M \in \mathbb{N}$ with $N, M \geq 1$, $A \in \mathcal{L}(\mathbb{R}^N)$ and $B \in \mathcal{L}(\mathbb{R}^M; \mathbb{R}^N)$, then the linear ordinary differential system

$$\begin{aligned} Y'(t) &= AY(t) + Bu(t), \\ Y(0) &= Y^0 \in \mathbb{R}^N, \end{aligned}$$

is controllable at time $T > 0$ if and only if

$$\text{rank}[A|B] = \text{rank}[A^{N-1}B, A^{N-2}B, \dots, B] = N, \quad (4)$$

where $[A^{N-1}B, A^{N-2}B, \dots, B] \in \mathcal{L}(\mathbb{R}^{MN}; \mathbb{R}^N)$.

Recently, Liard and Lissy [14] gave a general Kalman condition for the indirect controllability of N coupled d -dimensional wave equations. Here, indirect controllability refers to having less control functions than equations.

In the framework of parabolic coupled equations, [4] gives a general Kalman rank condition for the null boundary controllability of N coupled one-dimensional parabolic equations. The aim of this research is to establish general results, as in [4], in the case of one-dimensional coupled wave equations.

To state our results, we provide the following definition:

Definition 1.1. Let S be a positive self-adjoint operator in a separable Hilbert space H with spectrum $\{\lambda_n\}_{n=1}^\infty$ and corresponding orthonormal eigenfunctions $\{\varphi_n\}_{n=1}^\infty$. We introduce a weighted space for $r \in \mathbb{R}$

$$\ell_r^2 = \left\{ (c_n) \mid \|c_n\|_r = \left[\sum_{n=1}^{\infty} |c_n|^2 |\lambda_n|^r \right]^{1/2} < \infty \right\}.$$

We then define the scale of spaces

$$W_r = \left\{ f \mid f = \sum_{n=1}^{\infty} c_n \varphi_n, \|f\|_{W_r} := \|(c_n)\|_r < \infty \right\}.$$

For $r > 0$, we set $W_r = \text{Dom}(S^{r/2})$. In the case where $r = 0$, $W_0 = H$, and for $r < 0$, we set $W_r = (W_{-r})'$, where prime indicates the dual space.

Also, we recall that the operator $-\partial_x^2$ in $L^2(0, \pi)$ with zero Dirichlet boundary conditions admits a sequence of eigenvalues $\{\mu_k = k^2\}_{k=1}^\infty$ and eigenfunctions $\{\sin kx\}_{k=1}^\infty$. This family of eigenfunctions is an orthogonal basis in $L^2(0, \pi)$.

For $S = -\partial_x^2 I_N$ in $L^2(0, \pi; \mathbb{R}^N)$ with zero Dirichlet boundary conditions, we set $W_r = \text{Dom}(S^{r/2})$. So, $W_0 = L^2(0, \pi; \mathbb{R}^N)$, $W_1 = H_0^1(0, \pi; \mathbb{R}^N)$, and $W_2 = H^2(0, \pi; \mathbb{R}^N) \cap H_0^1(0, \pi; \mathbb{R}^N)$.

Our main result is the following:

Theorem 1.2. *For a given matrix A with eigenvalues $\{\lambda_i\}$, suppose that the following conditions hold:*

- (i) $[A|b]$ satisfies the Kalman rank condition,
- (ii)

$$\mu_k - \mu_l \neq \lambda_i - \lambda_j, \quad \forall k, l \in \mathbb{N}, \forall 1 \leq i, j \leq N \text{ with } k \neq l \text{ and } i \neq j,$$

$$(iii) \quad T \geq 2N\pi.$$

Then System (1)–(3) is exactly controllable in $\mathcal{H} = W_{N-1} \times W_{N-2}$.

If (i) or (iii) does not hold, then the codimension of the reachable set of System (1)–(3) in $L^2(0, \pi; \mathbb{R}^N) \times H^{-1}(0, \pi; \mathbb{R}^N)$ is infinite. On the other hand, if (ii) fails, the sequence $\{k^2 + \lambda_l\}$, $k \in \mathbb{N}$, $l = 1, \dots, N$, only contains a finite number of multiple points, and so the codimension of the reachable set is finite. Hence, if any of (i), (ii), or (iii) is not satisfied, then System (1)–(3) is not approximately controllable, i.e., the closure of the reachable set is a proper subspace of \mathcal{H} .

In order to prove this theorem, we begin by considering two subcases: when A has N distinct eigenvalues and when A has a single eigenvalue with algebraic multiplicity N . For each of these subcases, we have the following theorems.

Theorem 1.3. *Suppose that A has N distinct eigenvalues $\lambda_1, \dots, \lambda_N$. Assuming that Conditions (i), (ii), and (iii) of Theorem 1.2 hold, then System (1) is exactly controllable in $\mathcal{H} = W_{N-1} \times W_{N-2}$.*

Theorem 1.4. *Suppose that A has a single eigenvalue, λ , with algebraic multiplicity N . Assuming that Conditions (i) and (iii) of Theorem 1.2 hold, then System (1) is exactly controllable in $\mathcal{H} = W_{N-1} \times W_{N-2}$.*

The proof of Theorem 1.3 was presented in [6], the proof of Theorem 1.4 was described in [15].

Remark 1. With respect to Theorem 1.2, we have the following remarks.

- Conditions (i) and (ii) are also necessary conditions that appear in [4] for the null controllability of N coupled one-dimensional parabolic equations. The hyperbolicity of the equations in our case requires a minimal control time, namely $T \geq 2N\pi$.
- In general, the reachable space \mathcal{H} is not optimal. In some particular situations it is possible to give an optimal description of the space. Examples include the cases when $N = 2$ or the coupling matrix is cascade, i.e., when A is triangle inferior, or when A is given in canonical form. Some comments on the optimal reachable space are given in the last section.

The structure of the paper is as follows: we begin by proving Theorems 1.3 and 1.4. Using these theorems, we then prove Theorem 1.2. Additionally, we consider a particular case where $N = 2$ to demonstrate that we can obtain controllability in the sharp space of regularity.

2. Proof of Theorem 1.3 - The case of distinct eigenvalues.

2.1. The Fourier method and existence of solutions. In this section, we use the Fourier method and apply it to the case where the coupling matrix A has N distinct eigenvalues. On the assumptions of Theorem 1.3, we denote $\{\varphi_i\}_{i=1}^N$ to be the family of eigenvectors of A with corresponding eigenvalues $\{\lambda_i\}_{i=1}^N$. We denote by $\langle \cdot, \cdot \rangle$ the inner product in \mathbb{R}^N and so A^* has eigenvalues $\{\overline{\lambda_i}\}_{i=1}^N$ and eigenvectors $\{\psi_i\}_{i=1}^N$ with

$$\langle \varphi_i, \psi_j \rangle = \delta_{ij}.$$

As a result of Condition (i) of Theorem 1.3, we have the following lemma.

Lemma 2.1. *Eigenvectors $\{\varphi_i\}_{i=1}^N$ and $\{\psi_i\}_{i=1}^N$ may be chosen such that $\langle b, \psi_j \rangle = 1$ while maintaining $\langle \varphi_i, \psi_j \rangle = \delta_{ij}$.*

Proof. We first claim that $\langle b, \psi_j \rangle \neq 0$. Indeed, if there exists $1 \leq k \leq N$ such that $\langle b, \psi_k \rangle = 0$, then for all $1 \leq n \leq N - 1$

$$\langle A^n b, \psi_k \rangle = \langle b, (A^*)^n \psi_k \rangle = \langle b, (\overline{\lambda_k})^n \psi_k \rangle = \lambda_k^n \langle b, \psi_k \rangle = 0.$$

This implies that the columns of the matrix $[A|b]$ are linearly dependent, which is a contradiction to A and b satisfying the Kalman rank condition. Hence, we can construct the sets $\{\tilde{\varphi}_i\}_{i=1}^N$ and $\{\tilde{\psi}_i\}_{i=1}^N$ where

$$\begin{aligned} \tilde{\varphi}_i &= \langle b, \psi_i \rangle \varphi_i, \\ \tilde{\psi}_i &= \frac{\psi_i}{\langle b, \psi_i \rangle}. \end{aligned}$$

It then follows that $\langle b, \tilde{\psi}_i \rangle = 1$ for $1 \leq i \leq N$ and $\langle \tilde{\varphi}_i, \tilde{\psi}_j \rangle = \delta_{ij}$ for $1 \leq i, j \leq N$. \square

So we may assume that $\langle b, \psi_i \rangle = 1$. Let us define $\Phi_{nj}(x) = \sin(nx)\varphi_j$. Then $\{\Phi_{nj}(x)\}$, $n \in \mathbb{N}$, $j = 1, \dots, N$, is a Riesz basis in $L^2(0, \pi; \mathbb{R}^N)$ with biorthogonal family $\{\Psi_{nj}(x)\}$ where

$$\Psi_{nj}(x) = \frac{2}{\pi} \sin(nx)\psi_j.$$

We then represent the solution u of System (1) in the form of the series

$$u(x, t) = \sum_{n,j} a_{nj}(t) \Phi_{nj}(x) \tag{5}$$

and set

$$v(x, t) = g(t)\Psi_{kl}(x), \quad (6)$$

for some $k \in \mathbb{N}$, $1 \leq l \leq N$, and where $g(t)$ is a smooth function, i.e., $g \in C_0^2(0, T)$. Below are standard routine manipulations to solve for the coefficients $a_{nj}(t)$:

$$\begin{aligned} 0 &= \int_0^T \int_0^\pi \langle u_{tt} - u_{xx} + Au, v \rangle dx dt \\ &= \int_0^T \int_0^\pi \langle u, v_{tt} - v_{xx} + A^*v \rangle dx dt + \int_0^\pi [\langle u_t, v \rangle - \langle u, v_t \rangle]_{t=0}^T dx \\ &\quad - \int_0^T [\langle u_x, v \rangle - \langle u, v_x \rangle]_{x=0}^\pi dt \\ &= \int_0^T \int_0^\pi \langle u, \ddot{g}\Psi_{kl} + k^2 g\Psi_{kl} + \bar{\lambda}_l g\Psi_{kl} \rangle dx dt \\ &\quad - \frac{2}{\pi} \int_0^T k \langle b, \psi_l \rangle f(t) g(t) dt \\ &= \int_0^T a_{kl} [\ddot{g} + (k^2 + \bar{\lambda}_l)g] dt - \frac{2k}{\pi} \int_0^T f(t) g(t) dt \\ &= \int_0^T [\ddot{a}_{kl} + (k^2 + \bar{\lambda}_l)a_{kl}] g dt - \frac{2k}{\pi} \int_0^T f(t) g(t) dt. \end{aligned}$$

Thus we obtain the equations

$$\ddot{a}_{kl} + (k^2 + \bar{\lambda}_l)a_{kl} = \frac{2k}{\pi} f(t) \quad (7)$$

with zero initial conditions that follow from (3), i.e.

$$a_{kl}(0) = 0 = \dot{a}_{kl}(0). \quad (8)$$

We denote $k^2 + \bar{\lambda}_l$ by ω_{kl}^2 . In the formulas below we assume that $\omega_{kl} \neq 0$. In the case where $\omega_{kl} = 0$, we will set $\frac{\sin(\omega_{kl}t)}{\omega_{kl}} = t$ (see e.g. [8] Sec. III.2). We note the following properties of ω_{kl} .

Proposition 1. *Let $k \in \mathbb{K} = \{\pm 1, \pm 2, \dots\}$ and $1 \leq l, m \leq N$ with $l \neq m$. Provided Condition (ii) of Theorem 1.2, we have the following:*

- (1) $|\omega_{kl}| + 1 \asymp |k|$,
- (2) $|\omega_{kl} - \omega_{km}| \asymp |k|^{-1}$, and
- (3) For k fixed, the points ω_{kl} are asymptotically close, i.e., these N points lie inside an interval whose length tends to zero as k tends to infinity.

The solution of (7)–(8) is given by the formula

$$a_{kl}(t) = \frac{2k}{\pi} \int_0^t f(\tau) \frac{\sin \omega_{kl}(t - \tau)}{\omega_{kl}} d\tau. \quad (9)$$

By differentiating we obtain

$$\dot{a}_{kl}(t) = \frac{2k}{\pi} \int_0^t f(\tau) \cos \omega_{kl}(t - \tau) d\tau. \quad (10)$$

We now introduce the coefficients

$$c_{kl}(t) = i\omega_{kl}a_{kl}(t) + \dot{a}_{kl}(t). \quad (11)$$

We define $\omega_{-kl} = -\omega_{kl}$, $a_{-kl} = a_{kl}$, and $\dot{a}_{-kl} = \dot{a}_{kl}$ for $k \in \mathbb{K}$, $l \in \{1, \dots, N\}$, and rewrite (9) and (10) in the exponential form:

$$c_{kl}(t) = \frac{2k}{\pi} \int_0^t f(\tau) e^{i\omega_{kl}(t-\tau)} d\tau. \quad (12)$$

Taking into account that $\{\Phi_{nj}\}$ forms a Riesz basis in $L^2(0, \pi; \mathbb{R}^N)$ and Proposition 1 Property (1), we conclude that (by [8] Sec.III.1)

$$\sum_{k \in \mathbb{K}} \frac{|c_{kl}(t)|^2}{k^2} \asymp \|u(\cdot, t)\|_{L^2(0, \pi; \mathbb{R}^N)}^2 + \|u_t(\cdot, t)\|_{H^{-1}(0, \pi; \mathbb{R}^N)}^2. \quad (13)$$

On the other hand, from the explicit form for ω_{kl} , it follows that for any $T > 0$, the family $\{e^{i\omega_{kl}t}\}$ is either a finite union of Riesz sequences if $T < 2N\pi$ or a Riesz sequence in $L^2(0, T)$ if $T \geq 2N\pi$ (see [8] Section II.4). We recall that a Riesz sequence is a Riesz basis in the closure of its linear span. Therefore, from (12) it follows that for every fixed $t > 0$

$$\sum_{k,l} \frac{|c_{kl}(t)|^2}{k^2} \prec \|f\|_{L^2(0, t)}^2 \leq \|f\|_{L^2(0, T)}^2. \quad (14)$$

Recall that (13) and (14) refer, respectively, to two-sided and one-sided inequalities with constants independent of the sequences (c_{kl}) , (k) , and of the function f .

Additionally, it can be shown that the series in (14) is uniformly convergent by the Weierstrass criterion for uniform convergence. And by the uniform limit theorem, we obtain

$$\sum_{k,l} \frac{|c_{kl}(t+h) - c_{kl}(t)|^2}{k^2} \rightarrow 0, \quad h \rightarrow 0.$$

We combine our results in the following theorem.

Theorem 2.2. *For any $f \in L^2(0, T)$, there exists a unique generalized solution u^f of the IBVP (1)–(3) with coupling matrix $A \in \mathcal{L}(\mathbb{R}^N)$ with distinct eigenvalues such that*

$$(u^f, u_t^f) \in C([0, T]; L^2(0, \pi; \mathbb{R}^N) \times H^{-1}(0, \pi; \mathbb{R}^N)) =: \mathcal{V}$$

and

$$\|(u^f, u_t^f)\|_{\mathcal{V}} \prec \|f\|_{L^2(0, T)}.$$

2.2. Controllability results. In this section we will prove Theorem 1.3. We define γ_{kl} to be

$$\gamma_{kl} := c_{kl}(T) \left(\frac{2k}{\pi} e^{i\omega_{kl}T} \right)^{-1} \quad (15)$$

and rewrite (12) for $t = T$ in the form

$$\gamma_{kl} = (f, e_{kl})_{L^2(0, T)}, \quad (16)$$

where $e_{kl}(t) = e^{i\omega_{kl}t}$. We note that

$$\sum_{k,l} |\gamma_{kl}|^2 \asymp \sum_{k,l} \frac{|c_{kl}(T)|^2}{k^2}.$$

For any $T > 0$, the family $\{e_{kl}\}$ is not a Riesz basis as a result of Proposition 1 Property 3. Therefore, we need to use the so-called exponential divided differences (EDD). EDD were introduced in [9] and [10] for families of exponentials whose exponents are close, that is, the difference between exponents tends to zero. Under

precise assumptions, the family of EDD forms a Riesz sequence in $L^2(0, T)$. For each fixed k , we define

$$\tilde{e}_{k1} := [\omega_{k1}] = e^{i\omega_{k1}t},$$

and for $2 \leq l \leq N$

$$\tilde{e}_{kl} := [\omega_{k1}, \omega_{k2}, \dots, \omega_{kl}] = \sum_{j=1}^l \frac{e^{i\omega_{kj}t}}{\prod_{r \neq j} (\omega_{kj} - \omega_{kr})}.$$

Under Condition (ii) of our theorem, we are able to use this formula for divided differences in place of the formula for generalized divided differences (see e.g. [10]).

From asymptotics theory and the explicit formula for ω_{kl} , it follows that the generating function of the family of EDD $\{\tilde{e}_{kl}\}$ is a sine-type function (see [8, 9, 10]). Hence, the family of EDD $\{\tilde{e}_{kl}\}$ forms a Riesz sequence in $L^2(0, T)$ for $T \geq 2\pi N$. We then define

$$\tilde{\gamma}_{kl} = (f, \tilde{e}_{kl})_{L^2(0, T)}.$$

Since $\{\tilde{e}_{kl}\}$ is a Riesz sequence, $\{(\tilde{\gamma}_{kl}) \mid f \in L^2(0, T)\} = \ell^2$, i.e. any sequence from ℓ_2 can be obtained by a function $f \in L^2(0, T)$ and the family $\{\tilde{e}_{kl}\}$. Proposition 1 Property (2) implies that $|\tilde{\gamma}_{kl}| \prec |k^{N-1} \gamma_{kl}|$. Recalling Equations (15) and (16), we obtain

$$\{(\gamma_{kl}) \mid f \in L^2(0, T)\} \supseteq \ell_{N-1}^2 \quad (17)$$

where

$$\ell_{N-1}^2 = \left\{ (a_{kl}) \mid \sum_{k,l} |k^{N-1} a_{kl}|^2 < \infty \right\}.$$

Since $\{\Phi_{kl}\}$ forms a Riesz basis in $L^2(0, \pi; \mathbb{R}^N)$, it follows from (11), (15), and (17) that $(u(\cdot, t), u_t(\cdot, t)) \in W_{N-1} \times W_{N-2}$ and we have proved Theorem 1.3.

3. Proof of Theorem 1.4 - The case of a repeated eigenvalue.

3.1. Properties of root vectors and root vector adjustment. In this section, we investigate System (1) in the case where the coupling matrix A has only one eigenvalue, denoted λ , with algebraic multiplicity N and geometric multiplicity 1. We will assume that A and b satisfy the Kalman rank condition (4). We remark that λ is real since imaginary eigenvalues occur in conjugate pairs. We will define the vectors φ_N and ψ_1 to be the eigenvectors of A and A^* , respectively. Additionally, we let $\varphi_1, \varphi_2, \dots, \varphi_{N-1}$ and $\psi_2, \psi_3, \dots, \psi_N$ be root vectors of A and A^* , respectively. So we have the following:

$$\begin{aligned} (A - \lambda I)\varphi_i &= \varphi_{i+1}, \quad 1 \leq i \leq N-1 \\ (A - \lambda I)\varphi_N &= 0, \\ (A^* - \lambda I)\psi_1 &= 0, \\ (A^* - \lambda I)\psi_j &= \psi_{j-1}, \quad 2 \leq j \leq N. \end{aligned}$$

In particular, the collection $\{\varphi_i\}$ is listed in reverse order in comparison to $\{\psi_j\}$. This is intended to simplify indexing and reflect the construction of a biorthogonal family for $\{\varphi_i\}$. With this construction, we have the following propositions.

Proposition 2. $\langle \varphi_N, \psi_j \rangle = 0$ for $1 \leq j \leq N-1$.

Proof. Let $1 \leq j \leq N - 1$. We observe that

$$\lambda \langle \varphi_N, \psi_{j+1} \rangle = \langle \lambda \varphi_N, \psi_{j+1} \rangle = \langle A \varphi_N, \psi_{j+1} \rangle = \langle \varphi_N, A^* \psi_{j+1} \rangle$$

and

$$\lambda \langle \varphi_N, \psi_{j+1} \rangle = \langle \varphi_N, \lambda \psi_{j+1} \rangle = \langle \varphi_N, A^* \psi_{j+1} \rangle - \langle \varphi_N, \psi_j \rangle.$$

Comparing right hand sides yields

$$\langle \varphi_N, \psi_j \rangle = 0.$$

□

Proposition 3. $\langle \varphi_N, \psi_N \rangle \neq 0$.

Proof. Suppose on the contrary that $\langle \varphi_N, \psi_N \rangle = 0$. Together with Proposition 2, we have

$$\langle \varphi_N, \psi_j \rangle = 0, \quad 1 \leq j \leq N.$$

This implies that $\varphi_N = 0$, which is a contradiction because $\{\varphi_i\}_{i=1}^N$ is a basis in \mathbb{R}^N . □

Proposition 4. $\langle \varphi_i, \psi_j \rangle = \langle \varphi_{i+1}, \psi_{j+1} \rangle$ for $1 \leq i, j \leq N - 1$.

Proof. Let $1 \leq i, j \leq N - 1$. Observe that

$$\lambda \langle \varphi_i, \psi_{j+1} \rangle = \langle \lambda \varphi_i, \psi_{j+1} \rangle = \langle A \varphi_i, \psi_{j+1} \rangle - \langle \varphi_{i+1}, \psi_{j+1} \rangle$$

and

$$\lambda \langle \varphi_i, \psi_{j+1} \rangle = \langle \varphi_i, \lambda \psi_{j+1} \rangle = \langle \varphi_i, A^* \psi_{j+1} \rangle - \langle \varphi_i, \psi_j \rangle.$$

Comparing the right hand sides yields $\langle \varphi_i, \psi_j \rangle = \langle \varphi_{i+1}, \psi_{j+1} \rangle$, as desired. □

Proposition 5. $\langle b, \psi_1 \rangle \neq 0$.

Proof. Suppose on the contrary that $\langle b, \psi_1 \rangle = 0$. Then for $0 \leq n \leq N - 1$, observe

$$\langle A^n b, \psi_1 \rangle = \langle b, (A^*)^n \psi_1 \rangle = \langle b, \lambda^n \psi_1 \rangle = \lambda^n \langle b, \psi_1 \rangle = 0.$$

This implies that the columns of the matrix $[A|b]$ are linearly dependent. This is a contradiction because A and b satisfy the Kalman conditions and so $[A|b]$ has rank N . □

We now claim that given $\{\varphi_i\}_{i=1}^N$ and $\{\psi_j\}_{j=1}^N$, we can adjust them in a specific way.

Lemma 3.1. *Given ψ_1 , we can construct a collection $\{\tilde{\psi}_j\}_{j=1}^N$ such that*

- (i) $(A^* - \lambda I)\tilde{\psi}_1 = 0$,
- (ii) $(A^* - \lambda I)\tilde{\psi}_j = \tilde{\psi}_{j-1}$, for $2 \leq j \leq N$, and
- (iii) $\langle b, \tilde{\psi}_j \rangle = \delta_{1j}$.

Proof. From Proposition 5, $\langle b, \psi_1 \rangle \neq 0$. Define

$$\tilde{\psi}_1 = \frac{\psi_1}{\langle b, \psi_1 \rangle}.$$

Since $\tilde{\psi}_1$ is a scalar multiple of ψ_1 , $(A^* - \lambda I)\tilde{\psi}_1 = 0$ and $\langle b, \tilde{\psi}_1 \rangle = 1$, as desired.

From $\tilde{\psi}_1$, we obtain ψ_2 with $(A^* - \lambda I)\psi_2 = \tilde{\psi}_1$. We set

$$\tilde{\psi}_2 = \psi_2 - \langle b, \psi_2 \rangle \tilde{\psi}_1.$$

Observe that

$$\begin{aligned}(A^* - \lambda I)\tilde{\psi}_2 &= (A^* - \lambda I)(\psi_2 - \langle b, \psi_2 \rangle \tilde{\psi}_1) \\ &= (A^* - \lambda I)\psi_2 - \langle b, \psi_2 \rangle (A^* - \lambda I)\tilde{\psi}_1 \\ &= \tilde{\psi}_1.\end{aligned}$$

Also,

$$\langle b, \tilde{\psi}_2 \rangle = \langle b, \psi_2 \rangle - \langle b, \psi_2 \rangle \langle b, \tilde{\psi}_1 \rangle = 0.$$

We then proceed iteratively to build the collection $\{\tilde{\psi}_j\}_{j=1}^N$ that satisfies the conditions of the lemma. \square

Lemma 3.2. *Given φ_N and $\{\tilde{\psi}_j\}_{j=1}^N$, we can construct a collection $\{\tilde{\varphi}_i\}_{i=1}^N$ such that*

- (i) $(A - \lambda I)\tilde{\varphi}_N = 0$,
- (ii) $(A - \lambda I)\tilde{\varphi}_i = \tilde{\varphi}_{i+1}$ for $1 \leq i \leq N - 1$, and
- (iii) $\langle \tilde{\varphi}_i, \tilde{\psi}_j \rangle = \delta_{ij}$.

Proof. From Propositions 2 and 3, $\langle \varphi_N, \tilde{\psi}_j \rangle = 0$ for $1 \leq j \leq N - 1$ and $\langle \varphi_N, \tilde{\psi}_N \rangle \neq 0$. We define

$$\tilde{\varphi}_N = \frac{\varphi_N}{\langle \varphi_N, \tilde{\psi}_N \rangle}$$

so that $\langle \tilde{\varphi}_N, \tilde{\psi}_N \rangle = 1$.

From $\tilde{\varphi}_N$, we can obtain φ_{N-1} such that $(A - \lambda I)\varphi_{N-1} = \tilde{\varphi}_N$. From Proposition 4, $\langle \varphi_{N-1}, \tilde{\psi}_j \rangle = 0$ for $1 \leq j \leq N - 2$ and $\langle \varphi_{N-1}, \tilde{\psi}_{N-1} \rangle = 1$. We set

$$\tilde{\varphi}_{N-1} = \varphi_{N-1} - \langle \varphi_{N-1}, \tilde{\psi}_N \rangle \tilde{\varphi}_N.$$

With this construction, Condition (ii) is still maintained and $\langle \tilde{\varphi}_{N-1}, \tilde{\psi}_j \rangle = 0$ for $j \neq N - 1$ with $\langle \tilde{\varphi}_{N-1}, \tilde{\psi}_{N-1} \rangle = 1$.

Similarly, we obtain φ_{N-2} such that $(A - \lambda I)\varphi_{N-2} = \tilde{\varphi}_{N-1}$. Using $\tilde{\varphi}_N$ and $\tilde{\varphi}_{N-1}$ with Proposition 4 yields

$$\langle \varphi_{N-2}, \tilde{\psi}_j \rangle = \begin{cases} 1 & , \quad j = N - 2, \\ 0 & , \quad 1 \leq j \leq N - 3, \text{ or } j = N - 1. \end{cases}$$

We similarly define $\tilde{\varphi}_{N-2}$ by

$$\tilde{\varphi}_{N-2} = \varphi_{N-2} - \langle \varphi_{N-2}, \tilde{\psi}_N \rangle \tilde{\varphi}_N.$$

Continuing this process iteratively yields a collection $\{\tilde{\varphi}_i\}_{i=1}^N$ satisfying the conditions of the lemma. \square

We combine all our results into the following lemma.

Lemma 3.3. *Given System (1) with A and b given and satisfying the Kalman conditions, we may choose vectors $\{\varphi_i\}_{i=1}^N$, $\{\psi_j\}_{j=1}^N$ such that*

- (i) $(A - \lambda I)\varphi_i = \varphi_{i+1}$ for $1 \leq i \leq N - 1$,
- (ii) $(A - \lambda I)\varphi_N = 0$,
- (iii) $(A^* - \lambda I)\psi_1 = 0$,
- (iv) $(A^* - \lambda I)\psi_j = \psi_{j-1}$ for $2 \leq j \leq N$,
- (v) $\langle b, \psi_j \rangle = \delta_{1j}$, and
- (vi) $\langle \varphi_i, \psi_j \rangle = \delta_{ij}$.

3.2. The Fourier method and existence of solutions. In this section, we will be using the Fourier method. We will assume the conditions of Theorem 1.4 and use the results of Lemma 3.3. We define $\Phi_{nj}(x) = \sin(nx)\varphi_j$ and note that $\{\Phi_{nj}(x)\}$, $n \in \mathbb{N}$, $j = 1, \dots, N$, is a Riesz basis in $L^2(0, \pi; \mathbb{R}^N)$ with biorthogonal family $\{\Psi_{nj}(x)\}$ where

$$\Psi_{nj}(x) = \frac{2}{\pi} \sin(nx)\psi_j(x).$$

We thus write the solution to System (1) with zero Dirichlet boundary conditions in the form of the series

$$u(x, t) = \sum_{n,j} a_{nj}(t) \Phi_{nj}(x),$$

and set

$$v(x, t) = g(t)\Psi_{kl}(x),$$

with $g(t)$ being a smooth function. In the same way as in Section 2.2, we obtain the integral identity

$$\int_0^T \int_0^\pi \langle u, v_{tt} - v_{xx} + A^* v \rangle dx dt = \frac{2k}{\pi} \langle b, \psi_l \rangle \int_0^T f(t)g(t) dt. \quad (18)$$

For now, we set $l = 1$ and since ψ_1 is an eigenvector of A^* and $\langle b, \psi_1 \rangle = 1$, from (18) we obtain

$$\int_0^T [\ddot{a}_{k1}(t) + \omega_k^2 a_{k1}(t)]g(t) dt = \frac{2k}{\pi} \int_0^T f(t)g(t) dt,$$

where $\omega_k^2 = k^2 + \lambda$. We then obtain the differential equation

$$\ddot{a}_{k1}(t) + \omega_k^2 a_{k1}(t) = \frac{2k}{\pi} f(t) \quad (19)$$

with initial conditions

$$a_{k1}(0) = 0 = \dot{a}_{k1}(0). \quad (20)$$

As before, we are assuming that $\omega_k^2 \neq 0$ and if otherwise, we can make the same changes as prescribed in Section 2.2. The solution of (19) and (20) is given by

$$a_{k1}(t) = \frac{2k}{\pi} \int_0^t f(\tau) \frac{\sin \omega_k(t - \tau)}{\omega_k} d\tau.$$

We now let $l = 2, \dots, N$ in (18). We note that ψ_l is then a root vector and hence $A^* \psi_l = \lambda \psi_l + \psi_{l-1}$. Additionally, we have $\langle b, \psi_l \rangle = 0$. So, from Equation (18), we obtain

$$\int_0^T [\ddot{a}_{kl}(t) + \omega_k^2 a_{kl}(t)]g(t) dt = - \int_0^T a_{k,l-1}(t)g(t) dt.$$

Thus, we obtain the solution for $a_{kl}(t)$ to be

$$a_{kl}(t) = - \int_0^t a_{k,l-1}(\tau) \frac{\sin \omega_k(t - \tau)}{\omega_k} d\tau.$$

To motivate the following results, we compute $a_{kl}(t)$ and $\dot{a}_{kl}(t)$ for $l = 1, 2, 3$.

$$\begin{aligned} a_{k1}(t) &= \frac{2k}{\pi \omega_k} \int_0^t f(\tau) \sin \omega_k(t - \tau) d\tau \\ a_{k2}(t) &= \frac{k}{\pi \omega_k^2} \int_0^t f(\tau) \left[(t - \tau) \cos \omega_k(t - \tau) - \frac{1}{\omega_k} \sin \omega_k(t - \tau) \right] d\tau \end{aligned}$$

$$\begin{aligned}
a_{k3}(t) &= -\frac{k}{4\pi\omega_k^3} \int_0^t f(\tau) \left[(t-\tau)^2 \sin \omega_k(t-\tau) + \frac{3}{\omega_k} (t-\tau) \cos \omega_k(t-\tau) \right. \\
&\quad \left. - \frac{3}{\omega_k^2} \sin \omega_k(t-\tau) \right] d\tau. \\
\dot{a}_{k1}(t) &= \frac{2k}{\pi} \int_0^t f(\tau) \cos \omega_k(t-\tau) d\tau \\
\dot{a}_{k2}(t) &= -\frac{k}{\pi\omega_k} \int_0^t f(\tau) (t-\tau) \sin \omega_k(t-\tau) d\tau \\
\dot{a}_{k3}(t) &= -\frac{k}{4\pi\omega_k^2} \int_0^t f(\tau) \left[(t-\tau)^2 \cos \omega_k(t-\tau) - \frac{3}{\omega_k} (t-\tau) \sin \omega_k(t-\tau) \right] d\tau
\end{aligned} \tag{21}$$

We now introduce the functions $b_{kl}(t)$ and $\hat{b}_{kl}(t)$, for $k \in \mathbb{N}$, $l = 1, \dots, 3$, defined as

$$\begin{aligned}
b_{k1}(t) &= \int_0^t f(\tau) \sin \omega_k(t-\tau) d\tau &= \frac{\pi\omega_k}{2k} a_{k1}(t), \\
b_{k2}(t) &= \frac{1}{\omega_k} \int_0^t f(\tau) (t-\tau) \cos \omega_k(t-\tau) d\tau &= \frac{\pi\omega_k}{k} a_{k2}(t) + \frac{\pi}{2k\omega_k} a_{k1}(t), \\
b_{k3}(t) &= \frac{1}{\omega_k^2} \int_0^t f(\tau) (t-\tau)^2 \sin \omega_k(t-\tau) d\tau &= -\frac{4\pi\omega_k}{k} a_{k3}(t) - \frac{3\pi}{k\omega_k} a_{k2}(t), \\
\hat{b}_{k1}(t) &= \omega_k \int_0^t f(\tau) \cos \omega_k(t-\tau) d\tau &= \frac{\pi\omega_k}{2k} \dot{a}_{k1}(t), \\
\hat{b}_{k2}(t) &= \int_0^t f(\tau) (t-\tau) \sin \omega_k(t-\tau) d\tau &= -\frac{\pi\omega_k}{k} \dot{a}_{k2}(t), \\
\hat{b}_{k3}(t) &= \frac{1}{\omega_k} \int_0^t f(\tau) (t-\tau)^2 \cos \omega_k(t-\tau) d\tau &= -\frac{4\pi\omega_k}{k} \dot{a}_{k3}(t) - \frac{3\pi}{k\omega_k} \dot{a}_{k2}(t).
\end{aligned} \tag{22}$$

The purpose of this is to note that the transformation between the families $\{a_{kl}(t), \dot{a}_{kl}(t)\}$ and $\{b_{kl}(t), \hat{b}_{kl}(t)\}$ is both bounded and boundedly invertible in ℓ^2 . We proceed by considering the family $\{b_{kl}(t), \hat{b}_{kl}(t)\}$. We introduce the functions $c_{kl}(t)$ with $\omega_{-k} = -\omega_k$, $b_{-kl}(t) = b_{kl}(t)$, and $\hat{b}_{-kl}(t) = \hat{b}_{kl}(t)$ for $k \in \mathbb{K} = \{\pm 1, \pm 2, \dots\}$ in the form

$$c_{kl}(t) = \begin{cases} i\omega_k b_{kl}(t) + \hat{b}_{kl}(t) & , \text{ if } l \text{ odd,} \\ \omega_k b_{kl}(t) + i\hat{b}_{kl}(t) & , \text{ if } l \text{ even.} \end{cases}$$

Hence,

$$\omega_k^{l-2} c_{kl}(t) = \int_0^t f(\tau) (t-\tau)^{l-1} e^{i\omega_k(t-\tau)} d\tau, \tag{23}$$

for $1 \leq l \leq N$.

Recall that $\{\Phi_{nj}(x)\}$ forms a Riesz basis in $L^2(0, \pi; \mathbb{R}^N)$ and with Proposition 1 Property (1),

$$\sum_{\substack{k \in \mathbb{K} \\ l=1, \dots, N}} \frac{|c_{kl}(t)|^2}{k^2} \asymp \|u(\cdot, t)\|_{L^2(0, \pi; \mathbb{R}^N)}^2 + \|u_t(\cdot, t)\|_{H^{-1}(0, \pi; \mathbb{R}^N)}^2.$$

Additionally, from (23), it follows that for all $t > 0$,

$$\sum_{\substack{k \in \mathbb{K} \\ l=1, \dots, N}} \frac{|c_{kl}(t)|^2}{k^2} \prec \|f\|_{L^2(0,t)}^2 \leq \|f\|_{L^2(0,T)}^2.$$

This implies that the sequences $(a_{kl}(t))$ and $(k^{-1}a_{kl}(t))$ belong to ℓ^2 . Hence, $u(\cdot, t) \in L^2(0, \pi; \mathbb{R}^N)$ and $u_t(\cdot, t) \in H^{-1}(0, \pi; \mathbb{R}^N)$.

Similar to Section 2.1, we have

$$\sum_{\substack{k \in \mathbb{K} \\ l=1, \dots, N}} \frac{|c_{kl}(t+h) - c_{kl}(t)|^2}{k^2} \rightarrow 0, \quad h \rightarrow 0.$$

So, we obtain an analog of Theorem 2.2.

Theorem 3.4. *For any $f \in L^2(0, T)$, there exists a unique generalized solution u^f of the IVP (1) - (3) with coupling matrix $A \in \mathcal{L}(\mathbb{R}^N)$ with a single eigenvalue such that $(u^f, u_t^f) \in \mathcal{V}$ and*

$$\|(u^f, u_t^f)\|_{\mathcal{V}} \prec \|f\|_{L^2(0,T)}^2.$$

3.3. Controllability results. We will now prove Theorem 1.4. We denote $e_{kl}(t) = t^{l-1}e^{i\omega_k t}$ and define $f^T(t) = f(T-t)$. With (23), we have

$$\omega_k^{l-2}c_{kl}(T) = (f^T, e_{-kl})_{L^2(0,T)}.$$

Since $\{e_{k1}, \dots, e_{kN}\}$ is a Riesz sequence in $L^2(0, T)$ for $T \geq 2N\pi$ (see e.g. [8] II.4),

$$\{(k^{l-2}c_{kl}(T)) \mid f \in L^2(0, T)\} = \ell^2.$$

It follows that

$$\{(c_{kl}(T)) \mid f \in L^2(0, T)\} \supseteq \ell_{N-2}^2.$$

Hence, the space to guarantee exact controllability is $W_{N-1} \times W_{N-2}$ and Theorem 1.4 is proved.

4. Proof of Theorem 1.2 - The general case.

4.1. The Fourier method and existence of solutions. Equipped with Theorems 1.3, 1.4, 2.2, and 3.4, we now consider the case of a more general coupling matrix A . We now consider System (1) with A and b satisfying the assumptions of Theorem 1.2. Let $\lambda_1, \dots, \lambda_M$ be the distinct eigenvalues of A with corresponding algebraic multiplicities m_1, \dots, m_M . We note that

$$\sum_{i=1}^M m_i = N.$$

For each $i = 1, \dots, M$, we construct the collection of eigenvectors and root vectors corresponding to λ_i , denoted $\{\varphi_{ij}\}_{j=1}^{m_i}$. For each of these collections, we have a corresponding biorthogonal family $\{\psi_{ij}\}_{j=1}^{m_i}$. From Lemmas 2.1 and 3.3, these collections possess many specific properties as well as

$$\langle \varphi_{ij}, \psi_{kl} \rangle = \delta_{ik} \delta_{jl}.$$

We define $\Phi_{nij} = \sin(nx)\varphi_{ij}$. Then $\{\Phi_{nij}(x)\}$, $n \in \mathbb{N}$, $i = 1, \dots, M$, $j = 1, \dots, m_i$, is a Riesz basis in $L^2(0, \pi; \mathbb{R}^N)$ with biorthogonal family $\{\Psi_{nij}\}$ where

$$\Psi_{nij}(x) = \frac{2}{\pi} \sin(nx)\psi_{ij}.$$

We then use the Fourier method and express the solution $u(x, t)$ as

$$u(x, t) = \sum_{n,i,j} a_{nij}(t) \Phi_{nij}(x). \quad (24)$$

By doing so, for each $l = 1, \dots, n$, we obtain, as in the case of Section 2, coefficients $a_{klj}(t)$ defined below. We similarly denote $\omega_{kl}^2 = k^2 + \bar{\lambda}_l$ and recall Proposition 1.

$$\begin{aligned} a_{kl1}(t) &= \frac{2k}{\pi} \int_0^t f(\tau) \frac{\sin \omega_{kl}(t - \tau)}{\omega_{kl}} d\tau, \\ a_{klj}(t) &= - \int_0^t a_{kl,j-1}(\tau) \frac{\sin \omega_{kl}(t - \tau)}{\omega_{kl}} d\tau, \quad j = 2, \dots, m_l. \end{aligned}$$

Similar to Section 3.2, from $\{a_{klj}(t), \dot{a}_{klj}(t)\}$, we construct the collection $\{b_{klj}(t), \widehat{b}_{klj}(t)\}$. We then define the functions $c_{klj}(t)$ for $k \in \mathbb{K}$ with $\omega_{-kl} = -\omega_{kl}$, $b_{-klj}(t) = b_{klj}(t)$, and $\widehat{b}_{-klj}(t) = \widehat{b}_{klj}(t)$ by

$$c_{klj}(t) = \begin{cases} i\omega_{kl} b_{klj}(t) + \widehat{b}_{klj}(t) & , \text{ if } j \text{ even,} \\ \omega_{kl} b_{klj}(t) + i\widehat{b}_{klj}(t) & , \text{ if } j \text{ odd.} \end{cases}$$

So,

$$\omega_{kl}^{j-2} c_{klj}(t) = \int_0^t f(\tau) (t - \tau)^{j-1} e^{i\omega_{kl}(t-\tau)} d\tau. \quad (25)$$

Similar to Theorems 2.2 and 3.4, the collection $\{e_{klj}\}$, where $e_{klj}(t) = t^{j-1} e^{i\omega_{kl} t}$, is either a Riesz sequence in $L^2(0, T)$ if $T > 2N\pi$, or a finite union of Riesz sequences for $T < 2N\pi$. Hence, we obtain the following theorem.

Theorem 4.1. *For any coupling matrix $A \in \mathcal{L}(\mathbb{R}^N)$ and for any $f \in L^2(0, T)$, there exists a unique generalized solution u^f of the IVP (1)–(3) such that $(u^f, u_t^f) \in \mathcal{V}$ and*

$$\|(u^f, u_t^f)\|_{\mathcal{V}} \prec \|f\|_{L^2(0, T)}.$$

4.2. Controllability results. We define the sequence $\{\gamma_{kij}\}$ by

$$\gamma_{klj} := \omega_{kl}^{j-2} c_{klj}(T) = \int_0^T f(\tau) (T - \tau)^{j-1} e^{i\omega_{kl}(T-\tau)} d\tau,$$

so that

$$\gamma_{klj} = (f^T, e_{-klj})_{L^2(0, T)},$$

where $f^T(t) = f(T - t)$ and $e_{klj}(t) = t^{j-1} e^{i\omega_{kl} t}$. We note that $|\gamma_{klj}| \prec |k^{m_l-2} c_{klj}(T)|$.

In addition to Lemma 1 Property (2), we also need to account for eigenvalues with algebraic multiplicity greater than one, and thus we are required to use generalized divided differences (GDD). For each fixed k , we will list the sequence $\{\omega_{kl}\}$ including multiplicities.

GDD were described in [9] and [10]. For each fixed k , we define the divided difference (DD) of order zero of the point ω_{k1} is $[\omega_{k1}](t) = e^{i\omega_{k1} t}$. The DD of order $l-1$, $l \leq N$, corresponding to $\omega_{k1}, \dots, \omega_{kl}$ is

$$[\omega_{k1}, \dots, \omega_{kl}] := \begin{cases} \frac{[\omega_{k1}, \dots, \omega_{k,l-1}] - [\omega_{k2}, \dots, \omega_{kl}]}{\omega_{k1} - \omega_{kl}}, & \omega_{k1} \neq \omega_{kl} \\ \frac{\partial}{\partial \omega} [\omega, \omega_{k2}, \dots, \omega_{k,l-1}] \Big|_{\omega=\omega_{k1}}, & \omega_{k1} = \omega_{kl}. \end{cases}$$

We then denote $\tilde{e}_{kl}(t) = [\omega_{k1}, \dots, \omega_{kl}]$. We note that with this construction, the functions $\tilde{e}_{k1}, \dots, \tilde{e}_{km_l}$ are in fact the functions $e^{i\omega_{k1} t}, t e^{i\omega_{k1} t}, \dots, t^{m_l-1} e^{i\omega_{k1} t}$. In

addition, by the main theorems of [9] and [10], the collection of functions $\{\tilde{e}_{kl}\}_{l=1}^N$ is a Riesz sequence in $L^2(0, T)$ for $T \geq 2N\pi$. We then define

$$\tilde{\gamma}_{kl} = (f, \tilde{e}_{kl})_{L^2(0, T)}$$

and therefore $\{(\tilde{\gamma}_{kl}) \mid f \in L^2(0, T)\} = \ell^2$. From here, we note that the relation among γ_{klj} , $\tilde{\gamma}_{kl}$, and $c_{klj}(T)$ is

$$\sum |\tilde{\gamma}_{kl}|^2 \prec \sum |k^{N-m_i} \gamma_{klj}|^2 \prec \sum |k^{N-2} c_{klj}(T)|^2.$$

Hence, it follows that

$$\{(c_{klj}(T)) \mid f \in L^2(0, T)\} \supseteq \ell_{N-2}^2, \quad (26)$$

and since $\{\Phi_{nij}(x)\}$ forms a Riesz basis in $L^2(0, \pi; \mathbb{R}^N)$, we deduce from (26) the result of Theorem 1.2.

We will now prove the negative results in Theorem 1.2. We first assume that (i) and (iii) hold, but (ii) does not hold. Observe that this may only happen for a finite number of indices (see [4]). So we have

$$k_d^2 - l_d^2 = \lambda_{i_d} - \lambda_{j_d}, \quad 1 \leq d \leq m.$$

In this situation, the family given in (12), $\{e_{kl}\}$, is clearly linearly dependent since some function (or functions) is repeated twice in the family. Thus, according to Theorems I.2.1e and III.3.10e in [8], System (1) is not approximately controllable for any $T > 0$.

Let us now suppose that (i) does not hold. We will use the Hautus test, which is equivalent to the Kalman rank condition (see [17] Prop. 1.5.5 and [4] Prop. 3.1). If the Kalman rank condition does not hold, then there exists $\lambda \in \sigma(A^*)$ such that

$$\text{rank} \begin{bmatrix} A^* - \lambda I \\ b^* \end{bmatrix} < N.$$

Hence, there exists $1 \leq l \leq M$ such that ψ_l is an eigenvector of A^* corresponding to λ and $\langle b, \psi_l \rangle = 0$. This then implies that for all $k \in \mathbb{K}$, $1 \leq j \leq m_l$,

$$c_{klj}(T) = 0,$$

and thus the codimension of the reachable set is infinite.

If condition (iii) is not met, i.e. $T < 2N\pi$, then from [9] and [10], it follows that the family of EDD $\{\tilde{e}_{kl}\}$ is not a Riesz basis in $L^2(0, T)$. In particular, we can split $\{\tilde{e}_{kl}\}$ into two subfamilies \mathcal{E}_0 and \mathcal{E}_1 such that \mathcal{E}_0 is a Riesz sequence in $L^2(0, T)$ and \mathcal{E}_1 has infinite cardinality. This implies that $\{\tilde{e}_{kl}\}$ is not linearly independent and hence the reachable set has infinite codimension.

Thus we have proved the negative part of Theorem 1.2, and the proof is complete.

5. A particular case: $N = 2$. In the previous sections, we proved exact controllability with respect to a more regular space than the space of regularity for the system. This is typical of hybrid systems where clusters of close spectral points appear. However, in the case where $N = 2$, we are able to prove the sharp controllability result, i.e., to prove exact controllability in the space of sharp regularity of the system. To do this, we develop a new method based on the construction of a basis in a so-called asymmetric space. This method was proposed in [7] when investigating the controllability of another hybrid system of hyperbolic type – the string with point masses. In the present paper, we extend this method to the vector case.

We consider System (1)-(3) with $N = 2$ and

$$b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \quad (27)$$

In other words, the boundary control acts only on the first equation and the second equation is controlled through its connection with the first. From now on, we will refer to this system as \mathcal{S}_2 . The first question we ask is about the sharp regularity space. We claim that

$$u_1(\cdot, t) \in L^2(0, \pi), \quad u_2(\cdot, t) \in H_0^1(0, \pi).$$

From Theorem 4.1, $(u_1(\cdot, t), u_2(\cdot, t)) \in L^2(0, \pi)^2$. From the structure of the system, u_2 is the solution to a wave equation with zero Dirichlet boundary conditions where u_1 acts as the control over Q . Since $u_1(\cdot, t) \in L^2(0, \pi)$, we conclude that $u_2 \in H_0^1(0, \pi)$ (see for example [8] Sec.V.1).

The main result of this section is

Theorem 5.1. *Under conditions similar to those of Theorem 1.2, that is, with A and b given by (27) with $a_{21} \neq 0$ (so the Kalman rank condition for $[A|b]$ is fulfilled), that*

$$\mu_k - \mu_l \neq \lambda_1 - \lambda_2, \quad \forall k, l \in \mathbb{N}, \text{ with } k \neq l,$$

and that $T \geq 4\pi$, then the reachable set of System \mathcal{S}_2 , $\{(u^f(\cdot, T), u_t^f(\cdot, T)) \mid f \in L^2(0, T)\}$ is equal to \mathcal{H}_1 where

$$\mathcal{H}_1 := \begin{pmatrix} L^2(0, \pi) \\ H_0^1(0, \pi) \end{pmatrix} \times \begin{pmatrix} H^{-1}(0, \pi) \\ L^2(0, T) \end{pmatrix}$$

for $T \geq 4\pi$.

If $T < 4\pi$, then the reachable set has infinite codimension in \mathcal{H}_1 .

We will prove this theorem by considering the two possible cases, i.e., whether the matrix A has two distinct eigenvalues or a repeated eigenvalue.

5.1. Proof for distinct eigenvalues. As before, we construct the set of sequences $(\tilde{\gamma}_{kl})$ and for $T \geq 4\pi$, this set runs over ℓ^2 . This means that the set of the corresponding sequences $(\tilde{a}_{kl}(T))$ also runs over ℓ^2 .

We now return to the representation in (5):

$$u(x, T) = \sum_{n,j} a_{nj}(T) \Phi_{nj}(x). \quad (28)$$

Taking into account that for $N = 2$, we use EDD of order one, i.e.,

$$\tilde{a}_{n1} = a_{n1}, \quad \tilde{a}_{n2} = \frac{a_{n2} - a_{n1}}{\omega_{n2} - \omega_{n1}},$$

where we suppress the argument T . We can rewrite (28) in the form

$$u(x, T) = \sum_{n,j} \tilde{a}_{nj} \tilde{\Phi}_{nj}(x). \quad (29)$$

It is easy to verify that

$$\tilde{\Phi}_{n1}(x) = \Phi_{n1}(x) + \Phi_{n2}(x) = \sin(nx)(\varphi_1 + \varphi_2), \quad (30)$$

$$\tilde{\Phi}_{n2}(x) = \Phi_{n2}(x)(\omega_{n2} - \omega_{n1}) = \sin(nx)\varphi_2(\omega_{n2} - \omega_{n1}). \quad (31)$$

We note that $|\omega_{n2} - \omega_{n1}| \asymp n^{-1}$ (Lemma 1 Property 2). We present the following lemma.

Lemma 5.2. *Eigenvectors φ_1 and φ_2 can be chosen such that*

$$\varphi_1 + \varphi_2 = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}. \quad (32)$$

Proof. In particular, we claim that the second component of φ_1 and φ_2 are nonzero. If this is true, then by appropriate scaling, we can obtain eigenvectors φ_1 and φ_2 whose second components add to zero. Suppose on the contrary that φ_1 has a zero second component. By scaling, we can assume that

$$\varphi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

By the orthogonality of ψ_1, ψ_2 , this implies that ψ_2 has the form

$$\psi_2 = \begin{pmatrix} 0 \\ x \end{pmatrix},$$

for some nonzero x . However, this is a contradiction to the Kalman rank condition as

$$\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \psi_2 \right\rangle = 0.$$

Hence, φ_1 has a nonzero second component. Similarly, φ_2 has a nonzero second component and the lemma is proved. \square

We can now express (29) as

$$u(x, T) = \sum_n \sin(nx) \left[\tilde{a}_{n1} \begin{pmatrix} \alpha \\ 0 \end{pmatrix} + \tilde{a}_{n2} \begin{pmatrix} \beta \\ \gamma \end{pmatrix} (\omega_{n2} - \omega_{n1}) \right].$$

We note that $\gamma \neq 0$ since φ_2 has a nonzero second component.

We recall that (\tilde{a}_{n1}) and (\tilde{a}_{n2}) may be arbitrary ℓ^2 sequences (when f runs over $L^2(0, T)$). Taking into account that $\{\sin(nx)\}$ is an orthogonal basis in $L^2(0, \pi)$, we begin by choosing the second component of $u(x, T)$ to be any target function from $H_0^1(0, \pi)$, and thereby choosing \tilde{a}_{n2} .

After choosing \tilde{a}_{n2} , we can then choose \tilde{a}_{n1} so that the first component of $u(x, T)$ will coincide with any prescribed function from $L^2(0, \pi)$. We can treat $u_t(x, T)$ in a similar fashion. This is due to the relation of sine and cosine and their appearance in $u(x, T)$ and $u_t(x, T)$. It is this relation that allows us to obtain controllability in any time $T \geq 4\pi$. Thus, one of the cases for the positive part of Theorem 5.1 is proved. We note that the negative part of the theorem can be proved similar to Theorem 1.2.

As a result of this, we have the following corollary.

Corollary 1. *The family $\{\tilde{\Phi}_{nj}\}$ constructed in (30)–(32) forms a Riesz basis in the asymmetric space $L^2(0, \pi) \times H^1(0, \pi)$.*

Proof. We have proved that every function from $L^2(0, \pi) \times H^1(0, \pi)$ can be represented in the form of a series with respect to the family $\{\tilde{\Phi}_{nj}\}$ with ℓ^2 coefficients. Uniqueness of the representation follows from the basis property of $\{\sin(nx)\}$ and linear independence of the eigenvectors φ_1 and φ_2 . Finally, it is clear that

$$\|u_1(\cdot, T)\|_{L^2(0, \pi)}^2 + \|u_2(\cdot, T)\|_{H^1(0, \pi)}^2 \asymp \sum_{n,j} |a_{nj}|^2.$$

\square

As a remark, it may be shown that the latter sum is equivalent to $\|f\|^2$ where f is the corresponding control to $u(\cdot, T)$. Also, this control belongs to the closure of the linear span of $\{e^{i\omega_{nj}t}\}$ in $L^2(0, T)$, and hence has minimal norm.

5.2. Proof for a repeated eigenvalue. From Lemma 3.3, we can obtain eigenvectors φ_2, ψ_1 and root vectors φ_1, ψ_2 for A and A^* , respectively, with certain properties. We now express our solution $u(x, T)$ using the Fourier Method and observe that

$$\begin{aligned} u(x, T) &= \sum_{n,j} a_{nj}(T) \Phi_{nj}(x) \\ &= \sum_n [a_{n1}(T) \Phi_{n1}(x) + a_{n2}(T) \Phi_{n2}(x)] \\ &= \sum_n [a_{n1}(T) \sin(nx) \varphi_1 + a_{n2}(T) \sin(nx) \varphi_2] \\ &= \sum_n \sin(nx) [a_{n1}\varphi_1 + a_{n2}\varphi_2]. \end{aligned} \quad (33)$$

In the final equality, we suppress the argument of a_{n1} and a_{n2} for readability. By Lemma 3.3, $\langle b, \psi_2 \rangle = 0$ and thus the first component of ψ_2 is zero. It then follows from $\langle \varphi_1, \psi_2 \rangle = 0$ that φ_1 has the form

$$\varphi_1 = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}, \quad \alpha \in \mathbb{R}. \quad (34)$$

Thus, we can express φ_2 as

$$\varphi_2 = \begin{pmatrix} \beta \\ \gamma \end{pmatrix}, \quad \beta, \gamma \in \mathbb{R}, \quad (35)$$

and we can express $u(x, T)$ as

$$u(x, T) = \sum_n \sin(nx) \left[a_{n1} \begin{pmatrix} \alpha \\ 0 \end{pmatrix} + a_{n2} \begin{pmatrix} \beta \\ \gamma \end{pmatrix} \right]. \quad (36)$$

From Equation (21) it follows that $(a_{n2}) \in \ell_1^2$ and we can choose a_{n2} such that the second component of $u(x, T)$ is any target function from $H_0^1(0, \pi)$. After choosing a_{n2} , we can then choose a_{n1} so that the first component of $u(x, T)$ coincides with any prescribed target function from $L^2(0, \pi)$. Similar to the case of distinct eigenvalues, we approach $u_t(x, T)$ in the same way. We have thus proved Theorem 5.1.

6. Open problems and further results. When the coupling matrix A is in lower triangular form, it is not difficult to generalize the results for coupled hyperbolic equations. That is, it is possible to prove exact controllability under the same assumptions as Theorem 1.2 in the space $\mathcal{H} = \mathcal{H}^0 \times \cdots \times \mathcal{H}^{N-1}$ where $\mathcal{H}^N = W_N \times W_{N-1}$. On the other hand, given an arbitrary matrix A , if the Kalman rank condition holds, we can obtain a canonical version of the original system and obtain similar results for this *transformed system*. However, when converting back to the original system, we may be taking linear combinations of the components of $u(x, t)$ with respect to the transformed system. Thus, an optimal description of the controllability space is no longer possible.

In Section 5, we investigated the case where $N = 2$ and showed that we can obtain sharp exact controllability, i.e., we have exact controllability in the same space as the space of regularity. It remains to determine whether or not we can

repeat this for different values of N . For $N = 2$, this is simple as there are only two possibilities for the eigenvalues of the coupling matrix A . However, for $N > 2$, the number of possibilities grows, and hence it may not be possible to prove in the same way.

It remains an open problem to treat the boundary controllability of N coupled wave equations in \mathbb{R}^d . The methods in this paper are not of use in the general situation or when the matrix A depends on (x, t) .

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REFERENCES

- [1] F. Alabau-Boussoira, *A two-level energy method for indirect boundary observability and controllability of weakly coupled hyperbolic systems*, *SIAM J. Control Optim.*, **42** (2003), 871–906.
- [2] F. Alabau-Boussoira, *Insensitizing exact controls for the scalar wave equation and exact controllability of 2-coupled cascade systems of PDE's by a single control*, *Math. Control Signals Systems*, **26** (2014), 1–46.
- [3] F. Alabau-Boussoira and M. Léautaud, *Indirect controllability of locally coupled systems under geometric conditions*, *C. R. Acad. Sci. Paris*, **349** (2011), 395–400.
- [4] F. Ammar-Kohdja, A. Benabdallah, M. González-Burgos and L. de Teresa, *The Kalman condition for the boundary controllability of coupled parabolic systems. bounds on biorthogonal families to complex matrix exponentials*, *JMPA*, **96** (2011), 555–590, <https://doi.org/10.1016/j.matpur.2011.06.005>.
- [5] S. Avdonin, A. Choque and L. de Teresa, *Exact boundary controllability results for two coupled 1-d hyperbolic equations*, *Int. J. Appl. Math. Comput. Sci.*, **23** (2013), 701–710, <https://doi.org/10.2478/amcs-2013-0052>.
- [6] S. Avdonin and L. de Teresa, *The Kalman Condition for the Boundary Controllability of Coupled 1-d Wave Equations*, ArXiv E-Prints, [arXiv:1902.08682](https://arxiv.org/abs/1902.08682).
- [7] S. Avdonin and J. Edward, *Exact controllability for string with attached masses*, *SIAM J. Control Optim.*, **56** (2018), 945–980.
- [8] S. A. Avdonin and S. A. Ivanov, *Families of Exponentials: The Method of Moments in Controllability Problems for Distributed Parameter Systems*, Cambridge University Press, 1995.
- [9] S. A. Avdonin and S. A. Ivanov, Exponential Riesz bases of subspaces and divided differences, *St. Petersburg Mathematical Journal*, **13** (2002), 339–351.
- [10] S. Avdonin and W. Moran, Ingham type inequalities and Riesz bases of subspaces and divided differences, *Int. J. Appl. Math. Compt. Sci.*, **11** (2001), 803–820.
- [11] A. Benbour, F. Ammaar Khodja and D. Tenious, *Exact and approximate controllability of coupled one-dimensional hyperbolic equations*, *Ev. Eq. and Cont. Teho.*, **6** (2017), 487–516.
- [12] H. O. Fattorini, Estimates for sequences biorthogonal to certain complex exponentials and boundary control of the wave equation, *Lecture Notes in Control and Informat. Sci.*, **2** (1977), 111–124.
- [13] R. E. Kalman, P. L. Falb and M. A. Arbib, *Topics in Mathematical Control Theory*, New York-Toronto, Ont.-London, 1969.
- [14] T. Liard and P. Lissy, *A Kalman rank condition for the indirect controllability of coupled systems of linear operator groups*, *Math. Control Signals Syst.*, **29** (2017), Art. 9, 35 pp, <https://doi.org/10.1007/s00498-017-0193-x>.
- [15] J. Park, On the boundary controllability of coupled 1-d wave equations, *Proceedings of 3rd IFAC Workshop on Control of Systems Governed by Partial Differential Equations and XI Workshop Control of Distributed Parameter Systems*, Oaxaca, Mexico, May, 20–24.

- [16] L. Rosier and L. de Teresa, *Exact controllability of a cascade system of conservative equations, C. R. Acad. Sci. Paris, Ser. I*, **349** (2011), 291–296, <https://doi.org/10.1016/j.crma.2011.01.014>.
- [17] M. Tucsnak and G. Weiss, *Observation and Control of Operator Semigroups*, Advanced Texts, Birkhäuser, Basel-Boston-Berlin, 2009.

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E-mail address: saavdonin@alaska.edu

E-mail address: jcpark2@alaska.edu

E-mail address: 1deteresa@im.unam.mx