Distributed Leader-Follower Tracking Control for Multi-Agent Systems Subject to Disturbances

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Abstract— **This paper studies robust tracking control for a leader-follower multi-agent system (MAS) subject to disturbances. A challenging problem is considered here, which differs from those in the literature in two aspects. First, we consider the case when all the leader and follower agents are affected by disturbances, while the existing studies assume only the followers to suffer disturbances. Second, we assume the disturbances to be bounded only in rates of change rather than magnitude as in the literature. To address the new challenges, we propose a novel observer-based distributed tracking control design. As a distinguishing feature, the followers can cooperatively estimate the disturbance affecting the leader through to adjust their maneuvers accordingly, which is enabled by the design of first-of-its-kind distributed disturbance observer. We build a specific approach for MASs. Further, we prove that they lead to bounded-error tracking for the considered context and further, asymptotically convergent tracking under a mild relaxation of disturbance setting. We validate the proposed approach using a simulation example.**

I. Introduction

Distributed leader-follower tracking control is key for enabling a multi-agent system (MAS) in which a group of follower agents perform distributed control while interchanging information with their neighbors to collectively track the state of a leader agent. A large body of work has been developed recently to deal with the control design under diverse challenging situations, e.g., complex dynamics, communication delays, noisy measurements and switching topologies, see [1–8] and the references therein. However, a problem that has received inadequate attention to date is the case when the agents are subjected to disturbances. In a real world, disturbances can result from unmodeled dynamics, change in ambient conditions, inherent variability of the dynamic process, and sensor noises. They can cause degradation and even failure of tracking if not well addressed.

A lead is taken in [3] with the study of disturbance-robust leader-follower tracking. It presents a distributed control design that achieves bounded-error tracking when magnitude-bounded disturbances affect the followers. This notion is extended in [9] to make the followers affected by disturbances enter a bounded region centered around the leader in finite time. Another finite-time tracking control approach is offered in [10], where the sliding mode control technique is used to suppress the effects of disturbances. It is noted that, while the control designs in these works yield robustness, they intrinsically consider the disturbances to be unknown. By contrast, a different way is to capture the disturbances by designing some observers and then adjust the control run based on the disturbance estimation. Obtaining an explicit knowledge of disturbances, this approach can advantageously reduce conservatism in control and thus enhance the tracking performance further. In [11; 12], disturbance observers are developed and integrated into tracking controllers such that a follower can estimate and offset the local disturbance interfering with its dynamics during tracking. The results in both studies point to the effectiveness of disturbance observers for improving tracking accuracy—for instance, zero-error tracking can be attained despite non-zero disturbances under certain conditions. However, other than these two, there are no more studies on this subject to the best of our knowledge. This leaves many problems still open and the potential of the disturbance-observer-based approach far from being fully explored. It is noteworthy that observer-based tracking control has been investigated in a few works, e.g., [3; 9; 13–15], but observers in these studies are meant to infer various state variables rather than disturbances.

In this study, we uniquely focus on an open problem: can we enable distributed tracking control when not only the followers but also the leader are affected by unknown disturbances and when only the rates of change of the disturbances are bounded? The state of the art generally considers that disturbances plague just the followers and that they are bounded in magnitude or approach fixed values as time goes by [3; 9–12]. The leader's dynamics, however, can also involve disturbances from a practical viewpoint, and when this happens, they are more difficult to be rejected, because the leader cannot measure the disturbances and share the information with any of the followers. Furthermore, it can significantly relax the assumptions about disturbances if we simply require their rates of change to be bounded, which will be beneficial for dealing with large disturbances. Yet, this realistic relaxation comes a more complicated, harder-to-suppress influence on the system-wide tracking performance by the disturbances.

We propose this work to address the above problem, which presents a contribution as follows. We develop a novel observer-based distributed tracking control framework. Different from the previous studies, it takes into account a follower's estimation of the disturbance affecting the leader and thus allows the follower to apply the local control accordingly. To enable the estimation, we develop a new distributed disturbance observer, which allows the followers to collectively infer the leader's disturbance. Within this framework, we develop a tracking control approach for MASs, respectively. Then, we conduct theoretical analysis

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of the proposed approach. We show that, even though disturbances affect all the agents, bounded-error tracking can be guaranteed as long as their rates of change are bounded. Further, zero-error tracking can be produced if the followers' and leader's disturbances converge to fixed points. We finally present a simulation to validate the proposed approach.

II. Notation and Preliminaries

This section introduces notation and basic concepts about graph theory and nonsmooth analysis.

A. Notation

The notation used throughout this paper is standard. The n-dimensional Euclidean space is denoted as $\mathbb{R}^n$. For a vector, $\|\cdot\|_1$ denotes the 1-norm, and $\|\cdot\|$ stands for the 2-norm. The notation $\mathbf{1}$ represents a column vector of ones. We let $\mathrm{diag}(\ldots)$ and $\det(\cdot)$ represent a block-diagonal matrix and the determinant of a matrix, respectively. The eigenvalues of an $N\times N$ matrix are $\lambda_i(\cdot)$ for $i=1,2,\ldots,N$. The minimum and maximum eigenvalues of a real, symmetric matrix are denoted as $\underline{\lambda}(\cdot)$ and $\bar{\lambda}(\cdot)$. Matrices are assumed to be compatible for algebraic operations if their dimensions are not explicitly stated. A $\mathcal{C}^k$ function is a function with k continuous derivatives.

B. Graph Theory

We use a graph to describe the information exchange topology for a leader-follower MAS. First, consider a network composed of N independent followers, and model the interaction topology as an undirected graph. The follower graph then is expressed as $\mathcal{G}=(\mathcal{V},\mathcal{E})$, where $\mathcal{V}=\{1,2,\cdots,N\}$ is the vertex set and $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ is the edge set containing unordered pairs of vertices. A path is a sequence of connected edges in a graph. The follower graph is connected if there is a path between every pair of vertices. The neighbor set of agent i is denoted as $\mathcal{N}_i$, which includes all the agents in communication with it. The adjacency matrix $A=[a_{ij}]\in\mathbb{R}^{N\times N}$ is defined as $a_{ii}=0$ and $a_{ij}>0$ if $(i,j)\in\mathcal{E}$ where $i\neq j$. For the Laplacian matrix $L=[l_{ij}]\in\mathbb{R}^{N\times N}$, $l_{ij}=-a_{ij}$ if $i\neq j$ and $l_{ii}=\sum_{k\in\mathcal{N}_i}a_{ik}$. The leader is numbered as vertex 0 and can send information to its neighboring followers. Then, we have a graph $\bar{\mathcal{G}}$, which consists of graph $\mathcal{G}$, vertex 0 and edges from the leader to its neighbors. The leader is globally reachable in $\bar{\mathcal{G}}$ if there is a path in graph $\bar{\mathcal{G}}$ from every vertex i to vertex 0. To express the graph $\bar{\mathcal{G}}$ more precisely, we denote the leader adjacency matrix associated with $\bar{\mathcal{G}}$ by $B=\mathrm{diag}(b_1,\ldots,b_N)$, where $b_i>0$ if the leader is a neighbor of agent i and $b_i=0$ otherwise. The following lemma will be useful.

Lemma 1: [16] The matrix $H=L+B$ is positive stable if and only if vertex 0 is globally reachable in $\bar{\mathcal{G}}$.

C. Nonsmooth Analysis

Consider the following discontinuous dynamical system

$$\dot{x}=f(x),\ x\in\mathbb{R}^n,\ x(0)=x_0\in\mathbb{R}^n, \tag{1}$$

where $f(x):\mathbb{R}^n\to\mathbb{R}^n$ is defined almost everywhere (a.e.). In other words, it is defined everywhere for $x\in\mathbb{R}^n\setminus W$, where W is a subset of $\mathbb{R}^n$ of Lebesgue measure zero. Moreover, $f(x)$ is Lebesgue measurable in an open region and locally bounded. A vector variable $x(\cdot)\in\mathbb{R}^n$ is a Filippov solution of (1) on $[t_0,t_1]$ if $x(\cdot)$ is absolutely continuous on $[t_0,t_1]$ while for almost all $t\in[t_0,t_1]$, satisfying the following differential inclusion:

$$\dot{x}\in\mathcal{K}[f](x)\triangleq\bigcap_{\delta>0}\bigcap_{\mu(M)=0}co\{f(B(x,\delta)\setminus M)\}, \tag{2}$$

where $\bigcap_{\mu(M)=0}$ represents the intersection over all sets M of Lebesgue measure zero, $co(\cdot)$ is called by the closure of a convex hull and $B(x,\delta)$ denotes an open ball of radius δ centered at x. Let $V(x):\mathbb{R}^n\to\mathbb{R}$ be a locally Lipschitz continuous function. Its Clarke's generalized gradient is given by

$$\partial V(x)\triangleq co\left\{\lim_{i\to\infty}\nabla V(x_i)|x_i\to x,x_i\notin\Omega_V\cup M\right\},$$

where $\nabla V(x)$ is the conventional gradient, and Ω_V denotes a set of Lebesgue measure zero which includes all points where $\nabla V(x)$ does not exist. Moreover, the set-value of derivative of V associated with (1) is defined by

$$\begin{aligned}&\mathcal{L}\dot{V}(x)=\{a\in\mathbb{R}|\exists\ v\in\mathcal{K}[f](x)\text{ such that}\\&\zeta\cdot v=a,\forall\zeta\in\partial V(x)\}.\end{aligned}$$

The following lemma will be used later.

Lemma 2: [17] Let $x(t):[t_0,t_1]\to\mathbb{R}^n$ be a Filippov solution of (2). Let $V(x)$ be a locally Lipschitz and regular function. Then $d/dt(V(x(t)))$ exists a.e. and $dV(x(t))/dt\in\mathcal{L}\dot{V}(x)$ a.e.

III. Leader-Follower Tracking

This section studies leader-follower tracking with disturbances. We develop an observer-based control approach, pivoting the design on a set of observers to make a follower aware of the leader's and its own disturbances. We further analyze the closed-loop stability of the proposed approach.

A. Problem Formulation

Consider an MAS with $N+1$ agents, which are sequentially numbered. The agent numbered as 0 is the leader, and the others are followers. An agent's dynamics is given by

$$\dot{x}_i=u_i+f_i,\quad x_i,u_i,f_i\in\mathbb{R},\quad i=0,1,\ldots,N, \tag{3}$$

where x_i is the position, u_i the control input equivalent to the velocity maneuver, and f_i the unknown disturbance. Here, each follower is expected to control its dynamics to track the leader's trajectory *via* exchanging information with its neighbors. For this MAS, we consider the following assumptions.

Assumption 1: The input $u_0\in\mathcal{C}^1$ has a bounded first-order derivative, satisfying $|\dot{u}_0|\leq w$, where w is unknown.

Assumption 2: The external disturbance f_i for $i=0,1,\ldots,N$ has a bounded first-order derivative with

$\|\dot{f}_0\mathbf{1}_{N\times 1}\| \leq q_0$ and $\left\|\begin{bmatrix}\dot{f}_1 & \dot{f}_2 & \cdots & \dot{f}_N\end{bmatrix}^\top\right\| \leq q_1$, where $0 \leq q_0, q_1 < \infty$.

Remark 1: The problem setting here is generic and applicable to a wide range of practical scenarios. Below, we briefly outline a comparison with [3; 9–12], which are the main references about tracking control with disturbances and henceforth referred to as the existing literature. First, this work considers an input-driven leader, while the leader is usually assumed to be input-free in the literature. Assumption 1 only requires the leader's input to be bounded in rate of change (with the bound unknown), which can be easily satisfied since practical actuators only allow limited ramp-ups. Second, Assumption 2 imposes disturbances on all the leader and follower agents, compared with the literature assuming only followers to be affected by disturbances. Note that the case when a disturbance is inflicted on the leader is nontrivial. This is because the leader's disturbance is very difficult to be determined by the followers, especially in a distributed network where many followers cannot directly interact with the leader. Further, the disturbances are assumed to have only bounded rates of change rather than bounded magnitude as required in the literature, beneficial for dealing with very large disturbances. From the comparison, we conclude that the considered problem is less restrictive than the predecessors and still remains an open challenge. •

B. Proposed Algorithm

Given the above problem setting, we propose an observer-based tracking control approach. The development begins with the design of a distributed linear continuous controller for a follower (say, follower i). It crucially incorporates the estimation of three unknown variables, u_0, f_0 and f_i, enabling follower i to maneuver through simultaneously emulating the input and disturbance driving the leader and offsetting the local disturbance. We subsequently construct three observers to achieve the estimation and integrate them with the controller.

Considering follower i, we design the following controller:

$$u_i = -k\left[\sum_{j\in\mathcal{N}_i} a_{ij}(x_i - x_j) + b_i(x_i - x_0)\right] + \hat{u}_{0,i} + \hat{f}_{0,i} - \hat{f}_i, \tag{4}$$

where $k > 0$ is the controller gain, $\hat{f}_{0,i}$ and $\hat{u}_{0,i}$ are follower i's respective estimates of the leader's disturbance f_0 and input u_0, and $\hat{f}_i$ is follower i's estimate of its own disturbance f_i. In (4), the term $-\sum_{j\in\mathcal{N}_i} a_{ij}(x_i - x_j) - b_i(x_i - x_0)$ is employed to drive follower i approaching the leader; the term $\hat{u}_{0,i} + \hat{f}_{0,i}$ ensures that follower i applies maneuvers consistent with the leader's input and disturbance; the term $-\hat{f}_i$ is used to cancel the local disturbance. For this controller, we build a series of observers as shown below to estimate u_0, f_0 and f_i, respectively.

The observer to obtain $\hat{u}_{0,i}$ is proposed as follows:

$$\begin{aligned}\dot{\hat{u}}_{0,i} = &-\sum_{j\in\mathcal{N}_i} a_{ij}(\hat{u}_{0,i} - \hat{u}_{0,j}) - b_i(\hat{u}_{0,i} - u_0)\\ &- d_i\cdot\operatorname{sgn}\left[\sum_{j\in\mathcal{N}_i} a_{ij}(\hat{u}_{0,i} - \hat{u}_{0,j}) + b_i(\hat{u}_{0,i} - u_0)\right],\end{aligned} \tag{5a}$$

$$\dot{d}_i = \tau_i\left|\sum_{j\in\mathcal{N}_i} a_{ij}(\hat{u}_{0,i} - \hat{u}_{0,j}) + b_i(\hat{u}_{0,i} - u_0)\right|, \tag{5b}$$

for $i = 1, 2, \ldots, N$, where d_i is the observer gain and $\tau_i > 0$ a scalar coefficient. For (5a), the leading term on the right-hand side, $-\sum_{j\in\mathcal{N}_i} a_{ij}(\hat{u}_{0,i} - \hat{u}_{0,j}) - b_i(\hat{u}_{0,i} - u_0)$, is used to make $\hat{u}_{0,i}$ approach u_0; the signum function $\operatorname{sgn}(\cdot)$ is aimed to overcome the effect of u_0's first-order dynamics, i.e., $\dot{u}_0$, and ensure the convergence of $\hat{u}_{0,i}$ to u_0. It is noted that the observer gain, d_i, is adaptively adjusted through (5b), which permits a reasonable and effective gain to be determined even if the upper bound of u_0's rate of change, w, is unknown (see Assumption 1).

The following disturbance observer is proposed for follower i to estimate f_0:

$$\dot{z}_{f0,i} = -b_i z_i - b_i^2 x_0 - \sum_{j\in\mathcal{N}_i} a_{ij}(\hat{f}_{0,i} - \hat{f}_{0,j}) - b_i u_0, \tag{6a}$$

$$\hat{f}_{0,i} = z_{f0,i} + b_i x_0, \tag{6b}$$

where $z_{f0,i}$ is the internal state. The development of (6) is inspired by [18], in which a centralized disturbance observer is designed for a single plant. Here, we transform the original design and introduce the distributed observer as above, which allows follower i to estimate f_0 remotely and collectively along with its neighbors.

The last observer, designed as follows, can enable follower i to infer the disturbance f_i inherent in its own dynamics:

$$\dot{z}_{f,i} = -l z_{f,i} - l^2 x_i + u_i, \tag{7a}$$

$$\hat{f}_i = z_{f,i} + l x_i. \tag{7b}$$

Here, $l > 0$ is the observer gain, and $z_{f,i}$ is this observer's internal state.

Combining (4)-(7) will lead to a complete observer-based distributed tracking controller. Next, we will analyze the closed-loop stability of the MAS running on this controller.

C. Stability Analysis

Define $e_{u,i} = \hat{u}_{0,i} - u_0$, which is the input estimation error. According to (5), the closed-loop error dynamics of $e_{u,i}$ can be written as

$$\begin{aligned}\dot{e}_{u,i} = &-b_i e_{u,i} - \sum_{j\in\mathcal{N}_i} a_{ij}(e_{u,i} - e_{u,j})\\ &- d_i\cdot\operatorname{sgn}\left[\sum_{j\in\mathcal{N}_i} a_{ij}(e_{u,i} - e_{u,j}) + b_i e_{u,i}\right] - \dot{u}_0.\end{aligned}$$

Further, let us concatenate $e_{u,i}$ for $i = 1, 2, \ldots, N$ and define $e_u = \begin{bmatrix} e_{u,1} & e_{u,2} & \cdots & e_{u,N} \end{bmatrix}^\top$. The dynamics of e_u can be expressed as

$$\dot{e}_u = -He_u - D \cdot \operatorname{sgn}(He_u) - \dot{u}_0 \mathbf{1}, \tag{8}$$

where $H = B + L$ and $D = \operatorname{diag}(d_1, d_2, \ldots, d_N)$. It is noted that the signum-function-based term at the right-hand side of (8) is discontinuous, measurable and locally bounded. Hence, there exists a Filippov solution to (8), which is represented by a differential inclusion as follows:

$$\dot{e}_u \in^{a.e.} \mathcal{K}\left[-He_u - D \cdot \operatorname{sgn}(He_u) - \dot{u}_0 \mathbf{1}\right].$$

The following lemma illustrates the convergence of e_u to zero as $t \to \infty$.

Lemma 3: Suppose the leader is globally reachable in $\bar{\mathcal{G}}$ and Assumption 1 holds. The input estimate $\hat{u}_{0,i}$ approaches the input u_0 asymptotically, i.e.,

$$\lim_{t\to\infty} |\hat{u}_{0,i} - u_0| = 0, \tag{9}$$

for $i = 1, 2, \ldots, N$.

Proof: By Lemma 1, H is positive definite. For (8), consider the following functions:

$$\bar{V}_1(e_u) = \frac{1}{2} e_u^\top H e_u, \ \tilde{V}_1 = \sum_{i=1}^{N} \frac{(d_i - \beta)^2}{2\tau_i},$$

where $\beta \geq w$. Consider $V_1 = \bar{V}_1(e_u) + \tilde{V}_1$ as a Lyapunov functional candidate. For the set-valued Lie derivative of $\bar{V}_1(e_u)$, we have

$$\begin{aligned}
\mathcal{L}\dot{\bar{V}}_1 &= \mathcal{K}\left[-e_u^\top H^2 e_u - e_u^\top H D \cdot \operatorname{sgn}(He_u) - e_u^\top H \dot{u}_0 \mathbf{1}\right] \\
&= \mathcal{K}\Big[- \sum_{i=1}^{N} d_i \left(\sum_{j\in\mathcal{N}_i} a_{ij}(\hat{u}_{0,i} - \hat{u}_{0,j}) + b_i(\hat{u}_{0,i} - u_0) \right)^\top \\
&\quad \cdot \operatorname{sgn}\left(\sum_{j\in\mathcal{N}_i} a_{ij}(\hat{u}_{0,i} - \hat{u}_{0,j}) + b_i(\hat{u}_{0,i} - u_0) \right) \\
&\quad - e_u^\top H^2 e_u - e_u^\top H \dot{u}_0 \mathbf{1}\Big] \\
&\leq - \sum_{i=1}^{N} d_i \left| \sum_{j\in\mathcal{N}_i} a_{ij}(\hat{u}_{0,i} - \hat{u}_{0,j}) + b_i(\hat{u}_{0,i} - u_0) \right| \\
&\quad - e_u^\top H^2 e_u + w\|He_u\|_1
\end{aligned}$$

by the fact that $\mathcal{K}[f] = \{f\}$ if f is continuous. Invoking Lemma 2, we obtain that $\dot{\bar{V}}_1 \in \mathcal{L}\dot{\bar{V}}_1$. Then, the derivative of V_1 is given by

$$\begin{aligned}
\dot{V}_1 &= \dot{\bar{V}}_1 + \dot{\tilde{V}}_1 = \dot{\bar{V}}_1 + \sum_{i=1}^{N} \frac{(d_i - \beta)\dot{d}_i}{\tau_i} \\
&\leq - \sum_{i=1}^{N} d_i \left| \sum_{j\in\mathcal{N}_i} a_{ij}(\hat{u}_{0,i} - \hat{u}_{0,j}) + b_i(\hat{u}_{0,i} - u_0) \right| \\
&\quad + \sum_{i=1}^{N} (d_i - \beta) \left| \sum_{j\in\mathcal{N}_i} a_{ij}(\hat{u}_{0,i} - \hat{u}_{0,j}) + b_i(\hat{u}_{0,i} - u_0) \right| \\
&\quad - e_u^\top H^2 e_u + w\|He_u\|_1 \\
&= -e_u^\top H^2 e_u - (\beta - w)\|He_u\|_1.
\end{aligned}$$

It is noted that $e_u^\top H^2 e_u \geq 0$. This, in addition to the fact that there always exists a β such that $\beta \geq w$, ensures $\dot{V}_1 \leq 0$. As a result, $V_1(e_u)$ is nonincreasing, which implies that e_u and d_i are bounded. From (5b), it follows that d_i is monotonically increasing, indicating that d_i should converge to some finite value. In the meantime, since $V_1(e_u)$ is nonincreasing and lower-bounded by zero, it approaches a finite limit. Defining $s(t) = \int_0^t e_u^\top(\tau) H^2 e_u(\tau) d\tau$, we have $s(t) \leq V_1(0) - V_1(t)$ by integrating $\dot{V}_1(e_u) \leq -e_u^\top H^2 e_u$. Hence, $\lim_{t\to\infty} s(t)$ exists and is finite. Due to the boundedness of e_u and $\dot{e}_u$, $\ddot{s}$ is also bounded. This implies that $\dot{s}$ is uniformly continuous. By Barbalat's Lemma [19], $\lim_{t\to\infty} \dot{s}(t) = 0$, indicating that $\lim_{t\to\infty} e_u = 0$. Therefore, we conclude that (9) holds. •

Now, consider the observer for the estimation of f_0. Define $e_{0f,i} = \hat{f}_{0,i} - f_0$, which is follower i's estimation error for f_0. Using (6), the closed-loop dynamics of $e_{0f,i}$ is given by

$$\dot{e}_{0f,i} = -b_i e_{0f,i} - \sum_{j\in\mathcal{N}_i} a_{ij}(\hat{f}_{0,i} - \hat{f}_{0,j}) - \dot{f}_0.$$

Then, defining $e_{0f} = \begin{bmatrix} e_{0f,1} & e_{0f,2} & \cdots & e_{0f,N} \end{bmatrix}^\top$, we have

$$\dot{e}_{0f} = -He_{0f} - \dot{f}_0 \mathbf{1}. \tag{10}$$

The following lemma suggests the upper boundedness of e_{0f} when given Assumption 2.

Lemma 4: If Assumption 2 holds, e_{0f} satisfies

$$\|e_{0f}(t)\| \leq \|e_{0f}(0)\| + \frac{q_0}{\underline{\lambda}(H)}, \quad t > 0, \tag{11}$$

$$\lim_{t\to\infty} \|e_{0f}(t)\| \leq \frac{q_0}{\underline{\lambda}(H)}. \tag{12}$$

Proof: Consider the Lyapunov function candidate $V_2(e_{0f}) = \frac{1}{2} e_{0f}^\top e_{0f}$. Using (10), one can derive that

$$\begin{aligned}
\dot{V}_2(e_{0f}) &= - e_{0f}^\top H e_{0f} - e_{0f}^\top \dot{f}_0 \mathbf{1} \\
&\leq -\underline{\lambda}(H)\|e_{0f}\|^2 + \|e_{0f}\|\|\dot{f}_0 \mathbf{1}\| \\
&\leq -\underline{\lambda}(H)\|e_{0f}\|^2 + q_0\|e_{0f}\|.
\end{aligned}$$

The above inequality can be rewritten as

$$\dot{V}_2 \leq -2\underline{\lambda}(H) V_2 + \sqrt{2} q_0 \sqrt{V_2}.$$

It then follows that

$$\begin{aligned}
\sqrt{V_2(t)} &\leq \sqrt{V_2(0)} e^{-\underline{\lambda}(H)t} + \frac{\sqrt{2} q_0}{2\underline{\lambda}(H)} \left(1 - e^{-\underline{\lambda}(H)t}\right) \\
&\leq \sqrt{V_2(0)} + \frac{\sqrt{2} q_0}{2\underline{\lambda}(H)}.
\end{aligned} \tag{13}$$

Then, (11) can result from (13) because $\sqrt{V_2} = \frac{\sqrt{2}}{2}\|e_{0f}\|$. Meanwhile, for the first inequality in (13), taking the limits of both sides as $t \to \infty$ would yield (12). •

For the estimation of f_i, define the error as $e_{f,i} = \hat{f}_i - f_i$ and further the vector $e_f = \begin{bmatrix} e_{f,1} & e_{f,2} & \dots & e_{f,N} \end{bmatrix}^\top$. By (7), the dynamics of e_f is given by

$$\dot{e}_f = -le_f - \dot{f},$$

where $\dot{f} = \begin{bmatrix} \dot{f}_1 & \dot{f}_2 & \cdots & \dot{f}_N \end{bmatrix}^\top$. The next lemma shows that the error e_f is bounded under Assumption 2. Its proof is similar to that of Lemma 4 and thus omitted here.

Lemma 5: If Assumption 2 holds, e_f satisfies

$$\|e_f(t)\| \le \|e_f(0)\| + \frac{q_1}{l}, \quad t > 0$$
$$\lim_{t\to\infty} \|e_f(t)\| \le \frac{q_1}{l}.$$

The above results unveil the stability properties of the observers. Now, we are in a good position to analyze the stability of the entire system loop closed by the observer-based tracking controller in (4). Define follower i's position tracking error as $e_i = x_i - x_0$, and put together e_i for $i = 1, 2, \dots, N$ to form the vector $e = \begin{bmatrix} e_1 & e_2 & \cdots & e_N \end{bmatrix}^\top$. Using (3) and (4), it can be derived that the dynamics of e is governed by

$$\dot{e} = -kHe + e_{0f} + e_u - e_f. \tag{14}$$

The following theorem concerns about the stability of e.

Theorem 1: Suppose that Assumptions 1 and 2 hold. The system in (14) is stable and the tracking error e is bounded as follows:

$$\begin{aligned} &\|e(t)\| \le \|e(0)\| \\ &+ \frac{\|e_{0f}(0)\| + \|e_u(0)\| + \|e_f(0)\| + \frac{q_0}{\underline{\lambda}(H)} + \frac{q_1}{l}}{k\underline{\lambda}(H)}, \end{aligned} \tag{15}$$

$$\lim_{t\to\infty} \|e\| \le \frac{\frac{q_0}{\underline{\lambda}(H)} + \frac{q_1}{l}}{k\underline{\lambda}(H)}. \tag{16}$$

Proof: Select the Lyapunov function candidate $V_3(e) = \frac{1}{2}e^\top e$. Its time derivative can be expressed as

$$\begin{aligned} \dot{V}_3 &= -ke^\top He + e^\top e_{0f} + e^\top e_u - e^\top e_f \\ &\le -k\underline{\lambda}(H)\|e\|^2 + \|e\| \cdot \|e_{0f}\| + \|e\| \cdot \|e_u\| + \|e\| \cdot \|e_f\|, \end{aligned}$$

where $\underline{\lambda}(H) > 0$. Equivalently, one has

$$\dot{V}_3 \le -2k\underline{\lambda}(H)V_3 + \sqrt{2}(\|e_{0f}\| + \|e_u\| + \|e_f\|)\sqrt{V_3}.$$

Then,

$$\begin{aligned} &\sqrt{V_3(t)} \le \sqrt{V_3(0)}e^{-k\underline{\lambda}(H)t} \\ &+ \frac{\sqrt{2}(\|e_{0f}(t)\| + \|e_u(t)\| + \|e_f(t)\|)}{2k\underline{\lambda}(H)}(1 - e^{-k\underline{\lambda}(H)t}) \\ &\le \sqrt{V_3(0)} + \frac{\sqrt{2}(\|e_{0f}(t)\| + \|e_u(t)\| + \|e_f(t)\|)}{2k\underline{\lambda}(H)}, \end{aligned}$$

which would indicate (15)-(16). •

The following remarks summarize our discussion of the proposed tracking control approach.

Remark 2: Theorem 1 shows that, with the proposed distributed observer-based controller, each follower would keep tracking the leader as time goes by with bounded position errors despite the disturbances. As such, we can say that the influence of the disturbances is effectively suppressed and that tracking is achieved in a practically meaningful manner. In addition, the size of the upper bounds in (15)-(16) can be reduced if the observer gain l and control gain k are chosen to be large. Furthermore, perfect or zero-error tracking can be attained if the disturbances see their rates of change gradually settles down to zero, i.e., $\lim_{t\to\infty} \dot{f}_i(t) = 0$, for $i = 0, 1, \dots, N$. The proof can be developed following similar lines as above and is omitted here. •

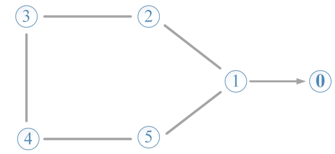


Fig. 1: Communication topology of the MAS.

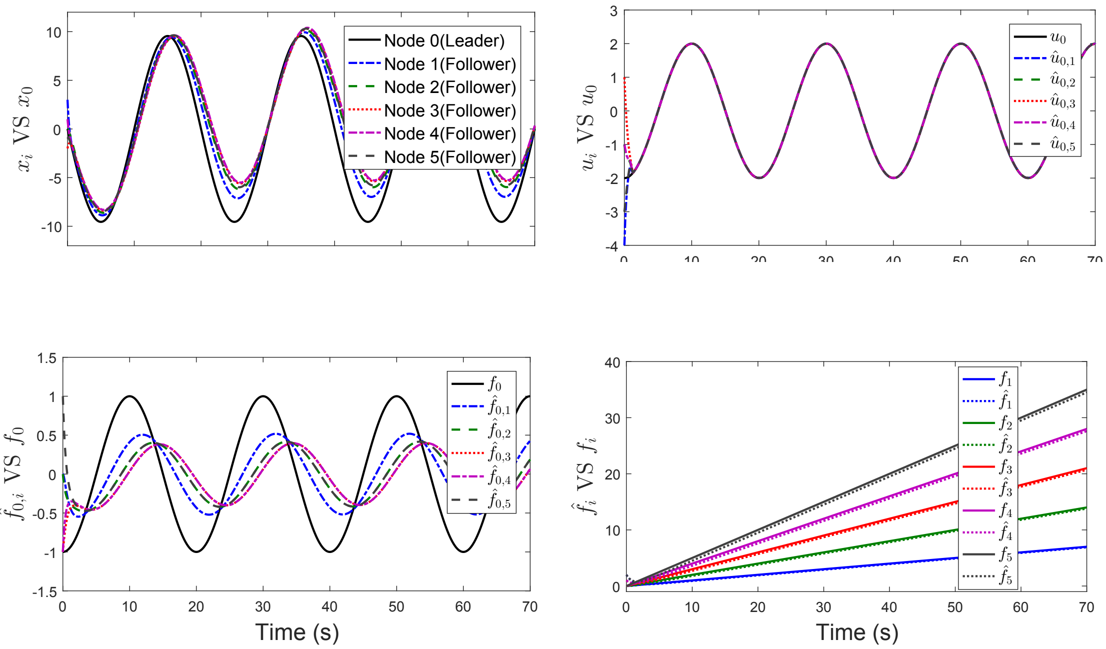
IV. NUMERICAL STUDY

This section presents an illustrative simulation example to validate the proposed distributed control approach. The example considers an MAS consisting of one leader and five followers, which share a communication topology shown in Figure 1. Vertex 0 is the leader, and vertices numbered from 1 to 5 are followers. The leader will only send information updates to follower 1, which is its only neighbor. The followers maintain bidirectional communication with their neighbors. For the topology graph, the edge-based weights are set to be unit for simplicity. Based on the communication topology, the leader adjacency matrix is $B = \mathrm{diag}(1, 0, 0, 0, 0)$. By the stability analysis in Lemmas 3, 5 and Theorem 1, we choose positive parameters as $\tau_i = 1$, $l = 1$ and $k = 0.5$, respectively. Furthermore, the upper bounds of e_f and e can decrease if larger gains of l and k are selected.

The initial positions of the leader and followers are set to be $x(0) = \begin{bmatrix} 0 & 3 & 0 & -2 & 1 & -1 \end{bmatrix}^\top$. We assume that the leader's input and disturbance and the followers' disturbances are given as

$$\begin{aligned} &u_0(t) = -2\cos(0.1\pi t), \ f_0(t) = -\cos(0.1\pi t), \\ &f(t) = \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.4 & 0.5 \end{bmatrix}^\top t. \end{aligned}$$

Note that the disturbances enforced on the followers are bounded in rates of change but linearly diverge through time. This extreme setting is used to illustrate the effectiveness of disturbance rejection here. Apply the observer-based control approach in Section III to the MAS. The simulation results are shown in Figure 2. Figure 2(a) presents the trajectories of the leader and followers. It is observed that all the followers keep tracking the leader with bounded position differences, even though they suffer growing local disturbances and the leader is also driven by an unknown disturbance. Let us now look at the performance of the observers. First, one can see from Figure 2(b) that the estimation of u_0 by each follower fast converges to the truth, agreeing with Lemma 3. Then, it is observed from Figures 2(c) and 2(d) that the observers for f_0 and f_i can track the changing disturbance overall though there are some differences. However, the



(c) (d)

Fig. 2: MAS tracking control: (a) leader's and followers' state trajectory profiles; (b) leader's input profile and the estimation by each follower; (c) leader's disturbance profile and the estimation by each follower; (d) followers' disturbance profiles and the estimation on their own.

differences or estimation errors are still bounded, matching the expectation as suggested by Lemmas 4 and 5.

V. Conclusion

MASs have attracted significant research interest in the past decade due to their increasing applications. In this paper, we have studied leader-follower tracking for MASs with unknown disturbances. Departing from the literature, we considered a much less restrictive setting about disturbances. Specifically, disturbances are applied to all the leader and followers and assumed to be bounded just in rates of change. This considerably relaxes the usual setting that only followers are affected by magnitude-bounded disturbances. To solve this problem, we have developed observer-based tracking control approach, which particularly included the design of distributed disturbance observer for followers to estimate the leader's unknown disturbance. We have proved that the proposed approach can enable bounded-error tracking in the considered disturbance setting. A simulation result further demonstrated the effectiveness of the proposed approach.

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