Simultaneous Diophantine approximation: sums of squares and homogeneous polynomials

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Abstract

Let f be a homogeneous polynomial with rational coefficients in d variables. We prove several results concerning uniform simultaneous approximation to points on the graph of f, as well as on the hypersurface $\{f(x_1,\dots,x_d)=1\}$. The results are first stated for the case $f(x_1,\dots,x_d)=x_1^2+\cdots+x_d^2$, which is of particular interest.

1 Diophantine exponents

Let $\Theta=(\theta_1,\dots,\theta_m)$ be a collection of real numbers. The *ordinary Diophantine exponent* $\omega=\omega(\Theta)$ for simultaneous rational approximation to Θ is defined as the supremum over all real γ such that the inequality

$$\max_{1\le j\le m}|q\theta_j-a_j|<q^{-\gamma}$$

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has infinitely many solutions in integer points $(q, a_1, \dots, a_m) \in \mathbb{Z}^{m+1}$ with $q > 0$.

The *uniform Diophantine exponent* $\hat{\omega} = \hat{\omega}(\Theta)$ for simultaneous approximation to Θ is defined as the supremum over all real γ such that the system of inequalities

$$\max_{1\le j\le m} |q\theta_j - a_j| < Q^{-\gamma}, \quad 1 \le q \le Q$$

has a solution $(q, a_1, \dots, a_m) \in \mathbb{Z}^{m+1}$ for every large enough real Q. It immediately follows from Minkowski's convex body theorem that $\hat{\omega}(\Theta) \ge \frac{1}{m}$ for any $\Theta \in \mathbb{R}^m$. Furthermore, let us say that Θ is *totally irrational* if $1, \theta_1, \dots, \theta_m$ are linearly independent over $\mathbb{Z}$. For such Θ it was first observed by Jarník [J38, Satz 9] that

$$\hat{\omega}(\Theta) \le 1.$$

(See also [M10, Theorem 17], as well as [W04, Theorem 5.2] for a proof based on homogeneous dynamics.) In particular for $m = 1$ one has

$$\hat{\omega}(\theta) = 1 \quad \text{for all } \theta \in \mathbb{R} \setminus \mathbb{Q}. \tag{1.1}$$

On the other hand, for $m \ge 2$ it is known that for arbitrary λ from the interval $\left[\frac{1}{m}, 1\right]$ there exists $\Theta \in \mathbb{R}^m$ with $\hat{\omega}(\Theta) = \lambda$.

Moreover it is clear from the definition that

$$\omega(\Theta) \ge \hat{\omega}(\Theta)$$

for any $\Theta \in \mathbb{R}^m$. Here we should mention that in [J54] Jarník gave an improvement of this bound for the collection of Θ such that there are at least two numbers θ_i, θ_j linearly independent over $\mathbb{Z}$ together with 1. In this case he proved the inequality

$$\frac{\omega}{\hat{\omega}} \ge \frac{\hat{\omega}}{1 - \hat{\omega}}.$$

This inequality is optimal for $m = 2$. For arbitrary m the optimal inequality was obtained recently by Marnat and Moshchevitin [MM18].

Theorem A. [MM18, Theorem 1] Let $\Theta \in \mathbb{R}^m$ be totally irrational, and let $\omega = \omega(\Theta)$ and $\hat{\omega} = \hat{\omega}(\Theta)$. Denote by G_m the unique positive root of the equation

$$x^{m-1} = \frac{\hat{\omega}}{1 - \hat{\omega}}(x^{m-2} + x^{m-3} + \dots + x + 1). \tag{1.2}$$

Then one has

$$\frac{\omega}{\hat{\omega}} \ge G_m.$$

In the present paper we study the bounds for the uniform exponent $\hat{\omega}$ for special collections of numbers. Theorem A will be an important ingredient of our proofs.

2 Approximation to several real numbers and sums of their squares

In [DS69] Davenport and Schmidt proved the following theorem:

Theorem B. [DS69, Theorem 1a] Suppose that $\xi \in \mathbb{R}$ is neither a rational number nor a quadratic irrationality. Then the uniform Diophantine exponent $\hat{\omega} = \hat{\omega}(\Xi)$ of the vector $\Xi = (\xi, \xi^2) \in \mathbb{R}^2$ satisfies the inequality

$$\hat{\omega} \leq \frac{\sqrt{5}-1}{2}.$$

Here we should note that $\frac{\sqrt{5}-1}{2}$ is the unique positive root of the equation

$$x^2 + x = 1.$$

It is known due to Roy [R03] that the bound of Theorem B is optimal. Davenport and Schmidt proved a more general result [DS69, Theorem 2a] involving successive powers $\xi, \xi^2, \dots, \xi^m$. However in the present paper we deal with another generalization.

In the sequel we will consider $m = d$ or $m = d + 1$ numbers. Namely, take

$$\boldsymbol{\xi} = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$$

and introduce the vector

$$\Xi = (\xi_1, \dots, \xi_d, \xi_1^2 + \dots + \xi_d^2) \in \mathbb{R}^{d+1}. \tag{2.1}$$

Also let H_d be the unique positive root of the equation

$$x^{d+1} + x^d + \dots + x = 1. \tag{2.2}$$

Note that $\frac{1}{2} < H_d < 1$, and $H_d \to \frac{1}{2}$ monotonically when $d \to \infty$.

In the present paper we prove the following two theorems dealing with sums of squares.

Theorem 2.1. *Let $d \geq 1$ be an integer. Suppose that Ξ as in* (2.1) *is totally irrational. Then the uniform Diophantine exponent of Ξ satisfies the inequality*

$$\hat{\omega}(\Xi) \leq H_d.$$

Note that in the case $d = 1$ Theorem 2.1 coincides with Theorem B. The next theorem can be proved by a similar argument.

Theorem 2.2. *Let $d \geq 2$. Suppose that $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_d)$ is totally irrational and*

$$\xi_1^2 + \cdots + \xi_d^2 = 1. \tag{2.3}$$

Then the uniform Diophantine exponent of $\boldsymbol{\xi}$ satisfies the inequality

$$\hat{\omega}(\boldsymbol{\xi}) \leq H_{d-1}.$$

Theorems 2.1 and 2.2 are particular cases of more general Theorems 1a and 2a, which we formulate in Section 4.

Remark 2.3. It is worth comparing Theorems 2.1 and 2.2, as well as their more general versions, with a lower bound obtained using the methods of [KW05, §5]. It is not hard to derive from [KW05, Corollary 5.2] that for any real analytic submanifold M of $\mathbb{R}^m$ of dimension at least 2 which is not contained in any proper rational affine hyperplane of $\mathbb{R}^m$ there exists totally irrational $\Theta \in M$ with

$$\hat{\omega}(\Theta) \geq \frac{1}{m} + \frac{2}{m(m^2-1)}.$$

It would be interesting to see if the above estimate could be improved, thus shedding some light on the optimality of our theorems.

3 Intrinsic approximation on spheres

Our study of vectors of the form (2.1) was motivated by problems related to intrinsic rational approximation on spheres. In [KM15] Kleinbock and Merrill proved the following result.

Theorem C. [KM15, Theorem 4.1] Let $d \geq 2$. There exists a positive constant C_d such that for any $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^d$ satisfying (2.3) and for any $T > 1$ there exists a rational vector

$$\boldsymbol{\alpha} = \left(\frac{a_1}{q}, \ldots, \frac{a_d}{q}\right) \in \mathbb{Q}^d \tag{3.1}$$

such that

$$|\boldsymbol{\alpha}|^2 = \left(\frac{a_1}{q}\right)^2 + \cdots + \left(\frac{a_d}{q}\right)^2 = 1$$

and

$$|\boldsymbol{\xi} - \boldsymbol{\alpha}| \leq \frac{C_d}{q^{1/2}T^{1/2}}, \quad 1 \leq q \leq T. \tag{3.2}$$

Here and hereafter by $|\cdot|$ we denote the Euclidean norm of a vector. In particular Theorem C implies that in the case $\boldsymbol{\xi} \notin \mathbb{Q}^d$ the inequality

$$|\boldsymbol{\xi} - \boldsymbol{\alpha}| \le \frac{C_d}{q}$$

has infinitely many solutions in rational vectors (3.1).

See [M16, M17] for effective versions of Theorem C, and [FKMS14, Theorem 5.1] for generalizations. Note that the formulation from [M16] involves sums of squares, while an effective version for an arbitrary positive definite quadratic form with integer coefficients can be found in [M17]. It is also explained in [FKMS14] how the conclusion of Theorem C can be derived from [SV95, Theorem 1] via a correspondence between intrinsic Diophantine approximation on quadric hypersurfaces and approximation of points in the boundary of the hyperbolic space by parabolic fixed points of Kleinian groups; see [FKMS14, Proposition 3.16].

In this paper we prove a result about uniform intrinsic approximation on the unit sphere. We need some notation. First of all, note that the inequality (3.2) can be rewritten as

$$\frac{|q\boldsymbol{\xi} - \boldsymbol{a}|^2}{q} \le \frac{C_d^2}{T}, \quad \boldsymbol{a} = (a_1, \dots, a_d) \in \mathbb{Z}^d, \quad a_1^2 + \dots + a_d^2 = q^2.$$

Now let us define the function

$$\Psi_{\boldsymbol{\xi}}(T) = \min_{(q,a_1,\dots,a_d)\in\mathbb{Z}^{d+1}\colon 1\le q\le T,\ a_1^2+\dots+a_d^2=q^2} \frac{|q\boldsymbol{\xi} - \boldsymbol{a}|^2}{q}.$$

Theorem C states that for any $\boldsymbol{\xi} \in \mathbb{R}^d$ under the condition (2.3) one has

$$T \cdot \Psi_{\boldsymbol{\xi}}(T) \le C_d^2 \quad \text{for} \quad T > 1.$$

Theorem 3.1. *Let $d \ge 2$. Let $\boldsymbol{\xi} \in \mathbb{R}^d \setminus \mathbb{Q}^d$ be such that* (2.3) *is satisfied. Then for any $\varepsilon > 0$ there exists arbitrary large T such that*

$$T \cdot \Psi_{\boldsymbol{\xi}}(T) \ge \frac{1}{4} - \varepsilon.$$

Theorem 3.1 is an analog of Khintchine's lemma on rational approximations to one real number (see [K26, Satz 1]). It admits the following corollary. One can try to define the *uniform Diophantine exponent* of $\boldsymbol{\xi}$ for the intrinsic approximation on the unit sphere as

$$\hat{\omega}_d^{\mathbf{i}}(\boldsymbol{\xi}) = \sup\left\{ \gamma \in \mathbb{R} \;\middle|\; \begin{array}{c} \text{the inequalities} \quad |\boldsymbol{\xi} - \boldsymbol{\alpha}| \le \dfrac{1}{q^{1/2}T^{\gamma/2}}, \quad 1 \le q \le T \\ \text{are solvable in } \boldsymbol{\alpha} \text{ of the form (3.1) for large enough } T \end{array} \right\}.$$

Then for all vectors $\boldsymbol{\xi} \notin \mathbb{Q}^d$ satisfying (2.3) we have

$$\hat{\omega}_d^{\mathbf{i}}(\boldsymbol{\xi}) = 1$$

by Theorem 3.1. So here we have an equality similar to (1.1) for the case of approximation to one real number. See also [BGSV16, Theorem 2], where a similar observation was made in the context of Kleinian groups. Theorem 3.1 follows from a more general Theorem 3a which we formulate in Section 4.

4 Results on homogeneous polynomials

Given integers $s \geq 2$ and $d \geq 1$, define $H_{d,s}$ to be the unique positive root of the equation

$$(1 - x) = x \cdot \sum_{k=1}^{d} \left(\frac{x}{s-1} \right)^k . \tag{4.1}$$

Note that for any s and d one has

$$\frac{s-1}{s} < H_{d,s} < 1.$$

Clearly $H_{d,2} = H_d$, and $H_{d,s}$ monotonically decreases to $\frac{s-1}{s}$ as $d \to +\infty$.

The results of this section deal with a homogeneous polynomial

$$f(\boldsymbol{x}) = \sum_{(s_1,\dots,s_d)\in\mathbb{Z}_+^d:\ s_1+\dots+s_d=s} f_{s_1,\dots,s_d} x_1^{s_1} \cdots x_d^{s_d}, \quad \text{where } f_{s_1,\dots,s_d} \in \mathbb{Q}, \tag{4.2}$$

of degree s in variables $x_1, \dots, x_d$ (here $\mathbb{Z}_+$ stands for the set of non-negative integers). Theorem 2.1 from the previous section is a corollary of the following general statement.

Theorem 1a. Let $s \geq 2$ be an integer, and let f as in (4.2) be such that

$$\#\{\boldsymbol{x} \in \mathbb{Q}^d : f(\boldsymbol{x}) = 0\} < \infty. \tag{4.3}$$

Suppose that

$$\Xi_f = \big(\xi_1, \dots, \xi_d, f(\xi_1, \dots, \xi_d)\big) \tag{4.4}$$

is totally irrational. Then $\hat{\omega}(\Xi_f) \leq H_{d,s}$.

We give a proof of Theorem 1a in Section 8.

Remark 4.1. Let us consider the case $d = 1$. In this case Theorem 1a states that the uniform exponent $\hat{\omega}$ of (ξ, ξ^s) is bounded from above by the positive root of the equation

$$x^2 + (s-1)x - (s-1) = 0,$$

that is

$$\hat{\omega} \leq \frac{\sqrt{(s-1)(s+3)} - (s-1)}{2}. \tag{4.5}$$

This result was obtained by Batzaya in [B15] for arbitrary vectors of the form (ξ^l, ξ^s) with $1 \leq l < s$. In the case $d = 1, s = 3$ much stronger inequality

$$\hat{\omega} \leq \frac{2(9+\sqrt{11})}{35}$$

is known due to Lozier and Roy (see [LR12] and the discussion therein). In [B17] Batzaya improved (4.5) and showed that for (ξ^l, ξ^s) with $1 \leq l < s$ one has

$$\hat{\omega} \leq \frac{s^2-1}{s^2-s-1}$$

for odd s. In the case of even s in the paper [B15] he had a better inequality

$$\hat{\omega} \leq \frac{(s-1)(s+2)}{s^2+2s-1}.$$

Also [B17] contains a better bound for $\hat{\omega}$ when $s = 5, 7, 9$. Thus the inequality of our Theorem 1a is not optimal for $s \geq 3$.

Theorem 2.2 from the previous section is a corollary of the following general statement.

Theorem 2a. *Let $s \geq 2$ be an integer, and let f as in (4.2) be such that (4.3) holds. Then*

$$\boldsymbol{\xi} \in \mathbb{R}^d \text{ is totally irrational and } f(\boldsymbol{\xi}) = 1 \quad \Longrightarrow \quad \hat{\omega}(\boldsymbol{\xi}) \leq H_{d-1,s}. \tag{4.6}$$

We give a proof of Theorem 2a in Section 7. To get Theorems 2.1, 2.2 from Theorems 1a, 2a one should put $s = 2$ and $f(\boldsymbol{x}) = x_1^2 + \cdots + x_d^2$.

Remark 4.2. The argument used in the proof of Theorem 2a yields (4.6) for any (not necessarily homogeneous) polynomial f with rational coefficients such that the number of rational points on the hypersurface $\{f = 1\}$ is finite. We state it as Theorem 2b in Section 7. For example (cf. (4.5) with $d = 1$ and $s = 6$) it follows that

$$\hat{\omega}(x, y) \leq \frac{\sqrt{45} - 5}{2}$$

for any $(x, y) \in \mathbb{R}^2$ such that $y^2 - x^2 - x^6 = 1$.

Now for $\boldsymbol{\xi} \in \mathbb{R}^d$ under the condition $f(\boldsymbol{\xi}) = 1$ consider the function

$$\Psi_{f,\boldsymbol{\xi}}(T) = \min_{(q,\boldsymbol{a})=(q,a_1,\dots,a_d)\in\mathbb{Z}^{d+1}:\ 1\le q\le T,\ f\left(\frac{a_1}{q},\dots,\frac{a_d}{q}\right)=1} \frac{|q\boldsymbol{\xi} - \boldsymbol{a}|^s}{q}.$$

It is clear that $\Psi_{f,\boldsymbol{\xi}}(T)$ is a non-increasing piecewise constant function. Here we do not suppose that it tends to zero as $T \to +\infty$.

Theorem 3a. Let $s \geq 2$ be an integer, and let f as in (4.2) be such that (4.3) holds. Take $\boldsymbol{\xi} \notin \mathbb{Q}^d$ with

$$f(\boldsymbol{\xi}) = 1,$$

and let $D = D(f) \in \mathbb{Z}_+$ be the common denominator of all rational numbers $f_{s_1,\dots,s_d}$. Also define

$$K = K(f) = \sup_{\boldsymbol{x}\in\mathbb{R}^d:\ |\boldsymbol{x}|=1} |f(\boldsymbol{x})|. \tag{4.7}$$

Then for any positive ε there exists arbitrary large T such that

$$T^{s-1} \cdot \Psi_{f,\boldsymbol{\xi}}(T) \geq \frac{1}{2^s DK} - \varepsilon.$$

For $f(\boldsymbol{x}) = x_1^2 + \cdots + x_d^2$ we have $s = 2$ and $D(f) = K(f) = 1$. Thus Theorem 3.1 is a direct corollary of Theorem 3a. We give a proof of Theorem 3a in Section 9.

5 The main lemma

The next lemma is a polynomial analogue of the classical simplex lemma in simultaneous Diophantine approximation going back to Davenport [D64]. See also [KS18] for a version for arbitrary quadratic forms, [BGSV16, Lemma 1] for a similar statement in the context of Kleinian groups, and [FKMS18, Lemma 4.1] for a general simplex lemma for intrinsic Diophantine approximation on manifolds.

Lemma 5.1. *Let $s \geq 2$ be an integer, and let f be as in* (4.2)*. Let $D = D(f)$ and $K = K(f)$ be defined as in Theorem 3a, and take two rational vectors*

$$\boldsymbol{\alpha} = \left(\frac{a_1}{q}, \dots, \frac{a_d}{q}\right) \quad \text{and} \quad \boldsymbol{\beta} = \left(\frac{b_1}{r}, \dots, \frac{b_d}{r}\right)$$

such that

$$f(\boldsymbol{\alpha} - \boldsymbol{\beta}) \neq 0. \tag{5.1}$$

(i) *Suppose that*

$$f(\boldsymbol{\alpha}) = \frac{A}{q}$$

with an integer A*. Then*

$$|\boldsymbol{\alpha}-\boldsymbol{\beta}|^s \geq \frac{1}{DKq^{s-1}r^s}\,. \tag{5.2}$$

(ii) *Suppose*

$$f(\boldsymbol{\alpha}) = \frac{A}{q}, \quad f(\boldsymbol{\beta}) = \frac{B}{r}$$

with integers A, B*. Then*

$$|\boldsymbol{\alpha}-\boldsymbol{\beta}|^s \geq \frac{1}{DKq^{s-1}r^{s-1}}\,. \tag{5.3}$$

Proof. **(i)** First of all we observe that

$$|f(\boldsymbol{\alpha}-\boldsymbol{\beta})| \geq \frac{1}{Dq^{s-1}r^s}. \tag{5.4}$$

Indeed, for any $s_1,\dots,s_d \in \mathbb{Z}_+$ under the condition $s_1+\dots+s_d = s$ consider the product

$$\Pi_{s_1,\dots,s_d} = \prod_{k=1}^{d}\left(\frac{a_k}{q}-\frac{b_k}{r}\right)^{s_k}.$$

It is clear that

$$\Pi_{s_1,\dots,s_d} = \frac{\prod_{k=1}^{d} a_k^{s_k}}{q^s} + \frac{W_{s_1,\dots,s_d}}{q^{s-1}r^s}$$

with an integer $W_{s_1,\dots,s_d}$. Now from (5.1) we see that

$$\begin{aligned} 0 \neq f(\boldsymbol{\alpha}-\boldsymbol{\beta}) &= \sum_{(s_1,\dots,s_d)\in\mathbb{Z}_+^d:\ s_1+\dots+s_d=s} f_{s_1,\dots,s_d}\Pi_{s_1,\dots,s_d} \\ &= f(\boldsymbol{\alpha}) + \frac{W}{Dq^{s-1}r^s} = \frac{A}{q} + \frac{W}{Dq^{s-1}r^s} = \frac{W_1}{Dq^{s-1}r^s}, \end{aligned}$$

with $W, W_1 \in \mathbb{Z}$, and (5.4) is proved. Then from the definition (4.7) we see that

$$|f(\boldsymbol{\alpha}-\boldsymbol{\beta})| \leq K|\boldsymbol{\alpha}-\boldsymbol{\beta}|^s. \tag{5.5}$$

Now (5.4) and (5.5) give (5.2).

(ii) The proof here is quite similar. From the conditions on $f(\boldsymbol{\alpha})$ and $f(\boldsymbol{\beta})$ we see that

$$0 \neq f(\boldsymbol{\alpha}-\boldsymbol{\beta}) = f(\boldsymbol{\alpha}) \pm f(\boldsymbol{\beta}) + \frac{W'}{Dq^{s-1}r^{s-1}} = \frac{A}{q} \pm \frac{B}{r} + \frac{W'}{Dq^{s-1}r^{s-1}} = \frac{W_1'}{Dq^{s-1}r^{s-1}},$$

with $W', W_1' \in \mathbb{Z}$. So we get

$$|f(\boldsymbol{\alpha} - \boldsymbol{\beta})| \geq \frac{1}{Dq^{s-1}r^{s-1}}. \tag{5.6}$$

Now (5.6) together with (5.5) give (5.3).

□

6 Best approximation vectors

If $\Theta = (\theta_1, \ldots, \theta_m) \in \mathbb{R}^m \setminus \mathbb{Q}^m$, let us say that a vector $(q, a_1, \ldots, a_m) \in \mathbb{Z}^{m+1}$ is a *best simultaneous approximation vector* of Θ if

$$\mathrm{dist}(q\Theta, \mathbb{Z}^m) < \mathrm{dist}(k\Theta, \mathbb{Z}^m) \quad \forall\, k = 1, \ldots, q-1,$$

and

$$\mathrm{dist}(q\Theta, \mathbb{Z}^m) = \max_{i=1,\ldots,m} |q\theta_i - a_i|.$$

Here 'dist' stands for the distance induced by the supremum norm on $\mathbb{R}^m$. Best approximation vectors of Θ form an infinite sequence $(q_\nu, a_{1,\nu}, \ldots, a_{m,\nu})$, $\nu \in \mathbb{N}$, and satisfy the inequalities

$$q_{\nu-1} < q_\nu, \quad \zeta_{\nu-1} > \zeta_\nu, \quad \nu \in \mathbb{N},$$

where one defines

$$\zeta_\nu = \max_{i=1,\ldots,m} |q_\nu\theta_i - a_{i,\nu}|.$$

It is important that

$$\text{g.c.d.}(q_\nu, a_{1,\nu}, \ldots, a_{m,\nu}) = 1, \quad \nu \in \mathbb{N}.$$

So for any two successive rational approximation vectors

$$\boldsymbol{\alpha}_j = \left(\frac{a_{1,j}}{q_j}, \ldots, \frac{a_{m,j}}{q_j}\right) \in \mathbb{Q}^m, \quad j = \nu - 1, \nu$$

we have

$$\boldsymbol{\alpha}_{\nu-1} \neq \boldsymbol{\alpha}_\nu.$$

Some detailed information about best approximation vectors may be found for example in papers [C13] and [M10]. In particular, the following property of the uniform exponent $\hat{\omega} = \hat{\omega}(\Theta)$ is well known (see e.g. [M10, Proposition 1]). Suppose that $\gamma < \hat{\omega}$. Then for all ν large enough one has

$$\zeta_{\nu-1} \leq q_\nu^{-\gamma}. \tag{6.1}$$

7 Proof of Theorem 2a

We take $m = d$ and consider best approximation vectors

$$\boldsymbol{z}_\nu = (q_\nu, a_{1,\nu}, \dots, a_{d,\nu}) \in \mathbb{Z}^{d+1}$$

of $\boldsymbol{\xi} = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$, together with distances

$$\zeta_\nu = \max_{1 \le j \le d} |q_\nu \xi_j - a_{j,\nu}|,$$

and the corresponding rational approximants

$$\boldsymbol{\alpha}_\nu = \left(\frac{a_{1,\nu}}{q_\nu}, \dots, \frac{a_{d,\nu}}{q_\nu} \right) \in \mathbb{Q}^d.$$

Under the condition $\gamma < \hat{\omega}(\boldsymbol{\xi})$ we have (6.1) for all large ν.

Here we should note that

$$\max_{1 \le j \le d} \left| \xi_j - \frac{a_{j,\nu}}{q_\nu} \right| = \frac{\zeta_\nu}{q_\nu}.$$

So for large ν we see that

$$\left(\frac{a_{1,\nu}}{q_\nu} \right)^{s_1} \cdots \left(\frac{a_{d,\nu-1}}{q_\nu} \right)^{s_d} = \xi_1^{s_1} \dots \xi_d^{s_d} + O\left(\frac{\zeta_\nu}{q_\nu} \right). \tag{7.1}$$

We consider two cases.

Case 1. $f(\boldsymbol{\alpha}_\nu) = 1$ for infinitely many ν.

Here, since $\boldsymbol{\alpha}_\nu \neq \boldsymbol{\alpha}_{\nu-1}$, we may apply Lemma 2.1**(i)** with $A = q_\nu$. Take $\boldsymbol{\alpha} = \boldsymbol{\alpha}_\nu$, $\boldsymbol{\beta} = \boldsymbol{\alpha}_{\nu-1}$; then (5.1) follows from (4.3) when ν is large enough, and from (5.2) we deduce that

$$\begin{aligned} \frac{1}{(DK)^{\frac{1}{s}} q_\nu^{\frac{s-1}{s}} q_{\nu-1}} &\le \sqrt{d} \max_{1 \le k \le d} \left| \frac{a_{k,\nu}}{q_\nu} - \frac{a_{k,\nu-1}}{q_{\nu-1}} \right| \\ &\le \sqrt{d} \left(\max_{1 \le k \le d} \left| \frac{a_{k,\nu}}{q_\nu} - \xi_k \right| + \max_{1 \le k \le d} \left| \frac{a_{k,\nu-1}}{q_{\nu-1}} - \xi_k \right| \right) \\ &\le \frac{2\sqrt{d}\zeta_{\nu-1}}{q_{\nu-1}} \le \frac{2\sqrt{d}}{q_{\nu-1} q_\nu^\gamma}. \end{aligned} \tag{7.2}$$

So with some positive c_1 we have $q_\nu^\gamma \le c_1 q_\nu^{\frac{s-1}{s}}$ for infinitely many ν and $\hat{\omega}(\boldsymbol{\xi}) \le \frac{s-1}{s} < H_{d-1,s}$.

Case 2. $f(\boldsymbol{\alpha}_\nu) \neq 1$ for all ν large enough.

For such ν the difference $f(\boldsymbol{\alpha}_\nu) - 1$ is a nonzero rational number with denominator Dq_ν^s. Therefore we have

$$|f(\boldsymbol{\alpha}_\nu) - 1| \geq \frac{1}{Dq_\nu^s}.$$

Now from (7.1) we see that

$$\frac{1}{Dq_\nu^s} \leq |f(\boldsymbol{\alpha}_\nu) - 1| = |f(\boldsymbol{\alpha}_\nu) - f(\boldsymbol{\xi})| = O\left(\frac{\zeta_\nu}{q_\nu}\right) \tag{7.3}$$

and

$$\zeta_\nu \geq \frac{c_2}{q_\nu^{s-1}} \tag{7.4}$$

with some positive constant c_2 depending on f and $\boldsymbol{\xi}$. As this inequality holds for all large ν, we conclude that $\omega(\boldsymbol{\xi}) \leq s-1$ and

$$\frac{s-1}{\hat{\omega}(\boldsymbol{\xi})} \geq \frac{\omega(\boldsymbol{\xi})}{\hat{\omega}(\boldsymbol{\xi})} \geq G_d.$$

Recall that G_d is a root of equation (1.2). This means that the upper bound for $\hat{\omega}(\boldsymbol{\xi})$ is given by the unique positive root of the equation

$$\left(\frac{s-1}{x}\right)^{d-1} = \frac{x}{1-x}\left(\left(\frac{s-1}{x}\right)^{d-2} + \left(\frac{s-1}{x}\right)^{d-3} + \cdots + \frac{s-1}{x} + 1\right)$$

which coincides with (4.1) if d is replaced by $d-1$. □

We close the section by observing that the argument used in the proof of Case 2 above does not rely on the homogeneity of f. Thus the following result can be established.

Theorem 2b. Suppose that f is an *arbitrary* polynomial of degree s in d variables with rational coefficients such that

$$\#\{\boldsymbol{x} \in \mathbb{Q}^d : f(\boldsymbol{x}) = 1\} < \infty,$$

and let $\boldsymbol{\xi} = (\xi_1, ..., \xi_d) \notin \mathbb{Q}^d$ be such that $f(\boldsymbol{\xi}) = 1$. Then:

(i) $\omega(\boldsymbol{\xi}) \leq s-1$;

(ii) if $\boldsymbol{\xi}$ is totally irrational, then $\hat{\omega}(\boldsymbol{\xi}) \leq H_{d-1,s}$.

The proof is left to the reader. In particular, the conclusion of Theorem 2b holds when $\{f = 1\}$ is an algebraic curve over $\mathbb{Q}$ of genus at least 2, such as the one mentioned in Remark 3.1.

8 Proof of Theorem 1a

The proof of Theorem 1a is similar to the proof of Theorem 2a. We take $m = d+1$ and consider a sequence of best simultaneous approximation vectors

$$\boldsymbol{z}_\nu = (q_\nu, a_{1,\nu}, \ldots, a_{d,\nu}, A_\nu) \in \mathbb{Z}^{d+2}, \quad \nu \in \mathbb{N},$$

of $\Theta = \Xi_f$ as in (4.4), and the corresponding distances from $q_\nu \Xi_f$ to $\mathbb{Z}^{d+1}$:

$$\zeta_\nu = \max\left(|q_\nu \xi_1 - a_{1,\nu}|, \ldots, |q_\nu \xi_d - a_{d,\nu}|, |q_\nu f(\boldsymbol{\xi}) - A_\nu|\right).$$

We also need “shortened" rational approximation vectors

$$\boldsymbol{\alpha}_\nu = \left(\frac{a_{1,\nu}}{q_\nu}, \ldots, \frac{a_{d,\nu}}{q_\nu}\right) \in \mathbb{Q}^d.$$

Note that now it may happen that

$$\boldsymbol{\alpha}_{\nu-1} = \boldsymbol{\alpha}_\nu \tag{8.1}$$

for some ν.

Lemma 8.1. *Suppose that* (8.1) *holds and*

$$f(\boldsymbol{\alpha}_\nu) = \frac{A_\nu}{q_\nu}. \tag{8.2}$$

Then

$$\Delta = \text{g.c.d.}(q_\nu, a_{1,\nu}, \ldots, a_{d,\nu}) = O\left(q_\nu^{\frac{s-1}{s}}\right). \tag{8.3}$$

Proof. We know that

$$\text{g.c.d.}(q_\nu, a_{1,\nu}, \ldots, a_{d,\nu}, A_\nu) = 1$$

and thus

$$\text{g.c.d.}(\Delta, A_\nu) = 1.$$

From (8.2) we see that

$$DA_\nu q_\nu^{s-1} = Dq_\nu^s f(\boldsymbol{\alpha}_\nu) \in \mathbb{Z}.$$

But $\Delta \,|\, a_{j,\nu}$ for any j. As $Dq_\nu^s f\left(\frac{\cdot}{q_\nu}\right)$ is a homogeneous polynomial of degree s with integer coefficients, we deduce that

$$\Delta^s \,|\, Dq_\nu^{s-1}.$$

This gives (8.3). □

To prove Theorem 1a we consider three cases.

Case 1.1. For infinitely many ν (8.1) and (8.2) hold.

In this case the vectors

$$(q_{\nu-1}, a_{1,\nu-1}, \dots, a_{d,\nu-1}), \quad (q_\nu, a_{1,\nu}, \dots, a_{d,\nu})$$

are proportional, but the vectors

$$(q_{\nu-1}, a_{1,\nu-1}, \dots, a_{d,\nu-1}, A_{\nu-1}), \quad (q_\nu, a_{1,\nu}, \dots, a_{d,\nu}, A_\nu)$$

are not proportional. This means that

$$\begin{vmatrix} q_{\nu-1} & A_{\nu-1} \\ q_\nu & A_\nu \end{vmatrix} \neq 0. \tag{8.4}$$

There exists a primitive vector

$$(q_*, a_{1,*}, \dots, a_{d,*}) \in \mathbb{Z}^{d+1}, \quad \text{g.c.d.}(q_*, a_{1,*}, \dots, a_{d,*}) = 1, \quad q_* \geq 1,$$

such that $(q_\nu, a_{1,\nu}, \dots, a_{d,\nu}) = \Delta \cdot (q_*, a_{1,*}, \dots, a_{d,*})$ and $(q_{\nu-1}, a_{1,\nu-1}, \dots, a_{d,\nu-1}) = \Delta' \cdot (q_*, a_{1,*}, \dots, a_{d,*})$, where

$$\Delta = \text{g.c.d.}(q_\nu, a_{1,\nu}, \dots, a_{d,\nu}), \quad \Delta' = \text{g.c.d.}(q_{\nu-1}, a_{1,\nu-1}, \dots, a_{d,\nu-1}).$$

In particular

$$q_\nu = \Delta q_*, \quad q_{\nu-1} = \Delta' q_*$$

and

$$\begin{vmatrix} q_{\nu-1} & A_{\nu-1} \\ q_\nu & A_\nu \end{vmatrix} \equiv 0 \pmod{q_*}.$$

Now from (8.4) we deduce

$$\frac{q_\nu}{\Delta} = q_* \leq \left| \begin{vmatrix} q_{\nu-1} & A_{\nu-1} \\ q_\nu & A_\nu \end{vmatrix} \right| \leq 2q_\nu |q_{\nu-1} f(\boldsymbol{\xi}) - A_{\nu-1}| \leq 2q_\nu \zeta_{\nu-1} \leq 2q_\nu^{1-\gamma}$$

by (6.1). Thus we get

$$q_\nu^\gamma \leq 2\Delta.$$

We apply Lemma 2.2 to see that $\gamma \leq \frac{s-1}{s}$, and hence $\hat{\omega}(\Xi_f) \leq \frac{s-1}{s} < H_{d,s}$.

Case 1.2. For infinitely many ν (8.2) holds with $\boldsymbol{\alpha}_{\nu-1} \neq \boldsymbol{\alpha}_\nu$.

We proceed similarly to Case 1 from the proof of Theorem 2a by applying Lemma 5.1**(i)** with $A = q_\nu$ for $\boldsymbol{\alpha} = \boldsymbol{\alpha}_\nu \neq \boldsymbol{\beta} = \boldsymbol{\alpha}_{\nu-1}$. From (5.2), similarly to (7.2), for large enough ν we get

$$\frac{1}{(DK)^{\frac{1}{s}} q_\nu^{\frac{s-1}{s}} q_{\nu-1}} \leq \sqrt{d} \max_{1 \leq k \leq d} \left| \frac{a_{k,\nu}}{q_\nu} - \frac{a_{k,\nu-1}}{q_{\nu-1}} \right| \leq \frac{2\sqrt{d}}{q_{\nu-1} q_\nu^\gamma}.$$

Again $q_\nu^\gamma = O\left(q_\nu^{\frac{s-1}{s}}\right)$ for infinitely many ν, and $\hat{\omega}(\Xi_f) \le \frac{s-1}{s} < H_{d,s}$.

Case 2. $f(\boldsymbol{\alpha}_\nu) \ne \frac{A_\nu}{q_\nu}$ for all ν large enough.

This case is similar to Case 2 from the proof of Theorem 2a. Now the difference $f(\boldsymbol{\alpha}_\nu) - \frac{A_\nu}{q_\nu}$ is a nonzero rational number with denominator Dq_ν^s. Therefore we have

$$\left| f(\boldsymbol{\alpha}_\nu) - \frac{A_\nu}{q_\nu} \right| \ge \frac{1}{Dq_\nu^s}.$$

Analogously to (7.3) we now get (7.4), which leads to $\omega(\Xi_f) \le s-1$. Then, applying Theorem A to the $(d+1)$-dimensional vector Ξ_f, we obtain

$$\frac{s-1}{\hat{\omega}(\Xi_f)} \ge \frac{\omega(\Xi_f)}{\hat{\omega}(\Xi_f)} \ge G_{d+1}.$$

This shows that the positive root of equation

$$\left(\frac{s-1}{x}\right)^d = \frac{x}{1-x}\left(\left(\frac{s-1}{x}\right)^{d-1} + \left(\frac{s-1}{x}\right)^{d-2} + \cdots + \frac{s-1}{x} + 1\right)$$

gives an upper bound for $\hat{\omega}(\Xi_f)$. Theorem 1a is proved. □

9 Proof of Theorem 3a

Let

$$q_1 < q_2 < \ \ldots \ < q_\nu < q_{\nu+1} < \ldots$$

be the sequence of points where the function $\Psi_{f,\boldsymbol{\xi}}(T)$ is not continuous. Without loss of generality we may suppose that this sequence is infinite. We consider the corresponding best approximation vectors

$$\boldsymbol{\alpha}_\nu = \left(\frac{a_{1,\nu}}{q_\nu}, \ldots, \frac{a_{d,\nu}}{q_\nu}\right) \in \mathbb{Q}^d,$$

where $a_{j,\nu}$ realize the minima in the definition of the function $\Psi_{f,\boldsymbol{\xi}}(T)$. They satisfy $f(\boldsymbol{\alpha}_\nu) = 1$. By definition of the function $\Psi_{\boldsymbol{\xi}}(T)$ and numbers q_ν we see that

$$\Psi_{\boldsymbol{\xi}}(T) = \Psi_{\boldsymbol{\xi}}(q_{\nu-1}) \ \text{ for } \ q_{\nu-1} \le T < q_\nu.$$

Now, since $\boldsymbol{\alpha}_\nu \ne \boldsymbol{\alpha}_{\nu-1}$, we may apply Lemma 5.1**(ii)** with $A = q_\nu$ and $B = q_{\nu-1}$. Indeed, $\boldsymbol{\alpha} = \boldsymbol{\alpha}_\nu$ and $\boldsymbol{\beta} = \boldsymbol{\alpha}_{\nu-1}$ satisfy (5.1) for large enough ν, and from (5.3) we have

$$\frac{1}{DKq_{\nu-1}^{s-1}q_\nu^{s-1}} \le |\boldsymbol{\alpha}_{\nu-1} - \boldsymbol{\alpha}_\nu|^s \le (|\boldsymbol{\alpha}_{\nu-1} - \boldsymbol{\xi}| + |\boldsymbol{\alpha}_\nu - \boldsymbol{\xi}|)^s \le 2^s|\boldsymbol{\alpha}_{\nu-1} - \boldsymbol{\xi}|^s,$$

since $|\boldsymbol{\alpha}_\nu - \boldsymbol{\xi}| \le |\boldsymbol{\alpha}_{\nu-1} - \boldsymbol{\xi}|$. Thus

$$\frac{1}{2^s DK} \le q_\nu^{s-1} q_{\nu-1}^{s-1} |\boldsymbol{\alpha}_{\nu-1} - \boldsymbol{\xi}|^s \le q_\nu^{s-1} \Psi_{f,\boldsymbol{\xi}}(q_{\nu-1}).$$

This means that

$$\lim_{T \to q_\nu - 0} T^{s-1} \cdot \Psi_{f,\boldsymbol{\xi}}(T) = q_\nu^{s-1} \Psi_{f,\boldsymbol{\xi}}(q_{\nu-1}) \ge \frac{1}{2^s DK}\,.$$

Theorem 3a is proved. □

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