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Integrability, exact reductions and special solutions of the KP–Whitham equations

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Abstract

Reductions of the KP–Whitham system, namely the (2+1)-dimensional hydrodynamic system of five equations that describes the slow modulations of periodic solutions of the Kadomtsev–Petviashvili (KP) equation, are studied. Specifically, the soliton and harmonic wave limits of the KP–Whitham system are considered, which give rise in each case to a four-component (2+1)-dimensional hydrodynamic system. It is shown that a suitable change of dependent variables splits the resulting four-component systems into two parts: (i) a decoupled, independent two-component system comprised of the dispersionless KP equation, (ii) an auxiliary, two-component system coupled to the mean flow equations, which describes either the evolution of a linear wave or a soliton propagating on top of the mean flow. The integrability of both four-component systems is then demonstrated by applying the Haantjes tensor test as well as the method of hydrodynamic reductions. Various exact reductions of these systems are then presented that correspond to concrete physical scenarios.

Keywords: hydrodynamic systems, integrability, Whitham modulation theory, Kadomtsev–Petviashvili equation

Mathematics Subject Classification numbers: 37K40, 74J25, 74J30.

1. Introduction

The study of infinite-dimensional integrable systems continues to be an essential part of contemporary theoretical physics and applied mathematics, for several reasons. On the one hand,

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these systems are amenable to analytical treatment; they possess a deep mathematical structure and admit a rich family of exact solutions. On the other hand, these systems often arise as model equations in a variety of concrete physical situations. It is often the case that, even when the governing equations in a given physical situation are not integrable, they are close enough to an integrable system that the properties and solutions of the nearby integrable system provide useful insight to study more general scenarios.

The majority of known integrable systems are one-dimensional, or, more precisely, (1+1)-dimensional, meaning systems with one spatial and one temporal dimension. An ongoing theme in the last fifty years has been to extend the theory of integrable systems to multi-dimensional systems, that is, to systems in more than one spatial dimension. Of course multi-dimensional systems are much more challenging than their one-dimensional counterparts [4–6, 19, 31, 41, 51]. As a distinguished example, consider the Kadomtsev–Petviashvili (KP) equation [32], which is arguably the prototypical (2+1)-dimensional integrable system, and which arises as a governing equation, for example, in asymptotic approximations of water waves, plasma physics, cosmology, and condensed matter [5, 31, 32].

While the inverse scattering transform for the Korteweg-de Vries equation (the one-dimensional counterpart of the KP equation) was originally developed in the 1960s, its development to solve the initial value problem for a class of functions broad enough to include soliton solutions required an extensive effort over a period spanning many decades [12–15, 20], and fundamental questions regarding the dynamics of solutions that are not simply localized perturbations of soliton solutions are still by and large unanswered. The KP equation also possesses a much richer family of exact solutions, with a more complex mathematical structure [8, 9, 19, 30, 37, 40] and more complicated physical behavior [7–9, 37, 49], than the KdV equation.

Another ongoing research theme in the last fifty years has been the study of dispersionless and semiclassical limits of dispersive evolution equations, which typically give rise to systems of equations of hydrodynamic type [21]. Here too, the attempt to extend our knowledge of one-dimensional systems to multi-dimensional ones has been an ongoing challenge. For example, with regard to multi-dimensional hydrodynamic systems, considerable effort has been devoted over the years to their classification [22, 25, 53], the development of effective tests for integrability [23, 24], the search for exact solutions [38, 39, 54], the recent development of a generalized inverse scattering transform for vector fields [46], and the study of the behavior of solutions, either numerically [36] or analytically [47, 48].

A useful tool in the study of small dispersion limits is nonlinear modulation theory, developed by G B Whitham in the late 1960s [55, 56]. Indeed, Whitham modulation theory has been applied with enormous success in a wide variety of situations (e.g. see the review article [21] and the references therein). With few exceptions [11, 31, 43], however, the vast majority of the works that use Whitham's approach have concerned effectively one-dimensional systems. A step forward towards applying Whitham modulation theory to multi-dimensional systems was recently presented in [2], where Whitham modulation theory was successfuly generalized to the KP equation, and a system of equations was derived that governs slow variations of the periodic solutions of the KP equation. The authors of [2] referred to these modulation equations as the KP–Whitham system. Importantly, the validity of the approach and the results of [2] were confirmed by comparing the predictions of the KP–Whitham system with direct numerical simulation of solutions of the KP equation, which confirmed that indeed the former correctly captures the dynamics of the latter in the small dispersion limit. The results of [2] were then extended to the two-dimensional Benjamin-Ono equation in [3] and to other (2+1)-dimensional equations of KP type in [1]. Several important questions were

left open in [1–3], however, including fundamental issues about the integrability of the resulting Whitham modulation systems and their solutions. The purpose of this work is to present several new results in this regard.

The rest of this work is organized as follows. We begin in section 2 by reviewing the relation of the one-phase (genus-1) KP–Whitham system to the solutions of the KP equation as well as its harmonic wave (elliptic parameter $m \to 0$) and soliton ($m \to 1$) reductions. We then show that, for both of these special cases, a suitable change of dependent variables decouples the mean flow from the rest of the system, thereby transforming these two special cases of the KP–Whitham system into the dispersionless KP (dKP) system together with two additional evolution equations that describe the soliton dynamics (in the case $m \to 1$) or that of a linear wave packet (in the case of $m \to 0$). In section 3, we study the question of the integrability of the KP–Whitham system by applying both the Haantjes tensor test [24] as well as the method of hydrodynamic reductions [23] to the $m \to 0$ and $m \to 1$ reductions. We show that the $m \to 0$ and $m \to 1$ cases of the KP–Whitham system are both completely integrable in the sense of hydrodynamic reductions. In section 4, we show how the system obtained in the limit $m \to 0$ can be completely integrated by the method characteristics once a solution of the dKP system is given. In section 5, we present several properties and exact reductions of the system obtained in the limit $m \to 1$, corresponding to different physical scenarios. We conclude this work in section 6 with some final remarks.

2. The KP–Whitham system and its harmonic and soliton limits

2.1. The KP equation and the KP–Whitham system

The KP equation in evolution form is the system

$$u_t + uu_x + u_{xxx} + \lambda v_y = 0\,, \tag{2.1a}$$

$$v_x = u_y, \tag{2.1b}$$

where subscripts x, y and t denote partial differentiation, the dependent variables $u(x, y, t)$ and $v(x, y, t)$ are real-valued, and $\lambda = \mp 1$ identifies respectively the KPI and KPII equations. Note that the authors of [2] used the 'IST-friendly' normalization in which the nonlinear term in (2.1) is replaced with $6uu_x$. They also considered the small-dispersion, hydrodynamic scaling in which order-one spatial and temporal scales are assumed slow. This results in a parameter ϵ^2 in front of the dispersive term u_{xxx} where $0 < \epsilon \ll 1$ quantifies the relative strength of dispersive effects compared to nonlinear ones. Here we use the 'physics-friendly' normalization without the coefficient 6 and where order-one length and time scales are considered fast. This is equivalent to a scaling of the modulation variables introduced below.

The exact, elliptic traveling wave solutions of (2.1) are given in terms of the cnoidal-wave expressions

$$u(x, y, t) = r_1 - r_2 + r_3 + 2(r_2 - r_1)\mathrm{cn}^2\left[(K_m/\pi)\theta; m\right], \tag{2.2a}$$

$$v(x, y, t) = qu + p, \tag{2.2b}$$

where r_1, r_2, r_3, q and p are constant parameters, $\mathrm{cn}(\cdot)$ denotes a Jacobian elliptic function, and the rapidly varying phase $\theta(x, y, t)$ is identified (up to an integration constant) by

$$\theta_x = k\,, \qquad \theta_y = l\,, \qquad \theta_t = -\omega\,, \tag{2.3a}$$

with

$$k = \frac{\pi\sqrt{r_3 - r_1}}{\sqrt{6}K_m}, \qquad q = l/k, \qquad \omega = (V + \lambda q^2)k, \tag{2.3b}$$

with the elliptic parameter m and the velocity parameter V respectively given by

$$m = \frac{r_2 - r_1}{r_3 - r_1}, \qquad V = \frac{1}{3}(r_1 + r_2 + r_3), \tag{2.3c}$$

and $K_m = K(m)$ and $E_m = E(m)$ denote, respectively, the complete elliptic integrals of the first and second kind [52].

Note that, in [2], the function θ was taken to have period 1; here we take it to have period 2π instead, which is equivalent to a rescaling of k, l and ω. The reason for the present choice is that, in the harmonic wave limit, k reduces to the usual wavenumber for trigonometric waves.

It was shown in [2], using a multiple scales expansion that slow modulations according to (2.1) of the above cnoidal wave solutions are governed by the KP–Whitham system

$$\frac{\partial r_j}{\partial t} + (V_j + \lambda q^2)\frac{\partial r_j}{\partial x} + 2\lambda q\frac{Dr_j}{Dy} + \lambda\nu_j\frac{Dq}{Dy} + \lambda\frac{Dp}{Dy} = 0, \qquad j = 1, 2, 3, \tag{2.4a}$$

$$\frac{\partial q}{\partial t} + (V_2 + \lambda q^2)\frac{\partial q}{\partial x} + 2\lambda q\frac{Dq}{Dy} + \nu_{4.1}\frac{Dr_1}{Dy} + \nu_{4.3}\frac{Dr_3}{Dy} = 0, \tag{2.4b}$$

$$\frac{\partial p}{\partial x} - (1-\alpha)\frac{Dr_1}{Dy} - \alpha\frac{Dr_3}{Dy} + \nu_5\frac{\partial q}{\partial x} = 0, \tag{2.4c}$$

where the 'convective' derivative is defined as

$$\frac{D}{Dy} = \frac{\partial}{\partial y} - q\frac{\partial}{\partial x}, \tag{2.5}$$

with

$$V_1 = V - b\frac{K_m}{K_m - E_m}, \quad V_2 = V - b\frac{(1-m)K_m}{E_m - (1-m)K_m}, \quad V_3 = V + b\frac{(1-m)K_m}{mE_m}, \tag{2.6a}$$

and V as above, as for the KdV equation, with $b = \frac{2}{3}(r_2 - r_1)$, and where the remaining coefficients are given by

$$\nu_1 = V + \frac{6b}{m}\frac{(1+m)E_m - K_m}{K_m - E_m}, \qquad \nu_2 = V + \frac{6b}{m}\frac{(1-m)^2K_m - (1-2m)E_m}{E_m - (1-m)K_m}, \tag{2.6b}$$

$$\nu_3 = V + \frac{6b}{m}\frac{(2-m)E_m - (1-m)K_m}{E_m}, \qquad \nu_4 = \frac{2mE_m}{E_m - (1-m)K_m}, \tag{2.6c}$$

$$\nu_{4.1} = \frac{1}{6}(4 - \nu_4), \quad \nu_{4.3} = \frac{1}{6}(2 + \nu_4), \qquad \nu_5 = r_1 - r_2 + r_3, \quad \alpha = E_m/K_m. \tag{2.6d}$$

When $p = q = 0$, the resulting system is the (1+1)-dimensional KdV-Whitham system written in diagonal, Riemann invariant form [55]. The reduction of (2.4a) and (2.4b) with $p = 0$

(but without the corresponding constraint (2.4c)) was also derived in [28] using Lagrangian averaging.

2.2. Harmonic limit, soliton limit and decoupling of the mean flow

It was shown in [2] that the KP–Whitham system (2.4) inherits the invariances of the KP equation under space-time translations, scaling transformations, Galilean boosts and pseudo-rotations. Some of these invariances will be useful in this work. It was also shown in [2] that the system (2.4) admits several distinguished limits and exact reductions. In particular, in the 'harmonic limit', that is, the limit $r_2 \to r_1$, we have $m \to 0$ in (2.3c) so that the cnoidal wave (2.2a) limits to a vanishing-amplitude trigonometric or harmonic wave, and the system (2.4) reduces to

$$\frac{\partial r_1}{\partial t} + (2r_1 - r_3 + \lambda q^2)\frac{\partial r_1}{\partial x} + 2\lambda q\frac{Dr_1}{Dy} + \lambda r_3\frac{Dq}{Dy} + \lambda\frac{Dp}{Dy} = 0, \tag{2.7a}$$

$$\frac{\partial r_3}{\partial t} + (r_3 + \lambda q^2)\frac{\partial r_3}{\partial x} + 2\lambda q\frac{Dr_3}{Dy} + \lambda r_3\frac{Dq}{Dy} + \lambda\frac{Dp}{Dy} = 0, \tag{2.7b}$$

$$\frac{\partial q}{\partial t} + (2r_1 - r_3 + \lambda q^2)\frac{\partial q}{\partial x} + 2\lambda q\frac{Dq}{Dy} + \frac{Dr_3}{Dy} = 0, \tag{2.7c}$$

$$\frac{\partial p}{\partial x} - \frac{Dr_3}{Dy} + r_3\frac{\partial q}{\partial x} = 0. \tag{2.7d}$$

Conversely, in the 'soliton limit', that is, the limit $r_2 \to r_3$, we have $m \to 1$ in (2.3c) so that the cnoidal wave (2.2a) limits to a sech^2 profile, and (2.4) reduces to the system

$$\frac{\partial r_1}{\partial t} + (r_1 + \lambda q^2)\frac{\partial r_1}{\partial x} + 2\lambda q\frac{Dr_1}{Dy} + \lambda r_1\frac{Dq}{Dy} + \lambda\frac{Dp}{Dy} = 0, \tag{2.8a}$$

$$\frac{\partial r_3}{\partial t} + \left(\frac{1}{3}(r_1 + 2r_3) + \lambda q^2\right)\frac{\partial r_3}{\partial x} + 2\lambda q\frac{Dr_3}{Dy} + \lambda\frac{4r_3 - r_1}{3}\frac{Dq}{Dy} + \lambda\frac{Dp}{Dy} = 0, \tag{2.8b}$$

$$\frac{\partial q}{\partial t} + \left(\frac{1}{3}(r_1 + 2r_3) + \lambda q^2\right)\frac{\partial q}{\partial x} + 2\lambda q\frac{Dq}{Dy} + \frac{1}{3}\frac{Dr_1}{Dy} + \frac{2}{3}\frac{Dr_3}{Dy} = 0, \tag{2.8c}$$

$$\frac{\partial p}{\partial x} - \frac{Dr_1}{Dy} + r_1\frac{\partial q}{\partial x} = 0. \tag{2.8d}$$

Note that both reductions are one-sided limits, and the restrictions $r_1 \leqslant r_2 \leqslant r_3$ $(0 \leqslant m \leqslant 1)$ are always maintained.

As we show next, both of these systems describe concrete physical scenarios. We first consider the limit $m \to 0$ of the KP–Whitham system. Since $r_2 \to r_1$ in this case, the mean flow is simply given by $\bar{u} = r_3$. Similarly, $\bar{v} = q\bar{u} + p$. Rewriting (2.7b) and (2.7d) in terms of $\bar{u}$ and $\bar{v}$, we obtain the two equations

$$\bar{u}_t + \bar{u}\bar{u}_x + \lambda\bar{v}_y = 0\,, \tag{2.9a}$$

$$\bar{v}_x = \bar{u}_y, \tag{2.9b}$$

which are decoupled from the rest of the system. We then note that, in the limit $m \to 0$, the wavenumber of the cnoidal oscillations in (2.2a) becomes $k = 2\sqrt{r_3 - r_1}/\sqrt{6}$. Rewriting the remaining equations in terms of $\bar{u}$, $\bar{v}$, q and k, we obtain

$$k_t + (\bar{u} - 3k^2 - \lambda q^2)\, k_x + 2\lambda q k_y + k\bar{u}_x = 0\,, \tag{2.9c}$$

$$q_t + (\bar{u} - 3k^2 - \lambda q^2)\, q_x + 2\lambda q q_y + \bar{u}_y - q\bar{u}_x = 0. \tag{2.9d}$$

The system of equations (2.9) is equivalent to the system (2.7).

Note, however, that (2.9a) and (2.9b) are simply the dispersionless KP (dKP) system. Therefore, the change of variables from (r_1, r_3, q, p) to $(\bar{u}, \bar{v}, k, q)$ is convenient not only because it decouples two of the equations from the rest of the system, but also because it clarifies the physical meaning of the system: equations (2.9a) and (2.9b) govern the dynamics of the mean flow, whereas (2.9c) and (2.9d) govern the dynamics of harmonic waves propagating on this mean flow. Note also that the homogeneous system associated to the forced system of equations (2.9c) and (2.9d) is in diagonal form. The decoupling of the mean flow from the remaining modulation variables in the soliton and harmonic limits is a general property of (1+1)-dimensional Whitham modulation systems [21]. Here, we have shown that this property persists for the (2+1)-dimensional KP–Whitham modulation system.

To further elucidate the physical meaning of equations (2.9c) and (2.9d), it is useful to go back to the derivation of the original KP–Whitham system (2.4). Recall that, when the KP–Whitham system is used to describe solutions of the KP equation, we have $\theta_{xy} = \theta_{yx}$. Hence, (2.9c) and (2.9d) are subject to the compatibility condition

$$k_y = (kq)_x. \tag{2.10a}$$

Combining (2.10a) with (2.9c) and (2.9d), we find that the latter two equations are equivalent to wave conservation (that is, to the conditions $\theta_{xt} = \theta_{tx}$ and $\theta_{yt} = \theta_{ty}$):

$$k_t + \omega_x = 0, \tag{2.10b}$$

$$(kq)_t + \omega_y = 0, \tag{2.10c}$$

where $\omega = (\bar{u} + \lambda q^2)k - k^3$ is the KP linear dispersion relation. Hence, (2.9c) and (2.9d) are simply equivalent to wave conservation for modulated linear waves.

A similar result can be obtained in the soliton limit of the KP–Whitham system, i.e. the limit $r_2 \to r_3$. In this case, $\bar{u} = r_1$, and similar operations as above on (2.8a) and (2.8d) yield

$$\bar{u}_t + \bar{u}\bar{u}_x + \lambda\bar{v}_y = 0, \tag{2.11a}$$

$$\bar{u}_y - \bar{v}_x = 0, \tag{2.11b}$$

$$a_t + \left(\bar{u} + \frac{1}{3}a - \lambda q^2\right) a_x + \lambda\left(2qa_y + \frac{4}{3}a(q_y - qq_x)\right) + \frac{2}{3}a\bar{u}_x = 0, \tag{2.11c}$$

$$q_t + \left(\bar{u} + \frac{1}{3}a - \lambda q^2\right) q_x + 2\lambda q q_y + \frac{1}{3}(a_y - qa_x) + \bar{u}_y - q\bar{u}_x = 0. \tag{2.11d}$$

In this case, the variable $a = 2(r_3 - r_1)$ describes the soliton amplitude, while q defines its inclination in the xy-plane according to the limit $r_2 \to r_3$ of equation (2.2a)

$$u(x, y, t) = \bar{u} + a\,\mathrm{sech}^2\left(\sqrt{\frac{a}{12}}(x + qy - ct)\right), \quad c = \lim_{k\to 0}\frac{\omega}{k} = \bar{u} + \frac{1}{3}a + \lambda q^2. \tag{2.12}$$

Thus, similarly to the harmonic limit, the first two equations, which are decoupled, describe the dynamics of the mean flow, governed by the dKP equation, while (2.11c) and (2.11d) describe the effect of the mean field on the soliton's properties.

Note that homogeneous equations associated to equations (2.11c) and (2.11d) (i.e. the reductions of those equations when $\bar{u}$ and $\bar{v}$ are constant) are not in diagonal form, unlike (2.9c) and (2.9d). On the other hand, in section 5 we will show that it is also possible to write (2.11c) and (2.11d) in diagonal form.

The systems (2.9) and (2.11) are the two-dimensional generalizations of the corresponding systems for the KdV equation that were recently studied theoretically in [17, 45] and also experimentally in [45]. The rest of this work is devoted to the study of the properties and solutions of the hydrodynamic systems of equations (2.9) and (2.11).

3. Integrability of the reduced KP–Whitham systems of equations

An important open question for the KP–Whitham system (2.4) concerns its possible integrability. On the one hand, as an exact asymptotic reduction of the KP equation, which is a completely integrable system, we expect it to also be completely integrable. It was stated in [2] that the full KP–Whitham system (2.4) fails the Haantjes test for integrability proposed in [24]. On the other hand, the formulation of the Haantjies test given in [24] does not directly apply to the KP–Whitham system and its reductions, as we discuss next.

3.1. Integrability test via the Haantjes tensor

In 2006, Ferapontov and Khusnutdinova, based on the observations of Gibbons and Tsarev in the context of the Benney moment equations [27], identified a necessary condition for the integrability of a (2+1)-dimensional system of hydrodynamic type [24]. A necessary condition for the integrability of an N-component (2+1)-dimensional quasi-linear system of equations in the form

$$\mathbf{u}_T + A(\mathbf{u})\,\mathbf{u}_X + B(\mathbf{u})\,\mathbf{u}_Y = \mathbf{0} \tag{3.1}$$

is the integrability of all of its (1+1)-dimensional hydrodynamic reductions. Based on a classical result by Haantjes [29], the (1+1)-dimensional strictly hyperbolic hydrodynamic type system

$$\mathbf{u}_\tau + M(\mathbf{u})\,\mathbf{u}_\xi = 0 \tag{3.2}$$

is diagonalizable if and only if the Haantjes tensor of the matrix M is identically zero. Recall that the Haantjes tensor $H(\mathbf{u})$ of the matrix $M(\mathbf{u}) = (M^i_j)_{i,j=1,\dots,N}$ is given by

$$H^i_{jk} = \sum_{p,q=1}^{N}\left(N^i_{pq}M^p_jM^q_k - N^p_{js}M^i_pM^s_k - N^p_{sk}M^i_pM^s_j + N^p_{jk}M^i_sM^s_p\right), \tag{3.3}$$

where $N(\mathbf{u})$ is the Nijenhuis tensor of $M(\mathbf{u})$, namely,

$$N^i_{jk} = \sum_{p=1}^{N} \left[M^p_j \frac{\partial M^i_k}{\partial u^p} - M^p_i \frac{\partial M^i_j}{\partial u^p} - M^i_p \left(\frac{\partial M^p_k}{\partial u^j} - \frac{\partial M^p_j}{\partial u^k} \right) \right]. \tag{3.4}$$

It was then shown in [24] that, as a result, a necessary condition for integrability of the system (3.1) is the vanishing of the Haantjes tensor of the matrix

$$M = (\lambda I + A)^{-1}(\mu I + B) \tag{3.5}$$

for arbitrary values of λ and μ, where I is the $N \times N$ identity matrix.

We emphasize, however, that the Haantjes tensor test cannot be directly applied to the KP–Whitham system (2.4) or to its reductions (2.9) and (2.11), because none of these systems are in evolutionary form with respect to the variable t. This is because no temporal derivative is present in neither (2.9b) nor (2.11b). In other words, introducing $\mathbf{u} = (\bar{u}, a, q, \bar{v})^T$, (2.9) and (2.11) yield systems of the form

$$\tilde{I}_4 \mathbf{u}_t + \tilde{A}(\mathbf{u})\, \mathbf{u}_x + \tilde{B}(\mathbf{u})\, \mathbf{u}_y = \mathbf{0}, \tag{3.6}$$

where $\tilde{I}_4 = \mathrm{diag}(1, 1, 1, 0)$. Thus, we cannot simply identify (X, Y, T) in (3.1) with (x, y, t). A similar issue arises with the full KP–Whitham system (2.4). Accordingly, the question of whether the full KP–Whitham system is integrable is still open. Such a question is outside the scope of this work, and we therefore leave it for future study.

Instead, in this work we limit ourselves to the study of the four-component reductions (2.9) and (2.11). One can obviate the above problems for the four-component reductions (2.9) and (2.11) by writing each of them as an evolutionary system with respect to either of the variables x or y. In the first case, we identify (X, Y, T) in (3.1) with (t, y, x) in (3.6), respectively, obtaining $A = \tilde{A}^{-1}\tilde{I}_4$ and $B = \tilde{A}^{-1}\tilde{B}$. In the second case, we identify (X, Y, T) with (t, x, y), respectively, obtaining $A = \tilde{B}^{-1}\tilde{I}_4$ and $B = \tilde{B}^{-1}\tilde{A}$. We find that the Haantjes tensor of the matrix M associated to either of these systems via (3.5) is identically zero, which suggests that these systems might be completely integrable. (The relevant calculations, which were performed with Mathematica, are relatively straightforward but rather tedious, and for this reason are omitted here.)

On the other hand, the Haantjes test is only a necessary condition for integrability. Next, we discuss a more conclusive test.

3.2. *Integrability test via hydrodynamic reductions*

A (2+1)-dimensional quasilinear system is said to be 'integrable' if it admits infinitely many reductions into a pair of compatible N-component one-dimensional systems in Riemann invariants [23]. Exact solutions described by these reductions, known as nonlinear interactions of planar simple waves, can be viewed as a natural dispersionless analogue of N-gap, quasi-periodic solutions.

In this section, we focus on the case corresponding to the soliton limit ($m \to 1$) of the KP–Whitham system, namely the system (2.11), in the case $\lambda = 1$ (i.e. for the KPII equation). The harmonic limit ($m \to 0$) is considered separately in the next section and, since the mean flow equations are integrable and the remaining equations can be solved in full generality, no integrability test is required. We look for N-component reductions, i.e. solutions in the form of

arbitrary functions of N variables $R^1, \ldots, R^N$:

$$\bar{u} = \bar{u}(R^1, \ldots, R^N), \qquad \bar{v} = \bar{v}(R^1, \ldots, R^N), \tag{3.7a}$$

$$a = a(R^1, \ldots, R^N), \qquad q = q(R^1, \ldots, R^N), \tag{3.7b}$$

where

$$R^i_y = \lambda^i(R^1, \ldots, R^N) R^i_x, \tag{3.8a}$$

$$R^i_t = \mu^i(R^1, \ldots, R^N) R^i_x \tag{3.8b}$$

(with no sum on repeated indices), subject to the compatibility conditions

$$\frac{\lambda^i_j}{\lambda^i - \lambda^j} = \frac{\mu^i_j}{\mu^i - \mu^j}, \tag{3.9}$$

where for brevity we used the notation $\lambda^i_j = \partial\lambda^i/\partial R^j$ and similarly for μ^i_j as well as $\bar{u}_j$ and $\bar{v}_j$. Substituting into (2.11a) and (2.11b), we obtain

$$\bar{v}_i = \lambda^i \bar{u}_i, \qquad \mu^i = -\bar{u} - (\lambda^i)^2. \tag{3.10}$$

The compatibility condition of the system (3.10) leads to the Gibbons–Tsarev system for the dKP equation [27], namely,

$$\bar{u}_{ij} = \frac{2\bar{u}_i \bar{u}_j}{(\lambda^j - \lambda^i)^2}, \tag{3.11a}$$

$$\lambda^i_j = \frac{\bar{u}_j}{\lambda^j - \lambda^i}, \tag{3.11b}$$

which is automatically in involution. Using the assumptions (3.7a) and (3.7b) in (2.11c) and (2.11d), we obtain

$$a_i = -\frac{2a\bar{u}_i}{a - q^2 + 2q\lambda^i - (\lambda^i)^2}, \tag{3.12a}$$

$$q_i = -\frac{(\lambda_i - q)\bar{u}_i}{a - q^2 + 2q\lambda^i - (\lambda^i)^2}. \tag{3.12b}$$

We verify by direct calculation that the compatibility conditions

$$a_{ij} = a_{ji}, \qquad q_{ij} = q_{ji} \tag{3.13}$$

are identically satisfied modulo all equations above. (As before, the relevant calculations were performed with Mathematica; they are relatively straightforward but rather tedious, and are omitted here for brevity.)

Therefore, the system (2.11) is integrable in the sense of hydrodynamic reductions. Moreover, the system (3.12) of ordinary differential equations (ODEs) allows us to obtain the solutions for a and q associated to any N-component reduction of the dKP equation.

4. General solution of the harmonic wave limit of the KP–Whitham system

We could perform similar calculations as in section 3 for the harmonic limit ($m \to 0$) of the KP–Whitham system. In this case, however, we next prove that the system is integrable by integrating it exactly. For convenience, let us rewrite the Whitham system in the harmonic wave limit (2.9):

$$\bar{u}_t + \bar{u}\bar{u}_x + \lambda \bar{v}_y = 0 \tag{4.1a}$$

$$\bar{v}_x = \bar{u}_y \tag{4.1b}$$

$$k_t + (\bar{u} - 3k^2 - \lambda q^2)\, k_x + 2\lambda q k_y + k\bar{u}_x = 0 \tag{4.1c}$$

$$q_t + (\bar{u} - 3k^2 - \lambda q^2)\, q_x + 2\lambda q q_y + \bar{u}_y - q\bar{u}_x = 0. \tag{4.1d}$$

Introducing the characteristic curves defined via the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \bar{u} - 3k^2 - \lambda q^2\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 2\lambda q, \tag{4.2}$$

we have that the functions $k(x(t), y(t), t)$ and $q(x(t), y(t), t)$ evaluated along the characteristics satisfy the ODEs

$$\frac{\mathrm{d}k}{\mathrm{d}t} = -k\bar{u}_x\,, \qquad \frac{\mathrm{d}q}{\mathrm{d}t} = -\bar{u}_y + q\bar{u}_x. \tag{4.3}$$

Hence, given any solution $(\bar{u}, \bar{v})$ to the integrable dKP equations (4.1a) and (4.1b), the general solution is obtained by direct integration of the equations (4.3) along the characteristics. There is a large class of known reductions and explicit solutions of the dKP system [33, 38, 39, 42], which could be used to obtain the corresponding solutions of the associated system (4.2).

As an illustrative example, let us consider a 'background' represented by the following solution to the dKP equation

$$\bar{u} = \alpha y\,, \qquad \bar{v} = \alpha x, \tag{4.4}$$

where α is a constant.

In this case, the characteristic system given by equations (4.2) and (4.3) can be integrated explicitly, providing the solution in parametric form

$$x(t) = -\frac{2}{3}\lambda\alpha^2 t^3 + 2\alpha\lambda q_0 t^2 + (\alpha y_0 - \lambda q_0^2 - 3k_0^2)t + x_0, \tag{4.5a}$$

$$y(t) = -\lambda\alpha t^2 + 2\lambda q_0 t + y_0, \tag{4.5b}$$

$$k(x(t), y(t), t) = f(x_0, y_0)\,, \tag{4.5c}$$

$$q(x(t), y(t), t) = g(x_0, y_0) - \alpha t, \tag{4.5d}$$

with initial conditions $x(0) = x_0$, $y(0) = y_0$ and

$$k(x_0, y_0, 0) = f(x_0, y_0)\,, \quad q(x_0, y_0, 0) = g(x_0, y_0), \tag{4.6}$$

and where, for brevity in the above formulae, we use the shorthand notation

$$k_0 = k_0(x_0, y_0)\,, \qquad q_0 = q_0(x_0, y_0). \tag{4.7}$$

Note that initial conditions must be taken in the class specified by the compatibility condition (2.10a) and therefore $f_y = (gf)_x$.

If desired, we could now solve (4.5a) and (4.5b) to obtain x_0 and y_0 explicitly as functions of (x, y, t) and then substitute the resulting expressions in (4.5c) and (4.5d) to obtain explicit expressions for $k(x, y, t)$ and $q(x, y, t)$.

It is important to realize that the key observations that allow us to find the general solution of the system (4.1) are that: (i) (4.1a) and (4.1b) are decoupled from (4.1c) and (4.1d), and (ii) the characteristic speeds in (4.1c) and those in (4.1d) coincide. The first of these properties is also common to the KP–Whitham system in the soliton limit (2.11). The second, however, is not, and this precludes the possibility of solving (2.11) in the same way.

5. Diagonalization and exact reductions of the soliton limit of the KP–Whitham system

As discussed above, it is not possible to find the general solution of the system in the soliton limit as in section 4. Nonetheless the $m \to 1$ system (2.11) admits various exact reductions that describe interesting physical scenarios. After studying the characteristic speeds, we discuss these reductions.

5.1. Hyperbolicity and characteristic speeds

The complete modulation system in the soliton limit is given by (2.11a)–(2.11d). As we mentioned before, this system, as well as the subsystem given by the dKP system (2.11a) and (2.11b), is not evolutionary with respect to the physical time t [16, 35], but they can be written in evolutionary form, for instance with respect to the variable y. Because dKP decouples from the soliton's evolution, it will be useful to consider the soliton evolution equations (2.11c) and (2.11d) as an inhomogeneous quasi-linear system for $\mathbf{u} = (a, q)^{\mathrm{T}}$

$$\mathbf{u}_t + A(\mathbf{u})\mathbf{u}_x + B(\mathbf{u})\mathbf{u}_y + \mathbf{b} = 0, \tag{5.1a}$$

with

$$A(\mathbf{u}) = \begin{pmatrix} \bar{u} + \frac{1}{3}a - \lambda q^2 & -\frac{4}{3}\lambda a q \\ -\frac{1}{3}q & \bar{u} + \frac{1}{3}a - \lambda q^2 \end{pmatrix}, \quad B(\mathbf{u}) = \begin{pmatrix} 2\lambda q & \frac{4}{3}\lambda a \\ \frac{1}{3} & 2\lambda q \end{pmatrix}, \tag{5.1b}$$

and inhomogeneity $\mathbf{b} = \left(\frac{2}{3}a\bar{u}_x, \bar{u}_y - q\bar{u}_x\right)^{\mathrm{T}}$ due to the independent mean flow $\bar{u}$ satisfying dKP (2.11a) and (2.11b). The variable t is a timelike variable for the homogeneous part of (5.1a) and we now analyze the characteristic speeds of this system.

We study plane wave solutions $\mathbf{v}\mathrm{e}^{\mathrm{i}(Kx+\Lambda y-\Omega t)}$ to (5.1a) linearized about the constant $\mathbf{u}$ and frozen mean flow $\bar{u}$, which yields the generalized eigenvalue problem

$$(KA(\mathbf{u}) + \Lambda B(\mathbf{u}) - \Omega I_2)\,\mathbf{v} = 0, \tag{5.2}$$

where I_2 is the 2×2 identity matrix. The roots of the characteristic polynomial are

$$\Omega_\pm = K\left(\bar{u} + \frac{1}{3}a - \lambda q^2\right) + 2\Lambda\lambda q \pm \frac{2}{3}\,|\Lambda - Kq|\,\sqrt{\lambda a} \tag{5.3}$$

and the associated eigenvectors are

$$\mathbf{v}_\pm = \begin{pmatrix} \pm 2\,|\Lambda - Kq|\,\sqrt{\lambda a} \\ \Lambda - Kq \end{pmatrix}. \tag{5.4}$$

Therefore, the homogeneous part of the modulation system (5.1a) is strictly hyperbolic if and only if $\lambda > 0$ and $\Lambda \neq Kq$. When $\lambda < 0$, the modulation system is elliptic, which is the modulation theory manifestation of the well-known fact that line soliton solutions to KPI exhibit a transverse instability [32]. From now onward, we will only consider the hyperbolic case $\lambda = 1$.

We now turn our attention to the characteristic speeds. Fixing the propagation direction $\mathbf{n} = (\cos\theta, \sin\theta)^{\mathrm{T}}$, the characteristic speeds $\lambda_\pm$ are determined from (5.3) with the identification $K = \cos\theta$, $\Lambda = \sin\theta$, and $\lambda_\pm = \Omega_\pm$ so that

$$\lambda_\pm = \left(\bar{u} + \frac{1}{3}a - q^2\right)\cos\theta + 2q\,\sin\theta \pm \frac{2}{3}\sqrt{a}\,|\sin\theta - q\,\cos\theta|\,. \tag{5.5}$$

We observe that $\lambda_+ = \lambda_-$ if and only if $q = \tan\theta$, i.e. the soliton propagation direction coincides with the characteristic direction. Thus, the homogeneous part of (5.1a) is strictly hyperbolic if and only if it has real characteristic speeds ($\lambda > 0$), $q \neq \tan\theta$ and $a \neq 0$. We compute

$$\nabla_{(a,q)}\lambda_\pm \cdot \mathbf{v}_\pm = \frac{8}{3}\,(\sin\theta - q\,\cos\theta)\,, \tag{5.6}$$

so that the soliton modulation system (5.1a) is both strictly hyperbolic and genuinely nonlinear if and only if $\lambda > 0$, $q \neq \tan\theta$ and $a \neq 0$.

5.2. Diagonalization of the homogeneous soliton KP–Whitham system

As mentioned earlier, the homogeneous system associated to the KP–Whitham subsystem (2.11c) and (2.11d) in the soliton limit is not in diagonal form, unlike that of the KP–Whitham subsystem (2.9c) and (2.9d) in the harmonic limit. On the other hand, this structural discrepancy can be resolved by performing the change of dependent variables $(a, q) \mapsto (w_1, w_2)$, with

$$a = \frac{1}{4}(w_1 - w_2)^2\,, \qquad q = \frac{1}{2}(w_1 + w_2), \tag{5.7}$$

which is inverted by

$$w_1 = q - \sqrt{a}\,, \qquad w_2 = q + \sqrt{a}. \tag{5.8}$$

This way, (2.11c) and (2.11d) take the following diagonal form

$$\frac{\partial w_j}{\partial t} + \mathbf{v}_j \cdot \nabla w_j + \bar{u}_y - b_j\bar{u}_x = 0\,, \qquad j = 1, 2, \tag{5.9}$$

where

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right), \quad \mathbf{v}_1 = (\bar{u} - w_1 b_2, 2b_1)^{\mathrm{T}}, \quad \mathbf{v}_2 = (\bar{u} - w_2 b_1, 2b_2)^{\mathrm{T}},$$
$$b_1 = \frac{1}{3}(2w_1 + w_2)\,, \quad b_2 = \frac{1}{3}(w_1 + 2w_2). \tag{5.10}$$

In these new variables, we identify the characteristic speeds $\lambda_{1,2} = \mathbf{v}_{1,2} \cdot (\cos\theta, \sin\theta)$.

We remark that the matrices A and B in (5.1a) and (5.1b) have the same eigenvectors and commute. This is a property of multidimensional hyperbolic systems that has been leveraged to obtain uniqueness [26] and stability [18] results. Despite having been studied mathematically, this is apparently the first example of a physically derived hyperbolic system in multiple dimensions in which the coefficient matrices exhibit this property [26].

5.3. Constant mean flow

Next we consider the $m \to 1$ system (2.11) in the case when the mean flow is constant, i.e. $\bar{u} = \text{const}$ and $\bar{v} = \text{const}$. Without loss of generality, we can then set $\bar{u} = \bar{v} = 0$ thanks to the invariances of the KP equation and of the KP–Whitham system. The system then reduces to the two-component (2+1)-dimensional hydrodynamic system of equation (5.1) with zero inhomogeneity $\mathbf{b} = 0$

$$\mathbf{u}_t + A(\mathbf{u})\mathbf{u}_x + B(\mathbf{u})\mathbf{u}_y = 0, \tag{5.11a}$$

with

$$A(\mathbf{u}) = \begin{pmatrix} \frac{1}{3}a - q^2 & -\frac{4}{3}aq \\ -\frac{1}{3}q & \frac{1}{3}a - q^2 \end{pmatrix}, \quad B(\mathbf{u}) = \begin{pmatrix} 2q & \frac{4}{3}a \\ \frac{1}{3} & 2q \end{pmatrix}. \tag{5.11b}$$

This system belongs to the class studied in [22] and can be written in the diagonal form (cf. equation (5.9))

$$\frac{\partial w_j}{\partial t} + \mathbf{V}_j \cdot \nabla w_j = 0, \qquad j = 1, 2, \tag{5.12a}$$

where $w_1 = q - \sqrt{a}$ and $w_2 = q + \sqrt{a}$ as in (5.8) and

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right), \quad \mathbf{V}_1 = (-w_1 b_2, 2b_1)^T, \quad \mathbf{V}_2 = (-w_2 b_1, 2b_2)^T,$$
$$b_1 = \frac{1}{3}(2w_1 + w_2), \quad b_2 = \frac{1}{3}(w_1 + 2w_2). \tag{5.12b}$$

This system inherits all of the properties (hyperbolicity, etc) discussed in section 5.1 for the homogeneous part of (5.1).

5.4. One-dimensional fields

Let us consider the modulation equation (2.11) in the case $m \to 1$ when no y dependence is present in the problem. In this case, the dependent variable $\bar{v}$ is constant and drops out of the system. Equation (2.11) then become

$$\mathbf{u}_t + A\,\mathbf{u}_x = 0, \tag{5.13a}$$

where

$$\mathbf{u}(x,t) = (\bar{u}, a, q)^{\mathrm{T}}, \qquad A = \begin{pmatrix} \bar{u} & 0 & 0 \\ \frac{2}{3}a & \bar{u} + \frac{a}{3} - q^2 & -\frac{4}{3}aq \\ -q & -\frac{1}{3}q & \bar{u} + \frac{a}{3} - q^2 \end{pmatrix}. \tag{5.13b}$$

It is important to realize that, even though the spatial dependence in (5.13) is one-dimensional, the geometry of the resulting solutions for the KP–equation is still two-dimensional. This is because the variable q describes the slope of the soliton in the xy-plane. Thus, solutions for which q is not constant describe changes in the soliton slope as a function of x.

In the previous two sections, we analyzed the quasi-linear soliton modulation equations in isolation from dKP evolution because dKP is a nonlocal equation. The full modulation system is degenerate as a hyperbolic system in time. Here, we can consider the full soliton-mean flow modulation system because the dKP equation reduces to the inviscid Burgers equation. The benefits of considering the full soliton-mean flow modulation system, despite Burgers equation being decoupled, have been recently identified [45].

Since the original system (2.11) is completely integrable, the reduced system (5.13) is too, and can therefore be diagonalized. The eigenvalues of the coefficient matrix in (5.13b) are given by $\mathbf{\Lambda} = (\lambda_{\bar{u}}, \lambda_+, \lambda_-)$, with

$$\lambda_{\bar{u}} = \bar{u}, \quad \lambda_+ = \bar{u} + \frac{a}{3} - q^2 - \frac{2}{3}\sqrt{aq^2}, \quad \lambda_- = \bar{u} + \frac{a}{3} - q^2 + \frac{2}{3}\sqrt{aq^2}. \tag{5.14}$$

In what follows, we assume that $q \geqslant 0$. The ordering of the characteristic velocities is determined by the relative magnitudes of a and q. In particular, the following special cases arise:

(a) $q^2 = a/9$, implying $\lambda_{\bar{u}} = \lambda_+$,
(b) $q = 0$, implying $\lambda_+ = \lambda_-$,
(c) $q^2 = a$, implying $\lambda_{\bar{u}} = \lambda_-$.

Thus, the system (5.13) is non-strictly hyperbolic. The left eigenvectors associated to each eigenvalue of A are

$$\mathbf{v}_{\bar{u}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_+ = \begin{pmatrix} \frac{2\sqrt{a}}{\sqrt{a}+q} \\ 1 \\ 2\sqrt{a} \end{pmatrix}, \quad \mathbf{v}_- = \begin{pmatrix} \frac{2\sqrt{a}}{\sqrt{a}-q} \\ 1 \\ -2\sqrt{a} \end{pmatrix}. \tag{5.15}$$

We therefore recognize $R_{\bar{u}} = \bar{u}$ as one Riemann invariant for the system (5.13). Taking the dot product of (5.13) with $\mathbf{v}_+$ yields the characteristic form

$$\mathrm{d}\bar{u} + \frac{1}{2}\,\mathrm{d}\big(q + \sqrt{a}\big)^2 = 0, \tag{5.16}$$

which can be integrated to give another Riemann invariant

$$R_+ = \bar{u} + \frac{1}{2}(q + \sqrt{a})^2. \tag{5.17}$$

Integrating the dot product of (5.13) with $\mathbf{v}_-$ gives the third Riemann invariant

$$R_- = \bar{u} + \frac{1}{2}(q - \sqrt{a})^2. \tag{5.18}$$

The Riemann invariants $R_\pm$ for this soliton mean-flow modulation system generalize the diagonalizing transformation (5.8) according to $R_\pm = \bar{u} + \frac{1}{2}w_{1,2}^2$ that was utilized for the isolated soliton modulation system (5.1a).

Note that we have the ordering

$$R_{\bar{u}} \leqslant R_- \leqslant R_+. \tag{5.19}$$

Moreover, (5.19) implies the following degenerate cases:

(a) $\sqrt{a} = \pm q$ if and only if $R_{\bar{u}} = R_\mp$,
(b) $q = 0$ if and only if $R_+ = R_-$.

Summarizing, and denoting $(R_1, R_2, R_3) = (R_{\bar{u}}, R_+, R_-)$, the system (5.13) can be written in Riemann invariant form as

$$\frac{\partial R_j}{\partial t} + \lambda_j \frac{\partial R_j}{\partial x} = 0, \qquad j = 1, 2, 3, \tag{5.20}$$

where

$$\lambda_1 = R_1, \tag{5.21a}$$

$$\lambda_2 = \frac{5}{3}R_1 - \frac{2}{3}\left(R_2 + 2\sigma\sqrt{(R_2 - R_1)(R_3 - R_1)}\right), \tag{5.21b}$$

$$\lambda_3 = \frac{5}{3}R_1 - \frac{2}{3}\left(R_3 + 2\sigma\sqrt{(R_2 - R_1)(R_3 - R_1)}\right), \tag{5.21c}$$

with $\sigma = \operatorname{sgn}(q^2 - a)$. Note that the mapping from the Riemann invariants R_1, R_2, R_3 to the modulation parameters u, a, and q is multivalued:

$$\bar{u} = R_1, \tag{5.22a}$$

$$a = -R_1 + \frac{1}{2}\left(R_2 + R_3 \pm 2\sqrt{(R_2 - R_1)(R_3 - R_1)}\right), \tag{5.22b}$$

$$q^2 = -R_1 + \frac{1}{2}\left(R_2 + R_3 \mp 2\sqrt{(R_2 - R_1)(R_3 - R_1)}\right). \tag{5.22c}$$

The branching occurs precisely when $a = q^2$ or $R_1 = R_3$. Note also how, once more, the dynamics of the mean flow (given by $\bar{u} = R_1$) is decoupled from the rest of the system.

In the regime of strict hyperbolicity, the system (5.13) is genuinely nonlinear because $\partial\lambda_j/\partial R_j \neq 0, j = 1, 2, 3$ if and only if $q^2 \notin \left\{0, \frac{1}{9}a, a\right\}$.

Finally, we note that the system (2.11) also admits a self-consistent reduction when all fields are independent of x. In this case, both $\bar{u}$ and $\bar{v}$ drop out of the system, which then reduces to the one-dimensional 2-component system for a and q

$$\begin{aligned} a_t + 2qa_y + \frac{4}{3}aq_y &= 0, \\ q_t + 2qq_y + \frac{1}{3}a_y &= 0, \end{aligned} \tag{5.23}$$

which is equivalent to the diagonalizable (1+1)-dimensional isentropic dynamics of a monoatomic gas with the ratio of specific heats $\gamma = 5/3$ after the velocity and density transformation $U = 2q$, $\rho = \frac{4}{27}a^{3/2}$, respectively

$$\begin{aligned} \rho_t + (\rho U)_y &= 0, \\ U_t + UU_y + \frac{1}{\rho}P(\rho)_y &= 0, \end{aligned} \tag{5.24}$$

with pressure law $P(\rho) = \frac{3}{5}\rho^{5/3}$. The modulation system (5.23) had been previously obtained in [44, 50] by an alternative approach using soliton perturbation theory.

6. Final remarks

In conclusion, we studied the $m \to 0$ and $m \to 1$ limits of the KP–Whitham system of equations. We showed that both of the resulting systems are completely integrable, and we discussed some of their exact reductions and some exact solutions. Both of the systems (2.9) and (2.11) are novel to the best of our knowledge. Indeed, the one-dimensional reduction (5.13) and the two-component reduction (5.11) are also novel to the best of our knowledge.

Recall that the first half of the systems (2.9) and (2.11) is comprised of the dKP equation. In section 4, we showed that the solution of the systems (2.9) can be obtained once $\bar{u}$ and $\bar{v}$ are given. Importantly, however, it does not appear to be possible in general to find solutions of the system (2.11) by expressing a and q simply in terms of $\bar{u}$ and $\bar{v}$. This suggests that the system of equation (2.11) is not simply a trivial extension of the dKP equation, but is instead a novel integrable system in its own right. Indeed, as we showed in section 5, even in the case when $\bar{u} = \bar{v} = 0$, it is necessary to solve a nontrivial 2-component (2+1)-dimensional system of hydrodynamic equations to obtain a and q.

The results of this work open up a number of interesting questions. What is the Lax pair of the (2+1)-dimensional hydrodynamic systems (2.9), (2.11) and (5.11)? A related question is whether one could use the recent generalization of the inverse scattering transform for vector fields developed by Manakov and Santini [46–48] to solve the initial value problem for these hydrodynamic systems. Another interesting question is whether there exist further exact reductions of the systems (2.9) and (2.11) and whether further exact solutions can be obtained, perhaps using similar methods as in [54]. Finally, an important unresolved question concerns the possible integrability of the full KP–Whitham system (2.4).

From a more practical point of view, an important avenue for further work is the use of the KP–Whitham system to study physically interesting scenarios. Even without using the full system, the reductions studied in this work appear to be very promising in this respect. For example, the soliton limit (2.11) can allow researchers to study the evolution of initial conditions for the KP equation that are not purely solitonic in nature, a problem that appears to still be beyond the capabilities of the state-of-the-art inverse scattering transform, and which, apart from some numerical attempts [10, 34], has therefore remained completely open so far. Indeed, note that even the one-dimensional reduction (5.13) can be very useful in this regard. For example, piecewise constant initial conditions in (5.13) can be used to study the evolution of scenarios in which an initial line soliton with a certain amplitude and slope, located in the first quadrant of the xy-plane, is connected at the origin to a soliton with a different amplitude and slope, located in the third quadrant of the xy-plane. More complicated scenarios that cannot be represented with the one-dimensional reduction can then be studied using the two-dimensional reduction (5.11).

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