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Affine representability results in $\mathbb{A}^1$-homotopy theory III: Finite fields and complements

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Abstract

We give a streamlined proof of $\mathbb{A}^1$-representability for G-torsors under "isotropic" reductive groups, extending previous results in this sequence of papers to finite fields. We then analyze a collection of group homomorphisms that yield fiber sequences in $\mathbb{A}^1$-homotopy theory, and identify the final examples of motivic spheres that arise as homogeneous spaces for reductive groups.

1. Introduction and statement of results

Suppose that k is a field. We study torsors under algebraic groups considered in the following definition.

Definition 1.1. If G is a reductive algebraic k-group scheme, we will say that G is "isotropic" if each of the almost k-simple components of the derived group of G contains a k-subgroup scheme isomorphic to $\mathbf{G}_m$.

Remark 1.2. In the above definition, the word "isotropic" is in quotes to distinguish the notion defined above from the standard definition of isotropic for reductive groups, which simply requires the existence of a k-subgroup scheme isomorphic to $\mathbf{G}_m$; see [Gil10, Définition 9.1.1]. Let us spell out what term "isotropic" as defined above means. The derived group of $\mathrm{G}^{\mathrm{der}} := [\mathrm{G}, \mathrm{G}]$ is a semi-simple k-subgroup scheme. As such it has a simply connected covering group G^{sc}, which is itself a product of almost k-simple factors. The condition above implies that each such factor is isotropic in the standard sense.

Write $\mathscr{H}(k)$ for the (unstable) Morel–Voevodsky $\mathbb{A}^1$-homotopy category [MV99]. Write $B\,\mathrm{G}$ for the usual bar construction of G (which can be thought of as a simplicial presheaf on the category of smooth k-schemes). If X is a smooth k-scheme, then write $[X, B\,\mathrm{G}]_{\mathbb{A}^1}$ for the set $\mathrm{Hom}_{\mathscr{H}(k)}(X, B\,\mathrm{G})$. The main goal of this paper is to establish the following representability result about Nisnevich-locally trivial G-torsors.

Theorem 1.3. *Suppose that k is a field and G is an "isotropic" reductive k-group. For every smooth affine k-scheme X, there is a bijection*

$$\mathrm{H}^1_{\mathrm{Nis}}(X, \mathrm{G}) \cong [X, B\,\mathrm{G}]_{\mathbb{A}^1}$$

that is functorial in X.

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In [AHW18, Theorem 4.1.3], Theorem 1.3 was proved under the more restrictive assumption that k is infinite. By [AHW18, Theorem 2.3.5], in order to establish Theorem 1.3, it suffices to prove that the functor $X \mapsto \mathrm{H}^1_{\mathrm{Nis}}(X, \mathrm{G})$ is $\mathbb{A}^1$-invariant on smooth affine schemes; that is, for every smooth affine k-scheme X, the pullback along the projection $X \times \mathbb{A}^1 \to X$ induces a bijection $\mathrm{H}^1_{\mathrm{Nis}}(X, \mathrm{G}) \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{Nis}}\left(X \times \mathbb{A}^1, \mathrm{G}\right)$.

Using a recent refinement of the Gabber presentation lemma over finite fields first stated by F. Morel [Mor12, Lemma 1.15] (where it is attributed to O. Gabber) and proven by A. Hogadi and G. Kulkarni [HK20], we establish affine homotopy invariance over finite fields in Theorem 2.4.

Remark 1.4. Over a finite field, one knows that all reductive k-group schemes are quasi-split by a result of S. Lang; cf. [Lan56]. In particular, semi-simple group schemes will automatically be "isotropic" in this case.

As immediate consequences, we may remove the assumption that k is infinite in many of the results stated in [AHW18]. In particular, we establish the following result.

Theorem 1.5. *Assume that k is a field. If $\mathrm{H} \to \mathrm{G}$ is a closed immersion of "isotropic" reductive k-group schemes and the H-torsor $\mathrm{G} \to \mathrm{G}\,/\,\mathrm{H}$ is Nisnevich-locally split, then for any smooth affine k-scheme X, there is a bijection*

$$\pi_0\left(\mathrm{Sing}^{\mathbb{A}^1}\mathrm{G}\,/\,\mathrm{H}\right)(X) \cong [X, \mathrm{G}\,/\,\mathrm{H}]_{\mathbb{A}^1}\,.$$

Theorem 2.7 contains a similar result for certain generalized flag varieties under "isotropic" reductive k-group schemes, and the remainder of the main results (for example, Theorem 2.15) contain some useful explicit examples.

Notation and conventions

Throughout the paper, k will be a field. Following [AHW17, AHW18], we use the following terminology:

- Sm_k is the category of smooth k-schemes.
- $\mathrm{sPre}(\mathrm{Sm}_k)$ is the category of simplicial presheaves on Sm_k; objects of this category will typically be denoted by script letters $\mathscr{X}$, $\mathscr{Y}$, etc.
- If t is a topology on Sm_k, we write R_t for the fibrant replacement functor for the injective t-local model structure on $\mathrm{sPre}(\mathrm{Sm}_k)$; see [AHW17, Section 3.1].
- $\mathrm{Sing}^{\mathbb{A}^1}$ is the singular construction; see [AHW17, Section 4.1].
- $\mathscr{H}(k)$ is the Morel–Voevodsky unstable $\mathbb{A}^1$-homotopy category; see [AHW17, Section 5].
- If $\mathscr{X}$ and $\mathscr{Y}$ are simplicial presheaves on Sm_k, we write $[\mathscr{X}, \mathscr{Y}]_{\mathbb{A}^1} := \mathrm{Hom}_{\mathscr{H}(k)}(\mathscr{X}, \mathscr{Y})$.

Throughout the text, we will speak of reductive group schemes; following SGA3 [DG70], by convention such group schemes have geometrically connected fibers.

2. Proofs

2.1 Homotopy invariance revisited

In [AHW18, Proposition 3.3.4], we developed a formalism for establishing the affine homotopy invariance of certain functors; this method was basically an extension of a formalism developed by J.-L. Colliot-Thélène and M. Ojanguren [CO92, Théorème 1.1] and relied on a refined Noether normalization result (a "presentation lemma") that held over infinite fields [CO92, Lemma 1.2]. In Theorem 2.1, we recall a version of a stronger "presentation lemma" due initially to Gabber. Then, in Proposition 2.2, we simplify and generalize [AHW18, Proposition 3.3.4].

Gabber's lemma

The following result was initially stated in [Mor12, Lemma 1.15], where it was attributed to private communication with Gabber. In the case that k is infinite, a detailed proof of a more general result is given in [CHK97, Theorem 3.1.1], while when k is finite, the result was established recently by Hogadi and Kulkarni [HK20, Theorem 1.1]. In fact, in what follows, we will not need the full strength of this result.

Theorem 2.1. *Suppose that F is a field, and suppose that X is a smooth affine F-variety of dimension $d \geqslant 1$. Let $Z \subset X$ be a principal divisor defined by an element $f \in \mathcal{O}_X(X)$, and let $p \in Z$ be a closed point. There exist* (i) *a Zariski-open neighborhood U of the image of p in X,* (ii) *a morphism $\Phi\colon U \to \mathbb{A}^d_F$, and* (iii) *an open neighborhood $V \subset \mathbb{A}^{d-1}_F$ of the composite*

$$\Psi\colon U \xrightarrow{\Phi} \mathbb{A}^d_F \xrightarrow{\pi} \mathbb{A}^{d-1}_F$$

(where π is the projection onto the first $d-1$ coordinates) such that

1. *the morphism Φ is étale;*
2. *setting $Z_V := Z \cap \Psi^{-1}V$, the morphism $\Psi|_{Z_V}\colon Z_V \to V$ is finite;*
3. *the morphism $\Phi|_{Z_V}\colon Z_V \to \mathbb{A}^1_V = \pi^{-1}(V)$ is a closed immersion;*
4. *there is an equality $Z_V = \Phi^{-1}\Phi(Z_V)$.*

In particular, the morphisms Φ and $j\colon \mathbb{A}^1_V \setminus Z_V \to \mathbb{A}^1_V$ yield a Nisnevich-distinguished square of the form

$$\begin{array}{ccc} U \setminus Z_V & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathbb{A}^1_V \setminus \Phi(Z_V) & \longrightarrow & \mathbb{A}^1_V\,. \end{array}$$

A formalism for homotopy invariance

The following result simplifies and generalizes [AHW18, Proposition 3.3.4]. By an essentially smooth k-scheme, we will mean a k-scheme that may be written as a filtered colimit of smooth schemes with affine étale transition morphisms.

Proposition 2.2. *Suppose that k is a field. Let $\mathbf{F}$ be a presheaf of pointed sets on the category $\mathbf{C}$ of essentially smooth affine k-schemes with the following properties:*

1. *If $\operatorname{Spec} A \in \mathbf{C}$ and $S \subset A$ is a multiplicative subset, the canonical map $\operatorname{colim}_{f \in S} \mathbf{F}(A_f) \to \mathbf{F}\big(S^{-1}A\big)$ has trivial kernel.*
2. *For every finitely generated separable field extension L/k and every integer $n \geqslant 0$, the restriction map*

$$\mathbf{F}(L[t_1,\ldots,t_n]) \longrightarrow \mathbf{F}(L(t_1,\ldots,t_n))$$

has trivial kernel.

3. *For every Nisnevich square*

$$\begin{array}{ccc} W & \hookrightarrow & V \\ \downarrow & & \downarrow \\ U & \hookrightarrow & X \end{array}$$

in $\mathbf{C}$*, where* $W \subset V$ *is the complement of a principal divisor, the map*

$$\ker(\mathbf{F}(X) \to \mathbf{F}(U)) \longrightarrow \ker(\mathbf{F}(V) \to \mathbf{F}(W))$$

is surjective.

If $\operatorname{Spec} B \in \mathbf{C}$ *is local, then for any integer* $n \geqslant 0$*, the restriction map*

$$\mathbf{F}(B[t_1, \ldots, t_n]) \longrightarrow \mathbf{F}(\operatorname{Frac}(B)(t_1, \ldots, t_n))$$

has trivial kernel.

Proof. We proceed by induction on the dimension d of B. The case $d = 0$ is immediate from property 2. Assume that we know the result in dimension at most $d - 1$. Suppose $\xi \in \ker(\mathbf{F}(B[t_1, \ldots, t_n]) \to \mathbf{F}(\operatorname{Frac}(B)(t_1, \ldots, t_n)))$. By property 2, the image of ξ in $\mathbf{F}(\operatorname{Frac}(B)[t_1, \ldots, t_n])$ is trivial. By property 1, we conclude that there is an element $g \in B \setminus 0$ such that ξ restricts to the trivial element in $\mathbf{F}(B_g[t_1, \ldots, t_n])$.

By Theorem 2.1 applied to $X = \operatorname{Spec} B$, the principal divisor Z defined by g, and the closed point p in $\operatorname{Spec} B$, we may find a Nisnevich square

$$\begin{array}{ccc} \operatorname{Spec} B_g & \hookrightarrow & \operatorname{Spec} B \\ \downarrow & & \downarrow \\ U & \hookrightarrow & \operatorname{Spec} A[x] \end{array}$$

with A an essentially smooth local ring of dimension $d - 1$. It follows immediately from the diagram above that the base change of $U \to \operatorname{Spec} A[x]$ along $\operatorname{Spec} B \to \operatorname{Spec} A[x]$ is affine. Likewise, since $U \to \operatorname{Spec} A[x]$ is an open immersion, its base change along $U \to \operatorname{Spec} A[x]$ is an isomorphism and therefore also affine. As a consequence, the base change of $U \subset \operatorname{Spec} A[x]$ along the surjective étale morphism $U \amalg \operatorname{Spec} B \to \operatorname{Spec} A[x]$ is affine; hence, the original morphism must be affine as well.

Now, by property 3, since ξ lies in the kernel of $\mathbf{F}(B[t_1, \ldots, t_n]) \to \mathbf{F}(B_g[t_1, \ldots, t_n])$, we may find a

$$\xi' \in \ker(\mathbf{F}(A[x][t_1, \ldots, t_n]) \to \mathbf{F}(U[t_1, \ldots, t_n]))$$

lifting ξ. In particular, the image of the class ξ' in $\mathbf{F}(\operatorname{Frac}(A)(x, t_1, \ldots, t_n))$ must also be trivial. However, $A[x][t_1, \ldots, t_n] = A[x, t_1, \ldots, t_n]$ and since A has dimension $d - 1$, we conclude that ξ' is trivial, which means that ξ must also be trivial, and we are done. □

Remark 2.3. The proof of Proposition 2.2 uses only assertions 1, 3, and 4 of Theorem 2.1, and it may be possible to give a shorter and more self-contained proof of these assertions.

Homotopy invariance for G-torsors over arbitrary fields

We now apply Proposition 2.2 in the case of the functor "isomorphism classes of Nisnevich-locally trivial G-torsors" under an "isotropic" reductive k-group G (see Definition 1.1).

Theorem 2.4. *If k is a field and G is an "isotropic" reductive k-group scheme, then for any smooth k-algebra A and any integer $n \geqslant 0$, the map*

$$\mathrm{H}^1_{\mathrm{Nis}}(\operatorname{Spec} A, \mathrm{G}) \longrightarrow \mathrm{H}^1_{\mathrm{Nis}}(\operatorname{Spec} A[t_1, \ldots, t_n], \mathrm{G})$$

is a pointed bijection.

Proof. Repeat the proof of [AHW18, Theorem 3.3.7], replacing appeals to [AHW18, Proposition 3.3.4] with a reference to Proposition 2.2. As the formulation of Proposition 2.2 differs slightly from that of [AHW18, Proposition 3.3.4], we include the argument here.

We want to show that every Nisnevich-locally trivial G-torsor $\mathscr{P}$ over the ring $A[t_1, \ldots, t_n]$ is extended from A. After [AHW18, Corollary 3.2.6], which is a local-to-global principle for torsors under a reductive group scheme, it suffices to show that for every maximal ideal $\mathfrak{m}$ of A, the G-torsor $\mathscr{P}_{\mathfrak{m}}$ over $A_{\mathfrak{m}}[t_1, \ldots, t_n]$ is extended from $A_{\mathfrak{m}}$. In fact, we will show that $\mathscr{P}_{\mathfrak{m}}$ is a trivial torsor.

We claim that the functor from k-algebras to pointed sets given by $A \mapsto \mathrm{H}^1_{\mathrm{Nis}}(\operatorname{Spec} A, \mathrm{G})$ satisfies the axioms of Proposition 2.2. The first point is an immediate consequence of the fact that G has finite presentation by [AHW18, Lemma 2.3.3]. Recall from [AHW18, Definition 2.3.1] that we write $B\mathbf{Tors}_{\mathrm{Nis}}(\mathrm{G})$ for the simplicial presheaf whose value on a smooth scheme U is the nerve of the groupoid of G-torsors over U. The third point is then a formal consequence of the fact that the functor $\mathrm{H}^1_{\mathrm{Nis}}(-, \mathrm{G})$ can be identified with the set of connected components $\pi_0(B\mathbf{Tors}_{\mathrm{Nis}}(\mathrm{G}))$ since $B\mathbf{Tors}_{\mathrm{Nis}}(\mathrm{G})$ satisfies Nisnevich excision essentially by definition (see [AHW18, § 2.3] for more details). Finally, the second point follows by appealing to results of M. S. Raghunathan [Rag78, Rag89], which are conveniently summarized in [CO92, Proposition 2.4 and Théorème 2.5]; this is where the assumption that G is "isotropic" is used.

The hypotheses of Proposition 2.2 having been satisfied, to conclude that $\mathscr{P}_m$ is trivial, it suffices to show that it becomes trivial over the field $\operatorname{Frac}(A_{\mathfrak{m}})(t_1, \ldots, t_n)$, but this follows immediately from the fact that a field has no non-trivial Nisnevich covering sieves. □

Representability results

Granted Theorem 2.4, we can immediately generalize a number of results from [AHW18]. For ease of reference, we restate the relevant results here. We begin by establishing Theorem 1.3 from the introduction.

If $\mathscr{F}$ is a simplicial presheaf on Sm_k and $\tilde{\mathscr{F}}$ is a Nisnevich-local and $\mathbb{A}^1$-invariant fibrant replacement of $\mathscr{F}$, then there is a canonical map $\mathrm{Sing}^{\mathbb{A}^1}\mathscr{F} \to \tilde{\mathscr{F}}$ that is well defined up to simplicial homotopy. Recall from [AHW18, Definition 2.1.1] that a simplicial presheaf $\mathscr{F}$ on Sm_k is called $\mathbb{A}^1$*-naive* if for every affine $X \in \mathrm{Sm}_k$, the map $\mathrm{Sing}^{\mathbb{A}^1}\mathscr{F}(X) \to \tilde{\mathscr{F}}(X)$ is a weak equivalence of simplicial sets. As observed in [AHW18, Remark 2.1.2], if $\mathscr{F}$ is $\mathbb{A}^1$-naive, then for every affine $X \in \mathrm{Sm}_k$, the map

$$\pi_0\big(\mathrm{Sing}^{\mathbb{A}^1}\mathscr{F}(X)\big) \longrightarrow [X, \mathscr{F}]_{\mathbb{A}^1}$$

is a bijection.

By [AHW18, Proposition 2.1.3], the simplicial presheaf $\mathscr{F}$ is $\mathbb{A}^1$-naive if and only if $\mathrm{Sing}^{\mathbb{A}^1}\mathscr{F}$ satisfies affine Nisnevich excision in the sense of [AHW17, Section 2.1]. In that case, $R_{\mathrm{Zar}}\,\mathrm{Sing}^{\mathbb{A}^1}\mathscr{F}$ is Nisnevich-local and $\mathbb{A}^1$-invariant.

Theorem 2.5. *If* G *is an "isotropic" reductive k-group scheme, then* $B_{\mathrm{Nis}}\,\mathrm{G}$ *is* $\mathbb{A}^1$*-naive. In particular, the canonical map*

$$\mathrm{H}^1_{\mathrm{Nis}}(X,\mathrm{G}) \longrightarrow [X, B\,\mathrm{G}]_{\mathbb{A}^1}$$

is a bijection for every affine $X \in \mathrm{Sm}_k$.

Proof. Combine [AHW18, Theorem 2.3.5] with Theorem 2.4. □

Suppose that $\mathrm{H} \to \mathrm{G}$ is a closed immersion of "isotropic" reductive k-group schemes. By [Ana73, Théorème 4.C], the quotient $\mathrm{G}\,/\,\mathrm{H}$ exists as a k-scheme. Since the map $\mathrm{G} \to \mathrm{G}\,/\,\mathrm{H}$ is an H-torsor, it follows that the quotient is smooth since G has the same property. That the quotient is affine follows from the fact that H is reductive and may be realized as $\operatorname{Spec}\Gamma(\mathrm{G}, \mathcal{O}_{\mathrm{G}})^{\mathrm{H}}$ (see [Alp14, Theorems 9.1.4 and 9.7.6]; for later use, observe that these statements hold over an arbitrary base). Since G and H are reductive, they are connected by assumption, and the connectness statement for the quotient follows. Granted these fact, we establish Theorem 1.5.

Theorem 2.6. *If* $\mathrm{H} \to \mathrm{G}$ *is a closed immersion of "isotropic" reductive k-group schemes, and if the* H*-torsor* $\mathrm{G} \to \mathrm{G}\,/\,\mathrm{H}$ *is Nisnevich-locally split, then* $\mathrm{G}\,/\,\mathrm{H}$ *is* $\mathbb{A}^1$*-naive.*

Proof. Combine [AHW18, Theorem 2.4.2] with Theorem 2.4. □

The following result generalizes [AHW18, Theorem 4].

Theorem 2.7. *Assume that* G *is an "isotropic" reductive k-group scheme and* $\mathrm{P} \subset \mathrm{G}$ *is a parabolic k-subgroup possessing an isotropic Levi factor* (*for example, if* G *is split*)*; then* $\mathrm{G}\,/\,\mathrm{P}$ *is* $\mathbb{A}^1$*-naive.*

Proof. Let L be a Levi factor for P. The quotients $\mathrm{G}\,/\,\mathrm{L}$ and $\mathrm{G}\,/\,\mathrm{P}$ exist; see, for example, [AHW18, Lemma 3.1.5]. Moreover, the map $\mathrm{G}\,/\,\mathrm{L} \to \mathrm{G}\,/\,\mathrm{P}$ induced by the inclusion is a composition of torsors under vector bundles. Under the assumption that L is "isotropic," $\mathrm{G}\,/\,\mathrm{L}$ is $\mathbb{A}^1$-naive by Theorem 2.6. The fact that $\mathrm{G}\,/\,\mathrm{P}$ is $\mathbb{A}^1$-naive then follows from [AHW18, Lemma 4.2.4] using the fact that $\mathrm{G}\,/\,\mathrm{L} \to \mathrm{G}\,/\,\mathrm{P}$ is a composition of torsors under vector bundles. □

2.2 Local triviality of homogeneous spaces

In order to apply Theorem 1.5, we need a criterion to establish that if $\mathrm{H} \subset \mathrm{G}$ is a group homomorphism, the quotient map $\mathrm{G} \to \mathrm{G}\,/\,\mathrm{H}$ is Nisnevich-locally trivial. In this section, we develop some criteria to guarantee that this condition holds.

Criteria for Nisnevich-local triviality

Lemma 2.8. *Assume that R is a commutative unital ring of finite Krull dimension, and suppose that* $\mathrm{H} \subset \mathrm{G}$ *is an inclusion of split reductive R-group schemes.*

1. *The quotient* $\mathrm{G}\,/\,\mathrm{H}$ *exists as a* (*connected*) *smooth affine scheme.*
2. *The* H*-torsor* $\mathrm{G} \to \mathrm{G}\,/\,\mathrm{H}$ *is Nisnevich-locally trivial if for any field K, the map* $\mathrm{H}^1_{\mathrm{fppf}}(K,\mathrm{H}) \to \mathrm{H}^1_{\mathrm{fppf}}(K,\mathrm{G})$ *has trivial kernel.*

If R is a field, the same results hold without the splitness assumptions.

Proof. We first treat the case with the splitness assumptions in place. In that case, split reductive group schemes are pulled back from $\mathbb{Z}$-group schemes. For both claims, it suffices to prove the result with $R = \mathbb{Z}$: the formation of quotients commutes with base change; affineness and

Nisnevich-local triviality will be preserved by base change as well. For $R = \mathbb{Z}$, the existence of the quotient and the relevant properties are established before the statement of Theorem 2.6.

Now, we establish the second statement. To show that the relevant torsor is Nisnevich-locally trivial, it suffices, by [Bia70, Proposition 2], to show that the H-torsor in question is rationally trivial, that is, trivial over the generic point of G/H (which is an integral affine $\mathbb{Z}$-scheme). To that end, note that the generic point is the spectrum of the fraction field K of the ring $\Gamma(\mathrm{G}/\mathrm{H}, \mathcal{O}_{\mathrm{G}/\mathrm{H}})$ and that it suffices to show that the restriction of $\mathrm{G} \to \mathrm{G}/\mathrm{H}$ admits a section upon restriction to K. However, the pullback of $\mathrm{G} \to \mathrm{G}/\mathrm{H}$ along the map $\operatorname{Spec} K \to \mathrm{G}/\mathrm{H}$ is an H-torsor on $\operatorname{Spec} K$ whose associated G-torsor is trivial. The condition that the map $\mathrm{H}^1_{\mathrm{fppf}}(K, \mathrm{H}) \to \mathrm{H}^1_{\mathrm{fppf}}(K, \mathrm{G})$ has trivial kernel precisely guarantees that this H-torsor over $\operatorname{Spec} K$ is trivial, that is, admits a section.

When R is a field, one proceeds in an analogous fashion: the existence and properties of the quotient follow exactly as above. To establish Nisnevich-local triviality, one replaces the reference to [Bia70, Proposition 2] above with a reference to [Nis84, Theorem 4.5] (note that Nisnevich's result is stated for semi-simple groups, but the argument works for reductive group schemes; this is mentioned, for example, in [FP15, Section 1.1]). □

The Rost invariant and Nisnevich-local triviality

Assume that G is a simple simply connected algebraic group over a field F. The Rost invariant of G is a natural transformation of functors on the category of field extensions of F:

$$\mathrm{H}^1_{\text{ét}}(-, \mathrm{G}) \xrightarrow{r_{\mathrm{G}}} \mathrm{H}^3_{\text{ét}}(-, \mathbb{Q}/\mathbb{Z}(2))\,;$$

see [GMS03, Appendix A] for more details regarding the group on the right (it will not be important here). What is important is that the Rost invariant is functorial for homomorphisms of simply connected groups [GMS03, Proposition 9.4]. In other words, if $\varphi\colon \mathrm{G}_1 \to \mathrm{G}_2$ is a homomorphism of simply connected reductive algebraic groups, then there is a commutative diagram of the form

$$\begin{array}{ccc} \mathrm{H}^1(F, \mathrm{G}_1) & \xrightarrow{r_{\mathrm{G}_1}} & \mathrm{H}^3(F, \mathbb{Q}/\mathbb{Z}(2)) \\ \downarrow & & \downarrow n_\varphi \\ \mathrm{H}^1(F, \mathrm{G}_2) & \xrightarrow[r_{\mathrm{G}_2}]{} & \mathrm{H}^3(F, \mathbb{Q}/\mathbb{Z}(2))\,, \end{array} \tag{2.1}$$

where n_φ is an integer called the *Dynkin index* or the *Rost multiplier* of the homomorphism φ.

If G is semi-simple and simply connected, then an n-dimensional k-rational representation ρ of G yields an embedding $\rho\colon \mathrm{G} \to \mathrm{SL}_n$; we refer to the Dynkin index of this homomorphism as the Dynkin index of the representation. The Dynkin index then has the following properties:

1. It is a non-negative integer that is 0 if and only if the homomorphism is trivial.
2. The Rost multiplier of a composite is the product of the Rost multipliers [GMS03, Proposition 7.9].
3. If ρ_1 and ρ_2 are two representations of G, then $n_{\rho_1 \oplus \rho_2} = n_{\rho_1} + n_{\rho_2}$.
4. The Dynkin index of the adjoint representation is the dual Coxeter number.

One then deduces the following criterion for detecting Nisnevich-local triviality.

Lemma 2.9. *Assume that $\varphi\colon \mathrm{H} \subset \mathrm{G}$ is a closed immersion group homomorphism of simply connected semi-simple k-group schemes. If* (i) *the Dynkin index for φ is* 1 *and* (ii) *for every* *extension K/k, the kernel of the Rost invariant for H_K is trivial, then the torsor $\mathrm{G} \to \mathrm{G}/\mathrm{H}$ is Nisnevich-locally trivial.*

Proof. By Lemma 2.8, it suffices to prove that for every extension K/k, if for an H-torsor P over $\operatorname{Spec} K$, the associated G-torsor P' over $\operatorname{Spec} K$ (obtained by extending the structure group via φ) is trivial, then P is already trivial.

Assume that the kernel of the Rost invariant for H is trivial for every extension K/k. Suppose that P is an H_K-torsor over $\operatorname{Spec} K$ and that the associated G_K-torsor P' over $\operatorname{Spec} K$ is trivial. Since the Rost invariant of P' is necessarily trivial, the assumption that φ has Rost multiplier 1 implies that P has trivial Rost invariant. However, since the Rost invariant for H_K was assumed to be injective, we conclude that P is trivial, which is precisely what we wanted to show. □

For quasi-split groups of low rank, the Rost invariant is frequently injective [Gar01b]. Indeed, R. S. Garibaldi shows [Gar01b, Theorems 0.1 and 0.5] that the Rost invariant is trivial in the following cases:

1. quasi-split groups of absolute rank $\leqslant 5$
2. quasi-split groups of type B_6, D_6, or E_6
3. quasi-split groups of type E_7 or split groups of type D_7.

Thus we obtain a number of Nisnevich-local triviality results by the computation of Dynkin indices.

Example 2.10. The Rost multiplier of the inclusion of split groups $\mathrm{Spin}_9 \hookrightarrow \mathrm{F}_4$ is 1, so Lemma 2.9 combined with [Gar01b, Theorems 0.1 and 0.5] imply that the Spin_9-torsor $\mathrm{F}_4 \to \mathrm{F}_4/\mathrm{Spin}_9$ is Nisnevich-locally trivial. Similar results hold for $\mathrm{F}_4 \subset \mathrm{E}_6$ and $\mathrm{E}_6 \subset \mathrm{E}_7$ (see [Gar01b] for more details). Thus, in each of these cases, Theorem 2.6 applies and guarantees that the relevant homogeneous space is $\mathbb{A}^1$-naive.

Remark 2.11. Following [AHW19], one can use the $\mathbb{A}^1$-fiber sequences associated with inclusions appearing in Example 2.10 to deduce results about the reduction of the corresponding structure groups for (Nisnevich-locally trivial) torsors over smooth affine schemes. Moreover, torsors under the various group schemes above are related to classical algebraic invariants (for example, F_4-torsors correspond to Albert algebras, E_6- and E_7-torsors correspond to certain structurable algebras [Gar01a]). In light of these applications, we pose the following question, which would be especially interesting to analyze in the cases mentioned in Example 2.10.

Question 2.12. Suppose that $\mathrm{H} \to \mathrm{G}$ is a closed immersion of "isotropic" reductive k-group schemes such that $\mathrm{G} \to \mathrm{G}/\mathrm{H}$ is Nisnevich-locally trivial.

- What is the $\mathbb{A}^1$-connectivity of G/H?
- What is the structure of the first non-vanishing $\mathbb{A}^1$-homotopy sheaf of G/H?

Motivic spheres as homogeneous spaces

In [Bor50], Borel completed the classification of homogeneous spaces that are spheres. We now establish a similar result for motivic spheres. To this end, we write Q_{2n-1} for the split smooth affine quadric defined by the equation $\sum_{i=1}^n x_i y_i = 1$, and Q_{2n} for the split smooth affine quadric defined by the equation $\sum_{i=1}^n x_i y_i = z(1-z)$. In [ADF17, Theorem 2], we showed that Q_{2n} is $\mathbb{A}^1$-weakly equivalent to $S^n {\wedge} \mathbf{G}_m{}^{\wedge n}$, and it is well known that Q_{2n-1} is $\mathbb{A}^1$-weakly equivalent to $S^{n-1} {\wedge} \mathbf{G}_m{}^{\wedge n}$.

Theorem 2.13. *Suppose that R is a commutative base ring. The following homogeneous spaces are isomorphic to odd-dimensional motivic spheres:*

1. *The quotients* $\mathrm{SL}_n / \mathrm{SL}_{n-1}$, $\mathrm{SO}_{2n} / \mathrm{SO}_{2n-1}$, *and* $\mathrm{Sp}_{2m} / \mathrm{Sp}_{2m-2}$ *(with* $n = 2m$*) are isomorphic to* Q_{2n-1}.
2. *The quotient* $\mathrm{Spin}_7 / \mathrm{G}_2$ *is isomorphic to* Q_7.
3. *The quotient* $\mathrm{Spin}_9 / \mathrm{Spin}_7$ *is isomorphic to* Q_{15}.

Furthermore, for each pair (G, H) *as above, the torsor* $\mathrm{G} \to \mathrm{G} / \mathrm{H}$ *is Zariski-locally trivial.*

Proof. All of these results are presumably well known. The first three appear in [AHW18, § 4.2], while the last one appears in [AHW19, Theorem 2.3.5]. It remains to identify $\mathrm{Spin}_9 / \mathrm{Spin}_7 \cong \mathrm{Q}_{15}$; this is essentially classical, so we provide an outline.

We use the notation of [AHW19, § 2]. Let O be the split octonion algebra over $\mathbb{Z}$, and consider the closed subscheme in the scheme $O \times O$ defined by $\mathrm{N}_O(x) - \mathrm{N}_O(y) = 1$ (see [AHW19, Definition 2.1.9] for an explicit formula for the norm); this scheme is isomorphic to Q_{15} by definition. The space $O \times O$ carries the split quadratic form of rank 16. However, there is an induced action of Spin_9 on Q_{15} coming from the spinor representation.

We now repeat the arguments at the beginning of the proof of [AHW19, Theorem 2.3.5]. We may first assume without loss of generality that $R = \mathbb{Z}$, and the result in general follows by base change. In that case, the relevant quotient exists by [Ana73, Théorème 4.C].

The action of Spin_9 on Q_{15} described above gives a morphism $\mathrm{Spin}_9 \to \mathrm{Q}_{15}$ by the choice of a point. It remains to show that this map induces an isomorphism of quotients. As in the proof of [AHW19, Theorem 2.3.5], we may reduce to the case of geometric points. Having reduced to geometric points, transitivity may be established and the stabilizer identified by a straightforward (and classical) computation using Clifford algebras (see [Con14, § C.4] for a discussion of the relevant groups).

For Zariski-local triviality, it suffices to show that if given a local ring R and a Spin_7-torsor $\mathscr{P}$ over R, triviality of the associated Spin_9-torsor implies the triviality of $\mathscr{P}$. Equivalently, if the quadratic space associated with the Spin_9-torsor is split, then the initial quadratic space must also be split; this follows from Witt's cancellation theorem [EKM08, Theorem 8.4]. □

Remark 2.14. Following [Bor50, Théorème 3], it seems reasonable to expect that the list above should be a complete list of homogeneous spaces that are isomorphic to odd-dimensional motivic spheres, at least over an algebraically closed field.

Theorem 2.15. *If k is a field having characteristic unequal to 2, then* Q_{2n} *is* $\mathbb{A}^1$*-naive.*

Proof. By [AHW18, Lemma 3.1.7], we know that under the hypotheses, $\mathrm{Q}_{2n} \cong \mathrm{SO}_{2n+1} / \mathrm{SO}_{2n}$ and the torsor $\mathrm{SO}_{2n+1} \to \mathrm{SO}_{2n+1} / \mathrm{SO}_{2n}$ is Zariski-locally trivial. Since SO_m is split, the result follows by Theorem 2.6. □

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