IISE Transactions

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/uiie21

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To cite this article: Erfan Mehmanchi , Hoda Bidkhori & Oleg A. Prokopyev (2020): Analysis of process flexibility designs under disruptions, IISE Transactions, DOI: 10.1080/24725854.2020.1759162

To link to this article: https://doi.org/10.1080/24725854.2020.1759162

IISE TRANSACTIONS
https://doi.org/10.1080/24725854.2020.1759162

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Analysis of process flexibility designs under disruptions

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ABSTRACT

Most previous studies concerning process flexibility designs have focused on expected sales and demand uncertainty. In this paper, we examine the worst-case performance of flexibility designs in the case of demand and supply uncertainties, where the latter can be in the form of either plant or arc disruptions. We define the Plant Cover Index under Disruptions (PCID) as the minimum required plants' capacity to supply a fixed number of products after the disruptions. By exploiting PCID, we establish that under symmetric uncertainty sets the worst-case performance can be expressed in terms of PCID, supply and demand uncertainties. Additionally, PCID enables us to make meaningful comparisons of different designs. In particular, we demonstrate that under disruptions the 2-long chain design is superior to a broad class of designs. Moreover, we identify a condition wherein both *Q*-short and *Q*-long chain designs have the same worst-case performance. We also discuss the notion of fragility that quantifies the impact of disruptions in the worst case and compare fragilities of *Q*-short and *Q*-long chain designs under different types of disruptions. Finally, by employing PCID, we develop an algorithm to generate designs that perform well under supply and demand uncertainties in both the worst case and in expectation.

ARTICLE HISTORY
Received 30 September 2018
Accepted 23 March 2020

KEYWORDS
Worst-case analysis; process flexibility design; disruptions; fragility; networks

1. Introduction

The problems associated with competing in a contemporary market cause many companies to face heightened levels of uncertainties. This issue is an area of growing concern and the subject of numerous discussions in the supply chain literature (Buzacott and Mandelbaum, 2008; Chou *et al.*, 2008). Flexibility is necessary to respond to such uncertainties (Simchi-Levi, 2010). Flexibility enables an expeditious response to changing demands, without raising inventory costs or increasing storage capacity. However, firms can also be exposed to a variety of low-probability high-impact risks that can disrupt operations. Therefore, the challenge is to maximize the benefits of flexibility despite supply and demand uncertainties.

Process flexibility, or the ability of each plant to produce multiple products, is one of the key strategies employed in modern industrial practice to respond to such uncertainties, see Jordan and Graves (1995) and Simchi-Levi (2010). In simpler terms, a design is more flexible if it responds to changes in supply and demand in an efficacious and cost-effective manner (Upton, 1994).

Many authors have related accounts of both the successes of flexibility and the failures of inflexibility. For example, the paper by Biller *et al.* (2006) reports that a failure by Chrysler to keep up with demand, despite having underutilized capacity at some plants, resulted in an estimated loss of \$240 000 000 in pretax profit. Mak and Shen (2009) relay media accounts stating that the Ford Motor Company made a \$485 000 000 investment in 2002 to boost flexibility at their engine and transmission plants worldwide; Chou *et al.* (2010) report that both GM and Nissan undertook similar initiatives. Process flexibility is an active research area and it has garnered notable attention in industries such as automobile, textile, and electronics (Chou *et al.*, 2010). We refer the reader to Buzacott and Mandelbaum (2008) and Chou *et al.* (2008) for overviews of process flexibility designs.

In process flexibility, designs are modeled as bipartite graphs wherein the vertex partitions correspond to plants and products. An arc links a plant to a product if the latter can be produced by the aforementioned plant. Supply-related uncertainty can manifest itself in the form of either arc or plant disruptions, or both. A plant-to-product arc disruption occurs when a plant can no longer produce a specific product; for example, due to the failure of suppliers and/or machines. Similarly, a plant disruption - for instance, a worker's strike or a natural disaster - forces a plant to shut down and causes massive damage to its production. Most related literature have taken cognizance of demand uncertainty in flexibility designs without paying enough attention to supply uncertainties. We refer the reader to, for instance, Jordan and Graves (1995), Chou *et al.* (2011), Simchi-Levi and Wei (2012), Désir *et al.* (2016), and our own brief literature review in Section 1.2.

The primary endeavor of this paper is to address this limitation and provide an analytical study to understand the worst-case performance of flexibility designs susceptible to

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supply and demand uncertainties simultaneously. We specifically focus on the worst-case analysis, as it provides insights on the effectiveness of flexibility designs and inspires a new method of generating flexibility designs that are effective with respect to both the worst-case and the expected-case performances. Studying the worst-case performance is particularly helpful whenever companies need to protect themselves in the face of extreme events. Moreover, our method requires very little information about customer demand and supply uncertainties, which makes it effective from the practical perspective, as manufacturers often cannot accurately estimate such types of uncertainties. Our main focus in this paper is on Q-long and Q-short chain designs that are especially structured connected and disconnect bipartite graphs, respectively. In these designs, the degree of each vertex is exactly Q, where $Q \in \{1, 2, \ldots\}$; see Section 2.1 for the formal definitions of these flexibility designs.

1.1. Results and contributions

The main contributions of this paper can be summarized as follows. In Section 3, we define the notion of the *Plant Cover Index under Disruptions* (PCID) as the first index defined for flexibility designs under disruptions. PCID allows us to characterize the worst-case performance of flexibility designs under both demand and supply uncertainties.

In Section 4, we exploit PCID to show that the 2-long chain is optimal for a wide-range of flexibility designs under the assumption of symmetric uncertainties for plant and arc disruptions as well as demand. Furthermore, we make meaningful comparisons between the performance of a Q-long chain against Q-short chains for any $Q \geqslant 2$. We demonstrate that the worst-case performances of Q-short chain and Q-long chain designs coincide whenever the number of arc disruptions is sufficiently large, regardless of plant disruptions.

In Section 5, we extend the notion of fragility – previously defined for the expected performance – in order to quantify the effect of disruptions on the worst-case performance of a flexibility design. In particular, we show that Q-long chain design is less fragile (sensitive) than Q-short chains in the case of a plant disruption for any $Q \geqslant 2$. In contrast, Q-short chain designs are less fragile than Q-long chains if the number of arc disruptions is sufficiently large, regardless of plant disruptions.

In Section 6, we employ PCID to develop an algorithm for constructing flexibility designs that outperform designs generated by other algorithms in the literature, with respect to the worst-case performance while maintaining a similar expected performance in both balanced and unbalanced as well as homogenous and non-homogenous systems. Finally, we note that most of our proofs are provided in the supplementary material.

1.2. Literature review

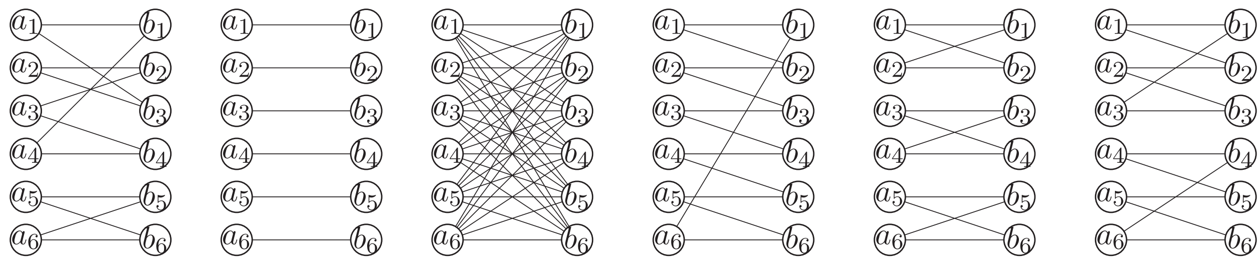
Many firms have integrated flexibility into their operation strategies; see, e.g., Fine and Freund (1990), Li and Tirupati (1994, 1997), Mieghem (1998), and Simchi-Levi (2010). Flexibility enables firms to increase the diversification of their product assortment in order to outperform competitors. Furthermore, it empowers them to shift between products at manufacturing facilities as a quick response to heightened variation in demand for a broader array of products. However, this flexibility comes at a cost, as a plant is often less costly to design if it is for the production of a single product than multiple ones (Feng *et al.*, 2017). Thus, the firms must decide upon a wide array of possible flexibilities that they may employ.

Jordan and Graves (1995) observed that the 2-long chain design on its own can offer numerous flexibility-related benefits. Motivated by this work, the concept of the 2-long chain and other sparse designs have found applications in multistage supply chains (Graves and Tomlin, 2003), serial production lines (Hopp *et al.*, 2004), and queueing networks (Iravani *et al.*, 2005), among others.

These applications prompted a number of subsequent works that analytically explored the effectiveness of the 2-long chain design from the expected performance point of view. In particular, Chou *et al.* (2010) were the first to provide a theoretical justification that the performance of a 2-long chain is comparable to the full flexibility design. They also showed that under some general conditions, the performance of a sparse design can be within (1-ϵ)-optimality of the full flexibility design. Simchi-Levi and Wei (2012) established the optimality of the 2-long chain among all 2-flexibility designs in the expected performance. However, by relaxing the 2-flexibility restriction Désir *et al.* (2016) showed that the 2-long chain is not optimal over all designs with the same cardinality ($2n$ arcs). Specifically, they provided a class of disconnected instances with $2n$ arcs that perform better than the 2-long chain. Wang and Zhang (2015) obtained a distribution-free lower-bound for the ratio of the expected performance of a Q-long chain over that with full flexibility. Similar in spirit results are shown by Bidkhori *et al.* (2016) for unbalanced designs.

In general, designing optimal flexibility is a challenging task, due to the combinatorial nature of the underlying problem. For this reason, various heuristics and guidelines have been proposed to construct effective sparse flexibility designs, we refer to, e.g., Chou *et al.* (2010, 2011).

Design indices can be used as an efficient way to compare the performance of different designs without complex simulations and the need for the detailed information on demand uncertainties. The reader can refer to Deng and Shen (2013) who provide a list of indices from the related literature. In particular, the Plant Cover Index (PCI) was proposed by Simchi-Levi and Wei (2015). They illustrated the relationship between PCI and JG index (Jordan and Graves, 1995) as well as graph expanders from Chou *et al.* (2011). The paper by Simchi-Levi and Wei (2015) can be viewed as the most related work to ours, in the sense that it also evaluates the performance of flexibility designs facing demand uncertainty by examining the worst-case scenario. However, neither this work nor the aforementioned papers take into account supply uncertainty.



(a) 2-flexibility (b) $\mathcal{LC}_1$ (c) full flexibility (d) $\mathcal{LC}_2$ (e) $\mathcal{SC}_2$ (f) $\mathcal{SC}_2$

Figure 1. Example of balanced designs with $m = n = 6$.

Lim *et al.* (2011) studied flexibility designs under supply uncertainty. They introduced the concept of fragility to quantify the change of expected performance of the 2-long and 2-short chain designs resulting from a disruption. Using an approximation scheme, they supported their simulation results and numerical study. In particular, they observed that for a single arc disruption, the expected fragility decreases as the size of chains decreases. On the other hand, in the case of a single plant disruption, the expected fragility decreases as the size of the chains increases.

Our work is different from Lim *et al.* (2011) in several aspects. First, we investigate the fragility with respect to the worst-case performance instead of the expected performance. Second, we consider general Q-long and Q-short chains with any degree $Q \geqslant 2$, whereas Lim *et al.* (2011) focus on 2-long and 2-short chains. Third, we take into consideration both single and multiple disruptions. Moreover, to develop our results we employ PCID that does not need any information about the demand. Finally, PCID is also exploited to develop an algorithm for constructing sparse designs that perform well under disruptions.

2. Definitions

2.1. Flexibility designs, long and short chains

Let sets $A = \{a_1, ..., a_m\}$ and $B = \{b_1, ..., b_n\}$ define plants and products, respectively, where m is the number of plants and n is the number of products. The process flexibility design $\mathcal{D}$ is the set of arcs that form a bipartite graph over the sets A and B, i.e., $\mathcal{D} \subseteq A \times B$. We use i, j and r to represent plant, product and arc indices, respectively. The set of neighbors for any $u \in A \cup B$ is denoted by $\mathcal{N}(u, \mathcal{D})$, i.e., $\mathcal{N}(u, \mathcal{D}) = \{v \mid (u, v) \text{ } or \text{ } (v, u) \in \mathcal{D}\}$. Let $deg_{\mathcal{D}}(u)$ represent the degree of vertex $u \in A \cup B$ in design $\mathcal{D}$, i.e., $deg_{\mathcal{D}}(u) = |\mathcal{N}(u, \mathcal{D})|$.

Plant capacities are denoted by vector $\mathbf{c}^{(p)} \in \mathbb{R}^m_+$. Arc capacities are assumed to be the same for all arcs and its value is defined as $c^{(a)} \in \mathbb{R}_+$. For the sake of simplicity, we use $c^{(a)}$ as opposed to $c^{(a),\mathcal{D}}$ whenever design $\mathcal{D}$ is specified. We say process flexibility design $\mathcal{D}$ is *homogenous* if all plant capacities are the same and *non-homogenous* when they are not.

Process flexibility design $\mathcal{D}$ is balanced if $m = n$ and it is unbalanced, otherwise. It is also assumed that there are no isolated vertices in $\mathcal{D}$, that is, $|\mathcal{N}(u, \mathcal{D})| \geqslant 1$ for all $u \in A \cup B$. A balanced design in which all plant and product vertices have the same degree of Q, $Q \leqslant m = n$, is referred to as a Q-flexibility design, such design corresponds to a Q-regular bipartite graph; see Figure 1(a) for an example of a 2-flexibility design.

Then we define Q-long and Q-short chain designs as particular forms of Q-flexibility designs in a balanced bipartite graph as follows:

- A Q-long chain design – that is also referred to as k-chain (Chou *et al.*, 2011; Chen *et al.*, 2015) and D-skill chaining (Hopp *et al.*, 2004; Iravani *et al.*, 2005) – has an arc set represented by $\mathcal{LC}_Q$ in which for all $a_i \in A$, $\mathcal{N}(a_i, \mathcal{LC}_Q) = \{b_j \mid j = i, i+1, \ldots, i+Q-1, \text{ take } j-n \text{ whenever } j > n\}$. Note that a Q-long chain design is always connected for $Q \geqslant 2$. The most popular example of Q-long chains is the 2-long chain design, which is defined as a cycle that connects plants and products in a balanced graph (Jordan and Graves, 1995). Other well-known examples of Q-long chains include the dedicated ($Q = 1$) and the full-flexibility ($Q = n$) designs (Feng and Shen, 2018). See Figures 1(b), 1(c), and 1(d) for examples of dedicated, full flexibility, and 2-long chain designs, respectively.
- A Q-short chain design is a disconnected graph that comprises of c, $c \geqslant 2$, connected components with sizes z_w, $w \in \{1, \ldots, c\}$, such that each component is a Q-long chain design and $\sum_{w=1}^{c} z_w = n$. An arc set of a Q-short chain is denoted by $\mathcal{SC}_Q$. It should be noted that $Q \leqslant \min\{z_1, \ldots, z_c\}$, and we assume that there are no components of size one. For given n and Q we can define a family of Q-short chain designs represented by $\{\mathcal{SC}_Q\}$ including all $\mathcal{SC}_Q$ with different numbers of components or components' sizes; we use $\mathcal{SC}_Q$ to denote any member of $\{\mathcal{SC}_Q\}$. Figures 1(e) and 1(f) show two members of $\{\mathcal{SC}_2\}$.

In the remainder of this paper, in any comparison of $\mathcal{SC}_Q$ and $\mathcal{LC}_Q$ we assume that they are of the same size.

Vectors $\mathbf{d} \in \mathbb{R}^n_+, \boldsymbol{g} \in \{0,1\}^m$ and $\mathbf{h} \in \{0,1\}^{|\mathcal{D}|}$ denote product demands, plant disruptions and arc disruptions, respectively. The set $\sum([n])$ includes all permutations for the set $\{1, ..., n\}$ and $\sigma \in \sum([n])$. Similarly, $\sum(\mathcal{D})$ is the set of all permutations for the index set of design $\mathcal{D}$, i.e., $\{(i,j) \mid (a_i, b_j) \in \mathcal{D}\}$, and $\rho \in \sum(\mathcal{D})$.

The operators $\min^j(\mathbf{x})$ and $\max^j(\mathbf{x})$ return the jth smallest and the jth largest elements of vector $\mathbf{x} = [x_1 \; x_2 \; \ldots \; x_n]$, respectively. For any $t \in \mathbb{R}$ define $t^+ = \max\{0, t\}$. The symbol $\mathbf{e}$ denotes the vector of all ones of appropriate dimension. Finally, $\mathbb{1}$ refers to an indicator function.

2.2. Demand and disruption uncertainty sets

One of the major modeling decisions that we have deferred until this point is the selection of appropriate uncertainty sets of demand and disruption scenarios. There are a variety of viable choices for selecting an uncertainty set, see, e.g., Bertsimas and Sim (2004), Bertsimas and Brown (2009), and Bertsimas *et al.* (2018).

As emphasized by Iravani *et al.* (2005), exact information in related applications is usually unavailable. Thus, this paper makes the assumption that the uncertainty sets are symmetric, which implies that the worst-case performance is not altered if products are relabeled. With respect to disruptions, this assumption implies that all arcs and all plants face the same disruption risks.

We note that symmetric demand uncertainty sets are frequently used in the flexibility design literature; see a recent work by Simchi-Levi and Wei (2015). Symmetric uncertainty sets in the worst-case performance correspond to the common assumption of symmetrical designs in the expected performance. Symmetrical designs are designs with exchangeable or identical demand distributions and identical plant capacities; see, e.g., Chou *et al.* (2010), Chen *et al.* (2015), Désir *et al.* (2016), Feng *et al.* (2017), and Feng and Shen (2018).

It is worth mentioning that our results are valid for the entire class of symmetric demand uncertainty sets and not just one. In addition, the analysis for symmetric uncertainty sets provides us insights for problems with asymmetric ones (Deng and Shen, 2013). In particular, with the results given in Section 3 we identify the key feature of the flexibility designs that perform well under any type of uncertainties. This leads to the development of an algorithm in Section 6 to generate flexibility designs robust against arbitrary types of supply and demand uncertainties.

Formally, given $\mathbf{x} \in \mathbb{R}^n$ and permutation $\sigma \in \sum([n])$, let $\mathbf{x}_\sigma$ be the rearrangement of the elements of $\mathbf{x}$ according to permutation σ, i.e., $\mathbf{x}_\sigma = [x_{\sigma(1)}\ x_{\sigma(2)}\ \dots\ x_{\sigma(n)}]$. Set $\mathcal{U}$ is symmetric if for any $\mathbf{u} \in \mathcal{U}$, then $\mathbf{u}_\sigma \in \mathcal{U}$ for any permutation of the index set of $\mathcal{U}$. In the remainder of this paper, we use the following demand, arc disruption, and plant disruption uncertainty sets.

Demand uncertainty set: Set $\mathcal{U}_d$ denotes the symmetric uncertainty set associated with demands, where $\mathbf{d} \in \mathcal{U}_d$ indicates a realization of this set. Examples of symmetric demand uncertainty sets include:

- budgeted uncertainty: $\mathcal{U}_d = \{\mathbf{d} \mid d_j = a + bz_j\ \forall j \in \{1, \dots, n\}, \|\mathbf{z}\|_1 \leqslant \Gamma, \|\mathbf{z}\|_\infty \leqslant 1\}$ for some $a, b,\ \Gamma \in \mathbb{R}_+$, where Γ is known as the *budget of uncertainty*;
- triangle uncertainty: $\mathcal{U}_d = \{\mathbf{d} \mid \sum_{j=1}^n d_j = t, d_j \geqslant 0, \forall j \in \{1, \dots, n\}\}$ for some $t \in \mathbb{R}_+$;
- box uncertainty: $\mathcal{U}_d = \{\mathbf{d} \mid \ell \leqslant d_j \leqslant u, \forall j \in \{1, \dots, n\}\}$ for some $\ell, u \in \mathbb{R}_+$;
- ellipsoidal uncertainty: $\mathcal{U}_d = \{\mathbf{d} \mid \sum_{j=1}^n (d_j - z)^2 \leqslant t, \forall j \in \{1, \dots, n\}\}$ for some $z, t \in \mathbb{R}_+$;

or the intersection of any symmetric uncertainty sets.

Arc disruption uncertainty set: We use $\mathcal{U}_a^\alpha$ to denote the symmetric uncertainty set associated with arc disruptions such that at most $\alpha \in \mathbb{Z}_+$ arcs can be disrupted. Specifically, set $\mathcal{U}_a^\alpha$ is given by

$$\mathcal{U}_a^\alpha = \{\mathbf{h} \in \{0,1\}^{|\mathcal{D}|} \mid \sum_{(i,j)\in\mathcal{D}} h_{ij} \geqslant |\mathcal{D}| - \alpha\},$$

where $\mathbf{h} \in \mathcal{U}_a^\alpha$ indicates a realization of this set; in particular, $h_{ij} = 0$ if the arc connecting plant i to product j is disrupted, and $h_{ij} = 1$ otherwise. This uncertainty set is similar to the budgeted uncertainty sets commonly used in the robust optimization literature, see, e.g., Bertsimas and Sim (2004); here α determines the budget of uncertainty and the level of arc disruptions.

Plant disruption uncertainty set: We denote by $\mathcal{U}_p^\gamma$ the symmetric uncertainty set associated with plant disruptions such that at most $\gamma \in \mathbb{Z}_+$ plants can be disrupted. Formally, set $\mathcal{U}_p^\gamma$ is given by

$$\mathcal{U}_p^\gamma = \{\mathbf{g} \in \{0,1\}^m \mid \sum_{i\in I} g_i \geqslant m - \gamma\},$$

where $g \in \mathcal{U}_p^\gamma$ indicates a realization of this set; in particular, $g_i = 0$ if plant i is disrupted, and $g_i = 1$ otherwise. Similar to the uncertainty set associated with arc disruptions, γ determines the budget of uncertainty and the level of plant disruptions.

For the sake of simplicity, we use $\mathcal{U}_a$ and $\mathcal{U}_p$ wherever there is no ambiguity with respect to disruption parameters α and γ, respectively. Note that if $\alpha = 0$ ($\gamma = 0$), then there are no arc (plant) disruptions. Throughout the paper, we consider supply uncertainties in the form of symmetric arc and plant disruption uncertainty sets defined in this section.

Remark 1. Recall that we seek for an optimal solution in the worst case. Clearly, this occurs when the lergest number of plants and arcs are disrupted. Thus, in order to evaluate any design in the worst case, we can assume that *exactly* γ plants and α arcs are disrupted, i.e., $\sum_{(i,j)\in\mathcal{D}} h_{ij} = |D| - \alpha$ and $\sum_{i\in I} g_i = m - \gamma$, respectively. □

Remark 2. In the worst case, for sufficiently small numbers of arc and plant disruptions the arc disruptions occur for those arcs that are incident to plants which are not disrupted. Specifically, in the worst-case scenario the budget of arc disruptions, α, is used to disrupt arcs that are still supplied by plants. For example, if for $\gamma = 1$ plant a_i with $deg_{\mathcal{D}}(a_i) = 2$ is disrupted and $\alpha = 3$, then in total five arcs are inactive in the design either due to failure in supply or in the production line. □

3. Robust measure and PCID

In this section, we formulate the robust measure to evaluate the worst-case performance of general flexibility designs under supply and demand uncertainties. To this end, first in

Section 3.1 we define PCID. Then in Section 3.2, we establish a relationship between PCID and the robust measure for flexibility designs.

Given vectors $\mathbf{d} \in \mathcal{U}_d, \boldsymbol{g} \in \mathcal{U}_p$ and $\mathbf{h} \in \mathcal{U}_a$, let $P(\mathbf{d}, \boldsymbol{g}, \mathbf{h}, \mathcal{D})$ denote the performance of design $\mathcal{D}$. The performance is measured by the maximum possible demand that can be supplied through $\mathcal{D}$ under plant and arc disruptions. Specifically, if f_{ij} denotes the amount of demand for product j satisfied by plant i, i.e., the product flow from i to j, then $P(\mathbf{d}, \boldsymbol{g}, \mathbf{h}, \mathcal{D})$ can be obtained by solving the following maximum flow problem:

$$P(\mathbf{d}, \boldsymbol{g}, \mathbf{h}, \mathcal{D}) = \max_{\mathbf{f}} \sum_{(a_i, b_j) \in \mathcal{D}} f_{ij}, \tag{1a}$$

$$s.t. \sum_{b_j \in \mathcal{N}(a_i, \mathcal{D})} f_{ij} \leqslant c_i^{(p)} g_i \quad \forall a_i \in A, \tag{1b}$$

$$\sum_{a_i \in \mathcal{N}(b_j, \mathcal{D})} f_{ij} \leqslant d_j \quad \forall b_j \in B, \tag{1c}$$

$$0 \leqslant f_{ij} \leqslant c^{(a)} h_{ij} \quad \forall (a_i, b_j) \in \mathcal{D}, \tag{1d}$$

where constraint (1b) provides an upper bound on the demand that can be satisfied by plant i under disruption g_i. Constraint (1c) enforces that the total production of product j does not exceed its demand. Finally, constraint (1d) ensures that the flow of product j from plant i does not exceed the arc capacity under disruption h_{ij}.

From the max-flow min-cut theorem, by taking the dual of problem (1) we obtain:

$$P(\mathbf{d}, \boldsymbol{g}, \mathbf{h}, \mathcal{D}) = \min_{\mathbf{p}, \mathbf{q}, \mathbf{t}} \left\{ \sum_{i=1}^{m} c_i^{(p)} g_i p_i + \sum_{j=1}^{n} d_j q_j + \sum_{(a_i, b_j) \in \mathcal{D}} c^{(a)} h_{ij} t_{ij} \right\}, \tag{2a}$$

$$s.t. \quad p_i + q_j + t_{ij} \geqslant 1 \quad \forall (a_i, b_j) \in \mathcal{D}, \tag{2b}$$

$$\mathbf{p} \in \{0, 1\}^m, \ \mathbf{q} \in \{0, 1\}^n, \tag{2c}$$

$$t_{ij} \in \{0, 1\} \quad \forall (a_i, b_j) \in \mathcal{D}, \tag{2d}$$

where dual variables p_i, q_j, and t_{ij} correspond to constraints (1b), (1c) and (1d), respectively. Furthermore, by using the total unimodularity property of the constraint matrix (Wolsey and Nemhauser, 1999, Corollary 2.8 and Proposition 2.1) it can be shown that the linear programming relaxation of problem (2) has a binary optimal solution.

Finally, for design $\mathcal{D}$ given uncertainty sets $\mathcal{U}_d$, $\mathcal{U}_p$ and $\mathcal{U}_a$, denote by $R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{D})$ the optimal objective function value of the following problem:

$$R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{D}) = \min_{\mathbf{d} \in \mathcal{U}_d, \boldsymbol{g} \in \mathcal{U}_p, \mathbf{h} \in \mathcal{U}_a} P(\mathbf{d}, \boldsymbol{g}, \mathbf{h}, \mathcal{D}).$$

Simply speaking, $R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{D})$ provides the worst-case performance of design $\mathcal{D}$ under supply and demand uncertainties.

3.1. PCID

A subset of plant and product vertices forms a vertex cover if every arc in the design has at least one of its endpoints in this subset. For any integers $k \in \{0, 1, \ldots, n\}$ and $\ell \in \{0, 1, \ldots, |\mathcal{D}|\}$, and any vector $\boldsymbol{g} \in \mathcal{U}_p$ we define the PCID at k, ℓ and $\boldsymbol{g}$ as the minimum plant capacity that is required to create a vertex cover on $\mathcal{D}$, given that the vertex cover contains exactly k products, exactly ℓ arcs are ignored (i.e., not required to be covered), and plants are disrupted according to $\boldsymbol{g}$.

Denote by $\delta^{k,\ell}(\boldsymbol{g}, \mathcal{D})$ PCID at k, ℓ and $\boldsymbol{g}$. Based on its definition, PCID can be computed as the objective function value of the following linear 0–1 program for any given k, ℓ and $\boldsymbol{g}$:

$$(\text{PCID}): \ \delta^{k,\ell}(\boldsymbol{g}, \mathcal{D}) = \min_{\mathbf{p}, \mathbf{q}, \mathbf{t}} \sum_{i=1}^{m} c_i^{(p)} g_i p_i, \tag{3a}$$

$$s.t. \sum_{j=1}^{n} q_j = k, \tag{3b}$$

$$\sum_{(a_i, b_j) \in \mathcal{D}} t_{ij} = \ell, \tag{3c}$$

$$p_i + q_j + t_{ij} \geqslant 1 \quad \forall (a_i, b_j) \in \mathcal{D}, \tag{3d}$$

$$\mathbf{p} \in \{0, 1\}^m, \mathbf{q} \in \{0, 1\}^n, \tag{3e}$$

$$t_{ij} \in \{0, 1\} \quad \forall (a_i, b_j) \in \mathcal{D}. \tag{3f}$$

Constraint (3b) restricts the number of products in the vertex cover to exactly k products. Constraint (3c) ensures that the number of arcs that are ignored (not covered) is exactly ℓ. Constraint (3d) satisfies the requirement that each arc either has at least one endpoint in the vertex cover or it is not covered. Note that in every feasible solution of problem (3) if $p_i = 1$ ($q_j = 1$), then plant a_i (product b_j) is in the vertex cover set; additionally, if $t_{ij} = 1$, then arc (a_i, b_j) is not required to be covered. The latter set of variables, t_{ij}, and the parameter ℓ allows us to capture arc disruptions, see further discussions after Proposition 1.

Next, we provide some basic properties of PCID.

Remark 3. For any design $\mathcal{D}$ and all $k \in \{0, \ldots, n\}$, $\ell \in \{0, \ldots, |\mathcal{D}|\}$, and $\boldsymbol{g} \in \mathcal{U}_p$ we have:

(*i*) $\delta^{0,0}(\boldsymbol{g}, \mathcal{D}) = \sum_{i=1}^{m} c_i^{(p)} g_i$

(*ii*) $\delta^{k,|\mathcal{D}|}(\boldsymbol{g}, \mathcal{D}) = 0$

(*iii*) $\delta^{n,\ell}(\boldsymbol{g}, \mathcal{D}) = 0$

(*iv*) $\delta^{k+1,\ell}(\boldsymbol{g}, \mathcal{D}) \leqslant \delta^{k,\ell}(\boldsymbol{g}, \mathcal{D})$

(*v*) $\delta^{k,\ell+1}(\boldsymbol{g}, \mathcal{D}) \leqslant \delta^{k,\ell}(\boldsymbol{g}, \mathcal{D})$

(*vi*) $\delta^{k,\ell}(\boldsymbol{g}, \mathcal{D}) = \min_{\substack{S \subseteq B, |S| = k, \\ E \subseteq \mathcal{D}, |E| = \ell}} \sum_{a_i \in \mathcal{N}(B \setminus S, \mathcal{D} \setminus E)} c_i^{(p)} g_i$

In particular, Equality (*vi*) illustrates that $\delta^{k,\ell}(\boldsymbol{g}, \mathcal{D})$ can be expressed as the minimum disrupted capacity of plants incident to $\mathcal{N}(B \setminus S, \mathcal{D} \setminus E)$ for any $S \subseteq B$ and $E \subseteq \mathcal{D}$ such that $|S| = k, |E| = \ell$ and $\boldsymbol{g} \in \mathcal{U}_p$, i.e., subset $S \cup \mathcal{N}(B \setminus S, \mathcal{D} \setminus E)$ creates a vertex cover on design $\mathcal{D} \setminus E$ such that $\mathcal{N}(B \setminus S, \mathcal{D} \setminus E)$ has the minimum disrupted capacity. □

If $\ell = 0$ and $\boldsymbol{g} = \mathbf{e}$, then PCID reduces to the PCI proposed by Simchi-Levi and Wei (2015) that corresponds to $\delta^{k,0}(\mathbf{e}, \mathcal{D})$. PCI is employed to characterize the worst-case performance under only demand uncertainties, whereas by defining PCID we attempt to take into account possible plant and arc disruptions in addition to demand uncertainties. Specifically, in the next subsection, we show that PCID provides a convenient tool for evaluating $R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{D})$ and comparing the worst-case performance of different designs.

3.2. Robust measure

Our aim in this subsection is to provide an explicit representation of $R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{D})$ under symmetric supply uncertainty sets defined in Section 2.2 and any symmetric demand uncertainty set by employing PCID given by problem (3). Thereafter, we exploit this representation to compare the worst-case performances of different designs. To this end, Lemma 1 gives us an upper-bound for $R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{D})$ when vectors $\mathbf{d}$, $\boldsymbol{g}$ and $\mathbf{h}$ are fixed.

Lemma 1. *Given design $\mathcal{D}$ and uncertainty sets $\mathcal{U}_d$, $\mathcal{U}_p$ and $\mathcal{U}_a$, for any $\mathbf{d}\in\mathcal{U}_d, \boldsymbol{g}\in\mathcal{U}_p$, and any $k\in\{0,\dots,n\}$, $\ell\in\{0,\dots,|\mathcal{D}|\}$:*

$$R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{D})\leqslant \delta^{k,\ell}(\boldsymbol{g},\mathcal{D})+\sum_{j=1}^{k}\min{}^j(\mathbf{d})+c^{(a)}\cdot(\ell-\alpha)^+. \tag{4}$$

Proof. Let vectors $\mathbf{p}'\in\{0,1\}^m, \mathbf{q}'\in\{0,1\}^n$ and $t'_{ij}\in\{0,1\}$ for $(a_i,b_j)\in\mathcal{D}$ be an optimal solution of problem (3). Thus, $\sum_{i=1}^m c_i^{(p)} g_i p'_i=\delta^{k,\ell}(\boldsymbol{g},\mathcal{D})$, $\sum_{j=1}^n q'_j=k$ and $\sum_{(a_i,b_j)\in\mathcal{D}} t'_{ij}=\ell$.
Let σ be a permutation in $\sum([n])$ such that $q'_j=1$ if and only if $d_{\sigma(j)}\in\{\min^z(\mathbf{d})\mid 1\leqslant z\leqslant k\}$. Similarly, let ρ be a permutation in $\sum(\mathcal{D})$ such that $h_{\rho(i,j)}=0$ for all $(i,j)\in\{(a_i,b_j)\in\mathcal{D}\mid t'_{ij}=1\}$ where $|\{(a_i,b_j)\in\mathcal{D}\mid t'_{ij}=1\}|\leqslant \min\{\ell,\alpha\}$, which implies $\mathbf{h}_{\rho(i,j)}\in\mathcal{U}_a$. Hence, we get

$$\sum_{i=1}^{m}c_i^{(p)}g_ip'_i+\sum_{j=1}^{n}d_{\sigma(j)}q'_j+\sum_{(a_i,b_j)\in\mathcal{D}}c^{(a)}h_{\rho(i,j)}t'_{ij}=\delta^{k,\ell}(\boldsymbol{g},\mathcal{D})+\sum_{j=1}^{k}\min{}^j(\mathbf{d})+c^{(a)}\cdot(\ell-\alpha)^+.$$

Note that $\mathbf{p}',\mathbf{q}'$ and $\mathbf{t}'$ is a feasible solution for problem (2). Thus, $P(\mathbf{d}_\sigma,\boldsymbol{g},\mathbf{h}_\rho,\mathcal{D})\leqslant\delta^{k,\ell}(\boldsymbol{g},\mathcal{D})+\sum_{j=1}^k\min^j(\mathbf{d})+c^{(a)}\cdot(\ell-\alpha)^+$. Recall that $\mathcal{U}_d$, $\mathcal{U}_p$ and $\mathcal{U}_a$ are symmetric sets. Therefore,

$$R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{D})\leqslant P(\mathbf{d}_\sigma,\boldsymbol{g},\mathbf{h}_\rho,\mathcal{D})\leqslant\delta^{k,\ell}(\boldsymbol{g},\mathcal{D})+\sum_{j=1}^{k}\min{}^j(\mathbf{d})+c^{(a)}\cdot(\ell-\alpha)^+.$$

□

Next, we provide an explicit formula for $R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{D})$. Specifically, we show that there always exist some integers k and ℓ as well as vectors $\mathbf{d}^\star\in\mathcal{U}_d, \boldsymbol{g}^\star\in\mathcal{U}_p$ and $\mathbf{h}^\star\in\mathcal{U}_a$ such that Lemma 1 holds at equality.

Proposition 1. *Let $(\boldsymbol{d}^\star,\boldsymbol{g}^\star,\boldsymbol{h}^\star)\in\operatorname{argmin}_{\boldsymbol{d}\in\mathcal{U}_d,\boldsymbol{g}\in\mathcal{U}_p,\boldsymbol{h}\in\mathcal{U}_a}P(\boldsymbol{d},\boldsymbol{g},\boldsymbol{h},\mathcal{D})$, then there exist some integers $0\leqslant k\leqslant n$ and $0\leqslant\ell\leqslant|\mathcal{D}|$ such that*

$$R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{D})=\delta^{k,\ell}(\boldsymbol{g}^\star,\mathcal{D})+\sum_{j=1}^{k}\min{}^j(\boldsymbol{d}^\star)+c^{(a)}\cdot(\ell-\alpha)^+. \tag{5}$$

Proof. The max-flow problem (1) is always feasible. Since the min-cut problem (2) is the dual of (1) by the strong duality property of linear programs their optimal solutions coincide and we have:

$$\begin{aligned}P(\mathbf{d}^\star,\boldsymbol{g}^\star,\mathbf{h}^\star,\mathcal{D})=\min_{\mathbf{p},\mathbf{q},\mathbf{t}}&\left\{\sum_{i=1}^{m}c_i^{(p)}g_i^\star p_i+\sum_{j=1}^{n}d_j^\star q_j+\sum_{(a_i,b_j)\in\mathcal{D}}c^{(a)}h_{ij}^\star t_{ij}\right\},\\ s.t.\quad & p_i+q_j+t_{ij}\geqslant1 \qquad \forall(a_i,b_j)\in\mathcal{D},\\ & \mathbf{p}\in\{0,1\}^m,\mathbf{q}\in\{0,1\}^n,\\ & t_{ij}\in\{0,1\} \qquad \forall(a_i,b_j)\in\mathcal{D}.\end{aligned}$$

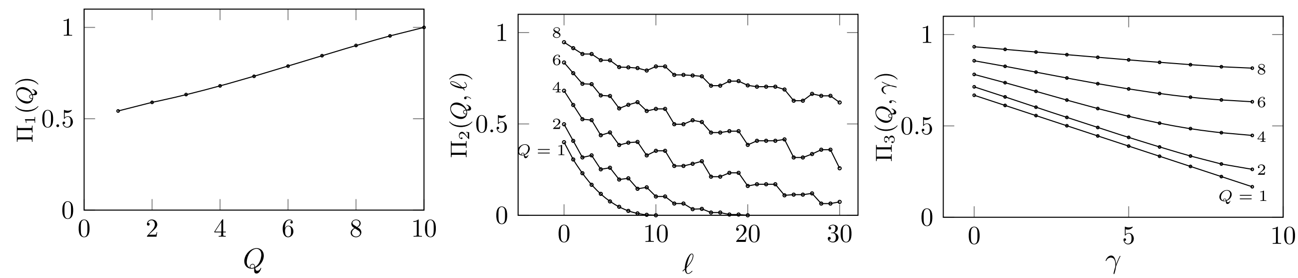
Let $\mathbf{p}^\star,\mathbf{q}^\star$, and $\mathbf{t}^\star$ denote an optimal solution to the optimization problem above, and let also $k:=\sum_{j=1}^n q_j^\star$ and $\ell:=\sum_{(a_i,b_j)\in\mathcal{D}}t_{ij}^\star$. Then we have that $\sum_{i=1}^m c_i^{(p)} g_i^\star p_i^\star\geqslant\delta^{k,\ell}(\boldsymbol{g}^\star,\mathcal{D})$, $\sum_{j=1}^n q_j^\star d_j^\star\geqslant\sum_{j=1}^k\min^j(\mathbf{d}^\star)$, and $\sum_{(a_i,b_j)\in\mathcal{D}}c^{(a)} h_{ij}^\star t_{ij}^\star\geqslant c^{(a)}\cdot(\ell-\alpha)^+$. Thus, we get

$$\begin{aligned}R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{D})&=P(\mathbf{d}^\star,\boldsymbol{g}^\star,\mathbf{h}^\star,\mathcal{D})\\&=\sum_{i=1}^{m}c_i^{(p)}g_i^\star p_i^\star+\sum_{j=1}^{n}d_j^\star q_j^\star+\sum_{(a_i,b_j)\in\mathcal{D}}c^{(a)}h_{ij}^\star t_{ij}^\star\\&\geqslant\delta^{k,\ell}(\boldsymbol{g}^\star,\mathcal{D})+\sum_{j=1}^{k}\min{}^j(\mathbf{d}^\star)+c^{(a)}\cdot(\ell-\alpha)^+.\end{aligned}\tag{6}$$

Therefore, by Lemma 1 and equation (6), Proposition 1 holds. □

Lemma 1 and Proposition 1 are extensions of the results derived in Simchi-Levi and Wei (2015), where the latter assumes that there are no disruptions and arcs are uncapacitated. More specifically, due to possible disruptions, our results are different in the following two aspects. First, in (4) and (5) we use PCID instead of PCI. Second, in (4) and (5) we have an additional third term, i.e., $c^{(a)}\cdot(\ell-\alpha)^+$, that is associated with arc disruptions. In fact, this term connects parameter ℓ of PCID to arc disruptions parameter α. It implies that by increasing ℓ up to α, PCID may decrease in the right-hand side of (4) and (5) while $c^{(a)}\cdot(\ell-\alpha)^+$ remains constant at zero and any further increase of ℓ may decrease PCID at the expense of increasing the right-hand side of (4) and (5) with a rate proportional to $c^{(a)}$. Therefore, as the number of arc disruptions (α) increases the worst-case performance of a given design deteriorates. Next, we exploit Lemma 1 and Proposition 1 to provide an explicit representation of the worst-case performance.

Proposition 2. *The worst-case performance of flexibility design $\mathcal{D}$ under uncertainty sets $\mathcal{U}_d,\mathcal{U}_p$ and $\mathcal{U}_a$ is given by*



(a) $\Pi_1(Q)$ (b) $\Pi_2(Q,\ell)$ (c) $\Pi_3(Q,\gamma)$

Figure 2. Comparing $\mathcal{LC}_Q$ and $\mathcal{LC}_n$ using ratios $\Pi_1(Q)$, $\Pi_2(Q,\ell)$, and $\Pi_3(Q,\gamma)$ as defined in Example 1, for $n = 10$ and $Q \in \{1, 2, ..., n\}$.

$$R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{D}) = \min_{\substack{0\leqslant k\leqslant n,\ 0\leqslant \ell\leqslant |\mathcal{D}| \\ \mathbf{d}\in\mathcal{U}_d,\ \boldsymbol{g}\in\mathcal{U}_p}} \left\{\delta^{k,\ell}(\boldsymbol{g},\mathcal{D}) + \sum_{j=1}^{k} d_j + c^{(a)}\cdot(\ell-\alpha)^+\right\}. \tag{7}$$

Proof. From Lemma 1 and Proposition 1 we get

$$R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{D}) = \min_{\substack{0\leqslant k\leqslant n,\\ 0\leqslant \ell\leqslant |\mathcal{D}|}} \left\{\min_{\boldsymbol{g}\in\mathcal{U}_p} \delta^{k,\ell}(\boldsymbol{g},\mathcal{D}) + \min_{\mathbf{d}\in\mathcal{U}_d}\sum_{j=1}^{k}\min{}^j(\mathbf{d}) + c^{(a)}\cdot(\ell-\alpha)^+\right\}.$$

The symmetric property of $\mathcal{U}_d$ implies that:

$$\min_{\mathbf{d}\in\mathcal{U}_d}\sum_{j=1}^{k}\min{}^j(\mathbf{d}) = \min_{\mathbf{d}\in\mathcal{U}_d}\sum_{j=1}^{k} d_j. \tag{8}$$

Thus,

$$R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{D}) = \min_{\substack{0\leqslant k\leqslant n,\ 0\leqslant \ell\leqslant |\mathcal{D}| \\ \mathbf{d}\in\mathcal{U}_d,\ \boldsymbol{g}\in\mathcal{U}_p}} \left\{\delta^{k,\ell}(\boldsymbol{g},\mathcal{D}) + \sum_{j=1}^{k} d_j + c^{(a)}\cdot(\ell-\alpha)^+\right\}. \quad \square$$

It is worth mentioning that equation (7) is valid for both balanced and unbalanced designs with either homogenous or non-homogenous plant capacities wherein arc capacities can be limited or uncapacitated.

Next, our aim is to establish conditions for comparing the worst-case performance of different flexibility designs. For this purpose, we first consider the following definitions.

Definition 1. Design $\mathcal{D}_1$ is *more symmetrically robust* than design $\mathcal{D}_2$ if and only if

$$R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{D}_1)\geqslant R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{D}_2),$$

for any symmetric uncertainty sets $\mathcal{U}_d$,$\mathcal{U}_p$, and $\mathcal{U}_a$ defined in Section 2.2.

Definition 2. Two different designs $\mathcal{D}_1$ and $\mathcal{D}_2$ are configured in equal conditions if they have the same number of plants, products and arcs, their plant capacities belong to the same vector $\mathbf{c}^{(p)}$, and arc capacities are equal to $c^{(a)}$.

Up to this point, we have considered a general form of the designs wherein arc capacities could be limited. Hereafter, throughout this paper, we narrow our attention to designs satisfying the following assumption.

Assumption 1. *Arc capacities are sufficiently large; specifically, $c^{(a)}\geqslant c_i^{(p)}$ for all $a_i \in A$.*

An uncapacitated arc is the extreme case of Assumption 1; it appears in the majority of related studies on process flexibility designs; see, e.g., Tomlin and Wang (2005), Chou *et al.* (2011), Chen *et al.* (2015), Désir *et al.* (2016), and Feng and Shen (2018). Note that under Assumption 1 in equation (7) we have $\ell^\star = \alpha$, where $\ell^\star$ is an optimal value of variable ℓ.

Based on the above definitions and assumption, in the following result, we show that the performance of different designs configured in equal conditions can be compared by only examining the minimums of their PCIDs over $\boldsymbol{g}\in\mathcal{U}_p$.

Theorem 1. *For any designs $\mathcal{D}_1$ and $\mathcal{D}_2$ configured in equal conditions, design $\mathcal{D}_1$ is more symmetrically robust than $\mathcal{D}_2$, i.e., $R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{D}_1)\geqslant R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{D}_2)$ for any symmetric uncertainty sets $\mathcal{U}_d$,$\mathcal{U}_p$ and $\mathcal{U}_a$ if and only if $\min_{\boldsymbol{g}\in\mathcal{U}_p}\delta^{k,\ell}(\boldsymbol{g},\mathcal{D}_1)\geqslant\min_{\boldsymbol{g}\in\mathcal{U}_p}\delta^{k,\ell}(\boldsymbol{g},\mathcal{D}_2)$ for any $0\leqslant k\leqslant n$ and $0\leqslant \ell\leqslant |\mathcal{D}_1| = |\mathcal{D}_2|$.*

Proof. From equation (7), we have

$$R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{D}_t) = \min_{\substack{0\leqslant k\leqslant n,\ 0\leqslant \ell\leqslant |\mathcal{D}_t|, \\ \mathbf{d}\in\mathcal{U}_d}} \left\{\min_{\boldsymbol{g}\in\mathcal{U}_p}\delta^{k,\ell}(\boldsymbol{g},\mathcal{D}_t) + \sum_{j=1}^{k} d_j + c^{(a)}\cdot(\ell-\alpha)^+\right\}\forall t\in\{1,2\}. \tag{9}$$

If $\min_{\boldsymbol{g}\in\mathcal{U}_p}\delta^{k,\ell}(\boldsymbol{g},\mathcal{D}_1)\geqslant\min_{\boldsymbol{g}\in\mathcal{U}_p}\delta^{k,\ell}(\boldsymbol{g},\mathcal{D}_2)$ for any $\mathcal{U}_p, 0\leqslant k\leqslant n$ and $0\leqslant \ell\leqslant |\mathcal{D}_1|$, then by equation (9), we get $R(\mathcal{U}_d,\ \mathcal{U}_p,\mathcal{U}_a,\mathcal{D}_1)\geqslant R(\mathcal{U}_d,\ \mathcal{U}_p,\ \mathcal{U}_a,\mathcal{D}_2)$.

Conversely, if there exist $\hat{k}, \hat{\ell}$, and $\hat{\mathcal{U}}_p$ such that

$$\min_{\boldsymbol{g}\in\hat{\mathcal{U}}_p}\delta^{\hat{k},\hat{\ell}}(\boldsymbol{g},\mathcal{D}_1) < \min_{\boldsymbol{g}\in\hat{\mathcal{U}}_p}\delta^{\hat{k},\hat{\ell}}(\boldsymbol{g},\mathcal{D}_2),$$

then we construct an example to show that $R(\hat{\mathcal{U}}_d,\hat{\mathcal{U}}_p,\hat{\mathcal{U}}_a,\mathcal{D}_1) < R(\hat{\mathcal{U}}_d,\hat{\mathcal{U}}_p,\hat{\mathcal{U}}_a,\mathcal{D}_2)$ for some uncertainty sets $\hat{\mathcal{U}}_d$ and $\hat{\mathcal{U}}_a$. To this end, define $C = \sum_{i=1}^{m} c_i^{(p)}$, and let $\hat{\mathbf{d}}$ be the vector such that $\hat{d}_j = 0$ for $1\leqslant j\leqslant \hat{k}$, and $\hat{d}_j = C$ for $\hat{k} < j\leqslant n$. Then let $\hat{\mathcal{U}}_d$ be the set of all permutations of vector $\hat{\mathbf{d}}$, i.e., $\hat{\mathcal{U}}_d = \sum(\hat{\mathbf{d}})$. Additionally, let $\hat{\mathcal{U}}_a$ be the arc disruption uncertainty set wherein $\alpha = \hat{\ell}$. Then based on Assumption 1 and Remark 3 parts (*iv*) and (*v*), we have that

Table 1. Expected (the second column as in Jordan and Graves, 1995) vs. the worst-case performances.

		Worst-case performance		
Design	Exp. Perf.	without disruptions $R(\mathcal{U}_d,\mathcal{U}_p^0,\mathcal{U}_a^0,\mathcal{D})$	with 2 arc disruptions, $R(\mathcal{U}_d,\mathcal{U}_p^0,\mathcal{U}_a^2,\mathcal{D})$	with 2 plant disruptions, $R(\mathcal{U}_d,\mathcal{U}_p^2,\mathcal{U}_a^0,\mathcal{D})$
$\mathcal{LC}_1$	853	200	160	160
$\mathcal{SC}_2$ (five components)	896	200	180	160
$\mathcal{LC}_2$	950	200	180	180
$\mathcal{LC}_{10}$	954	200	200	200

$R(\hat{\mathcal{U}}_d,\hat{\mathcal{U}}_p,\hat{\mathcal{U}}_a,\mathcal{D}) = \min_{\boldsymbol{g}\in\hat{\mathcal{U}}_p}\delta^{\hat{k},\hat{\ell}}(\boldsymbol{g},\mathcal{D})$. Thus, we get $R(\hat{\mathcal{U}}_d, \hat{\mathcal{U}}_p, \hat{\mathcal{U}}_a,\mathcal{D}_1) < R(\hat{\mathcal{U}}_d,\hat{\mathcal{U}}_p,\hat{\mathcal{U}}_a,\mathcal{D}_2)$. □

Theorem 1 implies that in order to compare the performance of two different designs configured in equal conditions under uncertainties we only need to obtain the information about minimums of their PCIDs over $\boldsymbol{g}\in\mathcal{U}_p$, i.e., $\min_{\boldsymbol{g}\in\mathcal{U}_p}\delta^{k,\ell}(\boldsymbol{g},\mathcal{D})$ at any $0\leqslant k\leqslant n$ and $0\leqslant \ell\leqslant |\mathcal{D}|$.

Example 1. We consider designs $\mathcal{LC}_Q$ with $n=10$ and $Q\in\{1,2,\ldots,n\}$. Let $\ell\in\{0,1,\ldots,n\cdot Q\}$ and $\gamma\in\{0,1,\ldots,n\}$. Then define:

$$\Pi_1(Q) = \frac{\left[\sum_{k=0}^{n}\sum_{\ell=0}^{n\cdot Q}\sum_{\gamma=0}^{n}\min_{\boldsymbol{g}\in\mathcal{U}_p^{\gamma}}\delta^{k,\ell}(\boldsymbol{g},\mathcal{LC}_Q)\right]\Big/(n\cdot Q+1)}{\left[\sum_{k=0}^{n}\sum_{\ell=0}^{n\cdot n}\sum_{\gamma=0}^{n}\min_{\boldsymbol{g}\in\mathcal{U}_p^{\gamma}}\delta^{k,\ell}(\boldsymbol{g},\mathcal{LC}_n)\right]\Big/(n^2+1)},$$

which is the ratio of the average value of $\min_{\boldsymbol{g}\in\mathcal{U}_p}\delta^{k,\ell}(\boldsymbol{g},\mathcal{LC}_Q)$ over k, ℓ and γ to the corresponding average value of the full flexibility design $\mathcal{LC}_n$. This ratio is plotted in Figure 2(a). As Q increases $\Pi_1(Q)$ increases to one, which is intuitive. On the other hand, for smaller values of Q, $\Pi_1(Q)$ is somewhat far from one, which indicates that under disruptions $\mathcal{LC}_Q$, $Q<n$, may have the worst-case performance that is inferior to that of the full-flexibility design. We further elaborate on this issue in Example 2 below.

Similarly, we define:

$$\Pi_2(Q,\ell) = \frac{\sum_{k=0}^{n}\sum_{\gamma=0}^{n}\min_{\boldsymbol{g}\in\mathcal{U}_p^{\gamma}}\delta^{k,\ell}(\boldsymbol{g},\mathcal{LC}_Q)}{\sum_{k=0}^{n}\sum_{\gamma=0}^{n}\min_{\boldsymbol{g}\in\mathcal{U}_p^{\gamma}}\delta^{k,\ell}(\boldsymbol{g},\mathcal{LC}_n)}$$

and

$$\Pi_3(Q,\gamma) = \frac{\left[\sum_{k=0}^{n}\sum_{\ell=0}^{n\cdot Q}\min_{\boldsymbol{g}\in\mathcal{U}_p^{\gamma}}\delta^{k,\ell}(\boldsymbol{g},\mathcal{LC}_Q)\right]\Big/(n\cdot Q+1)}{\left[\sum_{k=0}^{n}\sum_{\ell=0}^{n\cdot n}\min_{\boldsymbol{g}\in\mathcal{U}_p^{\gamma}}\delta^{k,\ell}(\boldsymbol{g},\mathcal{LC}_n)\right]\Big/(n^2+1)},$$

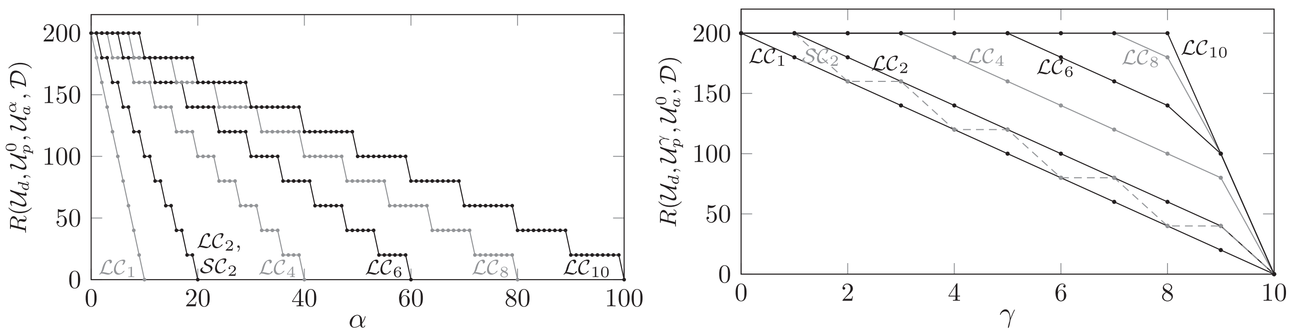
which are the ratios of the average values of $\min_{\boldsymbol{g}\in\mathcal{U}_p}\delta^{k,\ell}(\boldsymbol{g},\mathcal{LC}_Q)$ over k and γ as well as k and ℓ, respectively, to the corresponding average values of the full flexibility design $\mathcal{LC}_n$. These ratios are plotted in Figures 2(b) and 2(c). The values of these ratios decrease as either ℓ or γ increases, which indicates that the performance of the corresponding Q-long chains may be affected in the worst case as the number of possible disruptions increase. Naturally, we also observe that for larger values of Q the decrease in the performance is less pronounced. □

One should note that most of the related literature is mostly focused on the performance of 2-long chain design, $\mathcal{LC}_2$. Primarily, it has been observed that fewer, and longer chains are preferred for increasing the expected performance and $\mathcal{LC}_2$ has almost the same expected performance as the full flexibility design (Jordan and Graves, 1995). In the following example, we show that the aforementioned observations in the literature cannot be extended to the worst-case performance under disruptions, recall also the discussion in Example 1. In particular, we note that $\mathcal{LC}_2$ does not have the same worst-case performance as the full flexibility design. Moreover, $\mathcal{LC}_2$ and $\mathcal{SC}_2$ have the same worst-case performance whenever there is at least one arc disruption. These findings encourage us to explore the worst-case performance of chains with degrees higher than two and short-chain designs under supply uncertainty in the next section.

Example 2. We consider design $\mathcal{D}$ with 10 plants and 10 products, where $\mathbf{c}^{(p)} = 100\cdot\mathbf{e}$ and $c^{(a)} = 100$. The second column of Table 1 displays the simulation results for the expected performance of several designs computed by Jordan and Graves (1995). In particular, in their study the demand for each product is assumed to be from a truncated – with support $[20,180]$ – normal distribution with mean and variance 100 and 40^2, respectively, i.e., $N(100,40^2)$. Jordan and Graves (1995) concluded that: (*i*) the expected performance of $\mathcal{LC}_2$ is close to the performance of the full flexibility design, i.e., $\mathcal{LC}_{10}$, and thus, additional flexibility of the latter design does not provide significant improvements; (*ii*) longer chains with few connected components provide better expected performance than shorter ones. In other words, the expected performance of $\mathcal{LC}_2$ is at least as good as the expected performance of any $\mathcal{SC}_2$ in $\{\mathcal{SC}_2\}$. These observations provide primary justification for many of the related studies not to consider either designs with more than $2n$ arcs or short-chain designs, see, e.g., Deng and Shen (2013).

To evaluate the worst-case performances and to be consistent with Jordan and Graves (1995), we let $\mathcal{U}_d = \{\mathbf{d}\mid 20\leqslant d_j\leqslant 180,\ \forall j\in\{1,2,\ldots,n\}\}$. Based on our computations, the worst-case performances of all considered designs coincide if there are no disruptions (see the third column of Table 1). In particular, $\mathcal{SC}_2$ has the same worst-case performance as $\mathcal{LC}_2$ and the full flexibility design. However, for even a few disruptions the results differ. In particular, the worst-case performances of $\mathcal{LC}_2$ and $\mathcal{LC}_{10}$ can be significantly different (see the last two columns of Table 1), whereas $\mathcal{LC}_2$ and $\mathcal{SC}_2$ may have the same worst-case performance under arc disruptions (see the fourth column of Table 1).

To further illustrate these observations, we compare the performances of $\mathcal{LC}_Q$ for $Q\in\{1,2,4,6,8,10\}$ and $\mathcal{SC}_2$ under disruptions in Figure 3. First, we observe that the worst-case performance improves for Q-long chains as Q increases. Furthermore, Figure 3 demonstrates that the worst-case



(a) Performance of robustly optimal solutions under arc disruptions (α) without plant disruptions

(b) Performance of robustly optimal solutions under plant disruptions (γ) without arc disruptions

Figure 3. Worst-case performances of chains for different types of disruptions.

performances of the designs coincide when there are no disruptions and are non-increasing as the disruptions become more pronounced, i.e., values of α and γ increase. Finally, we point out that the worst-case performances of $\mathcal{SC}_2$ and $\mathcal{LC}_2$ coincide under arc disruptions (and no plant disruptions) for any value of α, see Figure 3(a). On the other hand, in the opposite scenario of plant disruptions without arc disruptions, the worst-case performance of $\mathcal{SC}_2$ is upper bounded by the performance of $\mathcal{LC}_2$, see Figure 3(b). □

4. Worst-case performance of chaining

Herein, we apply the results of the previous section to analyze the worst-case performance of flexibility designs under demand and disruptions uncertainties. In particular, we study the performance of $\mathcal{LC}_Q$ and any $\mathcal{SC}_Q$ in $\{\mathcal{SC}_Q\}$. In Section 4.1, we show that $\mathcal{LC}_2$ is superior to a wide-range of designs, configured in equal conditions, for any type of disruptions. In Section 4.2, we demonstrate that the worst-case performance of $\mathcal{LC}_Q$ is the same as that of any $\mathcal{SC}_Q$ in the presence of a sufficiently large number of arc disruptions. Finally, in Section 4.3 we consider the expected sales of chains (assuming some known demand distribution) under the worst-case supply disruptions.

Hereafter, throughout this paper unless it is specified otherwise, we make the following assumption, which is a common assumption in the related literature; see, e.g., Deng and Shen (2013), Chen *et al.* (2015), Wang and Zhang (2015), and Désir *et al.* (2016).

Assumption 2. *Designs are homogenous, i.e., all plants have equal capacity and without loss of generality* $\mathbf{c}^{(p)} = \mathbf{e}$.

Before we proceed with the discussion we need the following two technical results.

Lemma 2. *For any design* $\mathcal{D}, 0 \leqslant k \leqslant n$ *and* $0 \leqslant \ell \leqslant |\mathcal{D}|$, *we have*:

$$\min_{\boldsymbol{g} \in \mathcal{U}_p} \delta^{k,\ell}(\boldsymbol{g}, \mathcal{D}) = (\delta^{k,\ell}(\boldsymbol{e}, \mathcal{D}) - \gamma)^+.$$

Indeed, $\min_{\boldsymbol{g} \in \mathcal{U}_p} \delta^{k,\ell}(\boldsymbol{g}, \mathcal{D})$ is a quadratic problem; nevertheless, Lemma 2 indicates that it can still be solved as linear-binary program (3). It also should be noted that by using Lemma 2 we can rewrite equation (7) as

$$R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{D}) \tag{10}$$

$$= \min_{\substack{0 \leqslant k \leqslant n,\ 0 \leqslant \ell \leqslant |\mathcal{D}| \\ \mathbf{d} \in \mathcal{U}_d,}} \left\{ (\delta^{k,\ell}(\mathbf{e}, \mathcal{D}) - \gamma)^+ + \sum_{j=1}^{k} d_j + c^{(a)} \cdot (\ell - \alpha)^+ \right\}.$$

Therefore, the necessary and sufficient condition of Theorem 1 reduces to $\delta^{k,\ell}(\mathbf{e}, \mathcal{D}_1) \geqslant \delta^{k,\ell}(\mathbf{e}, \mathcal{D}_2)$ for any $0 \leqslant k \leqslant n$ and $0 \leqslant \ell \leqslant |\mathcal{D}|$.

Next, we evaluate $\min_{\boldsymbol{g} \in \mathcal{U}_p} \delta^{k,\ell}(\boldsymbol{g}, \mathcal{LC}_Q)$, which can be done via the following technical lemma.

Lemma 3. *For* $\mathcal{LC}_Q$ *and any* $0 \leqslant k \leqslant n$ *and* $0 \leqslant \ell \leqslant n \cdot Q$, *we have:*

$$\min_{\boldsymbol{g} \in \mathcal{U}_p} \delta^{k,\ell}(\boldsymbol{g}, \mathcal{LC}_Q) \geqslant \left(n - k - \left\lfloor \frac{\ell}{Q} \right\rfloor - \gamma \right)^+. \tag{11}$$

In particular, for any $0 \leqslant k \leqslant n-1$ *and* $\ell = 0$:

$$\min_{\boldsymbol{g} \in \mathcal{U}_p} \delta^{k,0}(\boldsymbol{g}, \mathcal{LC}_Q) = (\min\{n, n-k+Q-1\} - \gamma)^+. \tag{12}$$

Furthermore, for any $0 \leqslant k \leqslant n$ *and* $(Q-1)^2 \leqslant \ell \leqslant n \cdot Q$, *inequality (11) holds as equality*:

$$\min_{\boldsymbol{g} \in \mathcal{U}_p} \delta^{k,\ell}(\boldsymbol{g}, \mathcal{LC}_Q) = \left(n - k - \left\lfloor \frac{\ell}{Q} \right\rfloor - \gamma \right)^+. \tag{13}$$

4.1. Superiority of 2-long chain

We show that under plant and arc disruptions the 2-long chain design is superior to a broad class of designs in the same configuration. In particular, we establish the following two results. We first demonstrate the superiority of $\mathcal{LC}_2$ over all designs where each product vertex has degree two. Then we show that $\mathcal{LC}_2$ outperforms all connected designs with $2n$ arcs.

Theorem 2. *Let* $\mathcal{D}$ *be a design such that each product is supplied by exactly two plants, i.e.,* $|\mathcal{N}(u, \mathcal{D})| = 2$ *for any* $u \in B$. *Then the 2-long chain design,* $\mathcal{LC}_2$, *is more symmetrically robust than* $\mathcal{D}$. *That is,* $R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{LC}_2) \geqslant R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{D})$.

Although Theorem 2 shows the superiority of $\mathcal{LC}_2$, it restricts the optimality of $\mathcal{LC}_2$ over designs in which each product is produced by exactly two plants. In the next theorem, we relax this restriction for connected designs.

Theorem 3. *Let $\mathcal{D}$ be a connected design such that $|\mathcal{D}| = 2n$, then the 2-long chain design, $\mathcal{LC}_2$, is more symmetrically robust than $\mathcal{D}$. That is, $R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{LC}_2)\geqslant R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{D})$.*

The main idea of the proofs of Theorems 2 and 3 is based on Lemma 3 that provides the exact value of $\min_{g\in\mathcal{U}_p}\delta^{k,\ell}(g,\mathcal{LC}_2)$ at any k and ℓ. In order to establish the superiority of $\mathcal{LC}_2$ over any other class of designs it is sufficient, by Theorem 1, to show that $\min_{g\in\mathcal{U}_p}\delta^{k,\ell}(g,\mathcal{LC}_2) \geqslant\min_{g\in\mathcal{U}_p}\delta^{k,\ell}(g,\mathcal{D})$ at any $0\leqslant k\leqslant n$, $0\leqslant \ell\leqslant 2n$ for any $\mathcal{D}$ in that class. Theorems 2 and 3 are generalizations of the results in Simchi-Levi and Wei (2015), where the latter results do not take into account supply disruptions. Another key difference of our derivations is the following technical lemma that is employed in the proofs of Theorems 2 and 3, and establishes a special property of the class of flexibility designs in which each product is supplied by two plants.

Lemma 4. *Let $\mathcal{D}$ be a connected design over sets A and B such that for any $u\in B, |\mathcal{N}(u,\mathcal{D})| = 2$. Then for any $1\leqslant z\leqslant n$, and any $1\leqslant \ell\leqslant 2n$ there exist some $T\subseteq B, |T| = z$ and $E\subseteq\mathcal{D}$, $|E| = \ell$ such that $|\mathcal{N}(T,\mathcal{D}\setminus E)|\leqslant (z-\lfloor\frac{\ell}{2}\rfloor)^+$.*

By Theorem 2, we conclude that $\mathcal{LC}_2$ is more symmetrically robust than any $\mathcal{SC}_2$ in $\{\mathcal{SC}_2\}$. On the other hand, when there is no disruption some studies have shown that the performance of $\mathcal{LC}_2$ is at least as good as any $\mathcal{SC}_2$ in the worst-case scenario (Chou *et al.*, 2011) and in the expected performance (Simchi-Levi and Wei, 2012). Nonetheless, in the next subsection, we identify conditions under which any $\mathcal{SC}_Q$ has the same worst-case performance as $\mathcal{LC}_Q$ under supply and demand uncertainties for any $Q\geqslant 2$.

4.2. Higher chains

In this subsection, we evaluate the worst-case performance of Q-short chains, $\mathcal{SC}_Q\in\{\mathcal{SC}_Q\}$, versus Q-long chain, $\mathcal{LC}_Q$, for any $Q\geqslant 2$. We show that, in the absence of arc disruptions, the performance of $\mathcal{LC}_Q$ is superior to the performance of any $\mathcal{SC}_Q$. However, the worst-case performance of any $\mathcal{SC}_Q$ is the same as $\mathcal{LC}_Q$ when the number of arc disruptions is sufficiently large. First, we evaluate $\min_{g\in\mathcal{U}_p}\delta^{k,\ell}(g,\mathcal{SC}_Q)$, which can be done via the following technical result.

Lemma 5. *For any $\mathcal{SC}_Q$ in $\{\mathcal{SC}_Q\}$ and $0\leqslant k\leqslant n$ if $\ell = 0$, then*

$$(n-k-\gamma)^+\leqslant \min_{g\in\mathcal{U}_p}\delta^{k,0}(g,\mathcal{SC}_Q)\leqslant \min_{g\in\mathcal{U}_p}\delta^{k,0}(g,\mathcal{LC}_Q). \quad (14)$$

Additionally, if $(Q-1)^2\leqslant \ell\leqslant n\cdot Q$, then

$$\min_{g\in\mathcal{U}_p}\delta^{k,\ell}(g,\mathcal{SC}_Q) = \min_{g\in\mathcal{U}_p}\delta^{k,\ell}(g,\mathcal{LC}_Q) = \left(n-k-\left\lfloor\frac{\ell}{Q}\right\rfloor-\gamma\right)^+. \quad (15)$$

Note that there are examples that equations (13) and (15) do not hold for $\ell < (Q-1)^2$. Thus, the lower-bound $(Q-1)^2$ on the value of ℓ in the aforementioned equations is tight.

As a direct consequence of Lemma 5 and Theorem 1, we obtain the main result of this subsection. In particular, we demonstrate that $\mathcal{SC}_Q$ and $\mathcal{LC}_Q$ have the same performance for sufficiently large number of arc disruptions.

Proposition 3. *For any $\mathcal{SC}_Q$ in $\{\mathcal{SC}_Q\}$ and any plant disruption parameter γ if $\alpha = 0$, then*

$$R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{SC}_Q)\leqslant R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{LC}_Q). \quad (16)$$

Additionally, for any plant disruption parameter γ if $\alpha\geqslant(Q-1)^2$, then

$$R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{SC}_Q) = R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{LC}_Q). \quad (17)$$

It is worth mentioning that the worst-case performances in the right- and left-hand sides of equation (17) correspond, in general, to different realizations of the uncertainty sets.

Inequality (16) indicates that in the absence of arc disruptions (i.e., $\alpha = 0$), for any number of plant disruptions the worst-case performance of $\mathcal{LC}_Q$ is at least as good as the worst-case performance of any $\mathcal{SC}_Q$. This observation is consistent with the earlier results from the related literature that – without arc and plant disruptions – the performance of $\mathcal{LC}_Q$ is better than that of any $\mathcal{SC}_Q$ in both the worst case (Chou *et al.*, 2011) and in expectation (Jordan and Graves, 1995; Chou *et al.*, 2011; Simchi-Levi and Wei, 2012) with respect to the demand uncertainty. Hence, inequality (16) extends the existing observations in the literature to the worst-case performance of long and short chains under plant disruptions.

However, equation (17) implies that for sufficiently large number of arc disruptions any $\mathcal{SC}_Q$ in $\{\mathcal{SC}_Q\}$ has the same worst-case performance as $\mathcal{LC}_Q$. equation (17) has several noteworthy implications as we briefly outline below:

- For $Q = 2$, if there exists at least one arc disruption, i.e., $\alpha\geqslant 1$, then from the worst-case performance perspective any short chain $\mathcal{SC}_2\in\{\mathcal{SC}_2\}$ has the same performance as long chain $\mathcal{LC}_2$ for any number of plant disruptions. Therefore, any short chain $\mathcal{SC}_2$ is also an optimal design (from the worst-case performance perspective) over the design classes considered in Theorems 2 and 3.
- In many practical settings, the cost of flexibility increases with products dissimilarities. Thus, constructing longer chains is often more expensive than multiple shorter ones, e.g., producing two dissimilar products in one plant versus two similar ones (Lim *et al.*, 2011). To address this issue, by grouping similar products in short chains we can reduce the cost of flexibility and thus, simultaneously guarantee the optimality of the (worst-case) performance of the resulting short-chain design versus the long-chain design under sufficiently large number of arc disruptions.
- Finally, note that class $\{\mathcal{SC}_Q\}$ provides a broad-range of alternative designs (e.g., for $n = 10$ set $\{\mathcal{SC}_2\}$ includes 11 different designs) to the decision-maker, that all have the same worst-case performance as $\mathcal{LC}_Q$ provided that $\alpha\geqslant(Q-1)^2$. This opportunity is important, in particular, in the settings where either other criteria or some

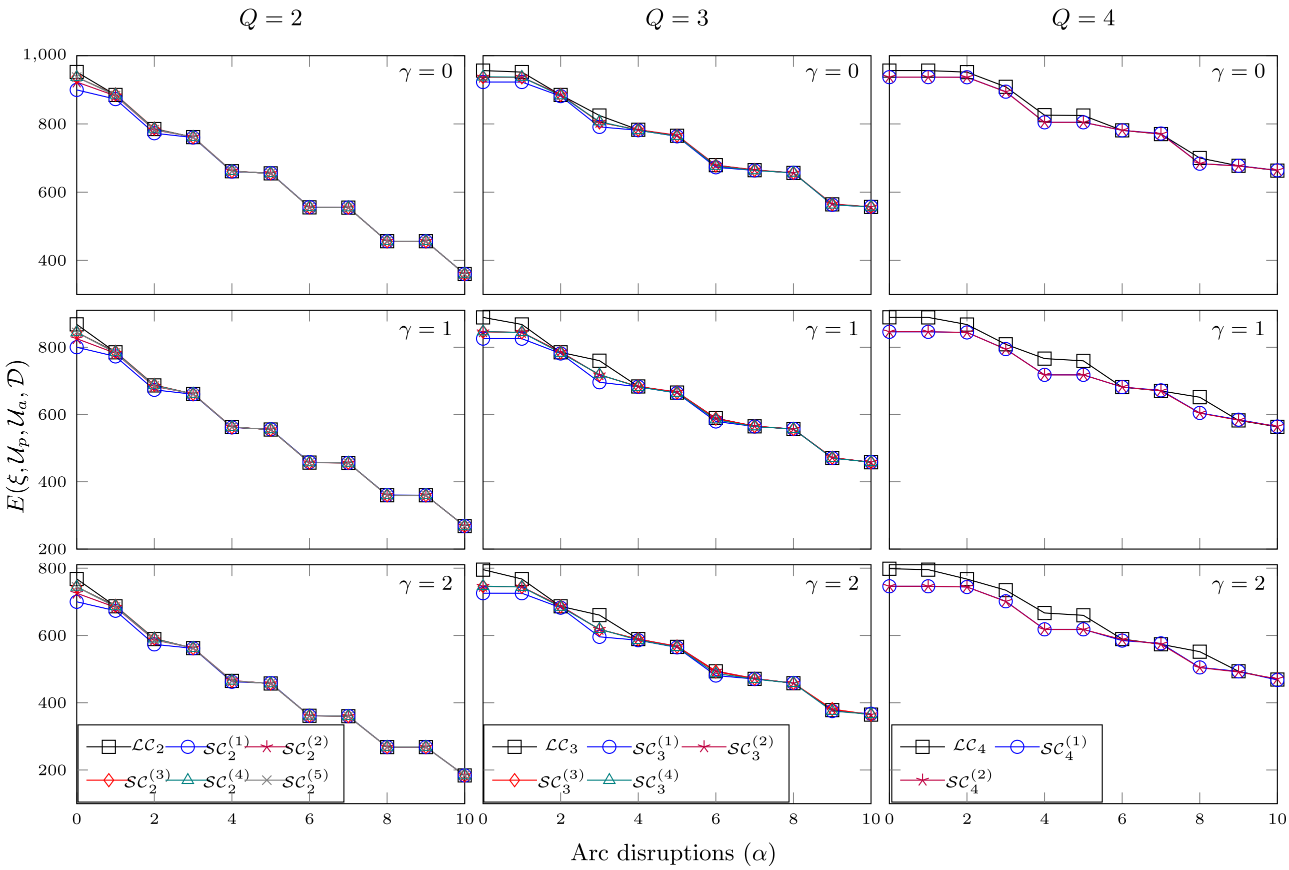


Figure 4. Performance of the designs from Example 3 with respect to $E(\xi,\mathcal{U}_p,\mathcal{U}_a,\mathcal{D})$. Subfigures in each column and row are for specific values of Q and γ, respectively. In each subfigure, the vertical axis represents $E(\xi,\mathcal{U}_p,\mathcal{U}_a,\mathcal{D})$ and the horizontal axis is reserved for the number of arc disruptions, α. Designs $\mathcal{SC}_2^{(1)}$ to $\mathcal{SC}_2^{(5)}$ have component sizes $\{2,2,2,2,2\}$, $\{3,3,4\}$, $\{5,5\}$, $\{6,4\}$, and $\{7,3\}$, respectively. Designs $\mathcal{SC}_3^{(1)}$ to $\mathcal{SC}_3^{(4)}$ have component sizes $\{3,3,4\}$, $\{5,5\}$, $\{6,4\}$, and $\{7,3\}$, respectively. Designs $\mathcal{SC}_4^{(1)}$ and $\mathcal{SC}_4^{(2)}$ have component sizes {5, 5} and {6, 4}, respectively.

limitations need to be taken into account for selecting a design.

Concluding the above discussion, we note that the results of Proposition 3 are also exploited in Section 5.

4.3. Expected performance for the worst-case supply disruptions

In this subsection, we take an alternative approach and combine the expected and the worst-case performance measures. Specifically, next we assume that the demand uncertainty is characterized by some known distribution ξ, i.e., $\mathbf{d}\sim\xi$, and thus, we can compute the expected performance of design $\mathcal{D}$ under the worst-case plant and arc disruption scenarios from $\mathcal{U}_p$ and $\mathcal{U}_a$, respectively. Formally, define:

$$E(\xi,\mathcal{U}_p,\mathcal{U}_a,\mathcal{D})=\mathbb{E}_{\mathbf{d}\sim\xi}\left[\min_{\mathbf{g}\in\mathcal{U}_p,\,\mathbf{h}\in\mathcal{U}_a}P(\mathbf{d},\mathbf{g},\mathbf{h},\mathcal{D})\right], \tag{18}$$

which we refer to as the expected performance of design $\mathcal{D}$ under the worst-case supply disruptions. In a sense, our main goal is to isolate the effect of low-probability high-impact disruption scenarios from the demand variability.

Recall that the key observation from Theorem 1 is that in order to compare two designs in equal conditions it is sufficient to consider their minimum PCID values. In particular, the derivation of Theorem 1 relies on Proposition 2, where the second term in the right-hand side of equation (7) captures the demand variability. When the demand distribution is known, the corresponding derivations seem to become somewhat more involved and thus, the results of $R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{D})$ cannot be generalized for $E(\xi,\mathcal{U}_p,\mathcal{U}_a,\mathcal{D})$ in a straightforward manner. We leave this research direction as a promising topic for future research.

Nevertheless, it is still interesting to compare the expected performances of $\mathcal{SC}_Q\in\{\mathcal{SC}_Q\}$ and $\mathcal{LC}_Q$ for the worst-case supply disruptions, and verify whether the results similar to (16) and (17) in Proposition 3 hold for the corresponding expected performances $E(\xi,\mathcal{U}_p,\mathcal{U}_a,\mathcal{SC}_Q)$ and $E(\xi,\mathcal{U}_p,\mathcal{U}_a,\mathcal{LC}_Q)$. We explore this issue numerically in the example below.

Example 3. We consider designs based on the configuration explored in Jordan and Graves (1995) as introduced in Example 2. For each $Q\in\{2,3,4\}$ we consider $\mathcal{LC}_Q$ and multiple $\mathcal{SC}_Q\in\{\mathcal{SC}_Q\}$. Moreover, we let $\alpha\in\{0,1,\ldots,10\}$ and $\gamma\in\{0,1,2\}$. The demands are assumed to be independent and identically distributed random variables from normal distribution $N(100,40^2)$ with support $[20,180]$. The normal distribution is often used in the related literature, see, e.g., the five benchmark test instances in Section 6 and the discussion in Jordan and Graves (1995). For each design and combinations of α and γ we simulate $E(\xi,\mathcal{U}_p,\mathcal{U}_a,\mathcal{D})$ for

5000 product demands drawn from the aforementioned normal distribution.

Figure 4 depicts the results for $E(\xi, \mathcal{U}_p, \mathcal{U}_a, \mathcal{D})$ and we make the following observations. For each Q, regardless of the number of plant disruptions γ, if $\alpha < (Q-1)^2$, then $\mathcal{LC}_Q$ is the superior design and in a few cases for specific values of α the short chains are competitive with $\mathcal{LC}_Q$, see for comparison (16). However, for $\alpha \geqslant (Q-1)^2$ most of the considered short chains – with one exception of $\mathcal{SC}_2^{(1)}$ – approximately match the performance of $\mathcal{LC}_Q$, see for comparison (17). These experimental observations are independent of the number of plant disruptions and can be viewed as consistent with the analytical results established in Proposition 3 for the worst-case performance $R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{D})$. □

5. Fragility

In this section, we analyze the sensitivity of chains to disruptions. *Fragility* is the concept originally proposed by Lim *et al.* (2011) to quantify the impact of disruptions on the expected-case performance of flexibility designs, in particular, in the context of analyzing the sensitivity of 2-long and 2-short chains. In the following, we extend the concept of fragility for the worst-case performance. We show that Q-short chain designs are less fragile (sensitive) than Q-long chain if the number of arc disruptions are sufficiently large regardless of the other disruption parameters for any $Q \geqslant 2$. In contrast, the fragility of a Q-long chain design is less than or equal to the fragility of Q-short chains in the case of a plant disruption.

Formally, let $R(\mathcal{U}_d, \mathcal{D})$ represent the worst-case performance of design $\mathcal{D}$ without disruptions and only subject to demand uncertainty. Then from Lemma 1, Proposition 1 and Assumption 1 we have:

$$R(\mathcal{U}_d, \mathcal{D}) = R(\mathcal{U}_d, \mathcal{U}_p^0, \mathcal{U}_a^0, \mathcal{D}) = \min_{0 \leqslant k \leqslant n, \mathbf{d} \in \mathcal{U}_d} \left\{ \delta^{k,0}(\mathbf{e}, \mathcal{D}) + \sum_{j=1}^{k} d_j \right\}. \quad (19)$$

The fragility of design $\mathcal{D}$, denoted by $Fr(\mathcal{D})$, with respect to uncertainty sets $\mathcal{U}_d, \mathcal{U}_p$ and $\mathcal{U}_a$ is the difference in the worst-case performance with and without disruptions, i.e.,

$$\begin{aligned} Fr(\mathcal{D}) &= R(\mathcal{U}_d, \mathcal{D}) - R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{D}) \\ &= \min_{0 \leqslant k \leqslant n, \mathbf{d} \in \mathcal{U}_d} \left\{ \delta^{k,0}(\mathbf{e}, \mathcal{D}) + \sum_{j=1}^{k} d_j \right\} \\ &\quad - \min_{\substack{0 \leqslant k \leqslant n,\ 0 \leqslant \ell \leqslant |\mathcal{D}| \\ \mathbf{d} \in \mathcal{U}_d}} \left\{ (\delta^{k,\ell}(\mathbf{e}, \mathcal{D}) - \gamma)^+ \right. \\ &\quad \left. + \sum_{j=1}^{k} d_j + c^{(a)} \cdot (\ell - \alpha)^+ \right\}. \end{aligned} \quad (20)$$

When the disruptions occur, the fragility of design $\mathcal{D}$ indicates the amount of reduction in $R(\mathcal{U}_d, \mathcal{D})$.

Disruptions can be low-probability contingencies and it is often difficult to accurately assess the uncertainties in supply disruptions. Hence, the supplier may consider $R(\mathcal{U}_d, \mathcal{D})$ as the worst-case performance of design $\mathcal{D}$ by simply ignoring the impact of disruptions. However, we envision that in some settings (for example, in the context of the the minimum supply agreements, see Sucky (2007) it may be necessary to quantify the maximum decrease in the supplied demands in case supply disruptions actually occur. This maximum decrease can be quantified by fragility, $Fr(\mathcal{D})$.

In this section, we make, to some extent, an interesting observation that if a system is subject to sufficiently large number of arc disruptions, then the worst-case performance of any $\mathcal{SC}_Q$ in $\{\mathcal{SC}_Q\}$ is less sensitive than $\mathcal{LC}_Q$, i.e., $Fr(\mathcal{SC}_Q) \leqslant Fr(\mathcal{LC}_Q)$. This result is independent of plant disruptions, as well as the number of short-chain design components. On the contrary, for a single plant disruption we show that $Fr(\mathcal{LC}_Q) \leqslant Fr(\mathcal{SC}_Q)$.

We start with the following result to demonstrate that under arc disruptions the fragility of any $\mathcal{SC}_Q$ is less than or equal to that of $\mathcal{LC}_Q$.

Proposition 4. *Let designs be subject to sufficiently large number of arc disruptions, i.e., $\alpha \geqslant (Q-1)^2$, then $Fr(\mathcal{SC}_Q) \leqslant Fr(\mathcal{LC}_Q)$ for any $\mathcal{SC}_Q$ in $\{\mathcal{SC}_Q\}$ and any plant disruption parameter γ.*

Proof. First, consider the case without disruptions, i.e., $\alpha = \gamma = 0$. Then by inequality (16) we have $R(\mathcal{U}_d, \mathcal{U}_p^0, \mathcal{U}_a^0, \mathcal{SC}_Q) \leqslant R(\mathcal{U}_d, \mathcal{U}_p^0, \mathcal{U}_a^0, \mathcal{LC}_Q)$, or equivalently:

$$R(\mathcal{U}_d, \mathcal{SC}_Q) \leqslant R(\mathcal{U}_d, \mathcal{LC}_Q).$$

When designs are subject to $\alpha \geqslant (Q-1)^2$ arc disruptions, based on equation (17) we get

$$R(\mathcal{U}_d, \mathcal{U}_p^\gamma, \mathcal{U}_a^\alpha, \mathcal{SC}_Q) = R(\mathcal{U}_d, \mathcal{U}_p^\gamma, \mathcal{U}_a^\alpha, \mathcal{LC}_Q),$$

for any γ. Therefore, based on the definition of fragility we conclude that Proposition 4 holds. □

Although Proposition 4 shows that in the case of sufficiently large number of arc disruptions $Fr(\mathcal{SC}_Q) \leqslant Fr(\mathcal{LC}_Q)$ for any $\mathcal{SC}_Q \in \{\mathcal{SC}_Q\}$, it does not determine how we can compare two different members of $\{\mathcal{SC}_Q\}$ with respect to fragility. In the following, we address this issue.

Consider two short chains $\mathcal{SC}_Q^{(1)}$ and $\mathcal{SC}_Q^{(2)}$ in $\{\mathcal{SC}_Q\}$, with component sizes $z_1^{(1)}, \ldots, z_{c_{(1)}}^{(1)}$ and $z_1^{(2)}, \ldots, z_{c_{(2)}}^{(2)}$, respectively. We say that the components of $\mathcal{SC}_Q^{(1)}$ are decomposition of the components of $\mathcal{SC}_Q^{(2)}$ if for every $k \in \{0, \ldots, n\}$ such that $k = \sum_{i \in I_2} z_i^{(2)}$ for some $I_2 \subseteq \{1, \ldots, c_{(2)}\}$, there exists some $I_1 \subseteq \{1, \ldots, c_{(1)}\}$ for which $\sum_{i \in I_1} z_i^{(1)} = k$. Next, we show that if this is the case, then $\mathcal{SC}_Q^{(1)}$ is less fragile than $\mathcal{SC}_Q^{(2)}$.

Proposition 5. *Consider two short chains $\mathcal{SC}_Q^{(1)}$ and $\mathcal{SC}_Q^{(2)}$ in $\{\mathcal{SC}_Q\}$ such that the components of $\mathcal{SC}_Q^{(1)}$ are the decomposition components of $\mathcal{SC}_Q^{(2)}$. For any plant disruption parameter γ if $\alpha \geqslant (Q-1)^2$, then $Fr(\mathcal{SC}_Q^{(1)}) \leqslant Fr(\mathcal{SC}_Q^{(2)})$.*

Based on equation (17) the worst-case performances of $\mathcal{SC}_Q^{(1)}$ and $\mathcal{SC}_Q^{(2)}$ under supply and demand uncertainties coincide, i.e., $R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{SC}_Q^{(1)}) = R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{SC}_Q^{(2)})$. Then in order to demonstrate that $Fr(\mathcal{SC}_Q^{(1)}) \leqslant Fr(\mathcal{SC}_Q^{(2)})$ the main effort is to show that $R(\mathcal{U}_d, \mathcal{SC}_Q^{(1)}) \leqslant R(\mathcal{U}_d, \mathcal{SC}_Q^{(2)})$. We ascertain the latter by comparing the PCIDs of two designs without disruptions, see the proof of Proposition 5 in the supplementary material.

The following example illustrates that if the condition of Proposition 5 does not hold, then the fragility of designs depends on the uncertainty sets considered.

Example 4. Let $\mathcal{SC}_3^{(1)}$ and $\mathcal{SC}_3^{(2)}$ include five and three equal size components as $\{3,3,3,3,3\}$ and $\{5, 5, 5\}$, respectively. By using equation (15), we can observe that

$$\min_{g\in\mathcal{U}_p} \delta^{k,\ell}(\mathbf{e},\mathcal{SC}_3^{(1)}) = \min_{g\in\mathcal{U}_p} \delta^{k,\ell}(\mathbf{e},\mathcal{SC}_3^{(2)})$$

for all $0\leqslant k\leqslant n$ and $4\leqslant \ell\leqslant 45$. Thus, Theorem 1 leads to $R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{SC}_3^{(1)}) = R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{SC}_3^{(2)})$ for $4\leqslant \alpha\leqslant 45$; however, for $\ell = 0$, we have $\delta^{5,0}(\mathbf{e},\mathcal{SC}_3^{(1)}) = 12 > \delta^{5,0}(\mathbf{e},\mathcal{SC}_3^{(2)}) = 10$ and $\delta^{6,0}(\mathbf{e},\mathcal{SC}_3^{(1)}) = 9 < \delta^{6,0}(\mathbf{e},\mathcal{SC}_3^{(2)}) = 10$. Therefore, none of the designs' fragility dominates the other because by Theorem 1, we can find demand uncertainty sets such that $R(\mathcal{U}_d,\mathcal{SC}_3^{(1)}) < R(\mathcal{U}_d,\mathcal{SC}_3^{(2)})$ or $R(\mathcal{U}_d,\mathcal{SC}_3^{(1)}) > R(\mathcal{U}_d,\mathcal{SC}_3^{(2)})$. □

Let again the components of $\mathcal{SC}_Q^{(1)}$ be the decomposition components of $\mathcal{SC}_Q^{(2)}$. For $\alpha\geqslant(Q-1)^2$, based on Propositions 4 and 5 we have:

$$Fr(\mathcal{SC}_Q^{(1)})\leqslant Fr(\mathcal{SC}_Q^{(2)})\leqslant Fr(\mathcal{LC}_Q).$$

Moreover, by equation (17):

$$\begin{aligned} R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{SC}_Q^{(1)}) &= R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{SC}_Q^{(2)}) \\ &= R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{LC}_Q). \end{aligned}$$

Hence, we conclude that:

$$R(\mathcal{U}_d,\mathcal{SC}_Q^{(1)})\leqslant R(\mathcal{U}_d,\mathcal{SC}_Q^{(2)})\leqslant R(\mathcal{U}_d,\mathcal{LC}_Q). \tag{21}$$

Relation (21) implies that, without disruptions, longer chains (i.e., chains with fewer number of components) have better performances than shorter ones. Moreover, the performance of chains is bounded above by $R(\mathcal{U}_d,\mathcal{LC}_Q)$. On the other hand, recall the discussion in Section 4.2 that the construction of shorter chains is less costly. Therefore, there is a trade-off between the worst-case performance of a chain without disruptions and the cost of flexibility.

In contrast with Proposition 4, we show in the following result that if there is only one plant disruption without any arc disruptions, then the fragility of $\mathcal{LC}_Q$ is less than that of any $\mathcal{SC}_Q$.

Proposition 6. *If a design is subject to only one plant disruption, i.e., $\alpha = 0$ and $\gamma = 1$, then $Fr(\mathcal{LC}_Q)\leqslant Fr(\mathcal{SC}_Q)$ for any $\mathcal{SC}_Q$ in $\{\mathcal{SC}_Q\}$.*

Proposition 6 is proved by either demonstrating that $Fr(\mathcal{SC}_Q) = 1$ and $Fr(\mathcal{LC}_Q)\leqslant 1$, hence $Fr(\mathcal{LC}_Q)\leqslant Fr(\mathcal{SC}_Q)$, or showing that $R(\mathcal{U}_d,\mathcal{SC}_Q) = R(\mathcal{U}_d,\mathcal{LC}_Q)$, which also leads to $Fr(\mathcal{LC}_Q)\leqslant Fr(\mathcal{SC}_Q)$ since $R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{SC}_Q) \leqslant R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{LC}_Q)$ based on inequality (16).

Proposition 6 implies that $\mathcal{LC}_Q$ is less sensitive than $\mathcal{SC}_Q$ under a single plant disruption. The intuition behind this result is that – under a single plant disruption without arc disruptions – a long chain allows us to better utilize the remaining capacity than localizing the effect of the disruption in short chains.

From Propositions 4, 5, and 6 we conclude that the impact of a plant disruption differs from that of arc disruptions. Losing a plant leads to a reduction in the plant's supply capacity, whereas losing arcs decreases the flexibility of a design. These results are important as the likelihoods of arc and plant disruptions and the costs of constructing different flexibility designs are not the same for each industry. Therefore, the preferred flexibility may differ depending on the company needs.

Remark 4. We can similarly define fragility as the impact of the worst-case disruptions on the expected performance, $E(\xi,\mathcal{D})$, i.e.,

$$Fr_\xi(\mathcal{D}) = E(\xi,\mathcal{D}) - E(\xi,\mathcal{U}_p,\mathcal{U}_a,\mathcal{D}).$$

It is worth mentioning that in our experiments (not reported here) the inequalities established in Propositions 4, 5, and 6 also hold with respect to $Fr_\xi(\mathcal{D})$ for the designs and configurations used in Example 3. □

6. Generating flexibility designs

The majority of the available algorithms for generating sparse flexibility designs are not intended for settings where the design is susceptible to disruptions, e.g., we refer to Deng and Shen (2013) and Chen *et al.* (2015). In this section, we propose an algorithm that exploits the notion of PCID to take into account possible supply disruptions to generate both balanced and unbalanced, as well as homogenous and non-homogenous, designs. We then evaluate the worst-case and the expected performances of these designs, and compare them against designs generated by other algorithms from the literature in settings with and without disruptions.

The idea of our approach is as follows. Consider an initial design $\mathcal{D}$, e.g., the dedicated design (Figure 1(b)), and arc and plant disruption parameters α and γ. We aim to add $\mathcal{E}$ arcs to this initial design. Our method is based on Theorem 1, which implies that a larger value for $\sum_{0\leqslant k\leqslant n-1} \min_{g\in\mathcal{U}_p}\delta^{k,\alpha}(\boldsymbol{g},\mathcal{D})$ may translate into a better performance. Therefore, an increase in the value of PCID, and subsequently, $\min_{g\in\mathcal{U}_p}\delta^{k,\alpha}(\boldsymbol{g},\mathcal{D})$, may lead to a better worst-case performance. Hence, our algorithm adds arc $(i,j)\notin\mathcal{D}$ to $\mathcal{D}$ at each iteration in order to increase PCID.

To elaborate further, from Assumption 1 we know that $\ell^\star = \alpha$ for $R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{D})$ in equation (10). Recall Remark 3 part (*vi*), then for given k and α, let $S^\star$ and $E^\star$ denote an optimal solution for the following problem:

$$\min_{g\in\mathcal{U}_p}\delta^{k,\alpha}(\boldsymbol{g},\mathcal{D}) = \min_{\substack{S\subseteq B,|S|=k,\\ E\subseteq\mathcal{D},|E|=\alpha,\\ g\in\mathcal{U}_p}} \sum_{a_i\in\mathcal{N}(B\setminus S,\mathcal{D}\setminus E)} c_i^{(p)} g_i. \tag{22}$$

Sets $S^\star$ and $\mathcal{N}(B\setminus S^\star,\mathcal{D}\setminus E^\star)$ create a minimum vertex cover for $\mathcal{D}\setminus E$. For a given k, we can increase $\min_{g\in\mathcal{U}_p}\delta^{k,\alpha}(\boldsymbol{g},\mathcal{D})$ by adding arc $(i,j)\notin\mathcal{D}$ in such a manner that none of its endpoints are part of the vertex cover, and plant i is not disrupted, i.e., $i \in A \setminus \mathcal{N}(B \setminus S^\star, \mathcal{D} \setminus E^\star), g_i \neq 0$ and $j \in B \setminus S^\star$. Thus, we compute $\min_{g \in \mathcal{U}_p} \delta^{k,\alpha}(g, \mathcal{D})$ in the algorithm for all $1 \leqslant k \leqslant n-1$. For each value of k, we obtain a minimum vertex cover. We then select the arc with endpoints belonging to a fewer number of minimum vertex covers over all $0 \leqslant k \leqslant n-1$. This arc has possibly the greatest effect in increasing $\sum_{0 \leqslant k \leqslant n-1} \min_{g \in \mathcal{U}_p} \delta^{k,\alpha}(g, \mathcal{D})$. By iterating this process for $\mathcal{E}$ times, we can create a highly flexible design under disruptions.

The formal pseudo-code is given in Algorithm 1. Since arc disruptions only impact α arcs of the output design, we consider the arc disruptions only for the last α added arcs. Thus, we set the number of arc disruptions as $\alpha' = \max\{0, r - (\mathcal{E} - \alpha)^+\}$ at the rth iteration of the algorithm. Then in STEP 1, we obtain an optimal solution of $\min_{g \in \mathcal{U}_p} \delta^{k,\alpha'}(g, \mathcal{D})$ for all $k \in \{1, ..., n-1\}$. In STEP 2, we define the function $\Psi(p_i^k, q_j^k, g_i^k)$, which yields one if none of the endpoints of arc (i, j) belong to the vertex cover and plant i is not disrupted. Otherwise, it returns zero. Then for each arc, we compute its weight $W(i, j)$ as the sum of $\Psi(p_i^k, q_j^k, g_i^k)$ for different values of k. In STEP 3, set M includes all candidate arcs with the maximum weight. For each arc in M, we compute $\Omega(i,j)$ as the sum of (weighted) degrees of its endpoints. Finally, we pick the arc in M with the minimum value of $\Omega(i,j)$.

Algorithm 1 PCID-based heuristic algorithm that adds $\mathcal{E}$ new arcs to initial design $\mathcal{D}$

For $1 \leqslant r \leqslant \mathcal{E}$
  Set $\alpha' = (r - (\mathcal{E} - \alpha)^+)^+$.
  STEP 1. For $1 \leqslant k \leqslant n-1$
    Find $\min_{g \in \mathcal{U}_p^\gamma} \delta^{k,\alpha'}(g, \mathcal{D})$ as well as its optimal solution $(\mathbf{p}^k, \mathbf{q}^k, \mathbf{t}^k, g^k)$.
   End
  STEP 2. Let $\Psi(p_i^k, q_j^k, g_i^k) = 1$ if $p_i^k = q_j^k = 0$ and $g_i^k = 1$. Otherwise, $\Psi(p_i^k, q_j^k, g_i^k) = 0$.
   For $1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n; (i,j) \notin \mathcal{D}$
    $W(i,j) = \sum_{k=1}^{n-1} \Psi(p_i^k, q_j^k, g_i^k)$.
   End
  STEP 3. $M = \{(i,j) \mid W(i,j) = \max\{W(i',j') \mid 1 \leqslant i' \leqslant m, 1 \leqslant j' \leqslant n, \ (i',j') \notin \mathcal{D}\}\}$.
   For $(i,j) \in M$
    If all products demands' means (μ_j) are known:
    $\Omega(i,j) = \sum_{b_j \in \mathcal{N}(a_i, \mathcal{D})} \frac{\mu_j}{c_i^{(p)}} + \sum_{a_i \in \mathcal{N}(b_i, \mathcal{D})} \frac{c_i^{(p)}}{\mu_j}$
    Else $\Omega(i,j) = deg_{\mathcal{D}}(i) + deg_{\mathcal{D}}(j)$.
   End
  Find arc $(i^\star, j^\star)$ in a manner that $\Omega(i^\star, j^\star) = \min\{\Omega(i,j) \mid \forall (i,j) \in M\}$ (in case of a tie, we randomly select an arc, albeit uniformly, with the minimum $\Omega(i,j)$).
  $\mathcal{D} = \mathcal{D} \cup (a_{i^\star}, b_{j^\star})$.
End

Both Algorithm 1 and the PCI algorithm introduced in Simchi-Levi and Wei (2015) take into account the vertex cover concept to generate a design. However, the key differences of Algorithm 1 are as follows:

- We take into account disruptions by using $\min_{g \in \mathcal{U}_p} \delta^{k,\alpha}(g, \mathcal{D})$ instead of PCI in STEP 1 and define function $\Psi(p_i^k, q_j^k, g_i^k)$ in STEP 2.
- If multiple arcs have the maximum weight (i.e., set M includes multiple arcs), then in STEP 3 we employ $\Omega(i,j)$ comprising of the information about the plants and products. If all products demands' means (μ_j) are known, then by computing the vertices' expansion ratios (Chou *et al.*, 2011) and selecting the arc that has the minimum value of $\Omega(i,j)$ we can better allocate unutilized plant capacities to unsatisfied products demands. If the means are unknown, then we can simply use the arc that has the minimum sum of its vertices degrees instead to more equalize vertices degrees; note that highly connected designs such as *Desargues graph* (Kutnar and Marušič, 2009, Figure 5) or *Levi graph* (Chou *et al.*, 2011, Figure 1) have the latter property. In contrast, in the PCI algorithm, when multiple arcs have the maximum weight, one arc is selected randomly in a uniform manner.

These features help us to generate designs that are less vulnerable to disruptions than those constructed by earlier methods from the literature.

With respect to the computational complexity of Algorithm 1, we first note that $\min_{g \in \mathcal{U}_p} \delta^{k,\alpha}(g, \mathcal{D})$ can be formulated as a Mixed-Integer Linear Program (MILP) with $O(m + n + |\mathcal{D}|)$ binary and continuous variables as well as $O(m + |\mathcal{D}|)$ constraints; see problem (A2) in the supplementary material for an MILP formulation. In STEP 1, we need to solve $O(\mathcal{E}n)$ problems of the form $\min_{g \in \mathcal{U}_p} \delta^{k,\alpha}(g, \mathcal{D})$ and thus, Algorithm 1 requires solution of $O(\mathcal{E}n)$ MILPs of size $O(m + n + |\mathcal{D}|)$. Clearly, the latter complexity dominates the complexity of STEPs 2 and 3 of Algorithm 1. Thanks to recent advances in commercial optimization solvers such as CPLEX (2017), our experiments (see Table 2 below) show that for reasonably sized designs MILPs for $\min_{g \in \mathcal{U}_p} \delta^{k,\alpha}(g, \mathcal{D})$ can be solved reasonably quickly.

Benchmarks. In order to evaluate the designs generated by Algorithm 1, we select PCI (Simchi-Levi and Wei, 2015) and Expander (Chou *et al.*, 2011) algorithms as the benchmarks among available algorithms. As reported by Simchi-Levi and Wei (2015) designs generated by the PCI algorithm outperform those generated by existing algorithms in the literature, including the algorithms proposed by Hopp *et al.* (2004), and Chou *et al.* (2011). However, under asymmetric demand, designs created by the Expander algorithm outperform those generated by the PCI algorithm.

Measures. To evaluate the performances of the constructed designs, we consider four measures:

- For given α and γ, measure $\Delta_{\alpha,\gamma}$ is the summation of $\min_{g \in \mathcal{U}_p^\gamma} \delta^{k,\alpha}(g, \mathcal{D})$ over all $0 \leqslant k \leqslant n$, i.e., $\Delta_{\alpha,\gamma} = \sum_{0 \leqslant k \leqslant n} \min_{g \in \mathcal{U}_p^\gamma} \delta^{k,\alpha}(g, \mathcal{D})$.
- For a given γ, measure $\bar{\Delta}_\gamma$ is the summation of $\Delta_{\alpha,\gamma}$ over all $0 \leqslant \alpha \leqslant |\mathcal{D}|$, i.e., $\bar{\Delta}_\gamma = \sum_{0 \leqslant \alpha \leqslant |\mathcal{D}|} \Delta_{\alpha,\gamma}$.
  Based on Theorem 1, for γ plant disruptions, higher values of $\Delta_{\alpha,\gamma}$ and $\bar{\Delta}_\gamma$ are indicators of better worst-case performances for a particular α and general $\alpha \in \{0, \ldots, |\mathcal{D}|\}$, respectively.

Table 2. Results for the designs generated by Expander, PCI, and PCID algorithms under different disruption settings.

		T1			*T2*			*T3*			*T4*			*T5*		
	Measure	*Expander*	*PCI*	*PCID*	*Expander*	*PCI*	*PCID*	*Expander*	*PCI*	*PCID*	*Expander*	*PCI*	*PCID*	*Expander*	*PCI*	*PCID*
								$\gamma=0$								
$\alpha=0$	$\Delta_{\alpha,\gamma}$	9.1	4.5	**0.0**	3.6	**0.0**	**0.0**	3.9	2.2	**0.0**	10.1	2.7	**0.0**	19.5	0.9	**0.0**
	$E(\xi,\mathcal{U}_p,\mathcal{U}_a,\mathcal{D})$	5.1	0.9	**0.0**	0.1	0.1	**0.0**	**0.0**	**0.0**	**0.0**	**0.0**	**0.0**	**0.0**	**0.0**	**0.0**	**0.0**
	Avg. Perf.	5.1	0.9	**0.0**	0.1	0.1	**0.0**	**0.0**	**0.0**	**0.0**	**0.0**	**0.0**	**0.0**	**0.0**	**0.0**	**0.0**
	Time (sec.)	0.0	22.4	26.1	0.0	37.3	51.7	0.0	43.3	41.1	0.0	29.4	29.6	0.0	54.2	58.9
$\alpha=3$	$\Delta_{\alpha,\gamma}$	**0.0**	4.4	**0.0**	2.1	4.1	**0.0**	7.5	1.1	**0.0**	7.4	4.4	**0.0**	27.5	2.6	**0.0**
	$E(\xi,\mathcal{U}_p,\mathcal{U}_a,\mathcal{D})$	0.2	3.1	**0.0**	**0.0**	3.0	0.1	1.7	5.8	**0.0**	**0.0**	2.9	1.5	**0.0**	5.9	2.3
	Avg. Perf.	3.4	1.1	**0.0**	0.1	0.4	**0.0**	**0.0**	0.7	0.2	**0.0**	0.6	0.3	**0.0**	3.5	4.2
	Time (sec.)	0.0	26.7	20.4	0.0	22.8	27.8	0.0	40.6	32.9	0.0	33.9	27.2	0.0	44.1	40.6
$\alpha=5$	$\Delta_{\alpha,\gamma}$	**0.0**	**0.0**	**0.0**	1.3	7.6	**0.0**	8.2	2.8	**0.0**	7.5	3.1	**0.0**	29.0	0.9	**0.0**
	$E(\xi,\mathcal{U}_p,\mathcal{U}_a,\mathcal{D})$	0.3	1.4	**0.0**	0.1	4.3	**0.0**	0.6	5.6	**0.0**	**0.0**	7.9	2.6	**0.0**	10.4	4.7
	Avg. Perf.	1.8	0.7	**0.0**	**0.0**	0.8	0.3	**0.0**	1.4	0.2	0.4	0.8	**0.0**	**0.0**	2.5	5.2
	Time (sec.)	0.0	17.1	10.8	0.0	28.5	24.3	0.0	40.4	29.8	0.0	19.3	14.9	0.0	52.8	36.0
$\alpha=7$	$\Delta_{\alpha,\gamma}$	**0.0**	**0.0**	**0.0**	1.5	12.1	**0.0**	13.0	4.7	**0.0**	7.6	7.6	**0.0**	31.6	4.8	**0.0**
	$E(\xi,\mathcal{U}_p,\mathcal{U}_a,\mathcal{D})$	0.2	0.6	**0.0**	**0.0**	4.2	0.3	1.3	2.6	**0.0**	**0.0**	11.0	2.6	**0.0**	17.6	6.5
	Avg. Perf.	**0.0**	0.1	0.5	0.7	0.9	**0.0**	**0.0**	0.3	**0.0**	**0.0**	1.9	1.7	**0.0**	3.7	4.4
	Time (sec.)	0.0	15.7	9.2	0.0	28.9	18.0	0.0	31.7	25.6	0.0	21.8	12.8	0.0	51.0	30.0
	$\bar{\Delta}_{\gamma}$	3.2	9.8	**0.0**	0.2	2.1	**0.0**	8.7	**0.0**	0.2	8.8	6.2	**0.0**	29.5	1.4	**0.0**
								$\gamma=1$								
$\alpha=0$	$\Delta_{\alpha,\gamma}$	12.3	7.0	**0.0**	2.1	**0.0**	1.1	7.6	4.4	**0.0**	13.7	1.9	**0.0**	22.1	1.1	**0.0**
	$E(\xi,\mathcal{U}_p,\mathcal{U}_a,\mathcal{D})$	7.3	2.6	**0.0**	0.1	**0.0**	0.1	0.7	0.7	**0.0**	1.0	**0.0**	**0.0**	2.2	0.4	**0.0**
	Avg. Perf.	5.6	1.2	**0.0**	**0.0**	0.9	1.0	**0.0**	1.0	0.8	**0.0**	1.6	1.6	**0.0**	0.5	0.3
	Time (sec.)	0.0	12.9	13.6	0.0	29.3	18.6	0.0	26.6	19.2	0.0	18.4	9.6	0.0	44.1	28.5
$\alpha=3$	$\Delta_{\alpha,\gamma}$	**0.0**	**0.0**	**0.0**	1.6	1.6	**0.0**	9.9	7.1	**0.0**	6.6	5.1	**0.0**	26.8	2.3	**0.0**
	$E(\xi,\mathcal{U}_p,\mathcal{U}_a,\mathcal{D})$	0.4	0.4	**0.0**	**0.0**	1.1	0.8	3.1	**0.0**	1.0	**0.0**	5.3	3.3	0.1	11.1	**0.0**
	Avg. Perf.	3.5	**0.0**	0.5	2.0	**0.0**	0.3	1.4	**0.0**	0.5	**0.0**	0.6	0.6	**0.0**	3.5	2.4
	Time (sec.)	0.0	17.8	10.4	0.0	22.1	21.6	0.0	27.4	20.4	0.0	16.8	11.8	0.0	47.2	29.6
$\alpha=5$	$\Delta_{\alpha,\gamma}$	**0.0**	**0.0**	**0.0**	**0.0**	4.3	2.2	5.2	3.8	**0.0**	11.5	8.0	**0.0**	30.2	4.0	**0.0**
	$E(\xi,\mathcal{U}_p,\mathcal{U}_a,\mathcal{D})$	**0.0**	1.0	0.3	**0.0**	4.4	0.6	0.9	0.2	**0.0**	**0.0**	3.2	4.2	**0.0**	18.6	1.8
	Avg. Perf.	2.3	**0.0**	0.5	1.5	0.4	**0.0**	0.5	**0.0**	1.2	**0.0**	2.5	3.1	**0.0**	5.3	4.1
	Time (sec.)	0.0	15.1	8.9	0.0	20.1	18.1	0.0	28.1	18.8	0.0	14.9	12.2	0.0	42.4	27.2
$\alpha=7$	$\Delta_{\alpha,\gamma}$	**0.0**	**0.0**	**0.0**	2.8	16.7	**0.0**	12.8	7.9	**0.0**	11.9	7.9	**0.0**	31.1	3.3	**0.0**
	$E(\xi,\mathcal{U}_p,\mathcal{U}_a,\mathcal{D})$	0.3	3.0	**0.0**	0.2	4.4	**0.0**	2.1	3.2	**0.0**	**0.0**	9.3	0.6	**0.0**	12.0	6.7
	Avg. Perf.	0.8	2.3	**0.0**	0.4	**0.0**	**0.0**	**0.0**	0.1	0.6	1.0	0.1	**0.0**	**0.0**	4.4	4.3
	Time (sec.)	0.0	11.7	8.2	0.0	19.5	16.9	0.0	24.4	18.4	0.0	14.9	12.2	0.0	30.3	25.1
	$\bar{\Delta}_{\gamma}$	2.4	13.0	**0.0**	4.4	**0.0**	1.8	12.3	6.1	**0.0**	8.4	2.2	**0.0**	33.2	2.7	**0.0**
								$\gamma=2$								
$\alpha=0$	$\Delta_{\alpha,\gamma}$	16.7	8.3	**0.0**	14.3	6.3	**0.0**	10.6	8.0	**0.0**	13.5	**0.0**	1.1	22.9	3.8	**0.0**
	$E(\xi,\mathcal{U}_p,\mathcal{U}_a,\mathcal{D})$	8.7	1.5	**0.0**	0.7	0.6	**0.0**	0.1	0.1	**0.0**	**0.0**	**0.0**	0.7	**0.0**	2.0	0.7
	Avg. Perf.	4.8	1.1	**0.0**	4.7	0.2	**0.0**	6.6	**0.0**	0.4	**0.0**	0.4	0.6	**0.0**	2.5	1.4
	Time (sec.)	0.0	11.3	6.6	0.0	16.8	13.0	0.0	20.6	13.7	0.0	13.1	9.1	0.0	28.9	23.4
$\alpha=3$	$\Delta_{\alpha,\gamma}$	**0.0**	17.9	**0.0**	20.5	7.7	**0.0**	14.3	4.8	**0.0**	10.9	1.0	**0.0**	31.1	5.4	**0.0**
	$E(\xi,\mathcal{U}_p,\mathcal{U}_a,\mathcal{D})$	0.4	6.1	**0.0**	5.7	2.3	**0.0**	5.3	5.1	**0.0**	**0.0**	1.8	0.5	**0.0**	11.5	6.7
	Avg. Perf.	3.9	1.5	**0.0**	0.4	**0.0**	0.3	**0.0**	2.7	2.5	**0.0**	2.8	3.1	**0.0**	2.2	3.1
	Time (sec.)	0.0	9.7	6.9	0.0	15.4	14.3	0.0	20.6	16.2	0.0	12.8	10.5	0.0	29.8	23.7
$\alpha=5$	$\Delta_{\alpha,\gamma}$	**0.0**	23.8	**0.0**	19.2	7.7	**0.0**	17.8	11.3	**0.0**	18.3	6.7	**0.0**	34.6	3.6	**0.0**
	$E(\xi,\mathcal{U}_p,\mathcal{U}_a,\mathcal{D})$	0.2	16.5	**0.0**	10.1	5.5	**0.0**	0.7	0.5	**0.0**	0.3	5.5	**0.0**	**0.0**	18.3	7.0
	Avg. Perf.	1.2	2.6	**0.0**	**0.0**	3.2	4.2	**0.0**	1.0	2.1	**0.0**	3.7	3.6	**0.0**	5.0	7.3
	Time (sec.)	0.0	8.1	6.8	0.0	14.1	14.0	0.0	19.1	14.6	0.0	12.5	9.5	0.0	28.9	24.3
$\alpha=7$	$\Delta_{\alpha,\gamma}$	**0.0**	**0.0**	**0.0**	16.7	11.1	**0.0**	14.1	1.3	**0.0**	25.6	15.1	**0.0**	41.1	**0.0**	8.5
	$E(\xi,\mathcal{U}_p,\mathcal{U}_a,\mathcal{D})$	**0.0**	1.1	1.7	12.6	5.9	**0.0**	5.7	1.5	**0.0**	1.6	4.0	**0.0**	**0.0**	25.3	6.5
	Avg. Perf.	2.7	**0.0**	0.9	**0.0**	1.1	1.0	**0.0**	2.7	4.4	**0.0**	2.7	2.6	**0.0**	5.6	3.6
	Time (sec.)	0.0	9.8	7.2	0.0	13.2	13.1	0.0	18.5	13.4	0.0	11.3	9.8	0.0	22.7	21.8
	$\bar{\Delta}_{\gamma}$	6.2	13.8	**0.0**	23.0	14.4	**0.0**	15.5	15.5	**0.0**	15.9	3.4	**0.0**	32.6	1.1	**0.0**
								Average								
	$\Delta_{\alpha,\gamma}$	4.9	5.4	**0.0**	4.7	4.7	**0.0**	8.8	4.3	**0.0**	10.5	3.9	**0.0**	26.8	2.2	**0.0**
	$E(\xi,\mathcal{U}_p,\mathcal{U}_a,\mathcal{D})$	2.4	2.7	**0.0**	1.3	2.4	**0.0**	1.5	2.1	**0.0**	**0.0**	3.8	1.2	**0.0**	9.4	2.9
	Avg. Perf.	2.8	0.8	**0.0**	0.3	0.1	**0.0**	**0.0**	0.1	0.3	**0.0**	1.3	1.2	**0.0**	3.2	3.3
	Time (sec.)	0.0	14.9	11.3	0.0	22.3	21.0	0.0	28.4	22.0	0.0	18.2	14.1	0.0	39.7	30.8
	$\bar{\Delta}_{\gamma}$	3.7	11.8	**0.0**	4.8	3.0	**0.0**	10.9	4.4	**0.0**	9.8	4.5	**0.0**	31.3	1.8	**0.0**

For each test instance, disruption setting, and measure ($M \in \{\Delta_{\alpha,\gamma},\ \bar{\Delta}_{\gamma},\ E(\xi,\mathcal{U}_p,\mathcal{U}_a,\mathcal{D}),$ Avg. Perf.$\}$), we present the results as the percentage in difference ($\frac{M^*-M_{Alg}}{M^*} \times 100\%$) between the value of the measure reported for the output design of an algorithm (M_{Alg}) and the best measure value among the outputs of the three considered algorithms (M^*). The running times (Time) of the algorithms are reported in seconds. The best result for each test instance, disruption setting, and measure is in **bold**.

- The expected performance for the worst-case disruptions, i.e., $E(\xi, \mathcal{U}_p, \mathcal{U}_a, \mathcal{D})$, when the demand distribution is known, as defined in equation (18).
- The average performance, that is the average of $P(\mathbf{d}, \boldsymbol{g}, \mathbf{h}, \mathcal{D})$ over all demands, plant and arc disruption scenarios, denoted as "Avg. Perf." in Table 2.

Test instances. We generate 5000 demand scenarios. We consider disruption parameters $\alpha \in \{0, 3, 5, 7\}$ and $\gamma \in \{0, 1, 2\}$. In order to compute the average performance (Avg. Perf.), for each combination of α and γ, we generate 40 arc disruptions and 15 plant disruptions scenarios (binary vectors), generated uniformly and randomly.

We use the following five test instances as initial designs for the algorithms. Each of these initial designs is then improved by each of the algorithms (Expander, PCI and PCID) by adding $\mathcal{E} = 15$ arcs:

- The test instances T1, T2 and T3 are considered in Simchi-Levi and Wei (2015). For T1, the initial design is a dedicated design, $\mathcal{LC}_1$, where $m = n = 10$. For T2 and T3, the initial design is an unbalanced design where $m = 7, n = 14$ and $\mathcal{D} = \{(1,1), (1,2), (1,3), (2,4), (2,5), (2,6), (3,7), (3,8), (4,9), (4,10), (5,11), (5,12), (6,13), (7,14)\}$. For all these three test instances products' demands are generated from independent normal distributions, where the demand for the jth product has mean μ_j, and standard deviation 0.5. In T1 and T2, $\mu_j = 1$, for $j \in B$, but in T3, μ_j is chosen uniformly and randomly from [0.5,1.5], for all $j \in B$. For all test instances $c_i^{(p)} = \sum_{j \in \mathcal{N}(i, \mathcal{D})} \mu_j$, for $i \in A$.
- As the fourth test instance, T4, we consider the design studied by Chou *et al.* (2008) with m = 7 and n = 9, where the initial design is $\mathcal{D} = \{(1,9), (2,8), (2,9), (3,2), (3,6), (4,3), (4,8), (5,1), (5,5), (6,7), (6,9), (7,4)\}$. Products' demands are normally distributed with means $\mu_1 = 7.6$, $\mu_2 = 5.7$, $\mu_3 = 7.4$, $\mu_4 = 5.1$, $\mu_5 = 4.5$, $\mu_6 = 2.1$, $\mu_7 = 3.6$, $\mu_8 = 7.3$, and $\mu_9 = 8.9$. The standard deviation of each product demand is 40% of its mean. Fixed plant capacities are $c_1^{(p)} = 1.3$, $c_2^{(p)} = 4.7$, $c_3^{(p)} = 8.6$, $c_4^{(p)} = 13.4$, $c_5^{(p)} = 11.7$, $c_6^{(p)} = 8.3$, and $c_7^{(p)} = 6.3$.
- For the fifth test instance, T5, we consider the design proposed by Jordan and Graves (1995) in the context of the auto industry that is also studied by Chou *et al.* (2011) and Feng *et al.* (2017). Instance T5 has an initial design with $m = 8$, $n = 16$, and $\mathcal{D} = \{(1,1), (1,2), (2,3), (3,4), (3,5), (4,5), (4,6), (5,7), (5,8), (5,9), (6,9), (6,10), (7,11), (7,12), (7,13), (8,14), (8,15), (8,16)\}$. The demand for each product is assumed to be from a truncated normal distribution $N(\mu_j, (0.4\mu_j)^2)$ with support $[0.2\mu_j, 1.8\mu_j]$ and means $\mu_1 = 320$, $\mu_2 = 150$, $\mu_3 = 270$, $\mu_4 = 110$, $\mu_5 = 220$, $\mu_6 = 110$, $\mu_7 = 120$, $\mu_8 = 80$, $\mu_9 = 140$, $\mu_{10} = 160$, $\mu_{11} = 60$, $\mu_{12} = 35$, $\mu_{13} = 40$, $\mu_{14} = 35$, $\mu_{15} = 30$, and $\mu_{16} = 180$. The plant capacities are $c_1^{(p)} = 380$, $c_2^{(p)} = 230$, $c_3^{(p)} = 250$, $c_4^{(p)} = 230$, $c_5^{(p)} = 240$, $c_6^{(p)} = 230$, $c_7^{(p)} = 230$, and $c_8^{(p)} = 240$.

Results and discussions. All computational experiments were performed on a PC, where we allocated eight threads (CPU 2.9 GHz) and 32 GB of RAM for each individual experiment. MILPs in PCI and PCID algorithms were solved using CPLEX 12.7.1 (2017) and all the algorithms were coded in C++ programming environment. Table 2 reports the performance of the designs generated by the considered algorithms. From these results we make the following observations.

First, the designs generated by the PCID algorithm have the largest values for $\Delta_{\alpha,\gamma}$ and $\bar{\Delta}_\gamma$ in most of our experiments. As a consequence, on average (see the bottom part of Table 2) the PCID-based designs outperform the other algorithms' designs for all of the considered test instances with respect to $\Delta_{\alpha,\gamma}$ and $\bar{\Delta}_\gamma$. Therefore, we expect that the designs generated by PCID have a better worst-case performance, $R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{D})$, under disruptions than designs constructed by the other algorithms (recall Theorem 1).

Furthermore, the PCID algorithm's designs, on average, have the best performances with respect to $E(\xi, \mathcal{U}_p, \mathcal{U}_a, \mathcal{D})$ and Avg. Perf. for T1 and T2; they are also superior designs with respect to $E(\xi, \mathcal{U}_p, \mathcal{U}_a, \mathcal{D})$ in T3. However, on average, the Expander algorithm's designs show slightly better performances than those generated by the PCID algorithm with respect $E(\xi, \mathcal{U}_p, \mathcal{U}_a, \mathcal{D})$ and Avg. Perf. for T4 and T5. Nevertheless, the designs generated by the Expander algorithm are relatively poor in our experiments with respect to $\Delta_{\alpha,\gamma}$ and $\bar{\Delta}_\gamma$, in particular, for T3, T4 and T5.

The designs generated by the PCI algorithm are competitive against the other designs in only a few experiments. Thus, on average, their performance is always dominated by at least one of the other two algorithms' designs with respect to the considered measures.

To summarize the discussion above, we conclude that the designs generated by the PCID algorithm are reasonably well protected against the worst-case demand and supply disruption scenarios. Furthermore, the performance of the PCID-based designs is competitive with the designs generated by the PCI and Expander algorithms with respect to all of the other performance measures in both disruption and no-disruption settings.

Finally, with respect to the running times the PCID algorithm is typically slower than Expander algorithm, but relatively close to the running time of PCI algorithm. This is reasonable to expect given that both the PCI and PCID algorithms involve solutions of MILPs. Nevertheless, in all of our experiments the overall running times are within one minute or less.

7. Conclusion

This paper studies the worst-case performance of process flexibility designs. In addition to the demand uncertainty, we assume that designs are susceptible to plant and arc disruptions. We define PCID, denoted by $\delta^{k,\ell}(\boldsymbol{g}, \mathcal{D})$, as the minimum required capacity of plants to create a vertex cover on $\mathcal{D}$, given that the vertex cover contains exactly k products, exactly ℓ arcs are ignored, and plants are disrupted according to vector $\boldsymbol{g}$. We show that the worst-case performance of any design can be formulated as a function of PCID and symmetric uncertainty sets.

PCID also allows us to compare the performance of different designs with no additional information on the demand. In particular, we demonstrate optimality of the well-known 2-long chain design over a broad class of designs in the worst-case performance under disruptions. This result is an extension of earlier studies in the literature that show the superiority of 2-long chain design with respect to the expected and the worst-case performances when no disruptions are present.

Furthermore, we show that, for $Q\geqslant 2$, any Q-short chain has the same performance as Q-long chain if the design is subject to a sufficiently large number of arc disruptions. This result holds regardless of the presence of plant disruptions and has noteworthy implications, e.g., if there exists at least one arc disruption, then 2-short chains (which are not often taken into account because of its poor expected performance) are optimal – in the worst case – over all designs for which 2-long chain is optimal.

In the second part of this paper, we consider the notion of fragility that quantifies impacts of disruptions on the worst-case performance. We show that, for $Q\geqslant 2$, Q-long chain is less fragile (sensitive) than any Q-short chain design under a single plant disruption. In contrast, we demonstrate that Q-short chain designs are less fragile than Q-long chain if the number of arc disruptions are sufficiently large regardless of the other disruption parameters.

By using the concept of PCID we also develop an algorithm for generating designs that are less vulnerable to supply and demand uncertainties than the designs generated by earlier methods from the literature. Our computational experiments demonstrate that the designs constructed by the PCID-based algorithm performs well under supply and demand uncertainties in both the worst and the expected cases.

Funding

For Hoda Bidkhori: HR001117S0039 Lagrange DARPA grant. For Oleg Prokopyev and Erfan Mehmanchi: NSF CBET 1803527.

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