Biangular Gabor frames and Zauner's conjecture

Mark Magsino and Dustin G. Mixon

Department of Mathematics, The Ohio State University, Columbus, OH 43210

ABSTRACT

Two decades ago, Zauner conjectured that for every dimension d, there exists an equiangular tight frame consisting of d^2 vectors in $\mathbb{C}^d$. Most progress to date explicitly constructs the promised frame in various dimensions, and it now appears that a constructive proof of Zauner's conjecture may require progress on the Stark conjectures. In this paper, we propose an alternative approach involving biangular Gabor frames that may eventually lead to an unconditional non-constructive proof of Zauner's conjecture.

1. INTRODUCTION

Let $F = \{f_j\}_{j=1}^n$ denote a finite sequence of vectors in $\mathbb{C}^d$. We say F is a **frame** for $\mathbb{C}^d$ if there exist $A, B > 0$ such that for every $x \in \mathbb{C}^d$, it holds that

$$A\|x\|_2^2 \leq \sum_{j=1}^n |\langle x, f_j \rangle|^2 \leq B\|x\|_2^2.$$

We say F is **tight** if one may take $A = B$, and we say F is **unit norm** if $\|f_j\|_2 = 1$ for every j. Finally, we say a unit norm F is **equiangular** if there exists $\alpha \geq 0$ such that $|\langle f_j, f_{j'} \rangle|^2 = \alpha$ whenever $j \neq j'$. The lines spanned by the vectors in an equiangular tight frame (ETF) happen to form an optimal packing of points in projective space, as they achieve equality in the so-called Welch bound.[1] As an artifact of this optimality, ETFs enjoy applications in compressed sensing,[2] digital fingerprinting,[3] multiple description coding,[4] and quantum state tomography.[5] The Gerzon bound[6] states that there exists an equiangular tight frame of n vectors in $\mathbb{C}^d$ only if $n \leq d^2$. Zauner conjectured in his doctoral thesis[7] that for every dimension d, there exists an equiangular tight frame that saturates the Gerzon bound. Such an equiangular tight frame is also known as a **symmetric informationally complete positive operator-valued measure (SIC)**.

In the sequel, we identify $\mathbb{C}^d$ with the space of complex-valued functions over $\mathbb{Z}_d := \mathbb{Z}/d\mathbb{Z}$. Put $\omega := e^{2\pi i/d}$, and define the **translation** and **modulation** operators $T, M \colon \mathbb{C}^d \to \mathbb{C}^d$ by

$$(Tv)(j) = v(j-1), \qquad (Mv)(j) = \omega^j \cdot v(j), \qquad v \in \mathbb{C}^d.$$

It is straightforward to verify that $G(v) := \{M^\ell T^k v\}_{k,\ell=0}^{d-1}$ is a tight frame with frame bound $d\|v\|^2$ for every choice of $v \neq 0$. We refer to $G(v)$ as the **Gabor frame** generated by v. When $G(v)$ is equiangular, we say that v is a **fiducial vector**. In his conjecture, Zauner actually predicted the existence of a fiducial vector (of a particular form) in $\mathbb{C}^d$ for every d. As a consequence of the theory of projective t-designs,[8] it holds that

$$\frac{1}{d} \sum_{k,\ell=0}^{d-1} |\langle v, M^\ell T^k v \rangle|^4 \geq \frac{2}{d+1}, \tag{1}$$

with equality precisely when v is a fiducial vector. As such, one can hunt for fiducial vectors by numerically minimizing the left-hand side of (1), and in fact, this approach has been used to identify putative fiducial vectors to machine precision for every $d \leq 151$, and for a handful of larger dimensions.[9] We say "putative fiducial vectors" because it is possible (albeit unlikely) that there is no solution to the defining system of polynomials that resides in a neighborhood of the numerical solution; a guarantee to the contrary would require a version of the Łojasiewicz inequality[10] with explicit constants.

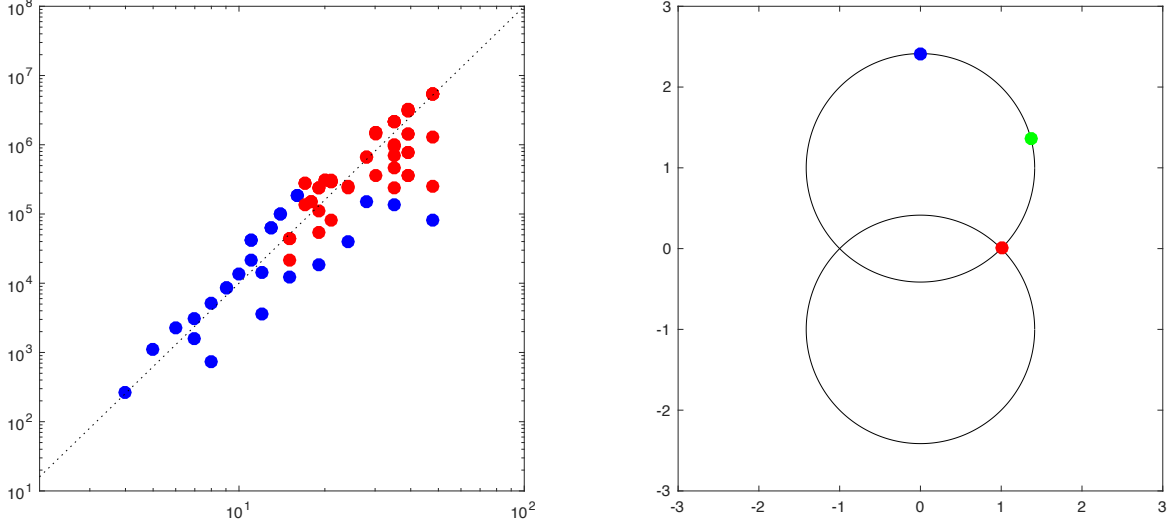Send correspondence to magsino.2@osu.edu and mixon.23@osu.edu

Figure 1. **(left)** Description lengths of naive expressions of fiducial vectors of SICs. For each fiducial reported,[16,17] count the number of characters used to describe the coordinates of the fiducial vector, and plot the results. Blue dots correspond to solutions obtained by Gröbner basis calculation, and red dots correspond to solutions obtained from promoting numerical solutions. The horizontal axis corresponds to the dimension d, while the vertical axis denotes the description length. The dotted line plots the curve d^4. This illustrates that the naive representation of fiducial coordinates in terms of radicals is inefficient, and so a more compact representation (in terms of Stark units,[14] say) is necessary before one can find a constructive proof of Zauner's conjecture. **(right)** Illustration of the proposed method. The set of (x, y) for which $G((1, x + yi))$ is biangular equals the union of two intersecting circles, plotted in black. The blue dot at $(x, y) = (0, 1 + \sqrt{2})$ corresponds to a Gabor MUB, while the red dot at $(x, y) = (1, 0)$ corresponds to the trivial biangular Gabor frame $G(\mathbf{1})$. The angles α and β satisfy $\alpha < \beta$ in the former case and $\alpha > \beta$ in the latter case. As such, for any curve connecting these two points, there exists a point $(x^\star, y^\star)$ at which $\alpha = \beta$, i.e., $G((1, x^\star + y^\star i))$ is a SIC. Such a point is plotted in green.

In order to identify honest fiducial vectors, one is inclined to solve the defining system of polynomials, and to this end, solutions have been obtained by Gröbner basis calculation.[11] However, calculating a Gröbner basis for even modest polynomial systems requires a substantial amount of memory and runtime, and so progress with this approach quickly stalled. Interestingly, the resulting fiducial vectors exhibit some predictable field structure,[12] and these observations have been leveraged to systematically promote numerical solutions to exact solutions[13] in dimensions that are much too large for Gröbner basis calculation. At this point, a new bottleneck has emerged: the naive description length of fiducial vectors grows quickly with the dimension. Case in point, the exact coordinates of one fiducial in dimension $d = 48$ "occupies almost a thousand A4 pages (font size 9 and narrow margins)".[13] Figure 1 illustrates that the description length appears to scale like d^4. Presumably, these coordinates enjoy a more compact description in some other representation. For example, the number fields in which the known fiducial coordinates reside are conjectured to be generated by Stark units.[14] Recently, Kopp[15] leveraged Stark units to formulate a conjectural construction of fiducial vectors in prime dimensions $d \equiv 2 \mod 3$, and this construction produced the first known exact solution in dimension $d = 23$.

Overall, the community appears to be converging towards a constructive proof of Zauner's conjecture that is conditional on the Stark conjectures. As an unconditional alternative, one might entertain the possibility of a non-constructive proof. One idea along these lines, posed by Peter Shor on MathOverflow,[18] is to leverage some sort of geometric fixed point theorem; sadly, no progress in this direction has been made public. In this paper, we propose another possible route towards a non-constructive proof. In particular, we relax the set of equiangular Gabor frames to a set of *biangular* Gabor frames. This larger set includes well-known constructions of *mutually unbiased bases*. We observe that this set is frequently one-dimensional, which opens up the possibility of a proof of Zauner's conjecture by the intermediate value theorem.

2. THE PROPOSED APPROACH

In this section, we outline an approach to prove Zauner's conjecture using the intermediate value theorem. We say $G(v)$ is **biangular** if there exists α and β such that

(i) $|\langle v, T^k v \rangle|^2 = \alpha$ for every $k \in \{1, \ldots, d-1\}$, and

(ii) $|\langle v, M^\ell T^k v \rangle|^2 = \beta$ for every $k \in \{0, \ldots, d-1\}$ and $\ell \in \{1, \ldots, d-1\}$.

In this case, we can be more precise by saying that $G(v)$ is (α, β)-biangular. We note that the angle parameters α and β depend on one another:

LEMMA 1. *If $G(v)$ is an (α, β)-biangular Gabor frame for $\mathbb{C}^d$, then $\alpha + d\beta = \|v\|_2^4$.*

Proof. By tightness, we have

$$d\|v\|_2^4 = \sum_{k,\ell=0}^{d-1} |\langle v, M^\ell T^k v \rangle|^2 = \|v\|_2^4 + (d-1)\alpha + (d^2 - d)\beta,$$

and so rearranging gives the result. □

It is helpful to consider a few examples of biangular Gabor frames:

EXAMPLE 2.

(a) Let $\mathbf{1}$ denote the all-ones vector in $\mathbb{C}^d$. Then $G(\mathbf{1})$ is biangular.

(b) Suppose $G(v)$ is a $(0, 1/d)$-biangular Gabor frame in $\mathbb{C}^d$. Then each $\{M^\ell T^k v\}_{k=0}^{d-1}$ is an orthonormal basis, and together, these bases are mutually unbiased.[19] For example, if $d \geq 5$ is prime and v is the Fourier transform of the corresponding Alltop sequence[20]

$$f(t) := \frac{1}{\sqrt{d}} \cdot e^{2\pi i t^3/d}, \qquad t \in \{0, \ldots, d-1\},$$

then $G(v)$ is $(0, 1/d)$-biangular. We refer to such $G(v)$ as **Gabor MUBs**.

(c) If $G(v)$ is equiangular, then $G(v)$ is biangular with $\alpha = \beta$.

(d) If $G(v)$ is biangular, then $G(cv)$ is also biangular for every $c \in \mathbb{C}^\times$.

Let B_d denote the real algebraic variety of $v \in \mathbb{C}^d$ for which $G(v)$ is biangular. Perhaps surprisingly, we observe that $B_d/\mathbb{C}^\times$ is at times one-dimensional even though B_d is defined by $\Omega(d^2)$ polynomials over $2d$ real variables. We suspect that this feature can be leveraged to prove the existence of SICs. For example, the following result allows us to promote MUBs to SICs:

LEMMA 3. *Suppose there exists a Gabor MUB in $\mathbb{C}^d$ and B_d is path-connected. Then there exists a SIC in $\mathbb{C}^d$.*

Proof. Select v_0 such that $G(v_0)$ is an MUB, put $v_1 = \frac{1}{\sqrt{d}}\mathbf{1}$. Then by path-connectivity, there exists a parameterized curve $v \colon [0,1] \to \mathbb{C}^d$ such that $v(0) = v_0$, $v(1) = v_1$, and $G(v(t))$ is biangular for every $t \in (0,1)$. Without loss of generality, it holds that $\|v(t)\|_2 = 1$ for every t. Define $\alpha, \beta \colon [0,1] \to \mathbb{R}$ such that

$$\alpha(t) := |\langle v(t), Tv(t) \rangle|^2, \qquad \beta(t) := |\langle v(t), MTv(t) \rangle|^2, \qquad t \in [0,1].$$

By Lemma 1, it holds that

$$\Delta(t) := \beta(t) - \alpha(t) = \frac{1 - \alpha(t)}{d} - \alpha(t) = \frac{1 - (d+1)\alpha(t)}{d}.$$

Considering $\alpha(0) = 0$ and $\alpha(1) = 1$, then the continuous function $\Delta \colon [0,1] \to \mathbb{R}$ satisfies $\Delta(0) = 1/d > 0$ and $\Delta(1) = -1 < 0$. The intermediate value theorem then guarantees the existence of $t^\star \in (0,1)$ such that $\Delta(t^\star) = 0$, i.e., $\alpha(t^\star) = \beta(t^\star)$. As such, $G(v(t^\star))$ is equiangular, i.e., the claimed SIC. □

Importantly, Gabor MUBs (unlike SICs) are known to exist in infinitely many dimensions. The bottleneck of applying Lemma 3 is demonstrating path-connectivity. The following provides a sufficient condition to this end:

LEMMA 4. *If $C_d := \{v \in B_d : v(0) = 1\}$ is path-connected, then B_d is path-connected.*

Proof. Suppose C_d is path-connected, and for each $j \in \mathbb{Z}_d$, denote $S_j := \{v \in B_d : v(j) = 1\}$ so that $S_0 = C_d$. Then by symmetry, every S_j is path-connected. To see that B_d is also path-connected, pick any $v_0, v_1 \in B_d$. For each $k \in \{0, 1\}$, we have that v_k is nonzero by assumption, and so one of its coordinates is nonzero, say, coordinate $j_k \in \mathbb{Z}_d$. Let $c_k \colon [0, 1] \to \mathbb{C}^\times$ denote any parameterized curve in $\mathbb{C}^\times$ from $c_k(0) = 1$ to $c_k(1) = v_k(j_k)$. Then $v_k(t) := v_k/c_k(t)$ is a curve in B_d such that $v_k(0) = v_k$ and $v_k(1) \in S_{j_k}$. As such, we can traverse from v_0 to $v_0/v_0(j_0)$ along $v_0(\cdot)$, and then from $v_0/v_0(j_0)$ to $\mathbf{1}$ by the path-connectivity of S_{j_0}, and then from $\mathbf{1}$ to $v_1/v_1(j_1)$ by the path-connectivity of S_{j_1}, and then from $v_1/v_1(j_1)$ to v_1 along the reversal of $v_1(\cdot)$. $\square$

As a proof of concept, we leverage the above results to prove the (well-known) existence of a SIC in $\mathbb{C}^2$. (Importantly, our proof is non-constructive, unlike the usual proof.)

COROLLARY 5. *There exists a SIC in $\mathbb{C}^2$.*

Proof. Put $u = (1, (1 + \sqrt{2})i)$. It is straightforward to verify that $G(u/\|u\|_2)$ is an MUB. We will demonstrate that C_2 is path-connected so that the result follows from Lemmas 3 and 4. To this end, note that $v \in C_2$ if and only if there exist $x, y, \alpha, \beta \in \mathbb{R}$ such that $v = (1, x + yi)$ and

$$4x^2 = \alpha, \qquad 4y^2 = \beta, \qquad (1 - x^2 - y^2)^2 = \beta.$$

In other words, C_2 is the set of all $(1, x + yi)$ such that $x^2 + (y \pm 1)^2 = 2$. Geometrically, this is the union of two circles of radius $\sqrt{2}$ centered at $(0, \pm 1)$; see Figure 1 for an illustration. Since these circles intersect, it follows that C_2 is path-connected, as desired. $\square$

To prove Zauner's conjecture, we would need to replicate this non-constructive proof technique in every dimension. This suggests the following:

PROBLEM 6. *For which dimensions d is B_d is path-connected?*

There has already been some work to prove path-connectivity of certain varieties of frames. Most work along these lines has focused on the variety of unit norm tight frames. Initial work[21] leveraged so-called *eigensteps*[22] to construct explicit paths that demonstrate path-connectivity, whereas a more recent treatment[23] exploits technology from symplectic geometry to obtain a non-constructive proof. It would be interesting if similar technology could be applied to tackle Problem 6.

Next, MUBs are only known to exist in prime power dimensions, and so we would need to improve Lemma 3 before we can hope to prove Zauner's conjecture. In fact, "Gabor MUB" in Lemma 3 can be replaced by any biangular Gabor frame with appropriately small α, suggesting the following:

PROBLEM 7. *For every d, find $v \in \mathbb{C}^d$ such that both $\|v\|_2 = 1$ and $G(v)$ is (α, β)-biangular with $\alpha < \frac{1}{d+1}$.*

Considering the successful instance of Gabor MUBs, it seems reasonable to suspect that Problem 7 can be solved in closed form, even though the $\alpha = \frac{1}{d+1}$ case of SICs has resisted such a solution. Finally, note that we do not require all of B_d to be path-connected, as it suffices to find v_0 and v_1 for which there exist α_0 and α_1 such that

  (i) $G(v_j)$ is $(\alpha_j, \frac{1}{d}(1 - \alpha_j))$-biangular for each $j \in \{0, 1\}$,

  (ii) v_0 and v_1 are path-connected in B_d, and

  (iii) $\alpha_0 < \frac{1}{d+1} < \alpha_1$.

Of course, it is likely easier to solve Problem 6.

To illustrate our observation that biangular Gabor frames enjoy path-connectivity, we run a simple numerical experiment: For each $d \in \{2, 4, 5\}$, we consider the numerical fiducial reported by Scott and Grassl[11] (when $d = 3$, the variety of SIC fiducials is already interesting). Call this vector v_0. We slightly perturb this fiducial and then locally minimize the sum of the squares of the polynomials that define the variety of biangular Gabor
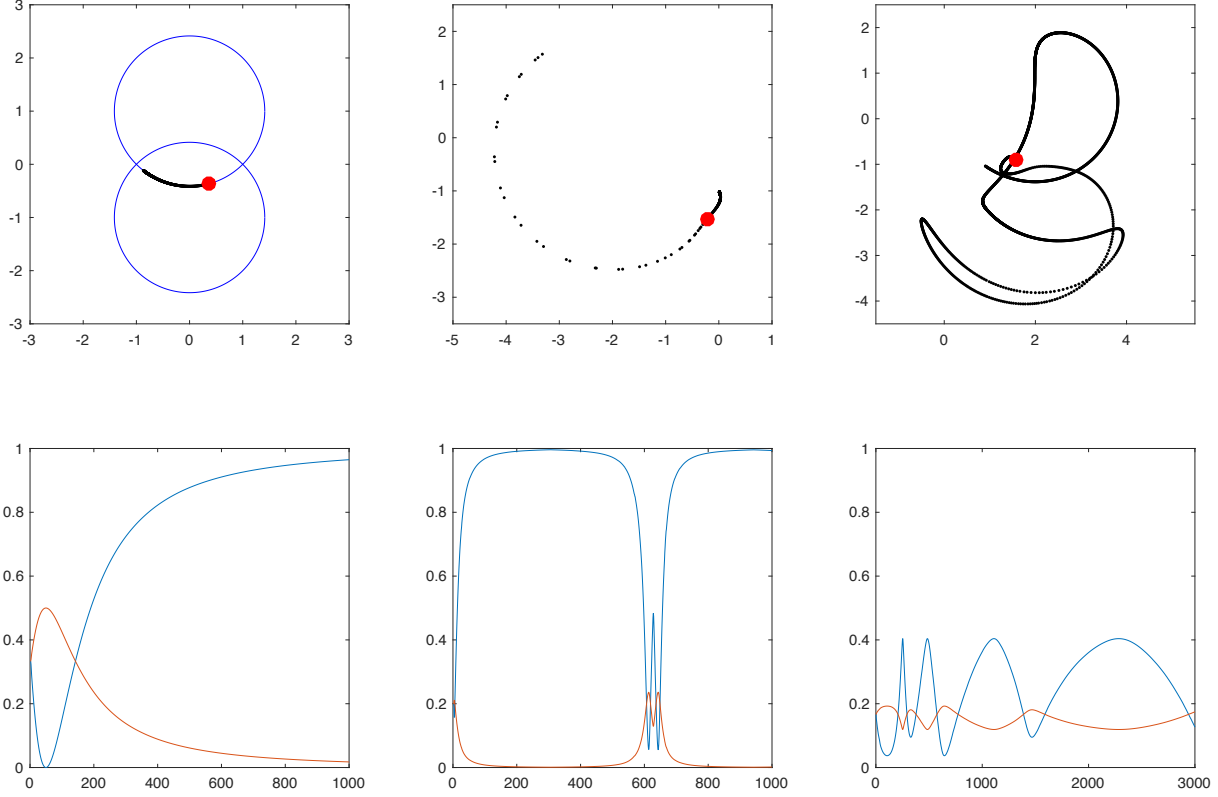
Figure 2. Numerical experiments to illustrate path-connectivity in the variety of biangular Gabor frames. **(left)** As a control, we first consider the case of $d = 2$, which we have already treated in the proof of Corollary 5. In blue, we plot the set of all (x, y) for which $(1, x + yi)$ generates a biangular Gabor frame. The red dot in this set corresponds to the numerical fiducial reported by Scott and Grassl.[11] We then traverse the variety using the numerical scheme discussed at the end of Section 2; we plot the corresponding trajectory in black. In the display below, we plot how (α, β) evolve over this trajectory (we compute these angles after normalizing v to have unit norm). The angles start at $\left(\frac{1}{3}, \frac{1}{3}\right)$, corresponding to the SIC, then pass through $\left(0, \frac{1}{2}\right)$, corresponding to an MUB, and then finally approach $(1, 0)$, corresponding to a trivial Gabor frame. We repeat this experiment for $d = 4$ **(middle)** and $d = 5$ **(right)**, fixing $v(0) = 1$, plotting the trajectory of $(\operatorname{Re} v(1), \operatorname{Im} v(1))$ above, and then plotting the angles (α, β) below. Unlike the $d = 2$ case, we do not have analytic expressions for the variety in these cases.

seed vectors. This produces a new point v_1 on the variety. Next, we locally minimize from the perturbation $v_j + c \cdot \frac{v_j - v_{j-1}}{\|v_j - v_{j-1}\|_2}$ to obtain v_{j+1} (with $j = 1$), and we iterate this procedure to identify a sequence of points on the variety. (Here, c is a small constant.) The results of this experiment are illustrated in Figure 2.

3. DISCUSSION

This paper proposed a new approach to tackle Zauner's conjecture. Specifically, we relax the set of SICs to a larger set of biangular Gabor frames, which appear to form a path-connected variety. This feature could very well allow for a non-constructive proof of Zuaner's conjecture, and we isolate Problems 6 and 7 as steps towards this end. In addition, it would also be interesting to leverage the variety of biangular Gabor frames to facilitate the search for numerical SICs. We leave these investigations for future work.

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REFERENCES

[1] L. Welch, "Lower bounds on the maximum cross correlation of signals (corresp.)," *IEEE Transactions on Information theory* **20**(3), pp. 397–399, 1974.

[2] A. S. Bandeira, M. Fickus, D. G. Mixon, and P. Wong, "The road to deterministic matrices with the restricted isometry property," *Journal of Fourier Analysis and Applications* **19**(6), pp. 1123–1149, 2013.

[3] D. G. Mixon, C. J. Quinn, N. Kiyavash, and M. Fickus, "Fingerprinting with equiangular tight frames," *IEEE Transactions on Information Theory* **59**(3), pp. 1855–1865, 2013.

[4] T. Strohmer and R. W. Heath Jr, "Grassmannian frames with applications to coding and communication," *Applied and computational harmonic analysis* **14**(3), pp. 257–275, 2003.

[5] J. M. Renes, R. Blume-Kohout, A. J. Scott, and C. M. Caves, "Symmetric informationally complete quantum measurements," *Journal of Mathematical Physics* **45**(6), pp. 2171–2180, 2004.

[6] P. W. Lemmens, J. J. Seidel, and J. Green, "Equiangular lines," in *Geometry and Combinatorics*, pp. 127–145, Elsevier, 1991.

[7] G. Zauner, "Grundz uge einer nichtkommutativen designtheorie," 1999.

[8] A. Roy and A. Scott, "Weighted complex projective 2-designs from bases: Optimal state determination by orthogonal measurements," *Journal of mathematical physics* **48**(7), p. 072110, 2007.

[9] C. Fuchs, M. Hoang, and B. Stacey, "The SIC question: History and state of play," *Axioms* **6**(3), p. 21, 2017.

[10] S. Ji, J. Kollár, and B. Shiffman, "A global Łojasiewicz inequality for algebraic varieties," *Transactions of the American Mathematical Society* **329**(2), pp. 813–818, 1992.

[11] A. J. Scott and M. Grassl, "Symmetric informationally complete positive-operator-valued measures: A new computer study," *Journal of Mathematical Physics* **51**(4), p. 042203, 2010.

[12] M. Appleby, S. Flammia, G. McConnell, and J. Yard, "SICs and algebraic number theory," *Foundations of Physics* **47**(8), pp. 1042–1059, 2017.

[13] M. Appleby, T.-Y. Chien, S. Flammia, and S. Waldron, "Constructing exact symmetric informationally complete measurements from numerical solutions," *Journal of Physics A: Mathematical and Theoretical* **51**(16), p. 165302, 2018.

[14] H. M. Stark, "*L*-functions at $s = 1$. III. Totally real fields and Hilbert's twelfth problem," *Advances in Mathematics* **22**(1), pp. 64–84, 1976.

[15] G. S. Kopp, "SIC-POVMs and the Stark conjectures," *arXiv preprint arXiv:1807.05877* , 2018.

[16] "SIC fiducials." http://www.gerhardzauner.at/sicfiducials.html.

[17] "Exact SIC fiducial vectors." http://www.physics.usyd.edu.au/~sflammia/SIC/.

[18] "Fixed point theorems and equiangular lines." https://mathoverflow.net/questions/30894/fixed-point-theorems-and-equiangular-lines.

[19] M. Planat, H. C. Rosu, and S. Perrine, "A survey of finite algebraic geometrical structures underlying mutually unbiased quantum measurements," *Foundations of Physics* **36**(11), pp. 1662–1680, 2006.

[20] W. Alltop, "Complex sequences with low periodic correlations (corresp.)," *IEEE Transactions on Information Theory* **26**(3), pp. 350–354, 1980.

[21] J. Cahill, D. G. Mixon, and N. Strawn, "Connectivity and irreducibility of algebraic varieties of finite unit norm tight frames," *SIAM Journal on Applied Algebra and Geometry* **1**(1), pp. 38–72, 2017.

[22] J. Cahill, M. Fickus, D. G. Mixon, M. J. Poteet, and N. Strawn, "Constructing finite frames of a given spectrum and set of lengths," *Applied and Computational Harmonic Analysis* **35**(1), pp. 52–73, 2013.

[23] T. Needham and C. Shonkwiler, "Symplectic geometry and connectivity of spaces of frames," *arXiv preprint arXiv:1804.05899* , 2018.