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Abstract

Given any pair (L, A) of Lie algebroids, we construct a differential graded manifold $(L[1] \oplus L/A, Q)$, which we call Fedosov dg manifold. We prove that the homological vector field Q constructed on $L[1] \oplus L/A$ by the Fedosov iteration method arises as a byproduct of the Poincaré–Birkhoff–Witt map established in [18]. Finally, using the homological perturbation lemma, we establish a quasi-isomorphism of Dolgushev–Fedosov type: the differential graded algebras of functions on the dg manifolds $(A[1], d_A)$ and $(L[1] \oplus L/A, Q)$ are homotopy equivalent.

Contents

Dedicated to the memory of our colleague and friend John Roe.

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1 Introduction

Fedosov resolutions—we call them Fedosov dg manifolds later in the paper—played a key role in globalizing Kontsevich's formality theorem to smooth manifolds [8]. One can expect deformation quantization of geometric objects other than smooth manifolds to require the development of analogues of Fedosov resolutions for these other geometric objects. The leaf space of a foliation on a smooth manifold is one instance of such other geometric objects. In general, the leaf space may not be a smooth manifold—it is in a certain sense a noncommutative manifold. However, it can be considered as a particular example of Lie pair.

By a Lie pair (L, A), we mean an inclusion $A \hookrightarrow L$ of Lie algebroids over a smooth manifold M. Lie pairs arise naturally in a number of areas of mathematics such as Lie theory, complex geometry, and foliation theory. For instance, a complex manifold X determines a Lie pair over $\mathbb{C}$ with $L = T_X \otimes \mathbb{C}$ and $A = T_X^{0,1}$. A foliation $\mathcal{F}$ on a smooth manifold M determines a Lie pair over $\mathbb{R}$: this time L is the tangent bundle to M and A is the integrable distribution $T_{\mathcal{F}}$ on M tangent to the foliation $\mathcal{F}$. A $\mathfrak{g}$-manifold also gives rise to a Lie pair in a natural way [20].

The purpose of this paper is to construct analogues of Fedosov resolutions for Lie pairs. More precisely, for any Lie pair (L, A), we present two equivalent constructions of a dg manifold, called Fedosov dg manifold, and we establish a quasi-isomorphism of Dolgushev–Fedosov type. The first construction relies on the Poincaré–Birkhoff–Witt map introduced in [18], a generalized symmetrization map, while the second construction is based on Fedosov's iteration method.

Given a Lie pair (L, A), the quotient L/A is naturally an A-module [7]. When L is the tangent bundle to a manifold M and A is an integrable distribution on M, the infinitesimal A-action on L/A reduces to the classical Bott flat connection [5]. In [18], together with Laurent-Gengoux, we showed that, for any Lie pair over $\mathbb{R}$, each choice of (1) a splitting of the short exact sequence of vector bundles

$$0 \to A \to L \to L/A \to 0$$

and (2) an L-connection ∇ on L/A extending the Bott A-connection determines an exponential map

$$\exp : L/A \to \mathscr{L}/\mathscr{A}.$$

Here $\mathscr{L}$ and $\mathscr{A}$ are local Lie groupoids corresponding to the Lie algebroids L and A, respectively. Considering the (fiberwise) infinite-order jet of this exponential map, we obtained an isomorphism of filtered R-coalgebras (with $R = C^\infty(M)$)

$$\text{pbw} : \Gamma(S(L/A)) \to \frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)},$$

which we called Poincaré–Birkhoff–Witt map. In particular, if L is a Lie algebra $\mathfrak{g}$ and A is the trivial Lie algebra of dimension 0, there exists a natural choice of connection and the resulting pbw map is precisely the symmetrization map $S(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$. These PBW maps arising from Lie pairs admit an explicit recursive characterization valid for Lie pairs over any field $\Bbbk$ of characteristic zero and not just $\mathbb{R}$. Hence these PBW maps can be considered as algebraic formal exponential maps.

Transferring the canonical infinitesimal action of L on the coalgebra $\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$—this is an infinitesimal action by coderivations—through the map pbw, we obtain a *flat* L-connection $\nabla^{\natural}$ on $S(L/A)$:

$$\nabla^{\natural}_l(s) = \text{pbw}^{-1}\big(l \cdot \text{pbw}(s)\big),$$

for all $l \in \Gamma(L)$ and $s \in \Gamma(S(L/A))$. The covariant Chevalley–Eilenberg differential

$$d_L^{\nabla^{\natural}} : \Gamma(\Lambda^\bullet L^\vee \otimes \hat{S}\big((L/A)^\vee\big)) \to \Gamma(\Lambda^{\bullet+1} L^\vee \otimes \hat{S}\big((L/A)^\vee\big))$$

of the induced flat L-connection on the dual bundle $\hat{S}\big((L/A)^\vee\big)$ is a derivation of degree $(+1)$ of the algebra $\Gamma(\Lambda^\bullet L^\vee \otimes \hat{S}\big((L/A)^\vee\big))$ of smooth functions on the graded manifold $L[1] \oplus L/A$. As a consequence, $(L[1] \oplus L/A, d_L^{\nabla^{\natural}})$ is a dg manifold. We prove that, when ∇ is torsion-free, the homological vector field $d_L^{\nabla^{\natural}}$ coincides with a homological vector field Q constructed by Fedosov's iteration method. We elect to call a dg manifold $(L[1] \oplus L/A, Q)$ constructed in this way a *Fedosov dg manifold*.

It is a well known theorem of Dolgushev [8] that, for a smooth manifold M, the Fedosov dg manifold $T_M[1] \oplus T_M$ (associated with the Lie pair (L, A) where L is the tangent bundle to M and A is its trivial subbundle of rank 0) gives rise to a resolution $\Omega^\bullet\big(M; \hat{S}(T_M^\vee)\big)$ of $C^\infty(M)$. Our second main theorem extends this result to Lie pairs. Note that, for a Lie pair (L, A), the space of functions on the Fedosov dg manifold $(L[1] \oplus L/A, Q)$ is the differential graded algebra $\big(\Gamma(\Lambda^\bullet L^\vee \otimes \hat{S}((L/A)^\vee)), Q\big)$ while the space of functions on the dg manifold $(A[1], d_A)$ is the differential graded algebra $\big(\Gamma(\Lambda^\bullet A^\vee), d_A\big)$. We construct an explicit quasi-isomorphism of Dolgushev–Fedosov type from $\big(\Gamma(\Lambda^\bullet A^\vee), d_A\big)$ to $\big(\Gamma(\Lambda^\bullet L^\vee \otimes \hat{S}((L/A)^\vee)), Q\big)$. More precisely, using homological perturbation, we establish a contraction of $\big(\Gamma(\Lambda^\bullet L^\vee \otimes \hat{S}((L/A)^\vee)), Q\big)$ onto $\big(\Gamma(\Lambda^\bullet A^\vee), d_A\big)$:

$$\big(\Gamma(\Lambda^\bullet A^\vee), d_A\big) \underset{\sigma}{\overset{\breve{\tau}}{\rightleftarrows}} \big(\Gamma(\Lambda^\bullet L^\vee \otimes \hat{S}((L/A)^\vee)), d_L^{\nabla^{\natural}}\big) \circlearrowright \breve{h}\,.$$

As an application, we obtain an alternative proof of a theorem of Emmrich–Weinstein [10, Theorem 1.6]. Given a smooth manifold M and a torsion-free affine connection ∇ on it, Emmrich–Weinstein [10] constructed a dg manifold $(T_M[1] \oplus T_M, Q)$ using Fedosov's iteration method (see [8,11]). Emmrich–Weinstein [10] explained that (1) the derivation

$$Q : \Omega^\bullet\big(M; \hat{S}(T_M^\vee)\big) \to \Omega^{\bullet+1}\big(M; \hat{S}(T_M^\vee)\big)$$

determines a formal flat (nonlinear) Ehresmann connection on some neighborhood of the zero section of $TM \to M$ and (2) the leaves of this flat Ehresmann connection are transversal to the zero section. Hence, this formal flat Ehresmann connection induces a 'formal exponential map' EXP—see [10, Section 7]. Emmrich–Weinstein proved that the map EXP coincides with the infinite-order jet of the geodesic exponential map exp associated with the affine connection ∇—see [10, Theorem 1.6]. Their proof resorted to an indirect argument involving analytic manifolds. In this paper, we present a simple and direct proof based on (1) our result that the homological vector fields Q and $d^{\nabla^\natural}$ are equal, (2) the contraction

$$C^\infty(M) \underset{\sigma}{\overset{\breve{\tau}}{\rightleftarrows}} \big(\Omega^\bullet(M; \hat{S}(T_M^\vee)), d^{\nabla^\natural}\big) \circlearrowright \breve{h}$$

mentioned earlier, and (3) the geometric interpretation of the PBW map described at length in [18]. Indeed, when $L = T_M$ and A is its trivial subbundle of rank 0, the map

$$\breve{\tau} : C^\infty(M) \to \Omega^0\big(M; \hat{S}(T_M^\vee)\big)$$

is precisely the pull-back by the formal exponential map EXP studied by Emmrich–Weinstein [10].

In fact, we obtain an extension of the Emmrich–Weinstein theorem to the context of matched pairs—see Theorem 5.6. A matched pair of Lie algebroids is a Lie algebroid L with two Lie subalgebroids A and B such that $L = A \oplus B$ as vector bundles. We use the notation $L = A \bowtie B$ to denote such a situation. In the special case of matched pairs, we obtain an explicit formula for the map $\breve{\tau}$—see Eq. (20)—generalizing Emmrich–Weinsteins's interpretation of $\breve{\tau}$ (the pull-back by EXP in the terminology of [10]) as the infinite-order jet of an exponential map.

The Dolgushev–Fedosov type resolutions for Lie pairs, which we establish in the present work, play a crucial role in the proof of two results expounded in a subsequent work [22]: a formality theorem and a Kontsevich–Duflo type theorem for Lie pairs. While the spaces of polyvector fields and polydifferential operators on a smooth manifold both carry obvious dgla structures, there is generally no such obvious L_∞ algebra structure on either of the spaces of polyvector fields and polydifferential operators associated with a Lie pair. However, there exist natural L_∞ algebra structures on the spaces of polyvector fields and polydifferential operators on a dg foliation of the Fedosov dg manifold arising from the Lie pair. Our Dolgushev–Fedosov resolutions for Lie pairs allow for the homotopy transfer of these L_∞ structures from the Fedosov dg manifold to the Lie pair. This was done in [2], where the dg foliation of the Fedosov dg manifold is called Fedosov dg Lie algebroid. The Fedosov dg manifold construction was recently extended to $\mathbb{Z}$-graded manifolds by Liao–Stiénon [19] (see also [21]).

2 Terminology and notations

Natural numbers We use the symbol $\mathbb{N}$ to denote the set of positive integers and the symbol $\mathbb{N}_0$ for the set of nonnegative integers.

Field $\Bbbk$ *and ring* R We use the symbol $\Bbbk$ to denote the field of either real or complex numbers. The symbol R always denotes the algebra of smooth functions on the manifold M with values in $\Bbbk$.

Completed symmetric algebra Given a module $\mathscr{M}$ over a ring, the symbol $\hat{S}(\mathscr{M})$ denotes the $\mathfrak{m}$-adic completion of the symmetric algebra $S(\mathscr{M})$, where $\mathfrak{m}$ is the ideal of $S(\mathscr{M})$ generated by $\mathscr{M}$.

Duality pairing For every vector bundle $E \to M$, we define a duality pairing

$$\Gamma(S(E)) \times \Gamma(\hat{S}(E^\vee)) \to R$$

by

$$\langle v_1 \odot \cdots \odot v_p | \nu_1 \odot \cdots \odot \nu_q \rangle = \begin{cases} \sum_{\sigma \in S_p} \prod_{k=1}^{p} \langle v_k | \nu_{\sigma(k)} \rangle & \text{if } p = q, \\ 0 & \text{otherwise.} \end{cases}$$

The symbol $\odot$ denotes the symmetric tensor product.

Multi-indices Let $E \to M$ be a smooth vector bundle of finite rank r, let $(\partial_i)_{i\in\{1,\dots,r\}}$ be a local frame of E and let $(\chi_j)_{j\in\{1,\dots,r\}}$ be the dual local frame of $E^\vee$. Thus, we have $\langle \partial_i | \chi_j \rangle = \delta_{i,j}$. Given a multi-index $J = (J_1, J_2, \dots, J_r) \in \mathbb{N}_0^r$, we adopt the following multi-index notations:

$$\begin{aligned} J! &= J_1! \cdot J_2! \cdots J_r! \\ |J| &= J_1 + J_2 + \cdots + J_r \\ \partial^J &= \underbrace{\partial_1 \odot \cdots \odot \partial_1}_{J_1 \text{ factors}} \odot \underbrace{\partial_2 \odot \cdots \odot \partial_2}_{J_2 \text{ factors}} \odot \cdots \odot \underbrace{\partial_r \odot \cdots \odot \partial_r}_{J_r \text{ factors}} \\ \chi^J &= \underbrace{\chi_1 \odot \cdots \odot \chi_1}_{J_1 \text{ factors}} \odot \underbrace{\chi_2 \odot \cdots \odot \chi_2}_{J_2 \text{ factors}} \odot \cdots \odot \underbrace{\chi_r \odot \cdots \odot \chi_r}_{J_r \text{ factors}} \end{aligned} \tag{1}$$

We use the symbol e_k to denote the multi-index all of whose components are equal to 0 except for the k-th which is equal to 1. Thus $\chi^{e_k} = \chi_k$.

Shuffles A (p, q)-shuffle is a permutation σ of the set $\{1, 2, \dots, p+q\}$ such that

$$\sigma(1) < \sigma(2) < \cdots < \sigma(p) \quad \text{and} \quad \sigma(p+1) < \sigma(p+2) < \cdots < \sigma(p+q).$$

The symbol $\mathfrak{S}_p^q$ denotes the set of (p,q)-shuffles.

Gradation shift Given a graded vector space $V = \bigoplus_{k\in\mathbb{Z}} V^{(k)}$, the notation $V[i]$ denotes the graded vector space obtained by shifting the grading on V according to the rule $(V[i])^{(k)} = V^{(i+k)}$. Accordingly, if $E = \bigoplus_{k\in\mathbb{Z}} E^{(k)}$ is a graded vector bundle over M, the notation $E[i]$ denotes the graded vector bundle obtained by shifting the degree in the fibers of E according to the above rule.

Dg manifolds A dg manifold is a $\mathbb{Z}$-graded manifold endowed with a homological vector field, i.e. a vector field Q of degree $(+1)$ such that $[Q,Q]=0$. Dg manifolds are also known as Q-manifolds. For details and further references, see [1,25,26].

3 Preliminaries

3.1 Lie algebroids and Lie pairs

Lie algebroids We use the symbol $\Bbbk$ to denote either of the fields $\mathbb{R}$ and $\mathbb{C}$. A *Lie algebroid* over $\Bbbk$ is a $\Bbbk$-vector bundle $L \to M$ together with a bundle map $\rho : L \to T_M \otimes_{\mathbb{R}} \Bbbk$ called *anchor* and a Lie bracket $[-,-]$ on sections of L such that $\rho : \Gamma(L) \to \mathfrak{X}(M) \otimes \Bbbk$ is a morphism of Lie algebras and

$$[X, fY] = f\,[X,Y] + \big(\rho(X)f\big)Y$$

for all $X, Y \in \Gamma(L)$ and $f \in C^\infty(M,\Bbbk)$. In this paper 'Lie algebroid' always means 'Lie algebroid over $\Bbbk$' unless specified otherwise. A $\Bbbk$-vector bundle $L \to M$ is a Lie algebroid if and only if $\Gamma(L)$ is a *Lie–Rinehart algebra* [30] over the commutative ring $C^\infty(M,\Bbbk)$.

Lie pairs By a *Lie pair* (L, A), we mean an inclusion $A \hookrightarrow L$ of a Lie subalgebroid A into a Lie algebroid L, both having the same smooth manifold M as base.

Examples 3.1 (1) Let $\mathfrak{h}$ be a Lie subalgebra of a Lie algebra $\mathfrak{g}$. Then $(\mathfrak{g}, \mathfrak{h})$ is a Lie pair over the one-point manifold $\{*\}$.
(2) Let X be a complex manifold. Then $(T_X \otimes \mathbb{C}, T_X^{0,1})$ is a Lie pair over X.
(3) Let $\mathcal{F}$ be a foliation on a smooth manifold M. Then $(T_M, T_{\mathcal{F}})$ is a Lie pair over M.

Matched pairs A *matched pair* of Lie algebroids is a Lie algebroid L with two Lie subalgebroids A and B such that $L = A \oplus B$ as vector bundles. We use the notation $L = A \bowtie B$ to denote such a situation—see [23,24,28] for more details.

Examples 3.2 (1) If X is a complex manifold, then $T_X \otimes \mathbb{C} = T_X^{0,1} \bowtie T_X^{1,0}$ is a matched pair of complex Lie algebroids over X.
(2) Let G be a Poisson Lie group and let P be a Poisson G-space, i.e. a Poisson manifold (P,π) endowed with a G-action $G \times P \to P$ which happens to be a Poisson map. According to Lu [23], the cotangent Lie algebroid $A = \big(T_P^\vee\big)_\pi$ and the transformation Lie algebroid $B = P \rtimes \mathfrak{g}$ form a matched pair of Lie algebroids over the manifold P.

3.2 Chevalley–Eilenberg differentials, connections, and representations

Let L be a Lie algebroid over a smooth manifold M, and R be the algebra of smooth functions on M valued in $\Bbbk$. The Chevalley–Eilenberg differential

$$d_L : \Gamma(\Lambda^k L^\vee) \to \Gamma(\Lambda^{k+1} L^\vee)$$

defined by

$$\begin{aligned}\big(d_L\omega\big)(v_0, v_1, \dots, v_k) &= \sum_{i=0}^{n}(-1)^i \rho(v_i)\big(\omega(v_0, \dots, \widehat{v_i}, \dots, v_k)\big) \\ &\quad + \sum_{i<j}(-1)^{i+j}\omega(\big[v_i, v_j\big], v_0, \dots, \widehat{v_i}, \dots, \widehat{v_j}, \dots, v_k)\end{aligned}$$

and the exterior product make $\bigoplus_{k\geq 0}\Gamma(\Lambda^k L^\vee)$ into a differential graded commutative algebra.

The following proposition is an immediate consequence of the definitions.

Proposition 3.3 *Let L be a Lie algebroid and let A and B be two vector subbundles of L such that $L = A \oplus B$. Let $p : L \twoheadrightarrow A$ and $q : L \twoheadrightarrow B$ denote the canonical projections, let $p^\vee : A^\vee \hookrightarrow L^\vee$ and $q^\vee : B^\vee \hookrightarrow L^\vee$ denote their respective dual maps, and set*

$$\Omega^{u,v} = \Gamma(p^\vee(\Lambda^u A^\vee) \wedge q^\vee(\Lambda^v B^\vee)).$$

The following assertions hold:

(1) *In general, we have*

$$d_L(\Omega^{u,v}) \subset \Omega^{u+2,v-1} \oplus \Omega^{u+1,v} \oplus \Omega^{u,v+1} \oplus \Omega^{u-1,v+2}.$$

(2) *The vector subbundle A is a Lie subalgebroid of L (i.e. (L, A) is a Lie pair) if and only if*

$$d_L(\Omega^{u,v}) \subset \Omega^{u+1,v} \oplus \Omega^{u,v+1} \oplus \Omega^{u-1,v+2}.$$

(3) *Both vector subbundles A and B are Lie subalgebroids of L (i.e. $L = A \bowtie B$ is a matched pair) if and only if*

$$d_L(\Omega^{u,v}) \subset \Omega^{u+1,v} \oplus \Omega^{u,v+1}.$$

Now let $E \xrightarrow{\varpi} M$ be a vector bundle over $\Bbbk$. The traditional description of a (linear) L-connection on E is in terms of a *covariant derivative*

$$\Gamma(L) \times \Gamma(E) \to \Gamma(E) : (l, e) \mapsto \nabla_l e$$

characterized by the following two properties:

$$\nabla_{f\cdot l}e = f\cdot\nabla_l e, \tag{2}$$
$$\nabla_l(f\cdot e) = \rho(l)f\cdot e + f\cdot\nabla_l e, \tag{3}$$

for all $l\in\Gamma(L)$, $e\in\Gamma(E)$, and $f\in R$. The covariant differential associated with an L-connection ∇ on a vector bundle $E\to M$ is the operator

$$d_L^\nabla : \Gamma(\Lambda^k L^\vee\otimes E)\to\Gamma(\Lambda^{k+1}L^\vee\otimes E)$$

that takes a section $\omega\otimes e$ of $\Lambda^k L^\vee\otimes E$ to

$$d_L^\nabla(\omega\otimes e) = (d_L\omega)\otimes e + \sum_{j=1}^n(\nu_j\wedge\omega)\otimes\nabla_{v_j}e,$$

where n is the rank of L, and $v_1, v_2, \dots, v_n$ and $\nu_1, \nu_2, \dots, \nu_n$ are any pair of dual local frames for the vector bundles L and $L^\vee$. Then $d_L^\nabla\circ d_L^\nabla = R^\nabla$, where $R^\nabla\in\Gamma(\Lambda^2L^\vee\otimes\operatorname{End}E)$ is the *curvature* of ∇.

Remark 3.4 An L-connection ∇ on E induces a covariant derivative

$$\nabla : \Gamma(L)\times\Gamma(S(E))\to\Gamma(S(E))$$

through the relation $\nabla_l 1 = 0$ and the Leibniz rule

$$\nabla_l(b_1\odot\cdots\odot b_n) = \sum_{k=1}^n b_1\odot\cdots\odot b_{k-1}\odot\nabla_l b_k\odot b_{k+1}\odot\cdots\odot b_n,$$

for all $l\in\Gamma(L)$, $n\in\mathbb{N}$ and $b_1,\dots,b_n\in\Gamma(E)$.

Remark 3.5 A covariant derivative

$$\nabla : \Gamma(L)\times\Gamma(S(E))\to\Gamma(S(E))$$

induces a covariant derivative

$$\nabla : \Gamma(L)\times\Gamma(\hat{S}(E^\vee))\to\Gamma(\hat{S}(E^\vee))$$

through the relation

$$\rho(l)\,\langle s|\sigma\rangle = \langle\nabla_l s|\sigma\rangle + \langle s|\nabla_l\sigma\rangle$$

for all $l\in\Gamma(L)$, $s\in\Gamma(SE)$, and $\sigma\in\Gamma(\hat{S}(E^\vee))$.

A *representation of a Lie algebroid* L on a vector bundle $E \to M$ is a flat L-connection ∇ on E, i.e. a covariant derivative $\nabla : \Gamma(L) \times \Gamma(E) \to \Gamma(E)$ satisfying

$$\nabla_{a_1}\nabla_{a_2}e - \nabla_{a_2}\nabla_{a_1}e = \nabla_{[a_1,a_2]}e, \tag{4}$$

for all $a_1, a_2 \in \Gamma(L)$ and $e \in \Gamma(E)$. A vector bundle endowed with a representation of the Lie algebroid L is called an L*-module*. More generally, given a left R-module $\mathscr{M}$, by an *infinitesimal action* of L on $\mathscr{M}$, we mean a $\Bbbk$-bilinear map $\nabla : \Gamma(L) \times \mathscr{M} \to \mathscr{M}$, $(a, e) \mapsto \nabla_a e$ satisfying Eqs. (2), (3), and (4). In other words, ∇ is a representation of the Lie–Rinehart algebra $(\Gamma(L), R)$ [30].

Examples 3.6 ([7]) Let (L, A) be a Lie pair. The *Bott representation* of A on the quotient L/A is the flat connection defined by

$$\nabla_a^{\text{Bott}} q(l) = q\big([a,l]\big), \quad \forall a \in \Gamma(A), l \in \Gamma(L),$$

where q denotes the canonical projection $L \twoheadrightarrow L/A$. Thus the quotient L/A of a Lie pair (L, A) is an A-module.

The Chevalley–Eilenberg differential associated with a representation ∇ of a Lie algebroid L on a vector bundle E is the covariant differential operator $d_L^\nabla : \Gamma(\Lambda^k L^\vee \otimes E) \to \Gamma(\Lambda^{k+1} L^\vee \otimes E)$ corresponding to the connection ∇. Because the connection ∇ is flat, d_L^∇ is a coboundary operator: $d_L^\nabla \circ d_L^\nabla = 0$.

3.3 Torsion-free connections

Let (L, A) be a Lie pair over $\Bbbk$. Consider the short exact sequence of vector bundles

$$0 \longrightarrow A \xrightarrow{\ i\ } L \xrightarrow{\ q\ } L/A \longrightarrow 0\ . \tag{5}$$

An L-connection ∇ on L/A is said to extend the Bott A-representation on L/A (see Example 3.6) if

$$\nabla_{i(a)} q(l) = \nabla_a^{\text{Bott}} q(l) = q\big([i(a),l]\big), \quad \forall a \in \Gamma(A), l \in \Gamma(L).$$

Given an L-connection ∇ on L/A, its torsion is the bundle map $T^\nabla : \Lambda^2 L \to L/A$ defined by

$$T^\nabla(l_1, l_2) = \nabla_{l_1} q(l_2) - \nabla_{l_2} q(l_1) - q\big([l_1,l_2]\big), \quad \forall l_1, l_2 \in \Gamma(L).$$

If ∇ is an L-connection on L/A extending the Bott A-representation on L/A, its torsion descends to a bundle map

$$\beta^\nabla : \Lambda^2(L/A) \to L/A,$$

making the diagram

$$\begin{array}{ccc} \Lambda^2 L & \xrightarrow{T^\nabla} & L/A \\ {\scriptstyle q}\downarrow & \nearrow {\scriptstyle \beta^\nabla} & \\ \Lambda^2(L/A) & & \end{array}$$

commute. According to [18, Lemma 5.2], if ∇ is torsion-free, it must be an extension of the Bott A-representation on L/A. Torsion-free L-connections on L/A always exist—see [18, Proposition 5.3].

3.4 Poincaré–Birkhoff–Witt isomorphisms

Let L be a Lie $\Bbbk$-algebroid over a smooth manifold M with anchor map ρ, and let R denote the algebra of smooth functions on M taking values in $\Bbbk$. By $\mathcal{U}(L)$ we denote the universal enveloping algebra of the Lie algebroid L—see [30]. Essentially, $\mathcal{U}(L)$ is the quotient of the (reduced) tensor algebra $\bigoplus_{n=1}^{\infty}\big(\bigotimes_{\Bbbk}^{n}((R\oplus\Gamma(L))\big)$ by the two-sided ideal generated by all elements of the following four types:

$$\begin{array}{ll} X\otimes Y - Y\otimes X - [X,Y] & \qquad f\otimes X - fX \\ X\otimes g - g\otimes X - \rho(X)(g) & \qquad f\otimes g - fg \end{array}$$

with $X, Y \in \Gamma(L)$ and $f, g \in R$.

The notion of the universal enveloping algebra $\mathcal{U}(L)$ of a Lie algebroid L unifies that of the universal enveloping algebra of a Lie algebra (when L is a Lie algebra) and that of the algebra of differential operators on M (when L is the tangent bundle Lie algebroid T_M). For a Lie algebroid L over $\mathbb{R}$, given any local Lie groupoid $\mathscr{L}$ having L as its associated Lie algebroid, the universal enveloping algebra $\mathcal{U}(L)$ of L can be identified in a canonical way with the algebra of target-fiberwise[1] differential operators on $\mathscr{L}$ invariant under left translations [27,29]. The universal enveloping algebra of L admits a natural filtration

$$R \hookrightarrow \big(\mathcal{U}(L)\big)^{\leq 1} \hookrightarrow \big(\mathcal{U}(L)\big)^{\leq 2} \hookrightarrow \big(\mathcal{U}(L)\big)^{\leq 3} \hookrightarrow \cdots \tag{6}$$

corresponding to the order filtration on differential operators—in particular, $\big(\mathcal{U}(L)\big)^{\leq 1} = R\oplus\Gamma(L)$. The universal enveloping algebra $\mathcal{U}(L)$ of the Lie algebroid $L\to M$ is a (left) coalgebra over R. Its comultiplication

$$\Delta : \mathcal{U}(L) \to \mathcal{U}(L)\otimes_R \mathcal{U}(L)$$

[1] We adopt the following convention for the multiplication in a groupoid $\mathscr{L} \rightrightarrows M$ with source map $s:\mathscr{L}\to M$ and target map $t:\mathscr{L}\to M$: given two elements g and h of $\mathscr{L}$, their product gh is defined only if the target of h coincides with the source of g, i.e. if $s(g)=t(h)$ in M. With this convention, we have $s(gh)=s(h)$ and $t(gh)=t(g)$. Hence, left translation by g maps the target-fiber $t^{-1}\big(s(g)\big)$ to the target-fiber $t^{-1}\big(t(g)\big)$. Consequently, the left invariant vector fields on $\mathscr{L}$ are necessarily tangent to the fibers of the target map.

is given explicitly by

$$\begin{aligned}\Delta(b_1 \cdot b_2 \cdot \cdots \cdot b_n) = {} & 1 \otimes (b_1 \cdot b_2 \cdot \cdots \cdot b_n) \\ & + \sum_{\substack{p+q=n \\ p,q \in \mathbb{N}}} \sum_{\sigma \in \mathfrak{S}_p^q} (b_{\sigma(1)} \cdot \cdots \cdot b_{\sigma(p)}) \otimes (b_{\sigma(p+1)} \cdot \cdots \cdot b_{\sigma(n)}) \\ & + (b_1 \cdot b_2 \cdot \cdots \cdot b_n) \otimes 1,\end{aligned}$$

for all $n \in \mathbb{N}$ and $b_1, \ldots, b_n \in \Gamma(L)$, and is compatible with its filtration (6). Indeed, $\mathcal{U}(L)$ is more than just an algebra and a coalgebra; it is a Hopf algebroid—see [31].

Now let (L, A) be a Lie pair over $\Bbbk$. Writing $\mathcal{U}(L)\Gamma(A)$ for the left ideal of $\mathcal{U}(L)$ generated by $\Gamma(A)$, the quotient $\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$ is automatically a filtered R-coalgebra since

$$\Delta\big(\mathcal{U}(L)\Gamma(A)\big) \subseteq \mathcal{U}(L) \otimes_R \big(\mathcal{U}(L)\Gamma(A)\big) + \big(\mathcal{U}(L)\Gamma(A)\big) \otimes_R \mathcal{U}(L),$$

and the filtration (6) on $\mathcal{U}(L)$ descends to a filtration

$$R \hookrightarrow \left(\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}\right)^{\leq 1} \hookrightarrow \left(\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}\right)^{\leq 2} \hookrightarrow \left(\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}\right)^{\leq 3} \hookrightarrow \cdots$$

of $\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$. We will use the symbol $\mathbf{1}$ to denote the image in $\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$ of the constant function $1 \in R$ under the canonical map $R \hookrightarrow \mathcal{U}(L) \twoheadrightarrow \frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$. We note that $\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$ is naturally a left module over the associative algebra $\mathcal{U}(L)$.

Similarly, $\Gamma(S(L/A))$ is an R-coalgebra with the deconcatenation

$$\Delta : \Gamma(S(L/A)) \to \Gamma(S(L/A)) \otimes_R \Gamma(S(L/A))$$

as its comultiplication:

$$\begin{aligned}\Delta(b_1 \odot b_2 \odot \cdots \odot b_n) = {} & 1 \otimes (b_1 \odot b_2 \odot \cdots \odot b_n) \\ & + \sum_{\substack{p+q=n \\ p,q \in \mathbb{N}}} \sum_{\sigma \in \mathfrak{S}_p^q} (b_{\sigma(1)} \odot \cdots \odot b_{\sigma(p)}) \otimes (b_{\sigma(p+1)} \odot \cdots \odot b_{\sigma(n)}) \\ & + (b_1 \odot b_2 \odot \cdots \odot b_n) \otimes 1,\end{aligned}$$

for all $n \in \mathbb{N}$ and $b_1, \ldots, b_n \in \Gamma(L/A)$.

The following theorem, which was obtained in [18], is an extension to Lie pairs of the Poincaré–Birkhoff–Witt isomorphism of classical Lie theory.

Theorem 3.7 *([18, Theorem 2.1]) Let (L, A) be a Lie pair. Given a splitting $j : L/A \to L$ of the short exact sequence $0 \to A \to L \to L/A \to 0$ and an L-connection ∇ on L/A extending the Bott A-representation, there exists a unique isomorphism of filtered R-coalgebras*

$$\mathrm{pbw} : \Gamma(S(L/A)) \to \frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$$

satisfying

$$\mathrm{pbw}(1) = \mathbf{1}, \tag{7}$$

$$\mathrm{pbw}(b) = j(b) \cdot \mathbf{1}, \tag{8}$$

$$\mathrm{pbw}(b^{n+1}) = j(b) \cdot \mathrm{pbw}(b^n) - \mathrm{pbw}\left(\nabla_{j(b)}(b^n)\right) \tag{9}$$

for all $b \in \Gamma(L/A)$ *and* $n \in \mathbb{N}$.

Remark 3.8 Equation (9) is equivalent to

$$\begin{aligned}\mathrm{pbw}(b_0 \odot \cdots \odot b_n) = \tfrac{1}{n+1} \sum_{i=0}^{n} \Big(& j(b_i) \cdot \mathrm{pbw}(b_0 \odot \cdots \odot \widehat{b_i} \odot \cdots \odot b_n) \\ & - \mathrm{pbw}\left(\nabla_{j(b_i)}(b_0 \odot \cdots \odot \widehat{b_i} \odot \cdots \odot b_n)\right)\Big)\end{aligned} \tag{10}$$

for all $b_0, \ldots, b_n \in \Gamma(L/A)$.

Note that Eqs. (7), (8), and (9) (or (10)) define inductively a unique R-linear map pbw.

The following lemma will be needed later on. Its proof is straightforward and is therefore omitted.

Lemma 3.9 *For all* Y, Z *in* $\Gamma(L/A)$*, we have*

$$\mathrm{pbw}(Y \odot Z) = j(Y) \cdot \mathrm{pbw}(Z) - \mathrm{pbw}\left(\nabla_{j(Y)} Z\right) + \tfrac{1}{2}\, \mathrm{pbw}\left(\beta^{\nabla}(Y, Z)\right),$$

where β^{∇} *is the bundle map defined in Sect. 3.3.*

Remark 3.10 When $L = T_M$ and A is the trivial Lie subalgebroid of L of rank 0, the pbw map of Theorem 3.7 is the inverse of the so-called 'complete symbol map,' which is an isomorphism from the space $\mathcal{U}(T_M)$ of differential operators on M to the space $\Gamma(S(T_M))$ of fiberwise polynomial functions on $T_M^{\vee}$. The complete symbol map was generalized to arbitrary Lie algebroids over $\mathbb{R}$ by Nistor–Weinstein–Xu [29]. It played an important role in quantization theory [13,16,17,29].

4 Fedosov dg manifolds for Lie pairs

Given a Lie pair (L, A) with quotient $B = L/A$, the graded manifold $L[1] \oplus B$ can be endowed with a homological vector field. We give two equivalent constructions of this homological vector field.

4.1 First construction by way of the PBW map

Making use of the Poincaré–Birkhoff–Witt isomorphism pbw of Theorem 3.7, one can endow the graded manifold $L[1] \oplus B$ with a homological vector field.

Every choice of a splitting $j : B \to L$ of the short exact sequence of vector bundles (5) and an L-connection ∇ on B extending the Bott A-connection determines an isomorphism of R-coalgebras

$$\mathrm{pbw} : \Gamma(SB) \to \frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}.$$

Being a quotient of the universal enveloping algebra $\mathcal{U}(L)$ by a left ideal, the R-coalgebra $\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$ is naturally a left $\mathcal{U}(L)$-module. Hence $\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$ is endowed with a canonical infinitesimal L-action by coderivations. Pulling back this infinitesimal action through pbw, we obtain an infinitesimal L-action on $\Gamma(S(B))$ by coderivations. The latter defines a *flat* L-connection $\nabla^{\natural}$ on $S(B)$:

$$\nabla^{\natural}_l(s) = \mathrm{pbw}^{-1}\big(l \cdot \mathrm{pbw}(s)\big), \tag{11}$$

for all $l \in \Gamma(L)$ and $s \in \Gamma(SB)$.

The L-connection $\nabla^{\natural}$ on $S(B)$ induces an L-connection on the dual bundle $\hat{S}(B^\vee)$—see Remark 3.5. We denote the corresponding Chevalley–Eilenberg differential by

$$d_L^{\nabla^{\natural}} : \Gamma(\Lambda^\bullet L^\vee \otimes \hat{S}(B^\vee)) \to \Gamma(\Lambda^{\bullet+1} L^\vee \otimes \hat{S}(B^\vee)). \tag{12}$$

Since the covariant derivative

$$\nabla^{\natural}_l : \Gamma(SB) \to \Gamma(SB)$$

is a *co*derivation of $\Gamma(SB)$ for all $l \in \Gamma(L)$, the covariant derivative

$$\nabla^{\natural}_l : \Gamma(\hat{S}(B^\vee)) \to \Gamma(\hat{S}(B^\vee))$$

is a derivation of the symmetric algebra $\Gamma(\hat{S}(B^\vee))$. Therefore, the operator $d_L^{\nabla^{\natural}}$ on $\Gamma(\Lambda^\bullet L^\vee \otimes \hat{S}(B^\vee))$ is a derivation of degree $(+1)$ satisfying $d_L^{\nabla^{\natural}} \circ d_L^{\nabla^{\natural}} = 0$, i.e. it is a homological vector field on $L[1] \oplus B$. Note however that $\nabla^{\natural}_l$ need not be a derivation of $\Gamma(SB)$ for any $l \in \Gamma(L)$.

Proposition 4.1 *Given a Lie pair (L, A) with quotient $B = L/A$, the choice of (1) a splitting $j : L/A \to L$ of the short exact sequence $0 \to A \to L \to B \to 0$ and (2) an L-connection ∇ on B extending the Bott representation determines an operator $d_L^{\nabla^{\natural}}$ as above making $(L[1] \oplus B, d_L^{\nabla^{\natural}})$ a dg manifold.*

Remark 4.2 The Kapranov dg manifolds of [18, Theorem 5.7] inspired the construction of the dg manifold of Proposition 4.1. Indeed, the Kapranov dg manifold $(A[1] \oplus B, D)$ constructed in [18, Theorem 5.7] is a dg submanifold of the dg manifold $(L[1] \oplus B, d_L^{\nabla^{\natural}})$ of Proposition 4.1 as can be readily observed by comparing Eq. (11) with [18, Equation (46)].

We will see in Theorem 4.7 that, when ∇ is torsion-free, the homological vector field $d_L^{\nabla^{\natural}}$ is exactly the homological vector field Q constructed by Fedosov's iteration method as described in Sect. 4.3.

4.2 Dependence of the construction on the choice of splitting and connection

Now we consider two different choices j_1, ∇_1 and j_2, ∇_2 of a splitting $B \to L$ and a torsion free L-connection on B as before, and the two induced homological vector fields $d_L^{\nabla_1^\sharp}, d_L^{\nabla_2^\sharp}$ on $L[1] \oplus B$. There are also two induced Poincaré–Birkhoff–Witt isomorphisms $\mathrm{pbw}_1, \mathrm{pbw}_2 : \Gamma(SB) \to \frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$, and the composition

$$\psi := \mathrm{pbw}_1^{-1} \circ \mathrm{pbw}_2 : \Gamma(SB) \to \Gamma(SB)$$

is an automorphism of the R-coalgebra $\Gamma(SB)$ intertwining the two induced L-module structures. Hence, the dual $\psi^\vee : \Gamma(\hat{S}(B^\vee)) \to \Gamma(\hat{S}(B^\vee))$ is an automorphism of the R-algebra $\Gamma(\hat{S}(B^\vee))$ intertwining the two induced L-module structures. It follows immediately that the isomorphism of differential graded algebras

$$\mathrm{id} \otimes \psi^\vee : \left(\Gamma(\Lambda^\bullet L^\vee \otimes \hat{S}(B^\vee)), d_L^{\nabla_1^\sharp}\right) \to \left(\Gamma(\Lambda^\bullet L^\vee \otimes \hat{S}(B^\vee)), d_L^{\nabla_2^\sharp}\right)$$

defines an isomorphism of dg manifolds $(L[1] \oplus B, d_L^{\nabla_2^\sharp}) \to (L[1] \oplus B, d_L^{\nabla_1^\sharp})$.

4.3 Second construction by way of Fedosov's iteration method

Consider the bundle of graded commutative algebras $\Lambda^\bullet B^\vee \otimes \hat{S}(B^\vee)$ and the fiberwise derivation $\dot{\delta}$ of degree $(+1)$ defined by its action on generators of $\Lambda^\bullet B^\vee \otimes \hat{S}(B^\vee)$ as follows:

$$\dot{\delta}(1 \otimes \chi) = \chi \otimes 1 \qquad \text{and} \qquad \dot{\delta}(\chi \otimes 1) = 0 \qquad \text{for all } \chi \in B^\vee.$$

It is clear that $\dot{\delta}^2 = 0$. Likewise, let $\dot{D}$ be the fiberwise derivation of degree (-1) defined on generators by

$$\dot{D}(1 \otimes \chi) = 0 \qquad \text{and} \qquad \dot{D}(\chi \otimes 1) = 1 \otimes \chi \qquad \text{for all } \chi \in B^\vee.$$

Obviously, we have $\dot{D}^2 = 0$. It is also clear that $[\dot{\delta}, \dot{D}]$ is a derivation of degree 0 of $\Lambda^\bullet B^\vee \otimes \hat{S}(B^\vee)$ satisfying $[\dot{\delta}, \dot{D}](1 \otimes \chi) = 1 \otimes \chi$ and $[\dot{\delta}, \dot{D}](\chi \otimes 1) = \chi \otimes 1$ for all $\chi \in B^\vee$. Thus, it follows that

$$[\dot{\delta}, \dot{D}] = (v + j)\,\mathrm{id} \qquad \text{on} \quad \Lambda^v B^\vee \otimes S^j(B^\vee).$$

Consider the operator $\dot{h}$ on $\Lambda^\bullet B^\vee \otimes \hat{S}(B^\vee)$ defined by the relation

$$\dot{h} = \begin{cases} \frac{1}{(v+j)} \dot{D} & \text{on} \quad \Lambda^v B^\vee \otimes S^j(B^\vee) \quad \text{with} \quad v + j > 0, \\ 0 & \text{on} \quad \Lambda^0 B^\vee \otimes S^0(B^\vee). \end{cases}$$

Then $\dot{\delta}\dot{h}+\dot{h}\dot{\delta}$ is the canonical projection of $\Lambda^\bullet B^\vee \otimes \hat{S}(B^\vee)$ onto $\Lambda^0 B^\vee \otimes S^0(B^\vee)$, which shows that the (fiberwise) Koszul[2] complex $\big(\Lambda^\bullet B^\vee \otimes \hat{S}(B^\vee), \dot{\delta}\big)$ is acyclic except in degree 0 where its cohomology is $\Lambda^0 B^\vee \otimes S^0(B^\vee)$.

Choose a splitting $i \circ p + j \circ q = \mathrm{id}_L$ of the short exact sequence

$$0 \longrightarrow A \underset{p}{\overset{i}{\rightleftarrows}} L \underset{j}{\overset{q}{\rightleftarrows}} B \longrightarrow 0 \tag{13}$$

and its dual

$$0 \longrightarrow B^\vee \underset{j^\vee}{\overset{q^\vee}{\rightleftarrows}} L^\vee \underset{p^\vee}{\overset{i^\vee}{\rightleftarrows}} A^\vee \longrightarrow 0 \ .$$

Tensoring the cochain complex $\big(\Lambda^\bullet B^\vee \otimes \hat{S}(B^\vee), \dot{\delta}\big)$ with the cochain complex $\big(\Lambda^\bullet A^\vee, 0\big)$ with trivial differential, and identifying $\Lambda A^\vee \otimes \Lambda B^\vee$ with $\Lambda L^\vee$ by the vector bundle isomorphism

$$\Lambda A^\vee \otimes \Lambda B^\vee \ni \alpha \otimes \beta \overset{\mu}{\longmapsto} p^\vee(\alpha) \wedge q^\vee(\beta) \in \Lambda L^\vee,$$

we obtain the cochain complex

$$\cdots \longrightarrow \Lambda^{k-1} L^\vee \otimes \hat{S}(B^\vee) \xrightarrow{\delta} \Lambda^k L^\vee \otimes \hat{S}(B^\vee) \xrightarrow{\delta} \Lambda^{k+1} L^\vee \otimes \hat{S}(B^\vee) \longrightarrow \cdots$$

whose coboundary operator δ is the operator $\mathrm{id}_{\Lambda A^\vee} \otimes \dot{\delta}$ conjugated by the isomorphism $\mu \otimes \mathrm{id}_{\hat{S}(B^\vee)}$. Let r be the rank of the bundle B. Given a local frame $\{\chi_k\}_{k=1}^r$ for the vector bundle $B^\vee$ and a multi-index $J = (J_1, J_2, \cdots, J_r) \in \mathbb{N}_0^r$, we make use of the notation χ^J defined by Eq. (1). The differential δ satisfies

$$\delta(\omega \otimes \chi^J) = \sum_{m=1}^r \big(q^\vee(\chi_m) \wedge \omega\big) \otimes J_m\, \chi^{J-e_m},$$

for all $\omega \in \Lambda L^\vee$ and $J \in \mathbb{N}_0^r$—we declare that $J_m \chi^{J-e_m} = 0$ if $J_m = 0$.

Now consider the pair of cochain maps

$$\tau : \Lambda^\bullet A^\vee \to \Lambda^\bullet L^\vee \otimes \hat{S}(B^\vee) \qquad \text{and} \qquad \sigma : \Lambda^\bullet L^\vee \otimes \hat{S}(B^\vee) \to \Lambda^\bullet A^\vee$$

respectively defined by

$$\tau(\alpha) = p^\vee(\alpha) \otimes 1,$$

[2] We are grateful to an anonymous referee for pointing out the relation between the Koszul complex—see [12]—and our initial construction as set forth in an earlier version of our manuscript.

for all $\alpha \in \Lambda^\bullet A^\vee$, and

$$\sigma(\omega \otimes \chi^J) = \begin{cases} i^\vee(\omega) \otimes \chi^J & \text{if } |J| = 0 \\ 0 & \text{otherwise,} \end{cases} \tag{14}$$

for all $\omega \in \Lambda^\bullet L^\vee$ and all multi-indices $J \in \mathbb{N}_0^r$. They realize a homotopy equivalence between the cochain complexes $\big(\Lambda^\bullet L^\vee \otimes \hat{S}(B^\vee), \delta\big)$ and $\big(\Lambda^\bullet A^\vee, 0\big)$ since we have $\sigma\tau = \mathrm{id}$ and $\mathrm{id} - \tau\sigma = \delta h + h\delta$, where the symbol h denotes the homotopy operator

$$h : \Lambda^\bullet L^\vee \otimes \hat{S}(B^\vee) \to \Lambda^{\bullet-1} L^\vee \otimes \hat{S}(B^\vee)$$

obtained by conjugating the operator $\mathrm{id}_{\Lambda A^\vee} \otimes \dot{h}$ by the isomorphism $\mu \otimes \mathrm{id}_{\hat{S}(B^\vee)}$. We note that, for all $\omega = \mu(\alpha \otimes \beta)$ with $\beta \in \Lambda^v B^\vee$ and all multi-indices $J \in \mathbb{N}_0^r$, we have

$$h(\omega \otimes \chi^J) = \begin{cases} \frac{1}{v+|J|} \sum_{k=1}^r (\iota_{j(\partial_k)}\omega) \otimes \chi^{J+e_k} & \text{if } v \geq 1, \\ 0 & \text{if } v = 0. \end{cases} \tag{15}$$

Clearly, the maps δ, σ, τ, and h respect the exhaustive, complete, descending filtrations

$$\mathcal{F}_0 \supset \mathcal{F}_1 \supset \mathcal{F}_2 \supset \mathcal{F}_3 \supset \cdots \qquad \text{and} \qquad \mathscr{F}_0 \supset \mathscr{F}_1 \supset \mathscr{F}_2 \supset \mathscr{F}_3 \supset \cdots$$

on $\Lambda^\bullet A^\vee$ and $\Lambda^\bullet L^\vee \otimes \hat{S}(B^\vee)$ defined respectively by

$$\mathcal{F}_m = \bigoplus_{k \geq m} \Lambda^k A^\vee \qquad \text{and} \qquad \mathscr{F}_m = \prod_{k+p \geq m} \big(\Lambda^k L^\vee \otimes S^p(B^\vee)\big).$$

Observing that $h\tau = 0$; $\sigma h = 0$; and $h^2 = 0$, we conclude that

Proposition 4.3 *The vector bundle maps δ, h, σ, and τ defined above determine a filtered contraction*

$$\big(\Gamma(\Lambda^\bullet A^\vee), 0\big) \underset{\sigma}{\overset{\tau}{\rightleftarrows}} \big(\Gamma(\Lambda^\bullet L^\vee \otimes \hat{S}(B^\vee)), -\delta\big) \circlearrowright h\,.$$

Remark 4.4 Unlike δ, the operator h is *not* a derivation of the bundle of graded commutative algebras $\Lambda^\bullet L^\vee \otimes \hat{S}(B^\vee)$.

Lemma 4.5 *Let (L, A) be a Lie pair, let d_L^∇ denote the Chevalley–Eilenberg differential associated with the covariant derivative $\nabla : \Gamma(L) \times \Gamma(\hat{S}(B^\vee)) \to \Gamma(\hat{S}(B^\vee))$ determined (as in Remarks 3.4 and 3.5) by an L-connection ∇ on B, and let T^∇ be the torsion of the latter (see Sect. 3.3). Then $T^\nabla = 0$ if and only if $\delta d_L^\nabla + d_L^\nabla \delta = 0$.*

Proof The operators δ and d_L^∇ being two derivations of degree $+1$ of the graded algebra $\Gamma(\Lambda^\bullet L^\vee \otimes \hat{S}(B^\vee))$, so is their graded commutator $\delta d_L^\nabla + d_L^\nabla \delta$.

This derivation $\delta d_L^\nabla + d_L^\nabla \delta$ vanishes on the subalgebra $\Gamma(\Lambda^\bullet L^\vee \otimes S^0(B^\vee))$ of $\Gamma(\Lambda^\bullet L^\vee \otimes \hat{S}(B^\vee))$ since $\delta(\omega \otimes 1) = 0$ and $d_L^\nabla(\omega \otimes 1) = (d_L\omega) \otimes 1$ for all $\omega \in \Gamma(\Lambda^\bullet L^\vee)$.

For all $\chi \in \Gamma(B^\vee)$, we have

$$\delta(1 \otimes \chi) = q^\vee(\chi) \otimes 1 \qquad \text{and} \qquad d_L^\nabla(1 \otimes \chi) = \sum_k \nu_k \otimes \nabla_{v_k}\chi$$

and hence

$$\left(d_L^\nabla \delta + \delta d_L^\nabla\right)(1 \otimes \chi) = \left(d_L(q^\vee \chi) + \sum_k q^\vee(\nabla_{v_k}\chi) \wedge \nu_k\right) \otimes 1.$$

Furthermore, we have

$$d_L(q^\vee \chi) + \sum_k q^\vee(\nabla_{v_k}\chi) \wedge \nu_k = \left(T^\nabla\right)^\vee(\chi),$$

where the symbol $\left(T^\nabla\right)^\vee$ denotes the vector bundle morphism $\left(T^\nabla\right)^\vee : B^\vee \to \Lambda^2 L^\vee$ dual to the torsion $T^\nabla : \Lambda^2 L \to B$. Indeed, for all $X, Y \in \Gamma(L)$, we have

$$\begin{aligned}
&\left\langle d_L(q^\vee \chi) + \sum_k q^\vee(\nabla_{v_k}\chi) \wedge \nu_k \middle| X \wedge Y \right\rangle \\
&= \rho(X)\left\langle q^\vee \chi \middle| Y\right\rangle - \rho(Y)\left\langle q^\vee \chi \middle| X\right\rangle - \left\langle q^\vee \chi \middle| [X,Y]\right\rangle \\
&\quad + \sum_k \left(\left\langle q^\vee(\nabla_{v_k}\chi) \middle| X\right\rangle \cdot \langle \nu_k | Y\rangle - \left\langle q^\vee(\nabla_{v_k}\chi) \middle| Y\right\rangle \cdot \langle \nu_k | X\rangle \right) \\
&= \rho(X)\,\langle \chi | q(Y)\rangle - \rho(Y)\,\langle \chi | q(X)\rangle - \langle \chi | q([X,Y])\rangle \\
&\quad + \left\langle \nabla_{\sum_k \langle \nu_k | Y\rangle \cdot v_k}\chi \middle| q(X)\right\rangle - \left\langle \nabla_{\sum_k \langle \nu_k | X\rangle \cdot v_k}\chi \middle| q(Y)\right\rangle \\
&= \rho(X)\,\langle \chi | q(Y)\rangle - \langle \nabla_X \chi | q(Y)\rangle - \rho(Y)\,\langle \chi | q(X)\rangle + \langle \nabla_Y \chi | q(X)\rangle - \langle \chi | q([X,Y])\rangle \\
&= \left\langle \chi \middle| \nabla_X\left(q(Y)\right)\right\rangle - \left\langle \chi \middle| \nabla_Y\left(q(X)\right)\right\rangle - \langle \chi | q([X,Y])\rangle \\
&= \left\langle \chi \middle| T^\nabla(X,Y)\right\rangle.
\end{aligned}$$

Thus, we obtain

$$\left(d_L^\nabla \delta + \delta d_L^\nabla\right)(1 \otimes \chi) = \left(T^\nabla\right)^\vee(\chi) \otimes 1, \qquad \forall \chi \in \Gamma(B^\vee).$$

The desired result now follows immediately since the algebra $\Gamma(\Lambda^\bullet L^\vee \otimes \hat{S}(B^\vee))$ is generated by its subalgebra $\Gamma(\Lambda^\bullet L^\vee \otimes S^0(B^\vee))$ and its elements of type $1 \otimes \chi$ with $\chi \in \Gamma(B^\vee)$. □

The sections of the vector bundle $\hat{S}(B^\vee)\otimes B$ may be interpreted as fiberwise formal vertical vector fields on B—they act as derivations of the algebra $\Gamma(\hat{S}(B^\vee))$ of fiberwise formal functions on B in a natural fashion. Tensoring the maps δ, h, σ, and τ with id_B, we obtain a filtered contraction

$$\big(\Gamma(\Lambda^\bullet A^\vee\otimes B),0\big)\underset{\sigma_\natural}{\overset{\tau_\natural}{\rightleftarrows}}\big(\Gamma(\Lambda^\bullet L^\vee\otimes\hat{S}(B^\vee)\otimes B),-\delta_\natural\big)\circlearrowleft h_\natural\,,$$

where $\tau_\natural=\tau\otimes\mathrm{id}_B$; $\sigma_\natural=\sigma\otimes\mathrm{id}_B$; $\delta_\natural=\delta\otimes\mathrm{id}_B$; and $h_\natural=h\otimes\mathrm{id}_B$.

We are now ready to present a second construction of a homological vector field Q on the graded manifold $L[1]\oplus B$, which relies on Fedosov's iteration method.

Proposition 4.6 *Let (L,A) be a Lie pair. Given a splitting $i\circ p+j\circ q=\mathrm{id}_L$ of the short exact sequence (13) and a torsion-free L-connection ∇ on B, there exists a* unique *1-form valued in the formal vertical vector fields on B:*

$$X^\nabla\in\Gamma(\Lambda^1L^\vee\otimes\hat{S}^{\geq 2}(B^\vee)\otimes B)$$

satisfying $h_\natural(X^\nabla)=0$ and such that the derivation

$$Q:\Gamma(\Lambda^\bullet L^\vee\otimes\hat{S}(B^\vee))\to\Gamma(\Lambda^{\bullet+1}L^\vee\otimes\hat{S}(B^\vee))$$

defined by

$$Q=-\delta+d_L^\nabla+X^\nabla$$

satisfies $Q^2=0$. Here X^∇ acts on the algebra $\Gamma(\Lambda^\bullet L^\vee\otimes\hat{S}(B^\vee))$ as a derivation in a natural fashion. As a consequence, $(L[1]\oplus B,Q)$ is a dg manifold.

Proof Suppose there exists such a X^∇ and consider its decomposition $X^\nabla=\sum_{k=2}^\infty X_k$, where $X_k\in\Gamma(\Lambda^1L^\vee\otimes S^k(B^\vee)\otimes B)$. Then $Q=-\delta+d_L^\nabla+X_2+X_{\geq 3}$ with $X_{\geq 3}=\sum_{k=3}^\infty X_k$. Since $\delta^2=0$; $\big[\delta,d_L^\nabla\big]=0$ according to Lemma 4.5; and $d_L^\nabla\circ d_L^\nabla=R^\nabla$, we have

$$\begin{aligned}Q^2&=\delta^2-\big(\delta d_L^\nabla+d_L^\nabla\delta\big)+\big\{d_L^\nabla d_L^\nabla-\delta X_2-X_2\delta\big\}\\&\quad+\big\{d_L^\nabla X^\nabla+X^\nabla d_L^\nabla+(X^\nabla)^2-\delta X_{\geq 3}-X_{\geq 3}\delta\big\}\\&=\big\{R^\nabla-[\delta,X_2]\big\}+\Big\{\Big[d_L^\nabla+\tfrac{1}{2}X^\nabla,X^\nabla\Big]-\big[\delta,X_{\geq 3}\big]\Big\}.\end{aligned}$$

With respect to the bigrading on $\Gamma(\Lambda^\bullet L^\vee\otimes S^\bullet B^\vee)$, the respective bidegrees of the operators δ, d_L^∇, and X_2 are $(1,-1)$, $(1,0)$, and $(1,1)$. Furthermore, the operator $X_{\geq 3}$ maps $\Gamma(\Lambda^kL^\vee\otimes S^pB^\vee)$ to $\Gamma(\Lambda^{k+1}L^\vee\otimes\hat{S}^{\geq p+2}(B^\vee))$. As a consequence, the operator $R^\nabla-[\delta,X_2]$ has bidegree $(2,0)$ while the operator $\big[d_L^\nabla+\frac{1}{2}X^\nabla,X^\nabla\big]-\big[\delta,X_{\geq 3}\big]$ maps $\Gamma(\Lambda^kL^\vee\otimes S^pB^\vee)$ to $\Gamma(\Lambda^{k+2}L^\vee\otimes\hat{S}^{\geq p+1}(B^\vee))$. Hence, since their sum Q^2 is zero by assumption, they must vanish separately.

In other words, the requirement $Q^2 = 0$ is equivalent to the pair of equations

$$[\delta, X_2] = R^\nabla \qquad \text{and} \qquad \left[\delta, X_{\geq 3}\right] = \left[d_L^\nabla + \tfrac{1}{2}X^\nabla, X^\nabla\right].$$

Note that $\sigma_\natural(X_2) = 0$ and $\sigma_\natural(X_{\geq 3}) = 0$ since $X_2, X_{\geq 3} \in \Gamma(\Lambda^1 L^\vee \otimes \hat{S}^{\geq 2}(B^\vee) \otimes B)$ and also that $h_\natural(X_2) = 0$ and $h_\natural(X_{\geq 3}) = 0$ as $h_\natural(X^\nabla) = 0$ by assumption. Therefore, since $\delta_\natural h_\natural + h_\natural \delta_\natural = \mathrm{id} - \tau_\natural \sigma_\natural$, we obtain $h_\natural \delta_\natural(X_2) = X_2$ and $h_\natural \delta_\natural(X_{\geq 3}) = X_{\geq 3}$.

It follows that

$$X_2 = h_\natural \delta_\natural(X_2) = h_\natural([\delta, X_2]) = h_\natural(R^\nabla)$$
$$X_{\geq 3} = h_\natural \delta_\natural(X_{\geq 3}) = h_\natural(\left[\delta, X_{\geq 3}\right]) = h_\natural\left[d_L^\nabla + \tfrac{1}{2}X^\nabla, X^\nabla\right].$$

Projecting the second equation onto $\Gamma(\Lambda^1 L^\vee \otimes S^{k+1}(B^\vee) \otimes B)$, we obtain

$$X_2 = h_\natural(R^\nabla) \tag{16}$$
$$X_{k+1} = h_\natural\Big(d_L^\nabla \circ X_k + X_k \circ d_L^\nabla + \sum_{\substack{p+q=k+1 \\ 2\leq p,q\leq k-1}} X_p \circ X_q\Big), \qquad \text{for } k \geq 2. \tag{17}$$

The successive terms of $X^\nabla = \sum_{k=2}^\infty X_k$ can thus be computed inductively starting from $X_2 = h_\natural(R^\nabla)$. Therefore, if it exists, the derivation X^∇ is uniquely determined by the torsion-free connection ∇ and the splitting $j : B \to L$.

Now, defining X_k inductively by the relations (16) and (17) and setting $X^\nabla = \sum_{k=2}^\infty X_k$, we have $h_\natural(X^\nabla) = h_\natural(X_2 + X_{\geq 3}) = h_\natural^2\big(R^\nabla + \delta_\natural(X_{\geq 3})\big) = 0$ since $h_\natural^2 = 0$. Moreover, we have $X_2 = h_\natural(R^\nabla) \in \Gamma(\Lambda^1 L^\vee \otimes S^2(B^\vee) \otimes B)$ as $R^\nabla \in \Gamma(\Lambda^2 L^\vee \otimes B^\vee \otimes B)$. Making use of Eq. (17), one proves by induction on k that $X_k \in \Gamma(\Lambda^1 L^\vee \otimes S^k(B^\vee) \otimes B)$. This completes the proof of the existence of X^∇. $\square$

We elect to call a dg manifold $(L[1] \oplus B, Q)$ constructed in this way a *Fedosov dg manifold*.

We note that Proposition 4.6 was proved independently by Batakidis–Voglaire [3].

4.4 Equivalence of the two constructions

The aim of this section is to prove the following theorem, which is one of the main results of this paper.

Theorem 4.7 *Let (L, A) be a Lie pair, let $i \circ p + j \circ q = \mathrm{id}_L$ be a splitting of the short exact sequence (13), and let ∇ be an L-connection on B extending the Bott A-connection. If ∇ is torsion-free, then the dg manifold $(L[1] \oplus B, d_L^{\nabla^\natural})$ described in Proposition 4.1 coincides with the dg manifold $(L[1] \oplus B, Q)$ constructed by the Fedosov iteration described in (the proof of) Proposition 4.6.*

Consider the bundle map

$$\Theta : L \otimes SB \to SB$$

defined in [18] by the relation

$$\Theta(l; s) = \nabla^{\natural}_{l} s - \nabla_{l} s - q(l) \odot s, \quad \forall l \in \Gamma(L), s \in \Gamma(SB).$$

Lemma 4.8 *([18, Lemma 5.13]) For all $l \in \Gamma(L)$, we have $\Theta(l; 1) = 0$.*

Proposition 4.9 *([18, Lemma 5.16]) For all $l \in \Gamma(L)$, the map $s \mapsto \Theta(l; s)$ is a coderivation of the R-coalgebra $\Gamma(SB)$ which preserves the filtration*

$$\cdots \hookrightarrow \Gamma(S^{\leq n-1}B) \hookrightarrow \Gamma(S^{\leq n}B) \hookrightarrow \Gamma(S^{\leq n+1}B) \hookrightarrow \cdots .$$

Lemma 4.10 *([18, Lemma 5.17]) For all $n \in \mathbb{N}$ and all $b_0, b_1, \ldots, b_n \in \Gamma(B)$, we have*

$$\sum_{k=0}^{n} \Theta\big(j(b_k); b_0 \odot \cdots \odot \widehat{b_k} \odot \cdots \odot b_n\big) = 0.$$

Lemma 4.11 *Provided the L-connection ∇ on B extends the Bott representation, the bundle map Θ associated with the connection ∇ and a splitting $i \circ p + j \circ q = \mathrm{id}_L$ satisfies the relation*

$$\Theta(l; b) = -\tfrac{1}{2}\beta^{\nabla}(q(l), b)$$

for all $l \in \Gamma(L)$ and $b \in \Gamma(B)$.

In particular, if the L-connection ∇ on B is torsion-free, we have $\Theta(l; b) = 0$ for all $l \in \Gamma(L)$ and $b \in \Gamma(B)$.

Proof We may rewrite Lemma 3.9 as

$$j(Y) \cdot \mathrm{pbw}(Z) = \mathrm{pbw}\left(Y \odot Z + \nabla_{j(Y)} Z - \tfrac{1}{2}\, \beta^{\nabla}(Y, Z)\right)$$

or

$$\nabla^{\natural}_{j(Y)} Z = Y \odot Z + \nabla_{j(Y)} Z - \tfrac{1}{2}\, \beta^{\nabla}(Y, Z)$$

for all $Y, Z \in \Gamma(B)$.

On the other hand, according to [18, Lemma 5.11], we have

$$\nabla^{\natural}_{i(X)} Z = \nabla^{\mathrm{Bott}}_{X} Z$$

for all $X \in \Gamma(A)$ and $Z \in \Gamma(B)$. Furthermore, since the L-connection ∇ on B is assumed to extend the Bott representation, we have

$$\nabla^{\text{Bott}}_X Z = \nabla_{i(X)} Z$$

for all $X \in \Gamma(A)$ and $Z \in \Gamma(B)$.

Therefore, for all $l \in \Gamma(L)$ and $b \in \Gamma(B)$, we have

$$\begin{aligned}\nabla^{\lightning}_l b &= \nabla^{\lightning}_{i\circ p(l)} b + \nabla^{\lightning}_{j\circ q(l)} b \\ &= \nabla_{i\circ p(l)} b + \left\{q(l) \odot b + \nabla_{j\circ q(l)} b - \tfrac{1}{2}\, \beta^{\nabla}(q(l), b)\right\} \\ &= q(l) \odot b + \nabla_l b - \tfrac{1}{2}\, \beta^{\nabla}(q(l), b).\end{aligned}$$

The result then follows from the definition of Θ. ◻

Let

$$\Gamma(SB) \otimes_R \Gamma(\hat{S}(B^\vee)) \xrightarrow{\langle -|-\rangle} R$$

be the duality pairing defined by

$$\left\langle b_1 \odot \cdots \odot b_p \middle| \beta_1 \odot \cdots \odot \beta_q \right\rangle = \begin{cases} \sum_{\sigma\in S_p} \iota_{b_1}\beta_{\sigma(1)} \cdot \iota_{b_2}\beta_{\sigma(2)} \cdots \iota_{b_p}\beta_{\sigma(p)} & \text{if } p = q \\ 0 & \text{if } p \neq q \end{cases}$$

for all $b_1, \ldots, b_p \in \Gamma(B)$ and $\beta_1, \ldots, \beta_q \in \Gamma(B^\vee)$.

A straightforward computation yields the following

Lemma 4.12 *Let $(\partial_i)_{i\in\{1,\ldots,r\}}$ be a local frame of B and let $(\chi_j)_{j\in\{1,\ldots,r\}}$ be the dual local frame of $B^\vee$. We have*

$$\left\langle \partial^I \middle| \chi^J \right\rangle = I!\, \delta_{I,J}, \quad \forall I, J \in \mathbb{N}_0^n$$

and

$$\sigma = \sum_{I\in\mathbb{N}_0^n} \frac{1}{I!} \left\langle \partial^I \middle| \sigma \right\rangle \chi^I, \quad \forall \sigma \in \Gamma\big(\hat{S}(B^\vee)\big).$$

Lemma 4.13 *For any $l \in \Gamma(L)$, and all $s \in \Gamma(SB)$, and $\sigma \in \Gamma\big(\hat{S}(B^\vee)\big)$, we have*

$$\langle s|\iota_l \delta(\sigma)\rangle = \langle q(l) \odot s|\sigma\rangle\,.$$

Proof It suffices to prove the relation for $s = \partial^I$ and $\sigma = \chi^J$. We have

$$\begin{aligned}\left\langle \partial^I \middle| \iota_l \delta(\chi^J) \right\rangle &= \left\langle \partial^I \middle| \sum_{k=1}^r \iota_{q(l)} \chi_k \cdot J_k \chi^{J-e_k} \right\rangle = \sum_{k=1}^r \iota_{q(l)} \chi_k \left\langle \partial^I \middle| J_k \chi^{J-e_k} \right\rangle \\ &= \sum_{k=1}^r \iota_{q(l)} \chi_k \; J_k \; I! \, \delta_{I,J-e_k} = \sum_{k=1}^r \iota_{q(l)} \chi_k \; J! \, \delta_{I+e_k,J} = \sum_{k=1}^r \iota_{q(l)} \chi_k \left\langle \partial^{I+e_k} \middle| \chi^J \right\rangle \\ &= \left\langle \sum_{k=1}^r \iota_{q(l)} \chi_k \cdot \partial_k \odot \partial^I \middle| \chi^J \right\rangle = \left\langle q(l) \odot \partial^I \middle| \chi^J \right\rangle.\end{aligned}$$

□

Consider the map

$$\Xi : \Gamma(S^k B^\vee) \to \Gamma(\Lambda^1 L^\vee \otimes \hat{S}^{\geq k+1}(B^\vee)), \quad \forall k \geq 0,$$

defined by

$$\langle s | \iota_l \Xi(\sigma) \rangle = \langle \Theta(l; s) | \sigma \rangle \,, \tag{18}$$

for all $l \in \Gamma(L)$, $s \in \Gamma(SB)$, and $\sigma \in \Gamma(\hat{S}(B^\vee))$.

The L-connections ∇ and $\nabla^{\natural}$ defined on $S(B)$ induce L-connections on the dual bundle $\hat{S}(B^\vee)$—see Remark 3.5.

Proposition 4.14 $d_L^{\nabla^{\natural}} = -\delta + d_L^{\nabla} - \Xi$

Proof According to Remark 3.5, we have

$$\left\langle \nabla_l^{\natural} s \middle| \sigma \right\rangle + \left\langle s \middle| \nabla_l^{\natural} \sigma \right\rangle = \rho(l) \, \langle s|\sigma\rangle = \langle \nabla_l s|\sigma\rangle + \langle s|\nabla_l \sigma\rangle \,,$$

for all $l \in \Gamma(L)$, $s \in \Gamma(SB)$, and $\sigma \in \Gamma(\hat{S}(B^\vee))$. From there, we obtain

$$\begin{aligned}\left\langle \nabla_l^{\natural} s - \nabla_l s \middle| \sigma \right\rangle &= \left\langle s \middle| \nabla_l \sigma - \nabla_l^{\natural} \sigma \right\rangle \\ \langle q(l) \odot s + \Theta(l; s) | \sigma \rangle &= \left\langle s \middle| \iota_l \left(d_L^{\nabla} \sigma - d_L^{\nabla^{\natural}} \sigma \right) \right\rangle\end{aligned}$$

and, making use of Lemma 4.13 and Eq. (18),

$$\left\langle s \middle| \iota_l \delta(\sigma) + \iota_l \Xi(\sigma) \right\rangle = \left\langle s \middle| \iota_l \left(d_L^{\nabla} \sigma - d_L^{\nabla^{\natural}} \sigma \right) \right\rangle$$

or, equivalently,

$$d_L^{\nabla^{\natural}} = -\delta + d_L^{\nabla} - \Xi.$$

□

Proposition 4.15 *For every $l \in \Gamma(L)$, the operator $\iota_l \Xi$ is a derivation of the R-algebra $\Gamma\big(\hat{S}(B^\vee)\big)$ which preserves the filtration*

$$\cdots \hookrightarrow \Gamma\big(\hat{S}^{\geq n+1}(B^\vee)\big) \hookrightarrow \Gamma\big(\hat{S}^{\geq n}(B^\vee)\big) \hookrightarrow \Gamma\big(\hat{S}^{\geq n-1}(B^\vee)\big) \hookrightarrow \cdots.$$

Proof The result follows immediately from Proposition 4.9 since the R-algebra $\Gamma\big(\hat{S}(B^\vee)\big)$ is dual to the R-coalgebra $\Gamma(S(B))$ and $\iota_l \Xi$ is the transpose of $\Theta(l; -)$ according to Eq. (18). □

Therefore, Ξ may be regarded as an element of the subspace $\Gamma\big(\Lambda^1 L^\vee \otimes \hat{S}(B^\vee) \otimes B\big)$ of the space of derivations of the algebra $\Gamma\big(\Lambda^\bullet L^\vee \otimes \hat{S}(B^\vee)\big)$.

Given any pair of dual local frames $(\partial_i)_{i\in\{1,\dots,r\}}$ and $(\chi_j)_{j\in\{1,\dots,r\}}$ for the vector bundles B and $B^\vee$ and any pair of dual local frames $(l_m)_{m=1}^{\mathrm{rk}(L)}$ and $(\lambda_m)_{m=1}^{\mathrm{rk}(L)}$ for the vector bundles L and $L^\vee$, we have

$$\Xi = \sum_{m=1}^{\mathrm{rk}(L)} \sum_{k=1}^{r} \lambda_m \otimes \iota_{l_m} \Xi(\chi_k) \otimes \partial_k,$$

since each $\iota_{l_m} \Xi$ is a derivation of the R-algebra $\Gamma\big(\hat{S}(B^\vee)\big)$, which is generated locally by $\chi_1, \dots, \chi_r$. Furthermore, for all $l \in \Gamma(L)$, we have

$$\begin{aligned}
\iota_l \Xi(\chi_k) &= \sum_{I \in \mathbb{N}_0^r} \frac{1}{I!} \Big\langle \partial^I \Big| \iota_l \Xi(\chi_k) \Big\rangle \chi^I && \text{by Lemma 4.12,} \\
&= \sum_{I \in \mathbb{N}_0^r} \frac{1}{I!} \Big\langle \Theta(l; \partial^I) \Big| \chi_k \Big\rangle \chi^I && \text{by Eq. (18).}
\end{aligned}$$

Lemma 4.16 *We have $\Xi \in \Gamma\big(\Lambda^1 L^\vee \otimes \hat{S}^{\geq 1}(B^\vee) \otimes B\big)$. Furthermore, if the L-connection ∇ on B is torsion-free, then $\Xi \in \Gamma\big(\Lambda^1 L^\vee \otimes \hat{S}^{\geq 2}(B^\vee) \otimes B\big)$.*

Proof If the connection ∇ is torsion-free, then $\iota_l \Xi \in \Gamma\big(\hat{S}^{\geq 2}(B^\vee) \otimes B\big)$ as $\Theta(l; \partial^I) = 0$ for $|I| \leq 1$ according to Lemma 4.8 and Lemma 4.11. However, if the torsion of ∇ is not zero, we can only say that $\iota_l \Xi \in \Gamma\big(\hat{S}^{\geq 1}(B^\vee) \otimes B\big)$ as $\Theta(l; \partial^I)$ need not vanish for $|I| = 1$. □

We note that, for every pair of dual local frames $(\partial_i)_{i\in\{1,\dots,r\}}$ and $(\chi_j)_{j\in\{1,\dots,r\}}$ for B and $B^\vee$, we have

$$\Xi = \sum_{k=1}^{r} \sum_{J \in \mathbb{N}_0^r} \frac{1}{J!} \Big\langle \partial^J \Big| \Xi(\chi_k) \Big\rangle \chi^J \partial_k.$$

Lemma 4.17 *For all $\lambda \in \Gamma(L^\vee)$ and $J \in \mathbb{N}_0^r$, we have*

$$h(\lambda \otimes \chi^J) = \frac{1}{1+|J|} \sum_{k=1}^{r} \iota_{j(\partial_k)} \lambda \otimes \chi^{J+e_k},$$

where $(\partial_i)_{i\in\{1,\dots,r\}}$ *is any local frame of* B *and* $(\chi_j)_{j\in\{1,\dots,r\}}$ *is the dual local frame of* $B^\vee$.

Proof For $\lambda \in \Gamma(q^\vee B^\vee)$, the result follows immediately from Eq. (15), the very definition of h. The result holds for $\lambda \in \Gamma(p^\vee A^\vee)$ as well since, for all $\alpha \in \Gamma(A^\vee)$, we have $h(p^\vee(\alpha)\otimes\chi^J) = 0$ by the very definition of h and $\iota_{j(\partial_k)} p^\vee(\alpha) = 0$ as $p \circ j = 0$. □

Proposition 4.18 $h_\natural(\Xi) = 0$

Proof Let $(\partial_i)_{i\in\{1,\dots,r\}}$ be a local frame of B and let $(\chi_j)_{j\in\{1,\dots,r\}}$ be the dual local frame of $B^\vee$.

From

$$\Xi = \sum_{k=1}^{r}\sum_{J\in\mathbb{N}_0^r}\frac{1}{J!}\left\langle \partial^J \middle| \Xi(\chi_k)\right\rangle \chi^J \partial_k,$$

we obtain, using Lemma 4.17,

$$\begin{aligned}
h_\natural(\Xi) &= \sum_{k=1}^{r}\sum_{J\in\mathbb{N}_0^r} h\left\{\frac{1}{J!}\left\langle \partial^J \middle| \Xi(\chi_k)\right\rangle \chi^J\right\}\partial_k \\
&= \sum_{k=1}^{r}\sum_{J\in\mathbb{N}_0^r}\frac{1}{1+|J|}\sum_{p=1}^{r}\frac{1}{J!}\left\langle \partial^J \middle| \iota_{j(\partial_p)}\Xi(\chi_k)\right\rangle \chi_p\chi^J\partial_k \\
&= \sum_{k=1}^{r}\sum_{J\in\mathbb{N}_0^r}\frac{1}{1+|J|}\sum_{p=1}^{r}\frac{1}{J!}\left\langle \Theta(j(\partial_p);\partial^J) \middle| \chi_k\right\rangle \chi^{J+e_p}\partial_k \\
&= \sum_{k=1}^{r}\sum_{M\in\mathbb{N}_0^r}\frac{1}{|M|}\frac{1}{M!}\left\langle \sum_{p=1}^{r} M_p\,\Theta(j(\partial_p);\partial^{M-e_p}) \middle| \chi_k\right\rangle \chi^M\partial_k.
\end{aligned}$$

It follows directly from Lemma 4.10 that

$$\sum_{p=1}^{r} M_p\,\Theta(j(\partial_p);\partial^{M-e_p}) = 0$$

for every $M = (M_1,\dots,M_r)\in\mathbb{N}_0^r$. □

We are now ready to complete the proof of Theorem 4.7.

Proof of Theorem 4.7 Since $\Xi \in \Gamma\big(\Lambda^1 L^\vee \otimes \hat{S}^{\geq 2}(B^\vee)\otimes B\big)$ provided $T^\nabla = 0$ (Proposition 4.16), $h_\natural(\Xi) = 0$ (Proposition 4.18), and $d_L^{\nabla^\natural} = -\delta + d_L^\nabla - \Xi$ (Proposition 4.14) satisfies $d_L^{\nabla^\natural}\circ d_L^{\nabla^\natural} = 0$, the uniqueness statement in Proposition 4.6 asserts that $X^\nabla = -\Xi$ and $Q = d_L^{\nabla^\natural}$. □

5 Dolgushev–Fedosov quasi-isomorphisms

5.1 Contraction of the Fedosov dg manifold

Our second main result, Theorem 5.1 below, extends the Dolgushev–Fedosov quasi-isomorphism [8, Theorem 3] to the context of Lie pairs. This section is devoted to its proof; Theorem 5.1 is an immediate consequence of Proposition 5.4 below (and Theorem 4.7).

Theorem 5.1 *Given a Lie pair (L, A), let $d_L^{\nabla^\natural}$ be the homological vector field on $L[1] \oplus B$ determined by the choice of a splitting $i \circ p + j \circ q = \mathrm{id}_L$ of the short exact sequence (13) and an L-connection ∇ on B as in Proposition 4.1. Then the natural inclusion $(A[1], d_A) \hookrightarrow (L[1] \oplus B, d_L^{\nabla^\natural})$ is a quasi-isomorphism of dg manifolds.*

Remark 5.2 In particular, if L is the tangent bundle to a smooth manifold M and A is its rank-zero subbundle, Theorem 5.1 reduces to the part of [8, Theorem 3] pertaining to functions on the manifold M.

Dolgushev established the quasi-isomorphism [8, Theorem 3] by a direct verification. Here we will prove the stronger result that the cochain complexes $\big(\Gamma(\Lambda^\bullet L^\vee \otimes \hat{S}(B^\vee)), d_L^{\nabla^\natural}\big)$ and $\big(\Gamma(\Lambda^\bullet A^\vee), d_A\big)$ are indeed homotopy equivalent. Homological perturbation—see Appendix A—provides a quick and easy proof of this result: the operator $\varrho = d_L^{\nabla^\natural} + \delta$ can be understood as a perturbation of the cochain complex $\big(\Gamma(\Lambda^\bullet L^\vee \otimes \hat{S}(B^\vee)), -\delta\big)$ appearing in the contraction of Proposition 4.3.

Lemma 5.3 *While the operator δ respects the filtration*

$$\cdots \subseteq \mathscr{F}_{m+1} \subseteq \mathscr{F}_m \subseteq \mathscr{F}_{m-1} \subseteq \cdots$$

defined by

$$\mathscr{F}_m = \prod_{k+p \geq m} \Gamma\big(\Lambda^k L^\vee \otimes S^p(B^\vee)\big),$$

the operator $\varrho = d_L^{\nabla^\natural} + \delta$ raises the filtration degree by 1, i.e. $\varrho(\mathscr{F}_m) \subseteq \mathscr{F}_{m+1}$. Moreover, the operator ϱ is a perturbation of the filtered cochain complex

$$\cdots \longrightarrow \Gamma(\Lambda^{k-1} L^\vee \otimes \hat{S}(B^\vee)) \xrightarrow{-\delta} \Gamma(\Lambda^k L^\vee \otimes \hat{S}(B^\vee)) \xrightarrow{-\delta} \Gamma(\Lambda^{k+1} L^\vee \otimes \hat{S}(B^\vee)) \longrightarrow \cdots$$

Proof According to Proposition 4.14, we have $\varrho = d_L^{\nabla^\natural} + \delta = d_L^\nabla - \Xi$. With respect to the bigrading on $\Gamma\big(\Lambda^\bullet L^\vee \otimes S^\bullet(B^\vee)\big)$, the operator δ has bidegree $(1, -1)$ while the operator d_L^∇ has bidegree $(1, 0)$. Furthermore, according to Lemma 4.16, the operator Ξ maps $\Gamma(\Lambda^k L^\vee \otimes S^p(B^\vee))$ to $\Gamma(\Lambda^{k+1} L^\vee \otimes S^{\geq p}(B^\vee))$. Therefore, the differential δ satisfies $\delta(\mathscr{F}_m) \subseteq \mathscr{F}_m$ and the operator ϱ satisfies $\varrho(\mathscr{F}_m) \subseteq \mathscr{F}_{m+1}$. Finally, we have $(-\delta + \varrho)^2 = (d_L^{\nabla^\natural})^2 = 0$ since the connection $\nabla^\natural$ is flat. $\square$

Proposition 5.4 *Given a Lie pair (L, A), let d_A denote the Chevalley–Eilenberg differential of the Lie algebroid A regarded as a homological vector field on $A[1]$ and let $d_L^{\nabla^\natural}$ be the homological vector field on $L[1] \oplus B$ determined by the choice of a splitting $i \circ p + j \circ q = \mathrm{id}_L$ of the short exact sequence (13) and an L-connection ∇ on B as in Proposition 4.1. Then, there exists a contraction*

$$\big(\Gamma(\Lambda^\bullet A^\vee), d_A\big) \underset{\sigma}{\overset{\breve{\tau}}{\rightleftarrows}} \big(\Gamma(\Lambda^\bullet L^\vee \otimes \hat{S}(B^\vee)), d_L^{\nabla^\natural}\big) \circlearrowright \breve{h}\,.$$

where

$$\breve{\tau} = \sum_{k=0}^{\infty} (h\varrho)^k \tau, \qquad \breve{h} = \sum_{k=0}^{\infty} (h\varrho)^k h, \qquad \varrho = d_L^{\nabla^\natural} + \delta,$$

and the maps δ, τ, σ, and h are those defined in Sect. 4.3. In particular, σ is the natural inclusion of graded manifolds $A[1] \hookrightarrow L[1] \oplus B$ defined by Eq. (14).

Proof We proceed by homological perturbation (see Lemma A.1). Starting from the filtered contraction of Proposition 4.3, it suffices to perturb the coboundary operator $-\delta$ by the operator ϱ (see Lemma 5.3) to obtain the new contraction

$$\big(\Gamma(\Lambda^\bullet A^\vee), \vartheta\big) \underset{\breve{\sigma}}{\overset{\breve{\tau}}{\rightleftarrows}} \big(\Gamma(\Lambda^\bullet L^\vee \otimes \hat{S}(B^\vee)), -\delta + \varrho\big) \circlearrowright \breve{h}\,.$$

We have $\sigma\varrho h = 0$ as, for all $n, p \in \mathbb{N}_0$,

$$\Gamma(\Lambda^n L^\vee \otimes S^p B^\vee) \xrightarrow{h} \Gamma(\Lambda^{n-1} L^\vee \otimes S^{p+1} B^\vee) \xrightarrow{\varrho} \Gamma(\Lambda^n L^\vee \otimes \hat{S}^{\geq p+1}(B^\vee)) \xrightarrow{\sigma} 0.$$

Therefore, we obtain

$$\breve{\sigma} := \sum_{k=0}^{\infty} \sigma(\varrho h)^k = \sigma$$

and

$$\vartheta := \sum_{k=0}^{\infty} \sigma(\varrho h)^k \varrho\tau = \sigma\varrho\tau = \sigma(d_L^\nabla - \Xi^\nabla)(p^\vee \otimes 1) = \sigma\big((d_L \circ p^\vee) \otimes 1\big) = d_A.$$

The result follows immediately since $-\delta + \varrho = d_L^{\nabla^\natural}$ (Proposition 4.14). □

We note that a similar construction of a Fedosov resolution of the algebra of smooth functions on a manifold based on homological perturbation was described by Hans-Christian Herbig in [14].

5.2 Matched pairs

In this section, we establish an explicit expression for the quasi-isomorphism

$$\breve{\tau} : \Gamma(\Lambda^\bullet A^\vee) \to \Gamma(\Lambda^\bullet L^\vee \otimes \hat{S}(B^\vee))$$

defined in Proposition 5.4 valid only in the special case of matched pairs. This formula can be considered as an extension to the matched pair case of the augmentation map

$$\breve{\tau} : C^\infty(M) \to \Omega^0\big(M; \hat{S}(T_M^\vee)\big)$$

arising from the Emmrich–Weinstein 'formal exponential map'—see Sect. 6 and [10, Theorem 1.6].

Suppose the short exact sequence (13) admits a splitting $j : B \to L$ whose image $j(B)$ is a Lie subalgebroid of L—i.e. $L = A \bowtie B$ is a matched pair. Then B is a Lie algebroid and the composition of the morphism of associative algebras $\mathcal{U}(B) \to \mathcal{U}(L)$ induced by j with the canonical projection $\mathcal{U}(L) \twoheadrightarrow \frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$ yields a canonical isomorphism of left R-coalgebras $\mathcal{U}(B) \cong \frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$.

Since $L = A \bowtie B$ is a matched pair, we have a Bott B-representation on A:

$$\nabla_b^{\text{Bott}} a = p\big([j(b), i(a)]\big), \quad \forall b \in \Gamma(B), a \in \Gamma(A).$$

The dual B-connection on $A^\vee$ extends to the exterior algebra $\Lambda A^\vee$ by derivation:

$$\nabla_b^{\text{Bott}} \alpha = i^\vee\big(\mathcal{L}_{j(b)}(p^\vee \alpha)\big), \quad \forall b \in \Gamma(B), \alpha \in \Gamma(\Lambda A^\vee). \tag{19}$$

The symbol $\mathcal{L}$ denotes the Lie derivative in the setting of the Lie algebroid L. Applying $p^\vee$ to both sides of Eq. (19), we conclude that the diagram

$$\begin{array}{ccc} \Gamma(\Lambda^\bullet A^\vee) & \xrightarrow{\nabla_b^{\text{Bott}}} & \Gamma(\Lambda^\bullet A^\vee) \\ \downarrow{\scriptstyle p^\vee} & & \downarrow{\scriptstyle p^\vee} \\ \Gamma(\Lambda^\bullet L^\vee) & \xrightarrow[\mathcal{L}_{j(b)}]{} & \Gamma(\Lambda^\bullet L^\vee) \end{array}$$

commutes for every $b \in \Gamma(B)$. The Bott representation of B on $\Lambda A^\vee$ extends to a $\mathcal{U}(B)$-representation on $\Gamma(\Lambda A^\vee)$. The action

$$\mathcal{U}(B) \times \Gamma(\Lambda A^\vee) \xrightarrow{\rtimes} \Gamma(\Lambda A^\vee)$$

satisfies

$$p^\vee\big(b_1 b_2 \cdots b_n \rtimes \alpha\big) = \mathcal{L}_{j(b_1)} \mathcal{L}_{j(b_2)} \cdots \mathcal{L}_{j(b_n)}(p^\vee \alpha),$$

for all $b_1, b_2, \ldots, b_n \in \Gamma(B)$ and $\alpha \in \Gamma(\Lambda A^\vee)$.

In the matched pair case, the chain map $\breve{\tau} : \Gamma(\Lambda^\bullet A^\vee) \to \Gamma(\Lambda^\bullet L^\vee \otimes \hat{S}(B^\vee))$ defined in Proposition 5.4 admits a simple description in terms of the splitting $j : B \hookrightarrow L$, the associated left $\mathcal{U}(B)$-module structure $\rtimes$ on $\Gamma(\Lambda A^\vee)$, and the map $\operatorname{pbw} : \Gamma(S(B)) \to \mathcal{U}(B)$ corresponding to the B-connection on B induced by j and ∇.

Consider the derivation $\mathscr{D}$ of the subalgebra $\Gamma(p^\vee(\Lambda A^\vee) \otimes \hat{S}(B^\vee))$ of $\Gamma(\Lambda L^\vee \otimes \hat{S}(B^\vee))$ defined on generators by the identities

$$\mathscr{D}(p^\vee\alpha \otimes 1) = \sum_{k=1}^{r} p^\vee(\nabla^{\mathrm{Bott}}_{\partial_k}\alpha) \otimes \chi_k, \qquad \forall \alpha \in \Gamma(A^\vee),$$
$$\mathscr{D}(1 \otimes \chi) = \sum_{k=1}^{r} 1 \otimes \chi_k \cdot \nabla_{j(\partial_k)}\chi, \qquad \forall \chi \in \Gamma(B^\vee).$$

In general, for all $\alpha \in \Gamma(\Lambda^\bullet A^\vee)$ and $\chi^J \in \Gamma(\hat{S}(B^\vee))$, with $J \in \mathbb{N}_0^r$, we have

$$\mathscr{D}(p^\vee\alpha \otimes \chi^J) = \sum_{k=1}^{r} \Big\{ p^\vee(\nabla^{\mathrm{Bott}}_{\partial_k}\alpha) \otimes \chi_k \cdot \chi^J + p^\vee\alpha \otimes \chi_k \cdot \nabla_{j(\partial_k)}(\chi^J) \Big\}.$$

Remark 5.5 An analogue of the derivation $\mathscr{D}$ was introduced recently in [4, Section 2.1 and Remark 2.3]. It would be interesting to understand the precise relation between these two derivations.

The remainder of this section is devoted to the proof of the following theorem.

Theorem 5.6 *If $L = A \bowtie B$ is a matched pair, then the cochain map*

$$\breve{\tau} : \Gamma(\Lambda^\bullet A^\vee) \to \Gamma(\Lambda^\bullet L^\vee \otimes \hat{S}(B^\vee))$$

defined in Proposition 5.4 satisfies

$$\breve{\tau} = \exp(\mathscr{D}) \circ \tau$$

and

$$\breve{\tau}(\alpha) = \sum_{J \in \mathbb{N}_0^r} \frac{1}{J!} p^\vee\big(\operatorname{pbw}(\partial^J) \rtimes \alpha\big) \otimes \chi^J, \qquad \forall \alpha \in \Gamma(\Lambda A^\vee). \tag{20}$$

The following lemma is an analogue of Lemma 4.18, which is proved *mutatis mutandis*.

Lemma 5.7 *For all $\alpha \in \Gamma(\Lambda A^\vee)$ and $J \in \mathbb{N}_0^r$, we have $h\Xi(p^\vee\alpha \otimes \chi^J) = 0$.*

Lemma 5.8 *For all $\alpha \in \Gamma(\Lambda A^\vee)$ and $J \in \mathbb{N}_0^r$, we have*

$$h\varrho(p^\vee\alpha \otimes \chi^J) = \frac{1}{1 + |J|}\mathscr{D}(p^\vee\alpha \otimes \chi^J).$$

Proof Since $L = A \bowtie B$, Proposition 3.3 asserts that, if $\alpha \in \Gamma(\Lambda^u A^\vee)$, then

$$d_L(p^\vee \alpha) \in \Omega^{u+1,0} \oplus \Omega^{u,1},$$

where $\Omega^{u,v} = \Gamma(p^\vee(\Lambda^u A^\vee) \wedge q^\vee(\Lambda^v B^\vee))$. Therefore, if $\alpha \in \Gamma(\Lambda^u A^\vee)$, we have

$$\begin{aligned} d_L^\nabla(p^\vee \alpha \otimes \chi^J) &= d_L(p^\vee \alpha) \otimes \chi^J \\ &\quad + \sum_t \nu_t \wedge (p^\vee \alpha) \otimes \nabla_{v_t}(\chi^J) \in (\Omega^{u+1,0} \oplus \Omega^{u,1}) \otimes_R \Gamma(S^{|J|}(B^\vee)), \end{aligned}$$

where $v_1, v_2, \ldots, v_n$ and $\nu_1, \nu_2, \ldots, \nu_n$ are any pair of dual local frames for the vector bundles L and $L^\vee$, and it follows from Eq. (15) that

$$\begin{aligned} hd_L^\nabla(p^\vee \alpha \otimes \chi^J) &= \frac{1}{1+|J|} \sum_k \Big\{ i_{j(\partial_k)} d_L(p^\vee \alpha) \otimes \chi^{J+e_k} + \sum_t (i_{j(\partial_k)} \nu_t) \cdot p^\vee \alpha \otimes \chi_k \cdot \nabla_{v_t}(\chi^J) \Big\} \\ &= \frac{1}{1+|J|} \sum_k \Big\{ \mathcal{L}_{j(\partial_k)}(p^\vee \alpha) \otimes \chi^{J+e_k} + p^\vee \alpha \otimes \chi_k \cdot \nabla_{\sum_t (i_{j(\partial_k)} \nu_t) v_t}(\chi^J) \Big\} \\ &= \frac{1}{1+|J|} \sum_k \Big\{ p^\vee(\nabla^{\mathrm{Bott}}_{\partial_k} \alpha) \otimes \chi^{J+e_k} + p^\vee \alpha \otimes \chi_k \cdot \nabla_{j(\partial_k)}(\chi^J) \Big\}. \end{aligned}$$

The desired result follows from Lemma 5.7. □

Proof of Theorem 5.6 Reasoning by induction on k, one proves that

$$\mathscr{D}^k \circ \tau(\alpha) \in \Gamma(p^\vee(\Lambda A^\vee) \otimes S^k(B^\vee))$$

for all $\alpha \in \Gamma(\Lambda A^\vee)$ and $k \in \mathbb{N}$. Using Lemma 5.8 and reasoning by induction on k once again, one proves that

$$(h\varrho)^k \circ \tau = \frac{1}{k!} \mathscr{D}^k \circ \tau$$

for all $k \in \mathbb{N}$. It follows that

$$\breve{\tau} = \left(\sum_{k=0}^\infty (h\varrho)^k \right) \circ \tau = \left(\sum_{k=0}^\infty \frac{1}{k!} \mathscr{D}^k \right) \circ \tau = \exp(\mathscr{D}) \circ \tau.$$

Set $\lceil \alpha \rfloor = \sum_{J \in \mathbb{N}_0^r} \frac{1}{J!} p^\vee \big(\mathrm{pbw}(\partial^J) \rtimes \alpha \big) \otimes \chi^J$ for all $\alpha \in \Gamma(\Lambda A^\vee)$. We claim that

$$(\mathrm{id} - h\varrho) \lceil \alpha \rfloor = \tau(\alpha).$$

It then follows from Proposition A.2 that $\breve{\tau}(\alpha) = \lceil \alpha \rfloor$—the desired result.

It remains to establish our claim. From

$$0 = \rho(j(\partial_k)) \left\langle \partial^K \middle| \chi^J \right\rangle = \left\langle \nabla_{j(\partial_k)}(\partial^K) \middle| \chi^J \right\rangle + \left\langle \partial^K \middle| \nabla_{j(\partial_k)}(\chi^J) \right\rangle,$$

we obtain

$$\nabla_{j(\partial_k)}(\chi^J) = \sum_K \frac{1}{K!} \left\langle \partial^K \middle| \nabla_{j(\partial_k)}(\chi^J) \right\rangle \chi^K = -\sum_K \frac{1}{K!} \left\langle \nabla_{j(\partial_k)}(\partial^K) \middle| \chi^J \right\rangle \chi^K. \quad (21)$$

From Lemma 5.7, Lemma 5.8, and Eq. (21), we obtain

$$\begin{aligned} h_Q \lceil\alpha\rfloor = \sum_J \tfrac{1}{J!} \tfrac{1}{1+|J|} \sum_k \Big\{ & p^\vee\Big(\nabla^{\text{Bott}}_{\partial_k}\big(\text{pbw}(\partial^J) \rtimes \alpha\big)\Big) \otimes \chi_k \cdot \chi^J \\ & - p^\vee\big(\text{pbw}(\partial^J) \rtimes \alpha\big) \otimes \chi_k \cdot \sum_K \tfrac{1}{K!} \left\langle \nabla_{j(\partial_k)}(\partial^K) \middle| \chi^J \right\rangle \chi^K \Big\} \end{aligned}$$

This can be rewritten as

$$\begin{aligned} h_Q \lceil\alpha\rfloor = & \sum_J \tfrac{1}{J!} \tfrac{1}{1+|J|} \sum_k p^\vee\big(j(\partial_k) \cdot \text{pbw}(\partial^J) \rtimes \alpha\big) \otimes \chi^{J+e_k} \\ & - \sum_K \tfrac{1}{K!} \tfrac{1}{1+|K|} \sum_k p^\vee\Big(\text{pbw}\left(\nabla_{j(\partial_k)}(\partial^K)\right) \rtimes \alpha \Big) \otimes \chi^{K+e_k} \end{aligned}$$

and then

$$\begin{aligned} h_Q \lceil\alpha\rfloor = & \sum_{\substack{M\in\mathbb{N}_0^r \\ |M|\geq 1}} \frac{1}{M!} p^\vee \\ & \times \left(\frac{1}{|M|} \sum_k M_k \Big(j(\partial_k) \cdot \text{pbw}\left(\partial^{M-e_k}\right) - \text{pbw}\left(\nabla_{j(\partial_k)} \partial^{M-e_k}\right) \Big) \rtimes \alpha \right) \otimes \chi^M. \end{aligned}$$

Finally, it follows from Eq. (10) that

$$h_Q \lceil\alpha\rfloor = \sum_{\substack{M\in\mathbb{N}_0^r \\ |M|\geq 1}} \frac{1}{M!} p^\vee\big(\text{pbw}(\partial^M) \rtimes \alpha\big) \otimes \chi^M = \lceil\alpha\rfloor - p^\vee(\alpha) \otimes 1 = \lceil\alpha\rfloor - \tau(\alpha).$$

□

6 Application: the 'formal exponential map' of Emmrich–Weinstein

In this section, we give a simple and direct proof of a result—see [10, Theorem 1.6 and Section 7] and Theorem 6.1 below—which Emmrich–Weinstein proved by way of an indirect argument involving real analytic manifolds.

According to Proposition 4.6, given a torsion-free affine connection ∇ on M, one can construct a homological vector field Q on the graded manifold $T_M[1] \oplus T_M$ by Fedosov's iteration method. This is essentially what [10, Theorem 1.1]—more precisely the special case when $a = 0$—and [8, Theorem 2] assert. One obtains a derivation Q of $\Omega^\bullet\big(M; \hat{S}(T_M^\vee)\big)$ such that $Q^2 = 0$. Identifying $\Gamma(\hat{S}(T_M^\vee))$ to the algebra of functions on the formal neighborhood $(T_M)_\infty$ of the zero section of the tangent bundle to M, Emmrich–Weinstein regard the derivation Q as vector fields determining a distribution on $(T_M)_\infty$ transverse to the fibers of $(T_M)_\infty \to M$, i.e. a (nonlinear) formal Ehresmann connection on $(T_M)_\infty$. Since $Q^2 = 0$, this distribution is involutive and the Ehresmann connection is flat. Those sections $\varsigma \in \Gamma(\hat{S}(T_M^\vee))$ such that $Q(\varsigma) = 0$ are interpreted as functions on $(T_M)_\infty$ which are constant along the leaves of the foliation tangent to the flat Ehresmann connection. As explained by Emmrich–Weinstein, the Ehresmann connection is transverse to the zero section and each one of its leaves intersects the zero section in a unique point. The leaves of the foliation given by the flat Ehresmann connection are the fibers of a 'mapping' $\mathrm{EXP} : (T_M)_\infty \to M$, which Emmrich–Weinstein call 'formal exponential map.' Identifying M with the zero section of $(T_M)_\infty$, functions defined on M can be extended to functions on $(T_M)_\infty$ constant along the leaves. The resulting map $C^\infty(M) \to \Gamma(\hat{S}(T_M^\vee))$ is the pull-back of functions through EXP.

Theorem 6.1 *(*[10, Theorem 1.6]*) Given a torsion-free affine connection ∇ on M, the 'formal exponential map'* EXP *described above coincides with the infinite-order jet of the geodesic exponential map* exp *determined by the connection ∇.*

Proof By definition, the 'formal exponential map' EXP is completely determined by Q in the following sense: the pull-back $\varsigma = \mathrm{EXP}^*(f)$ of a function $f \in C^\infty(M)$ by EXP is the unique solution $\varsigma \in \Gamma(\hat{S}(T_M^\vee))$ of the initial value problem

$$Q(\varsigma) = 0, \qquad \sigma(\varsigma) = f.$$

We think of the map $\sigma : \Gamma(\hat{S}(T_M^\vee)) \to C^\infty(M)$ as the pull-back of functions through the zero section of $(T_M)_\infty \to M$. On the other hand, Proposition 5.4 applied to the Lie pair (L, A) where L is the tangent bundle to M and A is its rank-zero subbundle yields the contraction

$$C^\infty(M) \underset{\sigma}{\overset{\breve{\tau}}{\rightleftarrows}} \big(\Omega^\bullet(M; \hat{S}(T_M^\vee)), d^{\nabla^\natural}\big) \circlearrowright \breve{h}$$

where $C^\infty(M)$ is seen as a cochain complex concentrated in degree 0. In particular, for all $f \in C^\infty(M)$, we have

$$\breve{\tau}(f) \in \Gamma(\hat{S}(T_M^\vee)), \qquad d^{\nabla^\natural}\big(\breve{\tau}(f)\big) = 0, \qquad \text{and} \qquad \sigma\big(\breve{\tau}(f)\big) = f.$$

According to Theorem 4.7, we have $Q = d^{\nabla^\natural}$. Therefore, $\mathrm{EXP}^* = \breve{\tau}$. Let n denote the dimension of the manifold M. It follows from Eq. (20) in Theorem 5.6 that

$$\breve{\tau}(f) = \sum_{J \in \mathbb{N}_0^n} \frac{1}{J!} \operatorname{pbw}(\partial^J) f \otimes \chi^J.$$

According to [18, Theorem 3.11], the Poincaré–Birkhoff–Witt isomorphism pbw (described in Theorem 3.7) is the infinite-order jet of the geodesic exponential map $\exp : T_M \to M$ arising from the connection ∇. Hence, we obtain

$$\mathrm{EXP}^*(f) = \breve{\tau}(f) = \sum_{J \in \mathbb{N}_0^n} \frac{1}{J!} \partial^J \big(\exp^*(f) \big) \otimes \chi^J .$$

This concludes the proof that EXP is the infinite-order jet of exp. □

In fact, Theorem 5.6 above can be seen as an extension of Emmrich–Weinstein's result [10, Theorem 1.6] to the case of matched pairs.

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Appendix A. Homological perturbation

A contraction of a cochain complex (N, δ) onto a cochain complex (M, d) consists of a pair of cochain maps $\sigma : N \to M$ and $\tau : M \to N$ and an endomorphism $h : N \to N[-1]$ of the graded module N satisfying the following five relations:

$$\sigma\tau = \mathrm{id}_M, \qquad \tau\sigma - \mathrm{id}_N = h\delta + \delta h,$$
$$\sigma h = 0, \qquad h\tau = 0, \qquad h^2 = 0.$$

We symbolize such a contraction by a diagram

$$(M, d) \underset{\sigma}{\overset{\tau}{\rightleftarrows}} (N, \delta) \circlearrowleft h \, .$$

If, furthermore, the cochain complexes N and M are filtered and the maps σ, τ, and h preserve the filtration, the contraction is said to be filtered [9, §12].

A descending filtration

$$\cdots \subseteq F_{p+1}N \subseteq F_p N \subseteq F_{p-1}N \subseteq \cdots$$

on a cochain complex N is said to be exhaustive if $N = \bigcup_p F_p N$ and complete if $N = \varprojlim_p \frac{N}{F_p N}$.

A *perturbation* of the filtered cochain complex

$$\cdots \longrightarrow N^{n-1} \xrightarrow{\delta} N^n \xrightarrow{\delta} N^{n+1} \longrightarrow \cdots$$

is an operator ϱ of degree $+1$ on N, which raises the filtration degree by 1 (i.e. $\varrho(F_pN) \subseteq F_{p+1}N$) and satisfies $(\delta + \varrho)^2 = 0$ so that $\delta + \varrho$ is a new coboundary operator on N.

We refer the reader to [15, §1] for a brief history of the following lemma.

Lemma A.1 (Homological Perturbation [6]) *Let*

$$(M, d) \underset{\sigma}{\overset{\tau}{\rightleftarrows}} (N, \delta) \circlearrowleft h$$

be a filtered contraction. Given a perturbation ϱ of the cochain complex (N, δ), if the filtrations on M and N are exhaustive and complete, then the series

$$\breve{\tau} := \sum_{k=0}^{\infty} (h\varrho)^k \tau \qquad \breve{h} := \sum_{k=0}^{\infty} (h\varrho)^k h = \sum_{k=0}^{\infty} h(\varrho h)^k$$

$$\breve{\sigma} := \sum_{k=0}^{\infty} \sigma(\varrho h)^k \qquad \vartheta := \sum_{k=0}^{\infty} \sigma(\varrho h)^k \varrho\tau = \sum_{k=0}^{\infty} \sigma\varrho(h\varrho)^k \tau$$

converge, ϑ is a perturbation of the cochain complex (M, d), and

$$(M, d + \vartheta) \underset{\breve{\sigma}}{\overset{\breve{\tau}}{\rightleftarrows}} (N, \delta + \varrho) \circlearrowleft \breve{h}$$

constitutes a new filtered contraction.

Proposition A.2 *Under the same hypothesis as in Lemma A.1, the chain map $\breve{\tau}$ is entirely determined by τ, $h\varrho$ and the relation $(\mathrm{id} - h\varrho)\breve{\tau} = \tau$. Likewise, the homotopy operator $\breve{h}$ is entirely determined by h, $h\varrho$ and the relation $(\mathrm{id} - h\varrho)\breve{h} = h$.*

Proof Since the filtration on N is complete and ϱ raises the filtration degree by 1 while h preserves it, the geometric series $\sum_{k=0}^{\infty}(h\varrho)^k$ converges and its sum is the inverse of the operator $\mathrm{id} - h\varrho$. The result follows immediately. □

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