GW/PT DESCENDENT CORRESPONDENCE VIA VERTEX OPERATORS

A. OBLOMKOV, A. OKOUNKOV, AND R. PANDHARIPANDE

ABSTRACT. We propose an explicit formula for the GW/PT descendent correspondence in the stationary case for nonsingular connected projective 3-folds. The formula, written in terms of vertex operators, is found by studying the 1-leg geometry. We prove the proposal for all nonsingular projective toric 3-folds. An application to the Virasoro constraints for the stationary descendent theory of stable pairs will appear in a sequel.

CONTENTS

0. INTRODUCTION

0.1. **Correspondences.** Let X be a nonsingular projective 3-fold. In [15, 16, 18] the correspondence between the *primary* Gromov-Witten and Donaldson-Thomas invariants was established in the toric case. As indicated in [16], the natural next step is to extend the map/sheaf correspondence to the full *descendent* theories of X. A basic compatibility for the map/sheaf correspondence is that the SL(2)-equivariant counts in geometries of the form

$$X = \mathsf{Curve} \times \mathbb{C}^2$$

should specialize, via Mumford's relation for Hodge classes and the analogous vanishing on the sheaf side, to the Gromov-Witten/Hurwitz correspondence for curves studied in [25, 26, 27]. This idea represents the technical starting point of the paper, and our formulas evolved from the formulas of [27].

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Our goal here is to a present a conjecture relating descendent integrals over the moduli of stable maps and sheaves. The conjecture is an explicit closed formula for the correspondence for all descendents of cohomology[1] classes

$$\gamma \in H^{\geq 2}(X) \tag{1}$$

of degree *at least 2*. We refer to the degree restriction (1) as the *stationary* case.[2]

Basic results on the map/sheaf descendent correspondence have been obtained in [32, 33] including general constructions, proofs in the toric and hypersurface cases, calculations of leading terms, and geometric applications. However, a closed formula for the descendent correspondence was *not* found in [32]. We have succeeded here in finding such a formula for descendents of classes of degree at least 2. For nonsingular projective toric 3-folds, we prove the stationary descendent correspondence formula (via the methods of [32]).

0.2. **Stable pairs.** In the years after the first papers, a better moduli space for the sheaf theory was introduced in [34]: the moduli of stable pairs on X. As the generating series of descendent integrals in the theory of stable pairs have much better analytic properties, we will work will the stable pairs theory on the sheaf side (instead of the moduli of ideal sheaves used in [15, 16, 18]).

Definition 1. *A* stable pair (F, s) *on a 3-fold* X *is a coherent sheaf* F *on* X *and a section* $s \in H^0(X, F)$ *satisfying the following stability conditions:*

- F *is* pure *of dimension 1,*
- *the section* $s : \mathcal{O}_X \to F$ *has cokernel of dimensional 0.*

Let C be the scheme-theoretic support of F. By the purity condition, all the irreducible components of C are of dimension 1 (no 0-dimensional components are permitted). By [34, Lemma 1.6], the kernel of s is the ideal sheaf of C,

$$\mathcal{I}_C = \ker(s) \subset \mathcal{O}_X \,,$$

and C has no embedded points. A stable pair

$$\mathcal{O}_X \to F$$

therefore defines a Cohen-Macaulay subcurve $C \subset X$ via the kernel of s and a 0-dimensional subscheme of C via the support of the cokernel of s.

To a stable pair, we associate the Euler characteristic and the class of the support C of the sheaf F,

$$\chi(F) = n \in \mathbb{Z} \quad \text{and} \quad [C] = \beta \in H_2(X, \mathbb{Z})\,.$$

For fixed n and β, there is a projective moduli space of stable pairs $P_n(X, \beta)$. Unless β is an effective curve class, the moduli space $P_n(X, \beta)$ is empty. An analysis of the deformation theory and the construction of the virtual cycle $[P_n(X, \beta)]^{vir}$ is given [34]. We refer the reader to [28, 36] for an introduction to the theory of stable pairs.

[1] We take singular cohomology always with $\mathbb{C}$-coefficients.

[2] The terminology agrees with the definition of stationary descendents in case X is a curve [25].

0.3. **Stable pairs descendents.** Stable pairs invariants are integrals of the form

$$\langle\omega\rangle_\beta^{\mathsf{PT}} = \sum_n q^n \int_{[P_n(X,\beta)]^{vir}} \omega_n\,,$$

where $\omega = \sum_{n\in\mathbb{Z}} \omega_n$ is an element of the formal sum $\bigoplus_{n\in\mathbb{Z}} H^*(P_n(X,\beta))$. For fixed β, the moduli space $P_n(X,\beta)$ is empty for all sufficiently negative n. Hence, $\langle\omega\rangle_\beta^{\mathsf{PT}}$ is a Laurent series in q.

Descendent classes are defined via universal structures over the moduli space of stable pairs. Let

$$\pi : X\times P_n(X,\beta) \to P_n(X,\beta)$$

be the projection to the second factor, and let

$$\mathcal{O}_{X\times P_n(X,\beta)} \to \mathbb{F}_n$$

be the universal stable pair on $X\times P_n(X,\beta)$. Let

$$\mathrm{ch}_k(\mathbb{F}) = \sum_{n\in\mathbb{Z}} \mathrm{ch}_k(\mathbb{F}_n) \in \bigoplus_{n\in\mathbb{Z}} H^*(X\times P_n(X,\beta))\,.$$

The following *descendent classes* are our main objects of study:

$$\mathrm{ch}_k(\gamma) = \pi_*\left(\mathrm{ch}_k(\mathbb{F})\cdot\gamma\right) \in \bigoplus_{n\in\mathbb{Z}} H^*(P_n(X,\beta)) \quad \text{for} \quad \gamma\in H^*(X)\,.$$

Conjecture 2. [34] *The stable pairs descendent series*

$$\Big\langle \mathrm{ch}_{k_1}(\gamma_1)\cdots\mathrm{ch}_{k_m}(\gamma_m)\Big\rangle_\beta^{\mathsf{PT}}$$

is the Laurent expansion of rational function of q for all $\gamma_i\in H^*(X)$ and all $k_i\geq 0$.

For Calabi-Yau 3-folds, Conjecture 2 reduces immediately to the rationality of the basic series $\langle 1\rangle_\beta^{\mathsf{PT}}$ proven via wall-crossing by [3, 40]. In the presence of descendent insertions, Conjecture 2 has been proven for rich class of varieties including toric varieties, hypersurface, and varieties admitting good degenerations [29, 30, 31, 32, 33].

The generating series for descendents in the DT theory of ideal sheaves have more complicated analytic properties. In particular, the descendent series are *not* always Laurent expansions of rational functions. Descendents in DT theory are discussed in Section 4, and a DT version of Conjecture 2 is presented there.

0.4. **Gromov-Witten descendents.** Let X be a nonsingular projective 3-fold. Gromov-Witten theory is defined via integration over the moduli space of stable maps.

Let C be a possibly disconnected curve with at worst nodal singularities. The genus of C is defined by $1-\chi(\mathcal{O}_C)$. Let $\overline{M}'_{g,m}(X,\beta)$ denote the moduli space of maps with possibly disconnected domain curves C of genus g with *no* collapsed connected components. The latter condition requires each connected component of C to represent a nonzero class in $H_2(X,\mathbb{Z})$. In particular, C must represent a nonzero class β. Let

$$\mathrm{ev}_i : \overline{M}'_{g,m}(X,\beta) \to X\,,$$

$$\mathbb{L}_i \to \overline{M}'_{g,m}(X,\beta)$$

denote the evaluation maps and the cotangent line bundles associated to the marked points. Let $\gamma_1,\ldots,\gamma_m \in H^*(X)$, and let

$$\psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{M}'_{g,m}(X,\beta))\,.$$

The *descendent insertions*, denoted by $\tau_k(\gamma)$, correspond to the classes $\psi_i^k \mathrm{ev}_i^*(\gamma)$ on the moduli space of stable maps. Let

$$\Big\langle \tau_{k_1}(\gamma_1)\cdots\tau_{k_m}(\gamma_m)\Big\rangle^{\mathsf{GW}}_{g,\beta} = \int_{[\overline{M}'_{g,m}(X,\beta)]^{vir}} \prod_{i=1}^m \psi_i^{k_i}\mathrm{ev}_i^*(\gamma_i)$$

denote the descendent Gromov-Witten invariants. The associated generating series is defined by

$$\Big\langle \tau_{k_1}(\gamma_1)\cdots\tau_{k_m}(\gamma_m)\Big\rangle^{\mathsf{GW}}_{\beta} = \sum_{g\in\mathbb{Z}} \Big\langle \prod_{i=1}^m \tau_{k_i}(\gamma_i)\Big\rangle^{\mathsf{GW}}_{g,\beta} u^{2g-2}. \tag{2}$$

Since the domain components must map nontrivially, an elementary argument shows the genus g in the sum (2) is bounded from below. Foundational aspects of the theory are treated, for example, in [2, 7, 12].

0.5. **Negative descendants.** To state our GW/PT descendent conjecture, we will require not only the usual Gromov-Witten descendant τ_k for $k \geq 0$ but also descendants τ_k with negative $k<0$ indices. While the negative subscripts have no geometric meaning for stable maps, negative descendents will drastically simplify the statement of the correspondence.

Negative descendents reflect the fact that descendent integrals can be interpreted as matrix coefficients of operators in a Fock space. The Fock space formalism for the study of ancestors in the GW theory of toric manifolds was developed by Givental [9], and his computations can be interpreted in terms of negative descendents. For another application of the negative descendents, see the undergraduate thesis of Pixton [37].

We introduce the negative descendants by means of an auxiliary algebra Heis_X with a linear functional which encodes the Gromov-Witten invariants.

Definition 3. Heis_X *is the* $\mathbb{C}(u)$*-algebra generated by the elements*

$$\left\{ \tau_k(\gamma) \;\middle|\; k\in\mathbb{Z},\ \gamma\in H^*(X) \right\}$$

and satisfying the relations

$$[\tau_k(\alpha),\tau_m(\beta)] = (-1)^k \frac{\delta_{k+m+1}}{u^2}\int_X \alpha\cdot\beta\,.$$

The standard (shifted) Heisenberg algebra Heis is generated by $\{\tau_k\}_{k\in\mathbb{Z}}$ with relations

$$[\tau_k,\tau_m] = (-1)^k\frac{\delta_{k+m+1}}{u^2}\,.$$

Normally ordered monomials

$$\tau_{i_1}\tau_{i_2}\ldots\tau_{i_k},\quad i_1\leq i_2\leq\cdots\leq i_k,$$

form a linear basis of Heis. To an element of Heis, we assign an element of $\mathrm{Hom}(H^*(X), \mathsf{Heis}_X)$ by the following rule on basis elements (with linear extension):

- every normally ordered monomial of positive degree[3] is assigned the $\mathbb{C}$-linear map
$$\tau_{i_1}\tau_{i_2}\dots\tau_{i_k} : H^*(X) \to \mathsf{Heis}_X$$
defined via coproduct (we use the Swidler coproduct convention [11]):
$$\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}(\gamma) = \tau_{i_1}(\gamma_{(1)})\tau_{i_2}(\gamma_{(2)})\cdots\tau_{i_k}(\gamma_{(k)})\,.$$
- the degree 0 monomial $1 \in \mathsf{Heis}$ is assigned to $0 \in \mathrm{Hom}(H^*(X), \mathsf{Heis}_X)$.

Furthermore, for $\alpha \in H^*(X)$, define the product
$$\alpha \cdot \tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}(\gamma) = \tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}(\alpha\cdot\gamma)\,.$$

We construct a linear functional $\langle\cdot\rangle_\beta$ on Heis_X via Gromov-Witten theory. The positive elements $\tau_{k\geq 0}(\gamma)$ generate a commutative[4] subalgebra $\mathsf{Heis}_X^+ \subset \mathsf{Heis}_X$. The linear functional
$$\left\langle \tau_{i_1}(\gamma_1)\tau_{i_2}(\gamma_2)\cdots\tau_{i_k}(\gamma_k)\right\rangle_\beta = \left\langle \tau_{i_1}(\gamma_1)\tau_{i_2}(\gamma_2)\cdots\tau_{i_k}(\gamma_k)\right\rangle_\beta^{\mathsf{GW}}$$
is well-defined on the basis elements of Heis_X^+. We extend the linear functional to the whole algebra Heis_X by imposing the condition
$$\left\langle \tau_k(\gamma)\Phi\right\rangle_\beta = \left\langle\Phi\right\rangle_\beta \cdot \frac{\delta_{k+2}}{iu}\int_X \gamma \tag{3}$$
for all $\Phi \in \mathsf{Heis}_X$ and $k < 0$. We will often denote $\langle\cdot\rangle_\beta$ on Heis_X by $\langle\cdot\rangle_\beta^{\mathsf{GW}}$ to emphasis the Gromov-Witten origins.

0.6. **Renormalized descendants.** The most convenient way to state our conjectural GW/PT correspondence is to introduce new classes $\mathrm{H}_k^{\mathsf{PT}}(\gamma)$ and $\mathrm{H}_k^{\mathsf{GW}}(\gamma)$ for $\gamma \in H^*(X)$. The required operators are introduced below.

- The classes $\mathrm{H}_k^{\mathsf{PT}}(\gamma)$ are linear combinations of descendents for stable pairs defined in Section 0.3. Let
$$\mathrm{H}_k^{\mathsf{PT}}(\gamma) = \pi_*\left(\mathrm{H}_k^{\mathsf{PT}}\cdot\gamma\right) \in \bigoplus_{n\in\mathbb{Z}} H^*(P_n(X,\beta))\,.$$

The classes $\mathrm{H}_k^{\mathsf{PT}} \in \bigoplus_{n\in\mathbb{Z}} H^*(X\times P_n(X,\beta))$ are defined by
$$\mathrm{H}^{\mathsf{PT}}(x) = \sum_{k=0}^{\infty} x^{k+1}\mathrm{H}_k^{\mathsf{PT}} = \mathcal{S}\left(\frac{x}{\theta}\right)\sum_{k=0}^{\infty} x^k \mathrm{ch}_k(\mathbb{F})\,,$$
where
$$\theta^{-2} = -c_2(T_X), \quad \mathcal{S}(x) = \frac{e^{x/2}-e^{-x/2}}{x}\,.$$

[3]Here, $\tau_{i_1}\tau_{i_2}\dots\tau_{i_k}$ has degree k.

[4]If X has odd cohomology, then supercommutative. For simplicity, our analysis will restricted to commutative case. The modifications for odd cohomology are not significant and are left to the reader.

In particular, we have

$$\mathrm{H}_k^{\mathsf{PT}} = \mathrm{ch}_{k+1}(\mathbb{F}) - \frac{c_2}{24}\mathrm{ch}_{k-1}(\mathbb{F}) + \frac{c_2^2}{1920}\mathrm{ch}_{k-3}(\mathbb{F}) - \frac{c_2^3}{322560}\mathrm{ch}_{k-5}(\mathbb{F}) + \dots$$

- The classes $\mathrm{H}_k^{\mathsf{GW}}(\gamma)$ are most naturally constructed in terms of linear combinations of

descendent operators introduced by Getzler [8] . These operators are

$$\sum_{n=-\infty}^{\infty} z^n \tau_n = \sum_{n>0} \frac{(iuz)^{n-1}}{(1+zc_1)_n}\mathfrak{a}_n + \frac{1}{c_1}\sum_{n<0}\frac{(iuz)^{n-1}}{(1+zc_1)_n}\mathfrak{a}_n\,, \tag{4}$$

where we use the standard Pochhammer symbol

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}\,.$$

Here, c_1 is treated as a formal symbol, but whenever there is an evaluation, c_1 becomes $c_1(T_X)$.

A straightforward computation shows that the relations for the operators τ_k imply the standard Heisenberg relations for the operators α_k:

$$[\mathfrak{a}_k(\alpha), \mathfrak{a}_m(\beta)] = k\delta_{k+m}\int_X \alpha\cdot\beta\,.$$

By definition (4), $\tau_{k\geq 0}(\gamma)$ is a linear combination of $\mathfrak{a}_i(\gamma\cdot c_1^{k+1-i})$, $i = k+1,\dots,1$,

$$\tau_k = \frac{(iu)^k}{(k+1)!}\mathfrak{a}_{k+1} - c_1\frac{(iu)^{k-1}}{k!}\left(\sum_{a=1}^{k}\frac{1}{a}\right)\mathfrak{a}_k + \frac{(iu)^{k-2}}{(k-1)!}c_1^2\left(\sum_{a=1}^{k-1}\frac{1}{a^2} + \sum_{1\leq a<b\leq k-1}\frac{1}{ab}\right)\mathfrak{a}_{k-1} + \dots\,.$$

For the first non-negative values of k, the formula yields

$$\tau_0 = \mathfrak{a}_1\,, \quad \tau_1 = \frac{iu}{2}\mathfrak{a}_2 - c_1\mathfrak{a}_1\,, \quad \tau_2 = -\frac{u^2}{6}\mathfrak{a}_3 - \frac{3iuc_1}{4}\mathfrak{a}_2 + c_1^2\mathfrak{a}_1\,. \tag{5}$$

Similarly, $\tau_{k<0}(\gamma)$ is a linear combination of $\mathfrak{a}_i(\gamma\cdot c_1^{k-i})$, $i = k, k-1, \dots,$

$$\tau_k = (-iu)^k(-k-1)!\mathfrak{a}_k + (-iu)^{k-1}(-k)!c_1\left(\sum_{a=1}^{-k}\frac{1}{a}\right)\mathfrak{a}_{k-1} + \dots\,.$$

If we invert the transition matrix from elements $\mathfrak{a}$ to τ, we obtain

$$\mathfrak{a}_{-2} = -u^2\tau_{-2} + \dots, \quad \mathfrak{a}_{-1} = -iu\tau_{-1} - iuc_1\tau_{-2} + \dots\,.$$

Thus the negative operators $\mathfrak{a}_{k<0}$ act inside the bracket in a nonstandard manner:

$$\big\langle \mathfrak{a}_k(\gamma)\Phi\big\rangle_\beta = \left[\int_X \big(-c_1\delta_{k+1} + \delta_{k+2}iu\big)\cdot\gamma\right]\big\langle\Phi\big\rangle_\beta\,, \quad k<0\,.$$

We assemble the operators $\mathfrak{a}$ in the following generating function:

$$\phi(z) = \sum_{n>0}\frac{\mathfrak{a}_n}{n}\left(\frac{izc_1}{u}\right)^{-n} + \frac{1}{c_1}\sum_{n<0}\frac{\mathfrak{a}_n}{n}\left(\frac{izc_1}{u}\right)^{-n}\,. \tag{6}$$

The main objects of our paper are the new operators

$$\mathrm{H}^{\mathsf{GW}}(x) = \sum_{k=0}^{\infty} \mathrm{H}_k^{\mathsf{GW}} x^{k+1} = \mathrm{Res}_{w=\infty} \left(\frac{\sqrt{dydw}}{y-w} : e^{\theta\phi(y)-\theta\phi(w)} : \right) \tag{7}$$

where y, w, and x satisfy the constraint

$$ye^y = we^w e^{-x/\theta} \,. \tag{8}$$

Here, $\mathrm{Res}_{w=\infty}$ denotes the integral along a small loop around $w=\infty$. The operators $\mathrm{H}_k^{\mathsf{GW}}$ are mutually commutative. To obtain explicit formulas for $\mathrm{H}_k^{\mathsf{GW}}$, we use the Lambert function to solve equation (8) and express y in terms of x, w. Then, the integral in the definition of $\mathrm{H}_k^{\mathsf{GW}}$ can be interpreted as an extraction of the coefficient in of w^{-1}. We provide an explicit method to compute $\mathrm{H}_k^{\mathsf{GW}}$ in Section 2.8. The descendent classes

$$\mathrm{H}_k^{\mathsf{GW}}(\gamma) \in \mathsf{Heis}_X$$

are then obtained using the Swidler coproduct as in Section 0.5. We also use Swidler coproduct conventions in

$$\mathrm{H}_{\vec{k}}^{\mathsf{GW}}(\gamma) = \prod_i \mathrm{H}_{k_i}^{\mathsf{GW}}(\gamma), \quad \vec{k} = (k_1, \ldots, k_m).$$

0.7. **Equivariant correspondence.** All the definitions and construction introduced in Sections 0.1-0.6 have canonical lifts to the equivariant setting with respect to a group action on the variety X. Our first result concerns the equivariant GW/PT descendent correspondence [32].

The most natural setting is the capped vertex formalism from [18, 32] which we review briefly here. Let the 3-dimensional torus

$$\mathsf{T} = \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$$

act on $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ diagonally The tangent weights of the T-action at the point

$$\mathsf{p} = 0 \times 0 \times 0 \in \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$$

are s_1, s_2, s_3. The T-equivariant cohomology ring of a point is

$$H_{\mathsf{T}}(\bullet) = \mathbb{C}[s_1, s_2, s_3] \,.$$

We have the following factorization of the restriction of class $c_1c_2 - c_3$ of X to p,

$$c_1c_2 - c_3 = (s_1+s_2)(s_1+s_3)(s_2+s_3) \,,$$

where $c_i = c_i(T_X)$.

Let $U \subset \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ be the T-equivariant 3-fold obtained by removing the three T-equivariant lines L_1, L_2, L_3 passing through the point $\infty \times \infty \times \infty$. Let $D_i \subset U$ be the divisor with i^{th} coordinate ∞. For a triple of partitions μ_1, μ_2, μ_3, let

$$\left\langle \prod_i \tau_{k_i}(\mathsf{p}) \,\middle|\, \mu_1, \mu_2, \mu_3 \right\rangle_{U,D}^{\mathsf{GW},\mathsf{T}}, \quad \left\langle \prod_i \mathrm{ch}_{k_i}(\mathsf{p}) \,\middle|\, \mu_1, \mu_2, \mu_3 \right\rangle_{U,D}^{\mathsf{PT},\mathsf{T}} \tag{9}$$

denote the generating series of the T-equivariant relative Gromov-Witten and stable pairs invariants of the pair

$$D = \cup_i D_i \subset U$$

with relative conditions μ_i along the divisor D_i. The stable maps spaces are always taken with no contracted connected components. The series (9) are the *capped descendent vertices* of [18].

Theorem 4. *After the change of variables $-q = e^{iu}$ the following correspondence between the 2-leg capped descendent vertices holds:*

$$\Big\langle \prod_i \mathrm{H}^{\mathsf{GW}}_{k_i}(\mathsf{p}) \,\Big|\, \mu_1, \mu_2, \emptyset \Big\rangle^{\mathsf{GW},\mathsf{T}}_{U,D} = q^{-|\mu_1|-|\mu_2|} \Big\langle \prod_i \mathrm{H}^{\mathsf{PT}}_{k_i}(\mathsf{p}) \,\Big|\, \mu_1, \mu_2, \emptyset \Big\rangle^{\mathsf{PT},\mathsf{T}}_{U,D} \mod (s_1+s_3)(s_2+s_3)\,.$$

The result of Theorem 4 has two defects. Since the third partition is empty, the result only covers the *2-leg* case. Moreover, the equality of the correspondence is not proven exactly, but only mod $(s_1 + s_3)(s_2 + s_3)$. For the 1-leg vertex with partitions $(\mu_1, \emptyset, \emptyset)$, Theorem 4 can be restricted in two ways to obtain the equality of the correspondence

$$\mod \ (s_1 + s_3)(s_1 + s_2)(s_2 + s_3)\,.$$

The analysis of the 1-leg geometry in Section 2 shows the relationship of the operators H^{GW} H^{PT} to the formulas of [25, 26, 27].

0.8. **Non-equivariant limit.** By following the proofs of [32], we derive a non-equivariant GW/PT descendent correspondence for stationary insertions. For our statements, we will follow as closely as possible the notation of [32].

Let Heis^c be the Heisenberg algebra with generators $\mathfrak{a}_{k\in\mathbb{Z}\setminus\{0\}}$, coefficients $\mathbb{C}[c_1, c_2]$, and relations

$$[\mathfrak{a}_k, \mathfrak{a}_m] = k\delta_{k+m}c_1c_2.$$

Let $\mathsf{Heis}^c_+ \subset \mathsf{Heis}^c$ be the subalgebra generated by the elements $\mathfrak{a}_{k>0}$, and define the $\mathbb{C}[c_1, c_2]$-linear map

$$\mathsf{Heis}^c \to \mathsf{Heis}^c_+\,, \quad \Phi \mapsto \widehat{\Phi} \tag{10}$$

by $\widehat{\mathfrak{a}}_k = \mathfrak{a}_k$ for $k > 0$ and

$$\widehat{\mathfrak{a}_k\Phi} = (-c_1\delta_{k+1} + \delta_{k+2}iu)\widehat{\Phi}\,, \quad \text{for } k < 0\,.$$

When restricted to the subalgebra Heis^c_+, the $\mathbb{C}[c_1, c_2]$-linear map (10) is an isomorphism.

For a nonsingular projective 3-fold X and classes $\gamma_1, \ldots, \gamma_l \in H^*(X)$, the hat operation make no difference inside the GW bracket,

$$\langle \mathrm{H}^{\mathsf{GW}}_{\vec{k}}(\gamma)\rangle^{\mathsf{GW}}_\beta = \langle \widehat{\mathrm{H}^{\mathsf{GW}}_{\vec{k}}(\gamma)}\rangle^{\mathsf{GW}}_\beta, \tag{11}$$

because the treatment of the negative descendents on the left side is compatible with the treatment of the negative descendents by the hat operation.

Let $\vec{k} = (k_1, \ldots, k_l)$ be a vector of non-negative integers. Following [32], we define the following element of Heis^c_+:

$$\widetilde{\mathrm{H}}_{\vec{k}} = \frac{1}{(c_1c_2)^{l-1}} \sum_{\text{set partitions } P \text{ of } \{1,\ldots,l\}} (-1)^{|P|-1}(|P|-1)! \prod_{S \in P} \widehat{\mathrm{H}}^{\mathsf{GW}}_{\vec{k}_S},$$

where $\mathrm{H}^{\mathsf{GW}}_{\vec{k}_S} = \prod_{i\in S} \mathrm{H}^{\mathsf{GW}}_{k_i}$ and the element $\mathrm{H}^{\mathsf{GW}}_k \in \mathsf{Heis}^c$ is a linear combination of monomials of $\mathfrak{a}_i$, the expression is given by (7). The polynomiality of $\widetilde{\mathrm{H}}_{\vec{k}}$ in c_1, c_2 is not obvious (and will be deduced in Section 3 from the results of [32]).

For classes $\gamma_1, \ldots, \gamma_l \in H^*(X)$ and a vector $\vec{k} = (k_1, \ldots, k_l)$ of non-negative integers, we define

$$\overline{\mathrm{H}_{k_1}(\gamma_1)\ldots\mathrm{H}_{k_l}(\gamma_l)} = \sum_{\text{set partitions } P \text{ of } \{1,\ldots,l\}} \prod_{S\in P} \widetilde{\mathrm{H}}_{\vec{k}_S}(\gamma_S),$$

where $\gamma_S = \prod_{i\in S} \gamma_i$.

Theorem 5. *Let X be a nonsingular projective toric 3-fold, and let $\gamma_i \in H^{\geq 2}(X, \mathbb{C})$. After the change of variables $-q = e^{iu}$, we have*

$$\left\langle \overline{\mathrm{H}_{k_1}(\gamma_1)\ldots\mathrm{H}_{k_l}(\gamma_l)} \right\rangle^{\mathsf{GW}}_{\beta} = q^{-d/2} \left\langle \mathrm{H}^{\mathsf{PT}}_{k_1}(\gamma_1)\ldots\mathrm{H}^{\mathsf{PT}}_{k_l}(\gamma_l) \right\rangle^{\mathsf{PT}}_{\beta},$$

where $d = \int_\beta c_1$.

The two main restrictions in Theorem 5 are that X is toric and that the classes γ_i are of degree at least 2. We conjecture the first restriction to be unnecessary.

Conjecture 6. *Let X be a nonsingular projective 3-fold, and let $\gamma_i \in H^{\geq 2}(X, \mathbb{C})$. After the change of variables $-q = e^{iu}$, we have*

$$\left\langle \overline{\mathrm{H}_{k_1}(\gamma_1)\ldots\mathrm{H}_{k_l}(\gamma_l)} \right\rangle^{\mathsf{GW}}_{\beta} = q^{-d/2} \left\langle \mathrm{H}^{\mathsf{PT}}_{k_1}(\gamma_1)\ldots\mathrm{H}^{\mathsf{PT}}_{k_l}(\gamma_l) \right\rangle^{\mathsf{PT}}_{\beta},$$

where $d = \int_\beta c_1$.

For the precise formula for our GW/PT correspondence, the second restriction (to the stationary theory) is required — the formula is not correct for descendents of the identity class.

0.9. **Plan of the paper.** After reviewing the *dressing operator* in Section 1, the goal of Section 2 is to establish the 1-leg version of Theorem 4 with

$$\mu_1 = \mu_2 = \emptyset$$

modulo s_1+s_2. We derive our formula for the 1-leg GW/PT descendent correspondence by an explicit analysis of the Gromov-Witten and stable pairs descendent theory (the modulo s_1+s_2 condition leads to drastic simplification). The results depend crucially on the earlier study of curves in [25, 26]. We then show our correspondence matches the correspondence of [32] modulo $c_3 - c_1c_2$ and use the results of [32] to conclude the proof of Theorem 4 in Section 3.

To prove the stationary non-equivariant result of Theorem 5, we must check that the non-equivariant limit formulation of the GW/PT descendent correspondence of [32] does not develop singularities under the specialization $c_3 = c_2c_1$. The matter is discussed in the Section 3. Examples are presented in Section 3.7.

We have conjectured Virasoro constraints for the stable pairs descendent theory for all nonsingular projective 3-folds (the precise formulas for $\mathbf{P}^3$ appear in [28]). In a sequel [21] to the present paper, we will apply Theorem 5 to obtain the Virasoro constraints for stable pairs on toric 3-folds in the stationary case from the proven Virasoro constraints in Gromov-Witten theory.

Section 4 contains results and conjectures concerning parallel questions about the descendent DT theory of ideal sheaves. The DT descendent series are not always rational functions in q, so a discussion of the analytic properties is necessary.

0.10. **Past and future directions.** The main formula and the method of the paper is quite old [22]. Since our first draft was written, many new approaches to understanding descendent integrals on both sides of the correspondence were developed. In particular, we now expect a geometric path to the GW/PT descendent correspondence for X should be possible via *relative* geometries X/D. For relative theories *without* higher descendent insertions, the correspondence is very simple [15]. After moving the descendents of the classes

$$\gamma \in H^{\geq 2}(X)$$

to the relative divisor D, the relative GW/PT descendent correspondence there implies a descendent correspondence for X.

For such a path to succeed, a detailed study of the bubble over D is required. On the sheaf side, there has been very good progress in explicitly relating the relative and descendent invariants in fully equivariant K-theory, see [1, 38].

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1. Dressing operator

1.1. **Summary.** We establish here properties of the dressing operator W which intertwines the operators $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ of [26]. These results are needed for the proofs of Theorems 4 and 5 of the introduction.

1.2. **Notation.** We recall the formulas for the operators $\mathcal{A}_k$ of [26, Section 3.2.2]:

$$\begin{aligned}
\mathcal{A} = \sum_{k\in\mathbb{Z}} \mathcal{A}_k z^k &= \frac{1}{u}\mathcal{S}(uz)^{tz} \sum_{k\in\mathbb{Z}} \frac{(e^{uz/2}-e^{-uz/2})^k}{(tz+1)_k} \mathcal{E}_k(uz)\,, \\
\mathcal{E}_r(z) &= \sum_{k\in\mathbb{Z}+\frac{1}{2}} e^{z(k-r/2)} E_{k-r,k} + \frac{\delta_{r,0}}{(e^{z/2}-e^{-z/2})}\,, \\
(a+1)_k &= \begin{cases} (a+1)(a+2)\dots(a+k), & k\geq 0 \\ (a(a-1)\dots(a+k+1))^{-1}, & k\leq 0 \end{cases}\,.
\end{aligned}$$

Here, E_{ij} are the matrix units of the Lie algebra[5] $\mathfrak{gl}(V)$ where V is the infinite dimensional $\mathbb{C}$-vector space with basis labeled by the shifted integers $\mathbb{Z}+\frac{1}{2}$. For a more detailed treatment, we refer the reader to [26, Section 2].

We will study the operator W which intertwines the operator $\mathcal{A}$ with

$$\widetilde{\mathcal{A}} = \sum_{k\in\mathbb{Z}} \widetilde{\mathcal{A}}_k z^k = \frac{1}{u} \sum_{k\in\mathbb{Z}} \frac{(uz)^k}{(tz+1)_k} \alpha_k\,,$$

the u-asymptotic expansion of $\mathcal{A}$ at $u \sim 0$. See [27, Section 4.4.2] for further[6] discussion. We have used here the operators

$$\alpha_k = \begin{cases} \mathcal{E}_k(0), & k\neq 0 \\ 1, & k=0 \end{cases}\,.$$

By definition, the matrix $\mathcal{A}$ can be written as a series in variables u, z, t with coefficients in the subalgebra of $\mathfrak{gl}(V)$ generated by the operators

$$H = \sum_{k\in\mathbb{Z}+\frac{1}{2}} kE_{kk} \quad \text{and} \quad S = \alpha_{-1}\,.$$

Let us denote the latter subalgebra by $\widetilde{\mathfrak{gl}}(V)$. The algebra $\widetilde{\mathfrak{gl}}(V)$ has a natural basis of ordered monomials

$$\left\{\, H^a S^b \,\middle|\, a, b \in \mathbb{Z} \,\right\}$$

with relations

$$SH = (H+1)S\,.$$

[5]Every operator in Section 1 is assumed to be an element of $\mathfrak{gl}(V)$ but not $\mathfrak{gl}(\Lambda^{\infty/2}V)$.

[6]In [27], the notation $\mathrm{A}_k = \mathcal{A}_{k+1}$ is used.

The coefficient in front of each monomial $H^a S^b$ in the formula for $\mathcal{A}$ is a Laurent polynomials of variables z, u, t. In other words,

$$\mathcal{A} \in \widetilde{\mathfrak{gl}}(V)[[z^{\pm 1}, u^{\pm 1}, t^{\pm 1}]] \,.$$

Moreover, $\mathcal{A}$ is homogeneous of degree -1 if we introduce the grading

$$\deg u = \deg t = -\deg z = 1 \,. \tag{12}$$

1.3. **The differential equation.** We consider first the intertwiner between the operators

$$\begin{aligned} D &= S^{-1} + H \,, \\ \widetilde{D} &= D - \frac{1}{2}\left(H \frac{1}{1-Z} + \frac{1}{(1-Z)} H \right) \quad \text{where} \quad Z = \frac{tS}{u} \,, \end{aligned}$$

and establish the following basic properties.

Lemma 7. *There is a unique solution W of the linear differential equation*

$$\frac{dW}{du} = \frac{1}{t} W B \quad \textit{where} \quad B = H^2 \frac{Z^2}{(1-Z)^2} + H \frac{Z^2}{(1-Z)^3} + \frac{2Z^3 + 3Z^2}{8(1-Z)^4} \,, \tag{13}$$

with the following properties:

(i) $W|_{u=0} = 1$,
(ii) $W^{-1} D W = \widetilde{D}$,
(iii) W *is upper-triangular.*

In fact, the unique solution W of Lemma 7 also intertwines $\mathcal{A}$ and $\widetilde{\mathcal{A}}$:

Theorem 8. *Let W be the operator of Lemma 7, then*

$$W^{-1} \mathcal{A} W = \widetilde{\mathcal{A}} \,. \tag{14}$$

Existence of the *dressing operator* W satisfying (14) is shown in [26] by slightly different methods, but the path via (13) is new (and very efficient).

Proof of Lemma 7. Let W be a solution of the differential equation (13). The equation has no singularity at $u = 0$, so there is a unique solution W satisfying $W|_{u=0} = 1$. A direct computation yields

$$\frac{d}{du}\left(W \widetilde{D} W^{-1} D^{-1} \right) = W \left(\frac{1}{t} \left[B, \widetilde{D} \right] + \frac{d\widetilde{D}}{du} \right) W^{-1} D^{-1} \,.$$

Then, after a lengthy but straightforward calculation, we find

$$\frac{1}{t} \left[B, \widetilde{D} \right] + \frac{d\widetilde{D}}{du} = 0.$$

Thus we obtain the first and second properties of W. The upper-triangularity follows from the upper-triangularity of B. $\square$

Proof of Theorem 8. As explained in Section 1.2, both $\mathcal{A}$ and W are sums of monomials $H^a S^b$ with coefficients in the ring of Laurent polynomials of variables u, t, z and, moreover, are homogeneous with respect to grading (12). Therefore, using Zariski density, we need only prove (14) at the values

$$z = m\,, \quad t = 1\,, \quad m \in \mathbb{Z}_{>0}\,.$$

We define the operators

$$\mathcal{A}^{(m)} = \frac{u^{m+1}m^m}{m!}\mathcal{A}|_{t=1,z=m}\,, \qquad \widetilde{\mathcal{A}}^{(m)} = \frac{u^{m+1}m^m}{m!}\widetilde{\mathcal{A}}|_{t=1,z=m}\,.$$

By [26, Lemma 2] in first case and a direct computation in second case, we find

$$\mathcal{A}^{(m)} = e^{1/S}e^{uH^2/2}S^m e^{-uH^2/2}e^{-1/S}\,, \qquad \widetilde{\mathcal{A}}^{(m)} = S^m e^{mu/S}\,.$$

Thus, to prove complete the proof of Theorem 8, we need only prove the equation

$$W^{-1}\mathcal{A}^{(1)}W = \widetilde{\mathcal{A}}^{(1)}\,. \tag{15}$$

Let us denote the operator on the LHS of equation (15) by O and the operator on the RHS by $\widetilde{\mathrm{O}}$. Equation (15) is satisfied at $u = 0$ since

$$\mathcal{A}^{(1)}|_{u=0} = S\,, \qquad \widetilde{\mathcal{A}}^{(1)}|_{u=0} = S\,, \qquad W|_{u=0} = 1\,.$$

Taking the u derivative of O, we find

$$\begin{aligned}
\frac{d\mathrm{O}}{du} &= [\mathrm{O}, B] + \frac{1}{2}W^{-1}e^{1/S}e^{uH^2/2}[H^2, S]e^{-uH^2/2}e^{-1/S}W \\
&= [\mathrm{O}, B] - \frac{1}{2}\mathrm{O} - W^{-1}e^{1/S}e^{uH^2/2}HSe^{-uH^2/2}e^{-1/S}W \\
&= [\mathrm{O}, B] - \frac{1}{2}\mathrm{O} - \widetilde{D}\mathrm{O}\,,
\end{aligned}$$

where we have used the intertwining relations for D and $\widetilde{D}$. A direct (lengthy) computation yields

$$\frac{d\widetilde{\mathrm{O}}}{du} = [\widetilde{\mathrm{O}}, B] - \frac{1}{2}\widetilde{\mathrm{O}} - \widetilde{D}\widetilde{\mathrm{O}}.$$

By the uniqueness of a solution of a linear ODEs, the proof of (15) is complete. ☐

2. 1-leg correspondence

2.1. Background. The 1-leg geometry concerns the space

$$\mathbb{C}^2 \times \mathbf{P}^1$$

with the action of the 3-dimensional torus

$$\mathsf{T} = (\mathbb{C}^*)^2 \times \mathbb{C}^*\,.$$

The first factor $(\mathbb{C}^*)^2$ acts on $\mathbb{C}^2$ with tangent weight s_1 and s_2 at the origin $0 \in \mathbb{C}^2$. The second factor $\mathbb{C}^*$ acts on $\mathbf{P}^1$ with tangent weights t and $-t$ at the respective fixed points $0, \infty \in \mathbf{P}^1$. For simplicity, we denote the two fixed points

$$0 \times 0\,,\ 0 \times \infty \in \mathbb{C}^2 \times \mathbf{P}^1$$

by 0 and ∞ respectively.

There is a 2-dimensional torus $\mathsf{T}_0 \subset \mathsf{T}$ which preserves the natural symplectic form $dz_1 \wedge dz_2$ on $\mathbb{C}^2$. Let

$$H_{\mathsf{T}_0}(\bullet) = \mathbb{C}[s,t]\,,$$

then the restriction to the action T_0 corresponds to the specialization

$$s_1 = -s_2 = s\,, \quad t = t\,.$$

By the Mumford identity for the Hodge classes, the T_0-equivariant Gromov-Witten invariants of $\mathbb{C}^2 \times \mathbf{P}^1$ are equal to the $\mathbb{C}^*$-equivariant Gromov-Witten invariants of $\mathbf{P}^1$ up to simple factors of s. The results of [26] solving the equivariant Gromov-Witten theory of $\mathbf{P}^1$ in terms of operators $\mathcal{A}$ are restated in Section 2.3 in the form we require here.

In [26], the Gromov-Witten invariants of $\mathbf{P}^1$ are computed in terms of the Fock space

$$\mathcal{F} = \Lambda^{\infty/2} V\,, \quad V = z^{1/2}\mathbb{C}[[z^{\pm 1}]]\,.$$

Before stating the results of [26], we give a quick overview of the basics about the Fock space and the related representation theory.

2.2. **Fock space.** The Fock space $\Lambda^{\infty/2}V$ has a natural basis of the form

$$\Lambda^{\infty/2}V = \bigoplus_S \mathbb{C} v_S, \quad v_S = z^{s_1} \wedge z^{s_2} \wedge z^{s_3} \ \ldots,$$

where $S = \{s_1 > s_2 > s_3 > \ldots\} \subset \mathbb{Z} + 1/2$ is an ordered sequence satisfying the properties

(i) $S_+ = S \setminus \left\{\mathbb{Z}_{\leq 0} - \frac{1}{2}\right\}$ is finite,
(ii) $S_- = \left\{\mathbb{Z}_{\leq 0} - \frac{1}{2}\right\} \setminus S$ is finite.

The fermionic operator ψ_k on $\Lambda^{\infty/2}V$ is defined by wedge product with the vector z^k,

$$\psi_k \cdot v = z^k \wedge v.$$

An inner product $(\cdot,\cdot)_z$ on V is defined by letting the monomials z^k be an orthonormal basis. We use the same notation $(\cdot,\cdot)_z$ for the induced inner product on $\Lambda^{\infty/2}V$. Let A^* denote the operator adjoint to A with respect to the inner product $(\cdot,\cdot)_z$. The adjoint operators ψ_k^* satisfy the canonical anti-commutation relations,

$$\psi_i \psi_j^* + \psi_j^* \psi_i = \delta_{ij}\,, \quad \psi_i \psi_j + \psi_j \psi_i = \psi_i^* \psi_j^* + \psi_j^* \psi_i^* = 0\,.$$

The projective representation $\pi_{\mathcal{F}}$ of $\mathfrak{gl}(V)$ is defined in terms of fermion operators by the formula

$$\pi_{\mathcal{F}}(E_{ij}) =: \psi_i \psi_j^* :,$$

where we have used normal order notation

$$:\psi_i\psi_j^* := \begin{cases} \psi_i\psi_j^*\,, & j>0 \\ -\psi_j^*\psi_i\,, & j<0\,. \end{cases}$$

The operators α_k of Section 1.2 commute as element of $\mathsf{End}(V)$, but the operators $\pi_{\mathfrak{F}}(\alpha_k)$ do *not* commute — the operators $\pi_{\mathfrak{F}}(\alpha_k)$ form an Heisenberg algebra. For simpler formulas, we will drop $\pi_{\mathfrak{F}}$ in our notation. That is, we use α_k for the operators $\pi_{\mathfrak{F}}(\alpha_k)$:

$$[\alpha_k, \alpha_l] = k\delta_{k+l}\,.$$

The action of the Heisenberg algebra preserves the eigenspaces of the charge operator

$$Cv_S = (|S_+| - |S_-|)v_S\,.$$

The operators $\mathcal{A}_k$, $\widetilde{\mathcal{A}}_k$ of Section 1.2 act on $\Lambda^{\infty/2}V$ via $\pi_{\mathfrak{F}}$. We obtain an alternative proof of Theorem 1 of [26].

Corollary 9. *As elements of* $\mathrm{End}(\Lambda^{\infty/2}V)$ *the operators* $\widetilde{\mathcal{A}}_k$ *satisfy*

$$\left[\widetilde{\mathcal{A}}_k, \widetilde{\mathcal{A}}_l\right] = (-1)^k\delta_{k+l-1}\frac{t}{u^2}\,.$$

Proof. Using the homogeneity of $\widetilde{\mathcal{A}}$, we set $u=1$ for the proof. The statement of Corollary 9 is equivalent to the equation

$$\left[\widetilde{\mathcal{A}}(z), \widetilde{\mathcal{A}}(w)\right] = tz\sum_{n\in\mathbb{Z}}\left(-\frac{z}{w}\right)^n$$

which we will derive from the Heisenberg relations for the operators α_k. By definition (after setting $u=1$),

$$\left[\widetilde{\mathcal{A}}(z), \widetilde{\mathcal{A}}(w)\right] = \sum_{n\neq 0} n\left(\frac{z}{w}\right)^n \frac{1}{(1+tz)_n(1+tw)_{-n}}\,.$$

On other hand, the summation over positive n after multiplication by $(1+\frac{z}{w})$ is equal to zt because

$$\begin{aligned}\left(\frac{z}{w}\right)^n&\left(\frac{n}{(1+tz)_n(1+tw)_{-n}} + \frac{z}{w}\frac{n}{(1+tz)_n(1+tw)_{-n}}\right) = \\ &\left(\frac{z}{w}\right)^n\left(\frac{1}{(1+tz)_{n-1}(1+tw)_{-n}} - \frac{tz}{(1+tz)_n(1+tw)_{-n}}\right) - \\ &\left(\frac{z}{w}\right)^n\left(\frac{z}{w}\frac{1}{(1+tz)_n(1+tw)_{-n-1}} - \frac{tz}{(1+tz)_n(1+tw)_{-n}}\right) = \\ &\left(\frac{z}{w}\right)^n\frac{1}{(1+tz)_{n-1}(1+tw)_{-n}} - \left(\frac{z}{w}\right)^{n+1}\frac{1}{(1+tz)_n(1+tw)_{-n-1}}\,.\end{aligned}$$

Analogously, the summation over negative n after multiplication by $1+\frac{w}{z}$ is equal $-zt$. □

The Fock space contains a special *vacuum vector*

$$v_\emptyset = v_{-1/2\,,\,-3/2\,,\,-5/2\,,\dots}$$

annihilated by the positive part of the Heisenberg algebra spanned by $\alpha_{k>0}$. The vectors

$$|\mu\rangle = \frac{1}{\mathfrak{z}(\nu)} \prod \alpha_{-\nu_i} v_\emptyset$$

form a basis of the Fock subspace of vectors of charge 0. For an operator $A \in \mathsf{End}(\mathcal{F})$, the shorthand notation

$$\langle A|\mu\rangle^{\mathcal{F}} = (v_\emptyset, A|\mu\rangle)_z$$

is commonly used. The Gromov-Witten bracket,

$$\left\langle \tau_{k_1}([0])\tau_{k_2}([0]) \dots \tau_{k_n}([0]) \,\middle|\, \mu \right\rangle_{\mathbf{P}^1}^{\mathsf{GW},\mathbb{C}^*},$$

denotes the $\mathbb{C}^*$-equivariant theory of $\mathbf{P}^1$ relative to $\infty \in \mathbf{P}^1$. A central result of [26] is the following matching.

Theorem 10. [26] *The $\mathbb{C}^*$-equivariant Gromov-Witten theory of $\mathbf{P}^1$ satisfies*

$$\left\langle \tau_{k_1}([0])\tau_{k_2}([0]) \dots \tau_{k_n}([0]) \,\middle|\, \mu \right\rangle_{\mathbf{P}^1}^{\mathsf{GW},\mathbb{C}^*} = \left\langle \mathcal{A}_{k_1+1}\mathcal{A}_{k_2+1} \dots \mathcal{A}_{k_n+1} e^{\alpha_1} \,\middle|\, \mu \right\rangle^{\mathcal{F}}.$$

2.3. GW **in terms of the Fock space.** To state the analogous formula for invariants of

$$X = \mathbb{C}^2 \times \mathbf{P}^1$$

with descendents placed at the fixed point $[0] \in \mathbb{C}^2 \times \mathbf{P}^1$, a slight modification $\bar{\mathcal{A}}_k$ of operator $\mathcal{A}_k$ is required. Let

$$\bar{\mathcal{A}}_k = s^2 \Psi(\mathcal{A}_k) + \delta_{k+1}/u\,,$$

where Ψ is the following homomorphism of $\mathbb{C}[t]$-algebras:

$$\Psi : \widetilde{\mathfrak{gl}}(V)[[u^{\pm1}, t^{\pm1}]] \to \widetilde{\mathfrak{gl}}(V)[[u^{\pm1}, t^{\pm1}, s^{\pm1}]], \quad \Psi(u) = ius, \quad \Psi(\alpha_k) = \alpha_k/s^k\,.$$

The above modification of operators is chosen in such way that the identification

$$\bar{\mathcal{A}}_{k+1} = \tau_k([0])\,, \quad k \in \mathbb{Z}$$

defines a homomorphism from the subalgebra of $\widetilde{\mathfrak{gl}}(V)[[u^{\pm1}, t^{\pm1}, s^{\pm1}]]$ generated $\mathcal{A}_k$ to the algebra Heis_X of Section 0.5. Moreover, the action of

$$\bar{\mathcal{A}}_k\,, \quad k < 0$$

on the vacuum is consistent with (3). After combining previous remarks with Theorem 10, we obtain the following formula for invariants with the relative condition $\mu_1([\infty]) \dots \mu_m([\infty])$ over $\infty \in \mathbf{P}^1$.

Proposition 11. *The T_0-equivariant Gromov-Witten theory of X satisfies*

$$\left\langle \tau_{k_1}([0])\tau_{k_2}([0]) \dots \tau_{k_n}([0]) \,|\, \mu \right\rangle_X^{\mathsf{GW},\mathsf{T}_0} = \left\langle \bar{\mathcal{A}}_{k_1+1}\bar{\mathcal{A}}_{k_2+1} \dots \bar{\mathcal{A}}_{k_n+1} e^{\alpha_1} \,|\, \mu \right\rangle^{\mathcal{F}}.$$

2.4. PT **in terms of Fock space.** The set of half-integers indexes the standard basis

$$\{ e_i \}_{i\in\mathbb{Z}+\frac{1}{2}}$$

of V. We identify the vector space V with $z^{1/2}\mathbb{C}[z][[z^{-1}]]$ by

$$e_i \mapsto z^i \,.$$

Then, the operator α_n acts as multiplication by z^{-n} on $z^{1/2}\mathbb{C}[z][[z^{-1}]]$. We define the operator H on V by $z\frac{d}{dz}$ on $z^{1/2}\mathbb{C}[z][[z^{-1}]]$. On the sheaf side, the insertion of the descendants is given by the following formula involving H.

Proposition 12. *The T_0-equivariant stable pairs theory of X satisfies*

$$q^{-|\mu|}\Big\langle \prod_j \mathrm{H}^{\mathsf{PT}}_{k_j}([0]) \,\Big|\, \mu\Big\rangle^{\mathsf{PT},\mathsf{T}_0} = \Big\langle \prod_j \frac{(sD)^{k_j}}{k_j!} e^{\alpha_1} \,\Big|\, \mu\Big\rangle^{\mathcal{F}},$$

where $D = H + \alpha_1$.

Proof. By standard arguments [15, 34], the moduli space $P_n(X,d)$ is empty if $n < d$. On the other hand if $n > d$ the virtual cycle on $P_n(X,n)$ vanishes in the T_0-theory [19]. If $n = d$,

$$P_n(X,n) = \mathsf{Hilb}_n(\mathbb{C}^2)$$

is nonsingular of expected dimension, and

$$\mathrm{ch}_k(\mathbb{F}) = (-1)^k \mathrm{ch}_k(\mathcal{J})$$

where $\mathcal{J}$ on $\mathbb{C}^2 \times \mathsf{Hilb}_n(\mathbb{C}^2)$ is the universal ideal sheaf associated to $\mathsf{Hilb}_n(\mathbb{C}^2)$,

$$\pi : \mathbb{C}^2 \times \mathsf{Hilb}_n(\mathbb{C}^2) \to \mathsf{Hilb}_n(\mathbb{C}^2)\,.$$

On the other hand, Nakajima's construction provides a natural identification between the Fock space and

$$\bigoplus_{n=0}^{\infty} H_{\mathsf{T}_0}(\mathsf{Hilb}_n(\mathbb{C}^2))\,.$$

By equivariant Grothendieck-Riemann-Roch, $\pi_*\mathrm{ch}(\mathcal{J})$ is expressible in terms of the Chern character of the tautological sheaf over the Hilbert scheme $\mathsf{Hilb}_n(\mathbb{C}^2)$. The operator multiplication by the latter Chern character is diagonal in the basis of torus fixed points and has a simple expression in terms of H (see, for example, [13]). In the Fock space model, we have,

$$(e^{xs/2} - e^{-xs/2})\pi_*\mathrm{ch}(\mathcal{J})(x) = \pi_{\mathcal{F}}(e^{xsH})\,.$$

Further discussion of the properties of the operator $\pi_{\mathcal{F}}(e^{xsH})$ can be found in [25, Section 2.2.1]. In other words, we have

$$\Big\langle \prod_j \mathrm{H}^{\mathsf{PT}}_{k_j}([0]) \,\Big|\, \mu\Big\rangle^{\mathsf{PT},\mathsf{T}_0} = q^{|\mu|}\Big\langle e^{\alpha_1} \prod_j \frac{(sH)^{k_j}}{k_j!} \,\Big|\, \mu\Big\rangle^{\mathcal{F}}.$$

The last claim of the Proposition follows from the formula $e^{\alpha_1}H = (H+\alpha_1)e^{\alpha_1}$. $\square$

2.5. **The dressing operator and the GW/PT operators.** The dressing operator

$$\overline{W} = \Psi(W)$$

drastically simplifies the formula for the Gromov-Witten invariants of X. Indeed, by the results of Section 1, we have:

$$\overline{W}^{-1}\left(\sum_{n\in\mathbb{Z}} x^n \bar{\mathcal{A}}_n\right)\overline{W} = \sum_{n>0}\frac{(iu)^{n-1}x^n}{(1+tx)_n}\bar{\alpha}_n + \frac{1}{t}\sum_{n<0}\frac{(iu)^{n-1}x^n}{(1+tx)_n}\bar{\alpha}_n\,, \tag{16}$$

$$\overline{W}^{-1}(\alpha_1 + H)\overline{W} = \mathrm{D}\,, \tag{17}$$

$$\mathrm{D} = \frac{\alpha_1}{s} - \sum_{n>0}(H+n/2)\left(\frac{t}{iu}\right)^n \alpha_{-n}. \tag{18}$$

where we define

$$\bar{\alpha}_k = s\alpha_k\,, \qquad \bar{\alpha}_{-k} = st\alpha_{-k} - t\delta_{k-1} + \delta_{k-2}iu\,, \qquad k>0\,.$$

Immediately from the formulas, we see that the operators $\bar{\alpha}_k$ satisfy the same relations as the operators $\mathfrak{a}_k([0])$ from Heis_X. Moreover, since $\overline{W}$ is upper-triangular (and thus preserves the vacuum), we have the following formulas for the invariants:

$$\Big\langle \prod_j \mathfrak{a}_{k_j}([0])\Big|\mu\Big\rangle^{\mathsf{GW},\mathsf{T}_0} = \Big\langle \prod_j \bar{\alpha}_{k_j}\overline{W}^{-1}e^{\alpha_1}\Big|\mu\Big\rangle^{\mathcal{F}}, \tag{19}$$

$$q^{-|\mu|}\Big\langle \prod_j \mathrm{H}^{\mathsf{PT}}_{k_j}([0])\Big|\mu\Big\rangle^{\mathsf{PT},\mathsf{T}_0} = \Big\langle \prod_j \frac{(s\mathrm{D})^{k_j}}{k_j!}\overline{W}^{-1}e^{\alpha_1}\Big|\mu\Big\rangle^{\mathcal{F}}. \tag{20}$$

To prove (19), we start from definition (4) which yields: The first formula is equivalent to the evaluation of the generating functions for $\mathfrak{a}_k$ from the definition (4):

$$\Big\langle \prod_i \sum_{n_i} \tau_{n_i}([0])x_i^{n_i}\Big|\mu\Big\rangle^{\mathsf{GW},\mathsf{T}_0} =$$

$$\Big\langle \prod_i \left(\sum_{n_i>0}\frac{(iux_i)^{n_i-1}}{(1+tx_i)_{n_i}}\mathfrak{a}_{n_i}([0]) + \frac{1}{t}\sum_{n_i<0}\frac{(iux_i)^{n_i-1}}{(1+tx_i)_{n_i}}\mathfrak{a}_{n_i}([0])\right)\Big|\mu\Big\rangle^{\mathsf{GW},\mathsf{T}_0}.$$

Then, using Proposition 11, the vacuum preservation of $\overline{W}^{-1}$, and (16), we find

$$\begin{aligned}\Big\langle \prod_i \sum_{n_i} \tau_{n_i}([0])x_i^{n_i}\Big|\mu\Big\rangle^{\mathsf{GW},\mathsf{T}_0} &= \Big\langle \prod_i \bar{\mathcal{A}}(x_i)e^{\alpha_1}\Big|\mu\Big\rangle^{\mathcal{F}} \\ &= \Big\langle \overline{W}^{-1}\prod_i \bar{\mathcal{A}}(x_i)e^{\alpha_1}\Big|\mu\Big\rangle^{\mathcal{F}} \\ &= \Big\langle \prod_i \left(\sum_{n_i>0}\frac{(iux_i)^{n_i-1}}{(1+tx_i)_{n_i}}\bar{\alpha}_{n_i} + \frac{1}{t}\sum_{n_i<0}\frac{(iux_i)^{n_i-1}}{(1+tx_i)_{n_i}}\bar{\alpha}_{n_i}\right)\overline{W}^{-1}e^{\alpha_1}\Big|\mu\Big\rangle^{\mathcal{F}}.\end{aligned}$$

Equation (19) follows from these two equations. The proof of (20) is simpler (and uses Proposition 12).

In order to approach Theorem 4 in the 1-leg case, we will require the following result.

Proposition 13. *The following identity holds in* $\mathsf{End}(\mathcal{F})[[u^{\pm 1}, t^{\pm 1}, s^{\pm 1}]]$*:*

$$e^{xs\mathrm{D}} = \oint_{|y|=1/\epsilon} \frac{\sqrt{dydw}}{y-w} \exp\left(\frac{iu}{2st}(w^2-y^2) + \frac{1}{s}(y-w)\right) : \exp\left(\sum_{n\neq 0} \frac{\alpha_n}{n}\left(y^{-n} - w^{-n}\right)\right) : ,$$

where the integral is taken on the surface defined by the equation

$$ye^{-iyt/u} = we^{-iwt/u}e^{sx}.$$

In the statement of Proposition 13, we have used normal ordering notation:

$$:\alpha_i\alpha_{-i} := \begin{cases} \alpha_i\alpha_{-i}\,, & i<0 \\ \alpha_{-i}\alpha_i\,, & i>0\,. \end{cases}$$

2.6. **Proof of Proposition 13.** We have seen that the operators H and α_n act respectively as $z\frac{d}{dz}$ and multiplication by z^{-n} on $z^{1/2}\mathbb{C}[z][[z^{-1}]]$. Thus, D, defined by equation (18), becomes a differential operator acting on the functions of z.

We view D as acting on functions of z from the left. Consider the eigenvalue problem

$$s\mathrm{D}f = \lambda f, \quad f \in z^{1/2}\mathrm{Hol}(\mathbb{C}^*)\,, \tag{21}$$

where $\mathrm{Hol}(\mathbb{C}^*)$ denotes holomorphic single-valued functions on $\mathbb{C}^*$. The Laurent series expansion provides a map from $z^{1/2}\mathrm{Hol}(\mathbb{C}^*)$ to the completion $\overline{V}$ of the space V.

Since α_n acts as multiplication by z^{-n}, the eigenvalue equation (21) is equivalent to the following ODE:

$$\left[\frac{1}{z} - \frac{d}{dz}\frac{stz^2}{iu-tz} - sa\left(\frac{tz}{iu}\right)\right] f = \lambda f\,, \qquad a(x) = \frac{1}{2}\frac{x}{(1-x)^2}\,,$$

with solution

$$f = z^{1/2-\lambda/s}\exp\left[\frac{iu}{2stz^2} - \left(\frac{1}{s} + \frac{iu\lambda}{st}\right)\frac{1}{z}\right]\left(1-\frac{tz}{iu}\right)^{-1/2}.$$

The condition $f \in z^{1/2}\mathbb{C}[z][[z^{-1}]]$ leads to the eigenfunctions:

$$f_k = (z^{-1}e^{-\frac{iu}{tz}})^{k+1/2}\exp\left[-\frac{iu}{2stz^2} + \left(\frac{1}{s}\right)\frac{1}{z}\right]\left(1-\frac{iu}{tz}\right)^{-1/2}, \qquad \lambda_k = s\left(k+\frac{1}{2}\right),$$

for $k \in \mathbb{Z}$.

For $h(z) \in 1 + z^{-1}\mathbb{C}[[z^{-1}]]$, the operator of multiplication by $h(z)$,

$$\mathrm{M}_h : z^k \mapsto z^k \cdot h(z)\,,$$

is an invertible endomorphism of V. Similarly, for $\theta(z) \in z^{-1} + z^{-2}\mathbb{C}[[z^{-1}]]$, the reparametrization operator

$$\mathrm{R}_\theta : z^k \mapsto \theta(z)^k$$

is invertible. We can therefore restate the above computation in terms of multiplication and reparametrization operators,

$$sH = \mathrm{R}_\theta^{-1}\mathrm{M}_{e^{\mathrm{g}}}^{-1}\mathrm{D}\mathrm{M}_{e^{\mathrm{g}}}\mathrm{R}_\theta \,, \tag{22}$$

where

$$\mathrm{g}(z) = -\frac{iu}{2stz^2} + \left(\frac{1}{s}\right)\frac{1}{z} - \frac{1}{2}\log\left(1 - \frac{iu}{zt}\right), \qquad \theta(z) = z^{-1}e^{-\frac{iu}{tz}} \,.$$

In the proof of Proposition 13 so far, we have studied operators in $\mathrm{End}(V)$. The claim of Proposition 13, on the other hand, is about the operators in $\mathrm{End}(\mathcal{F})$. For the remainder of the proof, we will work in $\mathrm{End}(\mathcal{F})$. We will use formula (22) to find an expression for D in terms of the operators

$$\alpha_n \in \mathrm{End}(\mathcal{F})$$

via the boson/fermion correspondence.

Let us quickly review the key points of the boson/fermion correspondence. It is customary to assemble fermionic operators in generating functions:

$$\psi(x) = \sum_{k\in\mathbb{Z}+1/2} \psi_k x^k \,, \qquad \psi^*(x) = \sum_{k\in\mathbb{Z}+1/2} \psi_k^* x^{-k} \,.$$

The zero mode of the product of two fermionic generating function gives the exponential of the operator sxH:

$$e^{sxH} = [y^0]\psi(y)\psi^*(ye^{-sx}) \,. \tag{23}$$

Thus, to express $e^{sx\mathrm{D}}$ in terms of the operators α_n using (22), we must compute the action of the reparametrization and scaling operators on $\psi(x)$ and $\psi^*(x)$.

Lemma 14. *For $g \in z^{-1} + z^{-2}\mathbb{C}[[z^{-1}]]$, we have:*

$$\mathrm{R}_g\psi(x)\mathrm{R}_g^{-1} = -\psi\left(\frac{1}{g^{inv}(1/x)}\right) x(\log(g^{inv}(1/x)))_x \,,$$

$$\mathrm{R}_g\psi^*(x)\mathrm{R}_g^{-1} = \psi\left(\frac{1}{g^{inv}(1/x)}\right) \,,$$

where f_x stands for the x-derivative of f and g^{inv} is the inverse function

$$g^{inv}(g(x)) = x \,.$$

Proof. The matrix coefficients of the operator R_g are given by the expansion

$$\mathrm{R}_g(z^k) = \sum_i r_{ik} z^i.$$

We then have the following formulas with summation indices i, k ranging in the set $1/2+\mathbb{Z}$:

$$\begin{aligned} \mathrm{R}_g\psi(y)\mathrm{R}_g^{-1}(f) &= \sum_k \mathrm{R}_g(y^k z^k \wedge \mathrm{R}_g^{-1}(f)) \\ &= \sum_k y^k \mathrm{R}_g(z^k) \wedge f \\ &= \sum_{k,i} r_{ik} y^k z^i \wedge g \\ &= \sum_i \mathrm{R}_g^*(y^i) z^i \wedge g\,, \end{aligned}$$

where R_g^* is the linear operator adjoint with respect to the scalar product $(\cdot,\cdot)_y$ with the orthonormal basis y^i. To complete the proof, we must compute the adjoint operator R_g^*:

$$\begin{aligned} (y^m, \mathrm{R}_g(y^k))_y &= \oint y^m g^k(1/y)\frac{dy}{y} \\ &= -\oint \left(\frac{1}{g^{inv}(1/w)}\right)^m w^{-k}\frac{g^{inv}(1/w)_w}{g^{inv}(1/w)}dw \\ &= -\Big(w(\log(g^{inv}(1/w)))_w \mathrm{R}_{1/g^{inv}(1/w)}(w^m), w^k\Big)_w\,. \end{aligned}$$

We conclude

$$\mathrm{R}_g^*(y^i) = y(\log(g^{inv}(1/y))_y)\mathrm{R}_{1/g^{inv}}(y^i)\,,$$

which implies the first equation of the Lemma. The second equation,

$$\mathrm{R}_g\psi^*(y)\mathrm{R}_g^{-1} = ((\mathrm{R}_g^*)^{-1}(\psi(1/y))\mathrm{R}_g^*)^* = (\psi(\mathrm{R}_g^{-1}(1/y)))^* = \psi^*\left(\frac{1}{\mathrm{R}_g^{-1}(1/y)}\right)\,,$$

then follows from the first. □

By applying Lemma 14, we obtain

$$\begin{aligned} [y^0]\mathrm{R}_\theta\psi(\xi)\psi^*(\xi e^{-sx})\mathrm{R}_\theta^{-1} &= -[\xi^0]\psi\left(\frac{1}{\theta^{inv}(1/\xi)}\right)\psi^*\left(\frac{1}{\theta^{inv}(e^{-sx}/\xi)}\right)\xi(\ln(\theta^{inv}(1/\xi)))_\xi \\ &= \oint_{|y|=1/\epsilon}\psi(y)\psi^*(w)\frac{dy}{y}\,, \end{aligned} \tag{24}$$

where $y=\frac{1}{\theta^{inv}(1/\xi)}$, $w=\frac{1}{\theta^{inv}(e^{-sx}/\xi)}$ and ϵ is close to zero. The variables y, w are subject to the constraint:

$$\theta(1/w) = \theta(1/y)e^{sx}.$$

The boson/fermion correspondence is written in terms of these generating functions and the following auxiliary operators:

$$\psi(x) = Tx^{C+1/2}\Gamma_+(x)\,, \quad \psi^*(x) = T^{-1}x^{-C+1/2}\Gamma_-(x)\,,$$
$$\Gamma_\pm(x) = e^{\pm\alpha_-(x)}e^{\pm\alpha_+(x)}\,, \quad \alpha_\pm(x) = \mp\sum_{k>0}\alpha_{\pm k}\frac{x^{\mp k}}{k}\,,$$

where C is the standard charge operator [25] and T is the shift operator

$$T(v_S) = v_{s_1+1,s_2+1,s_3+1,\dots}\,.$$

The boson/fermion correspondence (23) together with (22) and (24) yields:

$$e^{sx\mathrm{D}} = \oint_{|y|=1/\epsilon}\frac{dy}{y}\sqrt{\frac{w}{y}}\mathrm{S}\Gamma_+(y)\Gamma_-(w)\mathrm{S}^{-1}\,, \quad we^{-iuw/t} = ye^{-iuy/t}e^{-sx}\,,$$

where ϵ is very small and $S = M_{e^{\mathrm{g}}}$ is the operator of multiplication by e^{g},

$$\mathrm{g}(z) = -\frac{iu}{2stz^2} + \left(\frac{1}{s}\right)\frac{1}{z} - \frac{1}{2}\log\left(1-\frac{iu}{zt}\right)\,.$$

By Taylor expansion, we obtain

$$e^{c\alpha_k}\Gamma_\pm(x)e^{-c\alpha_k} = e^{\pm cx^k}\Gamma_\pm(x)\,.$$

Hence, the conjugation by S produces the following result:

$$\oint\frac{dy}{y}\sqrt{\frac{w}{y}}\exp(\mathrm{g}(1/y)-\mathrm{g}(1/w))\Gamma_+(y)\Gamma_-(w)\,.$$

Finally, we rewrite the integral in terms of semiforms. Indeed, the implicit equation for w and y implies:

$$dy\left(1-\frac{t}{iuy}\right) = dw\left(1-\frac{t}{iuw}\right)\,.$$

On the other hand, we have

$$e^{\mathrm{g}(1/z)} = \exp\left[-\frac{iuz^2}{2st}+\frac{z}{s}\right]\left(1-\frac{iuz}{t}\right)^{-1/2}\,.$$

We obtain

$$e^{xs\mathrm{D}} = \oint\frac{\sqrt{dydw}}{y}\Gamma_+(y)\Gamma_-(w)\,, \qquad we^{-iuw/t} = ye^{-iuy/t}e^{-sx}\,.$$

Combining the above equation with

$$\Gamma_+(y)\Gamma_-(w) =:\Gamma_+(y)\Gamma_-(w): /(1-w/y)$$

yield the formula in the statement of the Lemma. □

2.7. **The 1-leg case.** We prove here a weaker 1-leg version of Theorem 4.

Theorem 15. *After the change of variables $q = -e^{iu}$, we have for any collection of $k_i \geq 0$*

$$\left\langle \prod_i \mathrm{H}^{\mathsf{GW}}_{k_i}(\mathsf{p}) \middle| \mu, \emptyset, \emptyset \right\rangle^{\mathsf{GW},\mathsf{T}_0}_{U,D} = q^{-|\mu|} \left\langle \prod_i \mathrm{H}^{\mathsf{PT}}_{k_i}(\mathsf{p}) \middle| \mu, \emptyset, \emptyset \right\rangle^{\mathsf{PT},\mathsf{T}_0}_{U,D},$$

where $\mathsf{T}_0 \subset \mathsf{T}$ is the subtorus preserving the symplectic form on $\mathbb{C}^2$.

Proof. We start by rewriting Proposition 13 after the change of variables

$$y \mapsto i\frac{uy}{t}\,, \qquad w \mapsto i\frac{uw}{t}$$

in the form

$$e^{xs\mathrm{D}} = \oint_{|y|=1/\epsilon} \frac{\sqrt{dydw}}{y-w} : \exp\left(\frac{\bar{\phi}(y)}{s} - \frac{\bar{\phi}(w)}{s}\right) : \tag{25}$$

$$\bar{\phi}(z) = \sum_{n>0} \frac{\bar{\alpha}_n}{n}\left(\frac{izt}{u}\right)^{-n} + \frac{1}{t}\sum_{n<0}\frac{\bar{\alpha}_n}{n}\left(\frac{izt}{u}\right)^{-n}.$$

where y, w are constrained by $ye^y = we^we^{sx}$ and

$$\bar{\alpha}_k = s\alpha_k, \quad \bar{\alpha}_{-k} = st\alpha_{-k} - t\delta_{k-1} + \delta_{k-2}iu, \quad k > 0.$$

We define a homomorphism[7] $\mathsf{F} : \mathsf{Heis} \to \mathsf{Heis}_U$ by

$$\mathsf{F}(\bar{\alpha}_k) = \mathfrak{a}_k(\mathsf{p})\,, \qquad k \in \mathbb{Z} \setminus \{0\}\,.$$

The linear map F is a homomorphism of algebras because

$$[\mathsf{F}(\bar{\alpha}_k), \mathsf{F}(\bar{\alpha}_m)] = [\mathfrak{a}_k(\mathsf{p}), \mathfrak{a}_m(\mathsf{p})] = k\delta_{k+m}\int \mathsf{p}\cdot\mathsf{p} = k\delta_{k+m}(-s^2t)\,.$$

Moreover, F sends the LHS of (25) to the LHS of (7) since

$$t = c_1(T_U)\,, \qquad s^2 = -c_2(T_U)\,.$$

Let $\mathcal{F}$ be the standard Fock space for Heis with the vacuum vector $v_\emptyset$,

$$\alpha_k v_\emptyset = 0, \quad k < 0.$$

We denote by $\mathcal{F}_{geom}$ the Fock space space defined by the action of Heis_U on the vacuum:

$$\mathfrak{a}_k(\mathsf{p})u_\emptyset = \left[\int \big(-t\delta_{k+1} + \delta_{k+2}iu\big)\cdot\mathsf{p}\right]u_\emptyset\,, \quad k < 0\,.$$

The homomorphism F induces a canonical homomorphism of the Fock space,

$$\mathsf{F} : \mathcal{F} \to \mathcal{F}_{geom}\,, \tag{26}$$

by matching vacuum vectors $\mathsf{F}(v_\emptyset) = u_\emptyset$ since

$$\mathsf{F}(\bar{\alpha}_k)u_\emptyset = \mathfrak{a}_k u_\emptyset\,, \quad k < 0\,.$$

[7] The equivariant cohomology of U is generated over $\mathbb{Q}[s,t]$ by the class p of the fixed point.

The Fock space $\mathcal{F}_{geom}$ has a natural linear functional which evaluates Gromov-Witten invariants. Since the elements

$$\prod_{i>0} \mathfrak{a}_i^{k_i}(\mathsf{p})u_\emptyset$$

form a basis of $\mathcal{F}_{geom}$, we have a natural linear isomorphism between Heis_U^+ and $\mathcal{F}_{geom}$. The isomorphism allow us to define the following linear functional on $\mathcal{F}_{geom}$:

$$\Psi_\mu^{\mathsf{GW}}(\Phi) = \big\langle \Phi \big| \mu \big\rangle_\beta^{\mathsf{GW},\mathsf{T}_0}, \tag{27}$$

where $\beta = |\mu|\mathbf{P}^1$. Formula (19) implies that under the identification (26) of the Fock spaces $\mathcal{F}$ and $\mathcal{F}_{geom}$, the linear functional (27) corresponds to a pairing in $\mathcal{F}$ with the vector

$$v_\mu = \overline{W}^{-1} e^{\alpha_1}|\mu\rangle \in \mathcal{F}\,.$$

On the stable pairs side, equation (20) evaluates the the right side of the correspondence of Theorem 15:

$$q^{-|\mu|}\Big\langle \prod_j \mathrm{H}_{k_j}^{\mathsf{PT}}([0])\Big|\mu\Big\rangle^{\mathsf{PT},\mathsf{T}_0} = \Big\langle \prod_j \frac{(s\mathrm{D})^{k_j}}{k_j!}\overline{W}^{-1} e^{\alpha_1}\Big|\mu\Big\rangle^{\mathcal{F}}.$$

Since F sends the LHS of (25) to the LHS of (7), we obtain

$$\Big\langle \prod_j \frac{(s\mathrm{D})^{k_j}}{k_j!}\overline{W}^{-1} e^{\alpha_1}\Big|\mu\Big\rangle^{\mathcal{F}} = \Big\langle \prod_i \mathrm{H}_{k_i}^{\mathsf{GW}}(\mathsf{p})\Big|\mu,\emptyset,\emptyset\Big\rangle_{U,D}^{\mathsf{GW},\mathsf{T}_0}$$

via the evaluation of Ψ_μ^{GW} as a pairing in $\mathcal{F}$ with v_μ. □

Since the Gromov-Witten bracket is compatible with the hat operation (11), we can equivalently write the conclusion of Theorem 15 as:

$$\Big\langle \prod_i \widehat{\mathrm{H}}_{k_i}^{\mathsf{GW}}(\mathsf{p})\Big|\mu,\emptyset,\emptyset\Big\rangle_{U,D}^{\mathsf{GW},\mathsf{T}_0} = q^{-|\mu|}\Big\langle \prod_i \mathrm{H}_{k_i}^{\mathsf{PT}}(\mathsf{p})\Big|\mu,\emptyset,\emptyset\Big\rangle_{U,D}^{\mathsf{PT},\mathsf{T}_0}. \tag{28}$$

2.8. **Lambert function.** We explain how to convert the contour integral in definition (7) to an explicit formula. The first step is to solve the constraint equation (8),

$$ye^y = we^w e^{-x/\theta}\,.$$

We interpret both sides as formal power series in x, and then we can find the solution by induction on degree of x. In particular, the first few terms of the expansion are:

$$w(y) = y - \frac{xy}{\theta(y+1)} + \frac{x^2 y}{2\theta^2(y+1)^3} + \frac{x^3 y(2y-1)}{6\theta^3(y+1)^5} + O(x^4)\,. \tag{29}$$

We can therefore write explicit power series for the integrand in formula (7) and find an effective formula for $\mathrm{H}^{\mathsf{GW}}(x)$:

$$\mathrm{Res}_{y=\infty}\left(dy\left(\frac{dw(y)}{dy}\right)^{1/2}\frac{:e^{\theta(\phi(y)-\phi(w(y)))}:}{y-w(y)}\right), \tag{30}$$

where $w(y)$ is given by (29) and $\phi(z)$ is by (6).

3. Uniqueness of the correspondence

3.1. **Properties of the correspondence matrix.** We define an augmented partition size $|\cdot|^+$ by the formula

$$|\lambda|^+ = \sum_i (1+\lambda_i)\,.$$

Let $\mathcal{P}$ be the set of all partitions. Let $\mathcal{P}_d$ the set of partitions of augmented size d, and let $\mathcal{P}_{\leq d}$ be the set of partitions of augmented size less than or equal to d,

$$\mathcal{P}_d \subset \mathcal{P}_{\leq d} \subset \mathcal{P}\,.$$

As in Section 0.8, we set

$$\mathrm{H}^{\mathsf{GW}}_\mu = \prod_{i=1}^{\ell} \mathrm{H}^{\mathsf{GW}}_{\mu_i} \in \mathsf{Heis}^c\,, \qquad \widehat{\mathrm{H}}^{\mathsf{GW}}_\mu \in \mathsf{Heis}^c_+ \otimes \mathbb{C}[c_1, c_2^{1/2}]\,.$$

Lemma 16. *For every $\mu \in \mathcal{P}_d$, we have*

(i) $\widehat{\mathrm{H}}^{\mathsf{GW}}_\mu \in \mathsf{Heis}^c_+ \otimes \mathbb{C}[c_1, c_2]$,

(ii) $\widehat{\mathrm{H}}^{\mathsf{GW}}_\mu = \frac{\mathfrak{a}_\mu}{(\mu-1)!} + \sum_{\lambda\in\mathcal{P}_{<d}} b(\mu,\lambda)\mathfrak{a}_\lambda,$

with $\mathfrak{a}_\lambda = \prod_{i=1}^{\ell} \mathfrak{a}_{\lambda_i}$.

Proof. The operator $s\mathrm{D}$ defined by (18) is a linear combination of monomials in H and α_k with coefficients in $\mathbb{C}[s,t]$. The same holds for every power of $s\mathrm{D}$. Since the operator H is a quadratic expression[8] of α_k with coefficients in $\mathbb{C}[s,t]$, we conclude

$$\widehat{\mathrm{H}}^{\mathsf{GW}}(x) \in \mathsf{Heis}^c \otimes \mathbb{C}[c_1, c_2^{1/2}]\,.$$

The integral defining $\widehat{\mathrm{H}}^{\mathsf{GW}}(x)$ is invariant with respect to the sign change

$$\theta \mapsto -\theta\,.$$

Indeed, under the sign change the constraint equation turns into

$$ze^z = we^w e^{-x/\theta}$$

which is equivalent to the original constraint equation after switching y and w. On the other hand, the integral is unchanged after the switch. Thus, we have proven claim (i).

Definition (18) is homogeneous for the homological grading of the generators:

$$\deg \alpha_k = k+1, \quad \deg \alpha_{-k} = -k+2, \quad \deg s = \deg t = 1\,.$$

The powers of D are therefore also homogeneous, and claim (ii) follows. □

[8] $H = \sum_{k>0} \alpha_{-k}\alpha_k$.

3.2. **Uniqueness.** The 1-leg GW/PT descendent correspondence of Theorem 15 in the hat form of (28) is

$$\mathrm{H}_\mu^{\mathsf{PT}} \mapsto \widehat{\mathrm{H}}_\mu^{\mathsf{GW}} . \tag{31}$$

The correspondence rule (31) defines a $\mathbb{C}[c_1, c_2]$-linear operator

$$\mathcal{T} : \mathcal{P}^{\mathsf{PT}} \to \mathcal{P}^{\mathsf{GW}} ,$$

where $\mathcal{P}^{\mathsf{PT}}$ has $\mathbb{C}[c_1, c_2]$-basis $\mathrm{H}_\mu^{\mathsf{PT}}$ and $\mathcal{P}^{\mathsf{GW}}$ has $\mathbb{C}[c_1, c_2]$-basis $\mathfrak{a}_\mu$. By Lemma 16 part (ii), $\mathcal{T}$ restricts to

$$\mathcal{T}_d : \mathcal{P}_{\leq d}^{\mathsf{PT}} \to \mathcal{P}_{\leq d}^{\mathsf{GW}} ,$$

where the shifted size of the partitions are bounded in the bases on both sides.

There are two operators that encode 1-leg relative Gromov-Witten and stable pairs theories. The first operator

$$\mathcal{M}_d^{\mathsf{GW}} : \mathcal{P}_{\leq d}^{\mathsf{GW}} \to \mathcal{P}$$

has the Gromov-Witten invariants

$$\big\langle \mathfrak{a}_\mu(\mathsf{p}) \big| \lambda \big\rangle_{U,D}^{\mathsf{GW},\mathsf{T}_0} , \quad \mu \in \mathcal{P}_{\leq d} , \quad \lambda \in \mathcal{P}$$

as matrix entries. The second operator

$$\mathcal{M}_d^{\mathsf{PT}} : \mathcal{P}_{\leq d}^{\mathsf{PT}} \to \mathcal{P}$$

has the stable pairs invariants

$$\big\langle \mathrm{H}_\mu^{\mathsf{PT}}(\mathsf{p}) \big| \lambda \big\rangle_{U,D}^{\mathsf{PT},\mathsf{T}_0} , \quad \mu \in \mathcal{P}_{\leq d} , \quad \lambda \in \mathcal{P}$$

as matrix entries.

Lemma 17. *The operator $\mathcal{T}_d$ is an isomorphism and is the unique solution of the correspondence equation*

$$\mathcal{M}_d^{\mathsf{GW}} \mathcal{T}_d = \mathcal{M}_d^{\mathsf{PT}} .$$

Proof. That $\mathcal{T}_d$ is an isomorphism follows from Lemma 16 part (ii). The correspondence equation is exactly the statement of Theorem 15 in form (28).

To derive uniqueness, we will show that the operator $\mathcal{M}_d^{\mathsf{PT}}$ is injective. By the construction of the projective representation $\Lambda^{\infty/2} V$ (see for example [25, section 2.2.2]), we have:

$$\big\langle \mathrm{H}_\mu^{\mathsf{PT}}(p) \big| \lambda \big\rangle_{U,D}^{\mathsf{PT},\mathsf{T}_0} = \mathbf{p}_\mu(\lambda) ,$$

where $\mathbf{p}_\mu = \prod_i \mathbf{p}_{\mu_i}$ is the product of the shifted Newton polynomials from the ring of the shifted symmetric functions $\Lambda^* = \mathbb{Q}[\mathbf{p}_1, \mathbf{p}_2, \dots]$ and

$$\mathbf{p}_k(\lambda) = \sum_{i=1}^{\infty} \left[(\lambda_i - i + \frac{1}{2})^k - (-i + \frac{1}{2})^k \right] + (1 - 2^{-k})\zeta(-k),$$

is the evaluation of the shifted function at λ.

Since the products $\mathbf{p}_\mu$ span a basis of the ring of the shifted symmetric functions and the evaluation map

$$f \mapsto \{f(\lambda)\}_{\lambda \in \mathcal{P}}$$

is the Fourier transform in representation theory of S_∞ [24], $\mathcal{M}_d^{\mathsf{PT}}$ is injective. □

3.3. **Comparing correspondences.** The $\mathsf{GW/PT}$ correspondence for the standard descendents τ_k is studied in [32]. Since the descendents $\mathfrak{a}_k$ and $\mathrm{H}_k^{\mathsf{PT}}$ are the linear combinations of the standard descendents $\{\tau_m\}_{m\le k}$, the results of [32] hold in our setting here.

Theorem 18. [32] *There exists an invertible transformation $\underline{\mathfrak{T}} : \mathcal{P}^{\mathsf{PT}} \to \mathcal{P}^{\mathsf{GW}}$ linear over $\mathbb{C}[s_1, s_2, s_3]$ for which the correspondence equation*

$$\left\langle \underline{\mathfrak{T}}(\mu)(\mathsf{p}) \middle| \lambda_1, \lambda_2, \lambda_3 \right\rangle_{U,D}^{\mathsf{GW},\mathsf{T}} = \left\langle \mathrm{H}_\mu^{\mathsf{PT}}(\mathsf{p}) \middle| \lambda_1, \lambda_2, \lambda_3 \right\rangle_{U,D}^{\mathsf{PT},\mathsf{T}}$$

holds for all $\mu, \lambda_1, \lambda_2, \lambda_3 \in \mathcal{P}$. Moreover,

(i) *$\underline{\mathfrak{T}}$ sends $\mathcal{P}_{\le d}^{\mathsf{PT}}$ to $\mathcal{P}_{\le d}^{\mathsf{GW}}$,*

(ii) *the coefficients of $\underline{\mathfrak{T}}$ are polynomials in the symmetric functions*

$$c_1 = e_1(s_1, s_2, s_3)\,, \quad c_2 = e_2(s_1, s_2, s_3)\,, \quad c_3 = e_3(s_1, s_2, s_3)\,.$$

Corollary 19. *The coefficients of $\mathfrak{T}$ are polynomial in c_1, c_2 and*

$$\mathfrak{T} = \underline{\mathfrak{T}}|_{c_3 = c_1 c_2}\,.$$

Proof. The uniqueness of Lemma 17 implies that $\mathfrak{T} = \underline{\mathfrak{T}}|_{s_1=-s_2}$. Hence, the coefficients of $\mathfrak{T}$ are polynomials of $s = s_1$ and $t = s_3$. Since

$$c_1|_{s_1=-s_2} = t, \quad c_2|_{s_1=-s_2} = -s^2,$$

and since $\underline{\mathfrak{T}}$ is symmetric with respect to all permutations of s_i, the coefficients of $\mathfrak{T}$ must be polynomial in c_1, c_2. □

3.4. **Poles.** The following pole restriction result will play a crucial role in the proof of Theorem 4 in Section 3.5.

Lemma 20. *The descendent invariants*

$$\left\langle \tau_\mu \middle| \lambda_1, \lambda_2, \lambda_3 \right\rangle_{U,D}^{\mathsf{GW},\mathsf{T}} \quad \textit{and} \quad \left\langle \mathrm{ch}_\mu \middle| \lambda_1, \lambda_2, \lambda_3 \right\rangle_{U,D}^{\mathsf{PT},\mathsf{T}}$$

have no poles along the hyperplane $s_i + s_j = 0$ if either $\lambda_i = \emptyset$ or $\lambda_j = \emptyset$.

Proof. The invariants here are the capped vertices [18, 32]. The stated regularity property for Gromov-Witten invariants follows from the localization formula [10] for the capped vertex [18, section 2]. As explained in [18],

$$\left\langle \tau_\mu \middle| \lambda_1, \lambda_2, \lambda_3 \right\rangle_{U,D}^{\mathsf{GW},\mathsf{T}} = \sum_{\lambda_1', \lambda_2', \lambda_3'} \mathsf{V}_{\mathsf{GW}}(\tau_\mu | \lambda_1', \lambda_2', \lambda_3', u) \cdot H(\lambda_k', s_{k+1}, s_{k+2}, s_k) \cdot \prod_{k=1}^{3} \Psi_{\mathsf{GW}}(\lambda_k, \lambda_k', s_{k+1}, s_{k+2}, -s_k, u)\,, \tag{32}$$

where the partitions in the sum are constrained by $|\lambda'_i| = |\lambda_i|$, the half-edge term H is the edge-term for the local curve theory [4], the term

$$\Psi_{\mathsf{GW}}(\lambda, \mu, s_1, s_2, s_3, u) = \sum_g \left\langle \lambda \left| \frac{1}{s_3 - \psi_\infty} \right| \mu \right\rangle_{g,d}^{\sim\prime} u^{2g-2} ,$$

is the rubber integral, and $\mathsf{V}_{\mathsf{GW}}(\tau_\mu|\lambda_1, \lambda_2, \lambda_3, u)$ is the standard localization vertex [10] in Gromov-Witten theory.

The rubber integral is regular at $s_i + s_j = 0$, the half-edge term is the ratio of the explicit products of the linear expressions of s_i which can be easily checked to be regular at $s_i + s_j = 0$. The only potential source of poles at $s_i + s_j = 0$ is the standard localization vertex $\mathsf{V}_{\mathsf{GW}}(\tau_\mu|\lambda_1, \lambda_2, \lambda_3)$.

The standard Gromov-Witten localization vertex $\mathsf{V}_{\mathsf{GW}}(\tau_\mu|\lambda_1, \lambda_2, \lambda_3)$ is straightforward to analyze directly from the formula of [10]. In fact, the only source of poles at $s_i + s_j = 0$ is the tangent weight of the tangent space the space of smoothing of a nodal rational curve (which occurs in the Euler class of the virtual normal bundle to the T-fixed locus. If we are smoothing a node connecting the rational components with T-weights $\frac{s_i}{d_i}$ and $\frac{s_j}{d_j}$ at the node then the tangent space to the smoothing family is

$$\frac{s_i}{d_i} + \frac{s_j}{d_j} .$$

Since d_i and d_j are the degrees of the images of the corresponding rational components, we have $d_i \le |\lambda_i|$ and $d_j \le |\lambda_j|$. Thus the pole statement follows in the Gromov-Witten case since at least one of λ_i and λ_j are assume to be empty.

The PT case is shown by a computation similar to [17, Section 3.3] where the parallel DT statement is proven. In [18], the formula for

$$\left\langle \mathrm{ch}_\mu \middle| \lambda_1, \lambda_2, \lambda_3 \right\rangle_{U,D}^{\mathsf{PT},\mathsf{T}}$$

analogous to (32) is written. It immediately follows that the only possible source of poles at $s_i + s_j = 0$ is the standard localized vertex $\mathsf{V}_{\mathsf{PT}}(\tau_\mu|\lambda_1, \lambda_2, \lambda_3)$ for PT theory [35]. Thus, we must analyze the poles of

$$\mathsf{V}_{\mathsf{PT}}(\mathrm{ch}_\mu|\lambda_1, \emptyset, \lambda_3)$$

along $s_1 + s_2 = 0$. We will use the rim-hook technique of [15].

Let us recall the basic structure of the standard PT localization vertex from [35]. To a partition λ_i, we attach a monomial ideal $\lambda_i[x_{i-1}, x_{i+1}] \subset \mathbb{C}[x_{i-1}, x_{i+1}]$ and $\mathbb{C}[x_1, x_2, x_3]$-modules

$$M_i = \mathbb{C}[x_i, x_i^{-1}] \otimes \frac{\mathbb{C}[x_{i-1}, x_{i+1}]}{\lambda_i[x_{i-1}, x_{i+1}]}, \qquad M = \bigoplus_{i=1}^{3} M_i .$$

The T-fixed points of the moduli space of stable pairs $P_\bullet(U/D)_{\lambda_1,\lambda_2,\lambda_3}$ correspond to finitely generated T-invariant $\mathbb{C}[x_1, x_2, x_3]$-submodules:

$$Q \subset M/\langle(1,1,1)\rangle .$$

In the case at hand, $\lambda_2 = \emptyset$, so a[35] the T-invariant submodules Q as above form 0-dimensional families [35]. We can choose a monomial basis for each such Q. The combinatorics of the T-weights of a monomial basis of Q is discussed below.

The T-weights of the homogeneous monomials inside M_i form an infinite cylinder

$$\mathrm{Cyl}_i \subset \mathbb{Z}^3 .$$

Since $\lambda_2 = \emptyset$, the cylinder Cyl_2 is empty. Hence, the weights of Q form some subset of the union $\mathrm{Cyl}_1 \cup \mathrm{Cyl}_3$. The union has three types of weights:

$$\mathrm{Cyl}_1 \cup \mathrm{Cyl}_3 = \mathrm{I}^+ \cup \mathrm{II} \cup \mathrm{I}^-,$$

where $\mathrm{II} = \mathrm{Cyl}_1 \cap \mathrm{Cyl}_3$, I^+ consists of the weights that have only non-negative coordinates and lie in exactly one cylinder, and I^- are the rest of the weights.

The submodule Q is uniquely characterized by the associated set of weights $\mathsf{wt}(Q)$. Conversely, a subset $S \subset \mathbb{Z}^3$ is a set of weights of Q corresponding to a T-invariant element of $P_n(X/D)_{\lambda_1,\emptyset,\lambda_3}$ if and only if the following three conditions holds:

(i) $S \subset \mathrm{I}^- \cup \mathrm{II}$

(ii) $w \in S$ if any of the weights

$$(w_1 - 1, w_2, w_3), \quad (w_1, w_2 - 1, w_3), \quad (w_1, w_2, w_3 - 1)$$

are in S.

(iii) $|S| = n$.

Let us call the set of weights as above *geometric.* For given a geometric set of weights Q we introduce the generating functions:

$$\mathsf{F}_0(Q) = \sum_{(ijk)\in Q} s_1^i s_2^j s_3^k + \sum_{(ijk)\in \mathrm{I}^+} s_1^i s_2^j s_3^k,$$

$$\mathsf{F}_{12}(Q) = \sum_{(ij)\in\lambda_3} s_1^i s_2^j, \quad \mathsf{F}_{23}(Q) = \sum_{(ij)\in\lambda_1} s_2^i s_3^j.$$

In [35], the generating function of the redistributed virtual weights of the normal bundle to the corresponding T-fixed point of $P_n(U/D)_{\lambda_1,\emptyset,\lambda_3}$ is defined by:

$$\mathsf{V}_Q = \mathsf{F}_0 - \frac{\overline{\mathsf{F}}_0}{s_1 s_2 s_3} + \mathsf{F}_0 \overline{\mathsf{F}}_0 \frac{(1-s_1)(1-s_2)(1-s_3)}{s_1 s_2 s_3} + \frac{\mathsf{G}_{12}}{1-s_3} + \frac{\mathsf{G}_{23}}{1-s_1},$$

where $\overline{f}(s_1, s_2, s_3) = f(s_1^{-1}, s_2^{-1}, s_3^{-1})$, $\mathsf{F}_0 = \mathsf{F}_0(Q)$, and

$$\mathsf{G}_{ij} = -\mathsf{F}_{ij} - \frac{\overline{\mathsf{F}}_{ij}}{s_i s_j} + \mathsf{F}_{ij}\overline{\mathsf{F}}_{ij}\frac{(1-s_i)(1-s_j)}{s_i s_j},$$

with $\mathsf{F}_{ij} = \mathsf{F}_{ij}(Q)$.

The standard localized vertex $\mathsf{V}_{\mathsf{PT}}(\mathrm{ch}_\mu | \lambda_1, \emptyset, \lambda_3)$ is the sum over all geometric sets of weights Q of the expressions:

$$q^{|Q|} \prod_i \mathrm{ch}_{\mu_i}(\mathsf{F}_0(Q)) \cdot e(-\mathsf{V}_Q).$$

To prove the Lemma, we must analyze the poles of $e(-\mathsf{V}_Q)$, and we follow method of [16] in our argument. The order of the pole at $s_1+s_2=0$ is equal to the constant term of $\mathsf{V}_Q(x,x^{-1},t_3)$. Substituting[9]

$$\mathsf{F}_0 = \underline{\mathsf{F}}_0 + \frac{\mathsf{F}_{23}}{1-s_1}$$

into the the formula for V, we obtain

$$\underline{\mathsf{F}}_0 - \frac{\overline{\underline{\mathsf{F}}}_0}{s_1s_2s_3} + \underline{\mathsf{F}}_0\overline{\underline{\mathsf{F}}}_0\frac{(1-s_1)(1-s_2)(1-s_3)}{s_1s_2s_3} + \frac{\mathsf{G}_{12}}{1-s_3} + \overline{\underline{\mathsf{F}}}_0\mathsf{F}_{23}\frac{(1-s_2)(1-s_3)}{s_1s_2s_3} - \underline{\mathsf{F}}_0\overline{\mathsf{F}}_{23}\frac{(1-s_2)(1-s_3)}{s_2s_3}\,. \tag{33}$$

Since $\underline{\mathsf{F}}_0(x,x^{-1},s_3)$ has only strictly positive powers of x in its expansion and $\mathsf{F}_{23}(x,s_3)$ has only positive powers of x in its expansion, we conclude that the functions

$$\underline{\mathsf{F}}_0(x^{-1},x,s_3^{-1})\mathsf{F}_{23}(x^{-1},s_3)\frac{(1-x^{-1})(1-s_3)}{s_3}, \quad \underline{\mathsf{F}}_0(x,x^{-1},s_3)\mathsf{F}_{23}(x,s_3^{-1})\frac{(1-x^{-1})(1-s_3)}{x^{-1}s_3}$$

have only strictly negative and strictly positive, respectively, powers of x in their expansions. The expression G_{12} is the generating function for the tangent weights $\mathsf{Hilb}_{|\lambda_3|}(\mathbb{C}^2)$ at the corresponding monomial ideal, hence we can use well-known formula for the tangent weights to see that $\mathsf{G}_{12}(x,x^{-1})$ has no constant term.

To finish proof we must bound the constant term of

$$\underline{\mathsf{F}}_0(x,x^{-1},s_3) - \frac{\underline{\mathsf{F}}_0(x^{-1},x,s_3^{-1})}{s_3} + \underline{\mathsf{F}}_0(x,x^{-1},s_3)\underline{\mathsf{F}}_0(x^{-1},x,s_3^{-1})\frac{(1-x)(1-x^{-1})(1-s_3)}{s_3} \tag{34}$$

The function $\underline{\mathsf{F}}_0(x,x^{-1},s_3)$ can be expanded in Laurent power series of s_3, and the coefficients of the expansion are Laurent polynomials in x:

$$\underline{\mathsf{F}}_0(x,x^{-1},s_3) = \sum_{i,j} a_{ij}x^i s_3^j\,.$$

The formal computation of the proof of [16, Lemma 5] determines the constant term of (34) to be

$$-\frac{1}{2}\sum_{i,j}\left((a_{i,j}-a_{i+1,j})-(a_{i,j+1}-a_{i+1,j+1})\right)^2,$$

which is non-positive. □

3.5. **Proof of Theorem 4.** By Theorem 18, we have the correspondence

$$\left\langle \underline{\mathfrak{I}}(\mu)(\mathsf{p})\middle|\lambda_1,\lambda_2,\emptyset\right\rangle^{\mathsf{GW,T}}_{U,D} = \left\langle \mathrm{H}^{\mathsf{PT}}_\mu(\mathsf{p})\middle|\lambda_1,\lambda_2,\emptyset\right\rangle^{\mathsf{PT,T}}_{U,D}\,. \tag{35}$$

Theorem 4 will be derived from equation (35).

[9]For shorter formulas, we now drop Q from the notation.

The element $\widehat{\mathrm{H}}^{\mathsf{GW}}_\mu(\mathsf{p})$ is a linear combination of monomials of $\mathfrak{a}_i(\mathsf{p})$ with coefficients in $\mathbb{C}[c_1, c_2]$. The descendents $\mathrm{H}^{\mathsf{PT}}_\mu(\mathsf{p})$ are the linear combinations of monomials of $\mathrm{ch}_i(\mathsf{p})$ with coefficients in $\mathbb{C}[c_2]$. Lemma 20 therefore implies that for every λ_1, λ_2, the specializations

$$\big\langle \widehat{\mathrm{H}}^{\mathsf{GW}}_\mu(\mathsf{p}) \big| \lambda_1, \lambda_2, \emptyset \big\rangle^{\mathsf{GW},\mathsf{T}}_{U,D} \Big|_{s_3=-s_i} \quad \text{and} \quad \big\langle \mathrm{H}^{\mathsf{PT}}_\mu(\mathsf{p}) \big| \lambda_1, \lambda_2, \emptyset \big\rangle^{\mathsf{GW},\mathsf{T}}_{U,D} \Big|_{s_3=-s_i} \tag{36}$$

are well defined for both $i = 1$ and $i = 2$.

Consider first the $i = 1$ case. By Corollary 19, we have

$$\mathfrak{T} = \underline{\mathfrak{T}}|_{s_3=-s_1}$$

since the specialization $s_3 = -s_1$ implies $c_3 = c_1 c_2$. From (35), we conclude

$$\Big\langle \widehat{\mathrm{H}}^{\mathsf{GW}}_\mu(\mathsf{p}) \,\Big|\, \lambda_1, \lambda_2, \emptyset \Big\rangle^{\mathsf{GW},\mathsf{T}}_{U,D} = q^{-|\lambda_1|-|\lambda_2|} \Big\langle \mathrm{H}^{\mathsf{PT}}_\mu(\mathsf{p}) \,\Big|\, \lambda_1, \lambda_2, \emptyset \Big\rangle^{\mathsf{PT},\mathsf{T}}_{U,D} \mod (s_1 + s_3)\,.$$

By considering the $i = 2$ case, we obtain the above equality mod $(s_2 + s_3)$ also. □

3.6. **Proof of Theorem 5.** Theorem 5 follows almost immediately from the following reformulation of [32, Theorem 7]. We define

$$\widetilde{\underline{\mathrm{H}}}_\mu = \frac{1}{(c_3)^{l-1}} \sum_{\text{set partitions } P \text{ of} \{1,\ldots,l\}} (-1)^{|P|-1}(|P|-1)! \prod_{S \in P} \underline{\mathfrak{T}}(\mathrm{H}^{\mathsf{PT}}_{\mu_S})\,.$$

For classes $\gamma_i \in H^*(X)$ and a vector $\vec{k}$ of non-negative integers, we define

$$\overline{\underline{\mathrm{H}}_{k_1}(\gamma_1) \ldots \underline{\mathrm{H}}_{k_l}(\gamma_l)} = \sum_{\text{set partitions } P \text{ of} \{1,\ldots,l\}} \prod_{S \in P} \widetilde{\underline{\mathrm{H}}}_{\vec{k}_S}(\gamma_S),$$

where $\gamma_S = \prod_{i \in S} \gamma_i$.

Theorem 21. [32] *Let X be a nonsingular projective toric 3-fold, and let $\gamma_i \in H^*(X, \mathbb{C})$. After the change of variables $-q = e^{iu}$, we have*

$$\big\langle \overline{\underline{\mathrm{H}}_{k_1}(\gamma_1) \ldots \underline{\mathrm{H}}_{k_l}(\gamma_l)} \big\rangle^{\mathsf{GW}}_\beta = \big\langle \mathrm{H}^{\mathsf{PT}}_{k_1}(\gamma_1) \ldots \mathrm{H}^{\mathsf{PT}}_{k_l}(\gamma_l) \big\rangle^{\mathsf{PT}}_\beta\,,$$

where the non-equivariant limit is taken on both sides.

Theorem 5 follows because

$$\widetilde{\mathrm{H}}_\mu = \widetilde{\underline{\mathrm{H}}}_\mu|_{c_3=c_1c_2}$$

and the restriction $c_3 = c_1 c_2$ does not affect the non-equivariant limit if all γ_i have positive cohomological degree. □

3.7. **Examples for $X = \mathbf{P}^3$.** The prefactor in front of $\sum_{k=0}^{\infty} x^k \mathrm{ch}_k(\mathbb{F})$ in the definition of $\mathrm{H}^{\mathsf{PT}}(x)$ in Section 0.6 has an expansion that starts as:

$$\mathcal{S}\left(\frac{x}{\theta}\right) = 1 - \frac{c_2}{24}x^2 + \frac{c_2^2}{1920}x^4 - \frac{c_2^3}{322560}x^6 + \dots .$$

In particular, the non-equivariant limit of $\mathrm{H}_k^{\mathsf{PT}}(\gamma)$ is equal to

$$\mathrm{ch}_{k+1}(\gamma) - \frac{1}{24}\mathrm{ch}_{k-1}(\gamma \cdot c_2)\,.$$

On the Gromov-Witten side of the correspondence, we have

$$\begin{aligned}
\big\langle \mathrm{H}_1^{\mathsf{GW}}(\gamma)\Phi \big\rangle &= \big\langle \mathfrak{a}_1(\gamma)\Phi \big\rangle\,, \\
\big\langle \mathrm{H}_2^{\mathsf{GW}}(\gamma)\Phi \big\rangle &= \frac{1}{2}\big\langle \mathfrak{a}_2(\gamma)\Phi \big\rangle\,, \\
\big\langle \mathrm{H}_3^{\mathsf{GW}}(\gamma)\Phi \big\rangle &= \frac{1}{6}\big\langle \mathfrak{a}_3(\gamma)\Phi \big\rangle + \frac{1}{24u^2}\big\langle c_1^2 c_2 \cdot \Phi \big\rangle\,, \\
\big\langle \mathrm{H}_4^{\mathsf{GW}}(\gamma)\Phi \big\rangle &= \frac{1}{24}\big\langle \mathfrak{a}_4(\gamma)\Phi \big\rangle - \frac{i}{12u}\big\langle \mathfrak{a}_1^2(c_1\cdot\gamma)\Phi \big\rangle - \frac{5i}{144u^3}\big\langle c_1^3 c_2 \cdot \Phi \big\rangle\,, \\
\big\langle \mathrm{H}_5^{\mathsf{GW}}(\gamma)\Phi \big\rangle &= \frac{1}{120}\big\langle \mathfrak{a}_5(\gamma)\Phi \big\rangle - \frac{i}{24u}\big\langle \mathfrak{a}_1\mathfrak{a}_2(c_1\cdot\gamma)\Phi \big\rangle - \frac{1}{48u^2}\big\langle \mathfrak{a}_1^2(c_1^2\cdot\gamma)\Phi \big\rangle \\
&\quad + \frac{1}{24u^2}\big\langle \mathfrak{a}_1(c_1^2c_2\cdot\gamma)\Phi \big\rangle - \frac{1}{64u^4}\big\langle c_1^4 c_2 \cdot \Phi \big\rangle\,.
\end{aligned}$$

The operators $\mathfrak{a}_k$ are expressed in terms of standard descendents by inverting (5):

$$\begin{aligned}
\mathfrak{a}_1 &= \tau_0\,, \\
\frac{iu}{2}\mathfrak{a}_2 &= \tau_1 + c_1\cdot\tau_0\,, \\
-\frac{u^2}{3}\mathfrak{a}_3 &= 2\tau_2 + 3c_1\cdot\tau_1 + c_1^2\cdot\tau_0\,, \\
-\frac{iu^3}{4}\mathfrak{a}_4 &= 6\tau_3 + 11c_1\cdot\tau_2 + 6c_1^2\tau_1 + c_1^3\cdot\tau_0\,, \\
\frac{u^4}{5}\mathfrak{a}_5 &= 24\tau_4 + 50c_1\cdot\tau_3 + 35c_1^2\cdot\tau_2 + 10c_1^3\cdot\tau_1 + c_1^4\cdot\tau_0\,.
\end{aligned} \tag{37}$$

In particular, the GW/PT correspondence of Theorem 5 gives the following relations for the degree 1 invariants of $\mathbf{P}^3$:

$$\begin{aligned}
iq^{-2}\big\langle \mathrm{ch}_5(\mathsf{L})\big\rangle_1^{\mathsf{PT}} &= \frac{1}{u^3}\big\langle \tau_3(\mathsf{L})\big\rangle_1^{\mathsf{GW}} + \frac{22}{3u^3}\big\langle \tau_2(\mathsf{p})\big\rangle_1^{\mathsf{GW}} - \frac{1}{3u}\big\langle \tau_0\tau_0(\mathsf{p})\big\rangle_1^{\mathsf{GW}}\,, \\
-q^{-2}\big\langle \mathrm{ch}_6(\mathsf{H})\big\rangle_1^{\mathsf{PT}} + -\frac{q^{-2}}{4}\big\langle \mathrm{ch}_4(\mathsf{p})\big\rangle_1^{\mathsf{PT}} &= \frac{1}{u^4}\big\langle \tau_4(\mathsf{H})\big\rangle_1^{\mathsf{GW}} + \frac{25}{3u^4}\big\langle \tau_3(\mathsf{L})\big\rangle_1^{\mathsf{GW}} + \frac{70}{3u^4}\big\langle \tau_2(\mathsf{p})\big\rangle_1^{\mathsf{GW}} \\
&\quad - \frac{1}{3u^2}\big\langle \tau_0\tau_1(\mathsf{L})\big\rangle_1^{\mathsf{GW}} + \frac{5}{3u^2}\big\langle \tau_0\tau_0(\mathsf{p})\big\rangle_1^{\mathsf{GW}}\,.
\end{aligned} \tag{38}$$

Here, and below in Section 4,

$$\mathsf{p}\,,\ \mathsf{L}\,,\ \mathsf{H}\ \in H^*(\mathbf{P}^3)$$

are respectively the classes of a point, a line, and a plane. These formulas can verified numerically up to u^8 with the help of Gathmann's Gromov-Witten code and previously known complete calculations on the stable pairs side [28].

4. Concluding remarks: DT/PT/GW

4.1. **Stationary DT/PT correspondence.** The moduli space $I_n(X,\beta)$ parameterizes flat families of ideal sheaves $\mathcal{I} \subset \mathcal{O}_X$ with

$$\chi(\mathcal{I}) = n, \quad [\mathrm{Supp}(\mathcal{O}_X/\mathcal{I})] = \beta \in H_2(X,\mathbb{Z})\,.$$

There is a universal quotient sheaf $\mathbb{F}_n$ over $X \times I_n(X,\beta)$ with fibers

$$\mathbb{F}_n|_{\mathcal{I}\times X} = \mathcal{O}_X/\mathcal{I}\,.$$

We define

$$\mathrm{ch}_k(\gamma) = \pi_*\left(\mathrm{ch}_k(\mathbb{F})\cdot\gamma\right) \in \bigoplus_{n\in\mathbb{Z}} H^*(I_n(X,\beta)) \quad \text{for} \quad \gamma \in H^*(X)\,.$$

The moduli space $I_n(X,\beta)$ has a natural virtual cycle $[I_n(X,\beta)]^{vir}$. The integrals of the above descendents classes define generating series,

$$\left\langle \mathrm{ch}_{k_1}(\gamma_1)\dots\mathrm{ch}_{k_m}(\gamma_m)\right\rangle_\beta^{\mathsf{DT}} = \sum_{n\in\mathbb{Z}} q^n \int_{[I_n(X,\beta)]^{vir}} \mathrm{ch}_{k_1}(\gamma_1)\dots\mathrm{ch}_{k_m}(\gamma_m)\,,$$

just as for stable pairs. The normalized generating series of DT invariants have better properties:

$$\left\langle \mathrm{ch}_{k_1}(\gamma_1)\dots\mathrm{ch}_{k_m}(\gamma_m)\right\rangle_\beta^{\mathsf{DT}'} = \left\langle \mathrm{ch}_{k_1}(\gamma_1)\dots\mathrm{ch}_{k_m}(\gamma_m)\right\rangle_\beta^{\mathsf{DT}} / \langle 1\rangle_0^{\mathsf{DT}}\,. \tag{39}$$

We refer the reader to [15, 16] for a more detailed introduction.

The argument of [32] is valid if we replace the Gromov-Witten side by the DT theory of ideal sheaves. Since the 1-leg invariants in DT and PT theories are identical modulo s_1+s_2, our proof of Theorem 5 can be repeated to obtain the following non-equivariant result.

Theorem 22. *Let X be a nonsingular projective toric 3-fold, and let $\gamma_i \in H^{\geq 2}(X,\mathbb{C})$. The stationary descendent* DT/PT *correspondence holds:*

$$\left\langle \mathrm{ch}_{k_1}(\gamma_1)\dots\mathrm{ch}_{k_l}(\gamma_l)\right\rangle_\beta^{\mathsf{DT}'} = \left\langle \mathrm{ch}_{k_1}(\gamma_1)\dots\mathrm{ch}_{k_l}(\gamma_l)\right\rangle_\beta^{\mathsf{PT}}\,.$$

Finding a relation between DT and PT theories which includes the descendents of the identity class 1 is more subtle. A basic source of difficulty is that the DT generating series (39) are *not* always rational functions. We expect the non-equivariant DT descendent series to depend upon the function F_3,

$$F_3(q) = \sum_{i=1}^{\infty} n^2 \frac{q^n}{1-q^n}\,,$$

which arises as the logarithmic derivative of the McMahon function.

Conjecture 23. *Let X be a nonsingular projective 3-fold. For $\gamma_i \in H^*(X,\mathbb{C})$, the series*

$$\left\langle \mathrm{ch}_{k_1}(\gamma_1)\dots\mathrm{ch}_{k_m}(\gamma_m)\right\rangle_\beta^{\mathsf{DT}'}$$

is a polynomial in $(q\frac{d}{dq})^i F_3(-q)$ for $0 \le i \le m$ with coefficients in the ring of rational functions of q.

4.2. **Beyond the stationary case.** The GW/PT correspondence for the complete non-equivariant descendent theory is expected to be significantly more complex than the stationary case because of the analytic properties of the GW descendent series. In fact, we expect the analytic complexities of the GW descendent series to be very similar to those of the DT descendent series.

The Euler-Maclaurin formula provides an asymptotic expansion of $F_3(-q)$ at $u=0$:

$$F_3(u) \sim 2\zeta(3)/u^3 - \sum_{n=0}^{\infty} \frac{B_{2n+2}B_{2n}}{(2n)!(2n+2)}(iu)^{2n-1}\,.$$

For simplicity, we will use the notation $F_3(u)$ for the part of the expansion without the most singular term. In order words, we define

$$F_3(u) = -\sum_{n=0}^{\infty} \frac{B_{2n+2}B_{2n}}{(2n)!(2n+2)}(iu)^{2n-1}\,.$$

Let us denote by $\mathcal{R}$ the ring of rational functions of $q=-e^{iu}$. The following is a Gromov-Witten version of Conjecture 23.

Conjecture 24. *Let X be a nonsingular projective 3-fold. For $\gamma_i \in H^*(X,\mathbb{C})$, the series*

$$\left\langle \tau_{k_1}(\gamma_1)\dots\tau_{k_m}(\gamma_m)\right\rangle_\beta^{\mathsf{GW}}$$

is a polynomial in $(\frac{d}{du})^i F_3(u)$ for $0 \le i \le m$ with coefficients in the ring $\mathcal{R}[u^\pm]$.

The power series $F_3(q)$ does not converge at any point of a circle $|q|=1$. Hence, the q-derivatives of $F_3(q)$ are linearly independent over $\mathcal{R}$. Otherwise $F_3(q)$ would be a solution of a non-trivial linear differential equation with rational coefficients and hence analytic outside finite number of points. We can therefore define a homomorphism

$$\Theta : \mathcal{R}\left[u^{\pm 1}, F_3(-q), q\frac{d}{dq}F_3(-q), \dots\right] \to \mathbb{Q}[u^{-1}][[u]].$$

On the subring $\mathcal{R}[u^{\pm 1}]$, the homomorphism Θ is defined by the change of variable $e^{iu}=-q$, and, on the generators $(q\frac{d}{dq})^m F_3(-q)$, Θ is defined by:

$$\left(q\frac{d}{dq}\right)^m F_3(-q) \mapsto \frac{1}{2}\left(-i\frac{d}{du}\right)^m F_3(u)\,.$$

Note the factor of $\frac{1}{2}$ in the last formula: the homomorphism Θ is *not* merely a change of variable.

Conjecture 25. *We have*

$$\Theta\left(\Big\langle \prod_i \mathrm{H}^{\mathsf{DT}}_{k_i}(\mathsf{p}) \Big| \mu_1,\mu_2,\mu_3 \Big\rangle^{\mathsf{DT}'}_{U,D}\right) = \Big\langle \prod_i \mathrm{H}^{\mathsf{GW}}_{k_i}(\mathsf{p}) \Big| \mu_1,\mu_2,\mu_3 \Big\rangle^{\mathsf{GW}}_{U,D} \mod (c_1c_2-c_3)^2.$$

If Conjecture 25 were true, then we could also write a conjecture for the complete non-equivariant GW/DT descendent correspondence via the formulas of Section 0.8. The appearance of $\frac{1}{2}$ in the definition of Θ could be motivated by the computation of degree 0 GW and DT invariants [15]. Though Conjecture 25 is mysterious, the claimed equality has been supported by a large number of numerical experiments.

4.3. **Equivariant DT series.** Studying the GW/DT descendent correspondence in the general T-equivariant setting is difficult for many reasons.[10] A major unresolved question concerns the analytic properties of the generating series for T-equivariant Gromov-Witten descendent invariants. However, based on computer experiment, we propose conjectures controlling the behavior of the T-equivariant DT descendent series.

Let X be a nonsingular projective toric 3-fold equipped with an action of the 3 dimensional torus T. As in Section 0.7, let

$$H_{\mathsf{T}}(\bullet) = \mathbb{C}[s_1,s_2,s_3]\,.$$

We define the algebra $\mathfrak{Fr}$ generated by the series[11]

$$F_{2k+1}(-q) = \sum_{n=0}^{\infty} (-q)^n \sum_{d|n} d^{2k}\,, \quad k \geq 1\,,$$

[12]and their iterated $q\frac{d}{dq}$ derivatives.

Conjecture 26. *The* T*-equivariant* DT *descendent series of* X *satisfy*

$$\big\langle \mathrm{ch}_{k_1}(\gamma_1)\dots\mathrm{ch}_{k_l}(\gamma_l)\big\rangle^{\mathsf{DT},\mathsf{T}}_{\beta} \in H^*_{\mathsf{T}}(\bullet)\otimes\mathbb{Q}(q)\otimes\mathfrak{Fr}$$

for $\gamma_i \in H^*_{\mathsf{T}}(X,\mathbb{C})$ *and* $\beta \in H_2(X,\mathbb{Z})$.

Conjecture 26 fits into a web of conjectures about the analytic behavior of generating functions of equivariant integrals of tautological classes over moduli spaces of sheaves [23]. We refer the reader to [23] for more motivation, further conjectures, and future directions.

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A. Oblomkov, Department of Mathematics and Statistics, Univ. of Massachusetts, Amherst
E-mail address: `oblomkov@math.umass.edu`

A. Okounkov, Department of Mathematics, Columbia University, New York
Institute for Problems of Information Transmission, Moscow
Laboratory of Representation Theory and Mathematical Physics, HSE, Moscow
E-mail address: `okounkov@math.columbia.edu`

R. Pandharipande, Department of Mathematics, ETH Zürich
E-mail address: `rahul@math.ethz.ch`